Solving Relativistic Three-Body Integral Equations in the Presence of Bound States and Resonances

Taylor R. Powell[1,2], Raúl A. Briceño[1,2], Andrew W. Jackura[1,2]

[1]Thomas Jefferson National Accelerator Facility, [2]Old Dominion University

Abstract

Three-body interactions play an important role throughout modern-day particle, nuclear, and hadronic physics; many experimentally observed reactions of interest for testing the Standard Model result in final states composed of three particles or more. Due to these issues, a full description of three-body interactions from Quantum Chromodynamics is required. The focus of this project was to extend previous results for a two-body subsystem with a bound state to include resonance channels. We first derived a novel single-variable observable, denoted as an intensity distribution, which is proportional to the probability density of the three-body scattering amplitude. We explored this distribution in the context of established results for a two-body subsystem with a bound state. We then developed a model two-body scattering amplitude with both a resonant and a bound state and examined the three-body scattering intensity distribution for this system. For each of these two-body scattering subsystem models, intensity distributions were computed, resulting in novel graphs of relevant scattering behavior.

Integral Equation

The three-body scattering amplitude is taken as in Reference [1] as:

$$\mathcal{M}_3(\mathbf{p},\mathbf{k}) = \mathcal{D}(\mathbf{p},\mathbf{k}) + \mathcal{M}_{df,3}(\mathbf{p},\mathbf{k})$$

Where $\mathcal{D}$ represents the sum over all possible pairwise interactions between the three bodies with one-particle exchanges (Shown in Figure 1), $\mathcal{M}_{df,3}$ represents short-distance three-particle interactions, and (**p**,**k**) are momenta of one of the hadrons in the initial and final states respectively. In this work, the limit as $\mathcal{M}_{df,3} \to 0$ is taken, and $\mathcal{D}$ takes the form:

$$\mathcal{D}(\mathbf{p},\mathbf{k}) = -\mathcal{M}_2(p)G(\mathbf{p},\mathbf{k})\mathcal{M}_2(k) - \mathcal{M}_2(p)\int \frac{d^3\mathbf{k}'}{(2\pi)^3 2\omega_k} G(\mathbf{p},\mathbf{k}')\mathcal{D}(\mathbf{k}',\mathbf{k})$$

Where $\mathcal{M}_2$ is the two-body scattering amplitude and G is the exchange propagator. This integral equation is shown diagrammatically in Figure 1.

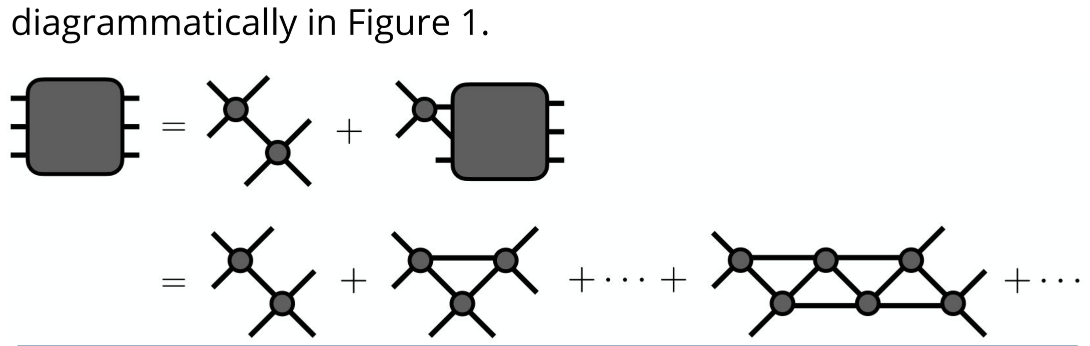


Figure 1: Diagrammatic representation of the three body scattering amplitude with no short-range three body forces as a ladder diagram.

In order to reduce pole singularities associated with the two-body amplitudes with bound states, we define an 'amputated' amplitude by:

$$\mathcal{D}(\mathbf{p},\mathbf{k}) \equiv \mathcal{M}_2(p)d(\mathbf{p},\mathbf{k})\mathcal{M}_2(k)$$

In the present case, we partial-wave project over the total angular momentum, $\mathbf{J}$, and take the S-wave component ($\mathbf{J}=0$). Finally, we discretize in terms of the initial and final momenta and solve for the amputated amplitude in the form of a matrix equation:

$$d_S = -[B^{-1}G_S]$$

$$B_{nn'} = \delta_{nn'} + \frac{\Delta k' \, k_{n'}^2}{(2\pi)^2 \omega_{k_{n'}}} G_S(k_{n'},k_n)\mathcal{M}_2(k_{n'})$$

Two-Body Amplitudes

In our analysis, we assume the three-body interactions are exchange dominated with no short-range three-particle forces. Therefore it is important to model two-body scattering amplitudes and understand their impact on the overall three-body scattering amplitudes. In this work, we primarily examine two models for two-body scattering: a bound state and a combined bound state and resonance channel.

Bound State

In the bound state case, we take the model presented in Reference [1] as:

$$\mathcal{M}_{2,BS}(E_2) = \frac{16\pi E_2}{-1/a - iq^*_{2k}(E_2)}$$

Where E_2 is the energy of the two-particle subsystem in the center-of-momentum frame, $q^*_{2k}(E_2)$ is the relative momentum between the two particles, and a is the scattering length. We show two specific cases below for $ma = \{2, 16\}$ in Figure 2:

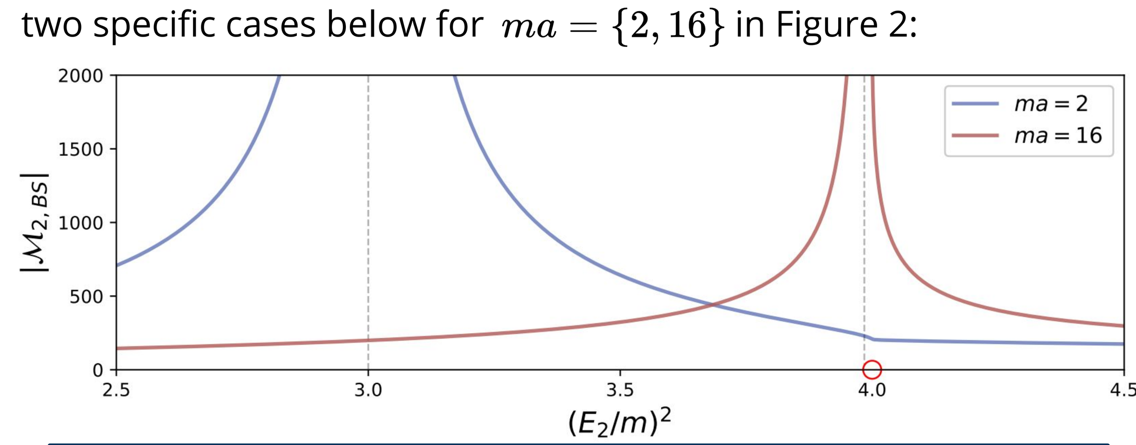


Figure 2: Magnitude of two-body scattering amplitude containing a single bound state, indicated by a pole singularity shown at the dotted line. The red circle indicates the two-body energetic threshold.

In each case, the location of the bound state is indicated by a dashed vertical line, which appears as a pole singularity in the two-body amplitude. Here, $ma = 2$ is a deeply bound state since the pole is located further from the two-particle threshold indicated by the red circle on the x-axis whereas $ma = 16$ is a shallow bound state.

Combined Bound State and Resonance Channel

For the combined case, we developed a model two-particle amplitude with a deeply bound state along with a resonance channel defined by:

$$\mathcal{M}_{2,RS+BS}(E_2) = \frac{\mathcal{K}_2(E_2)}{1 - i\rho\mathcal{K}_2(E_2)} \qquad \mathcal{K}_2(E) = \frac{g_1^2}{m_1^2 - E^2} + \frac{g_2^2}{m_2^2 - E^2}$$

Where ρ is the phase space factor and $\mathcal{K}_2$ is the K-matrix. Inside the K-matrix definition, the first term roughly corresponds to the bound state pole and the second term roughly corresponds to the resonance peak. Below is a graph of a sample two-body amplitude for an appropriate choice of parameters:

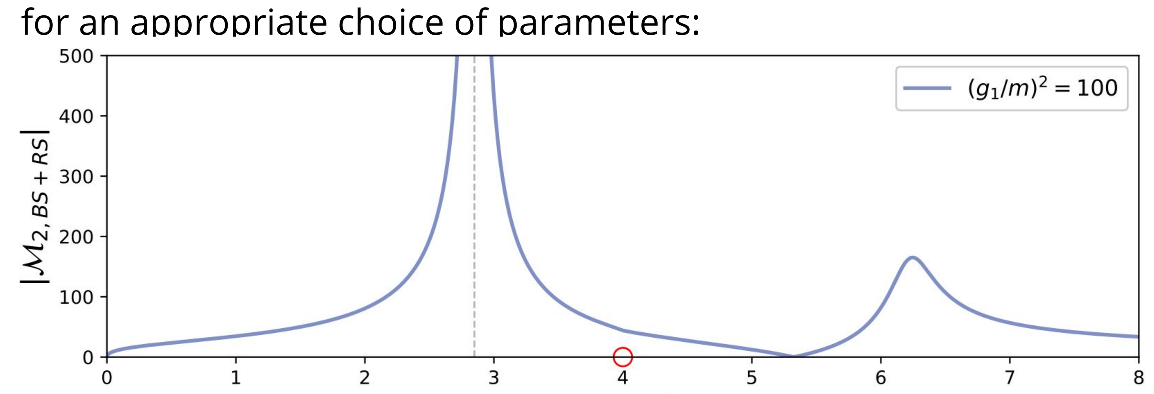


Figure 3: Magnitude of two-body scattering amplitude containing a single bound state, indicated by a pole singularity shown at the dotted line, as well as a resonance peak. The red circle indicates the two-body energetic threshold

Three-Body Intensity Distribution

Bound State

For the bound state, we examine each of the intensity distributions in turn. From Figure 4(a), the individual contributions from each of the components of the intensity distribution are seen for the deeply bound state. In particular, the contributions from the bound state to bound state scattering remain significant to high energies and die off slowly. In contrast, Figure 4(b), shows the contributions from the bound state to bound state scattering is much less relevant and the three-particle to three-particle contribution dominates.

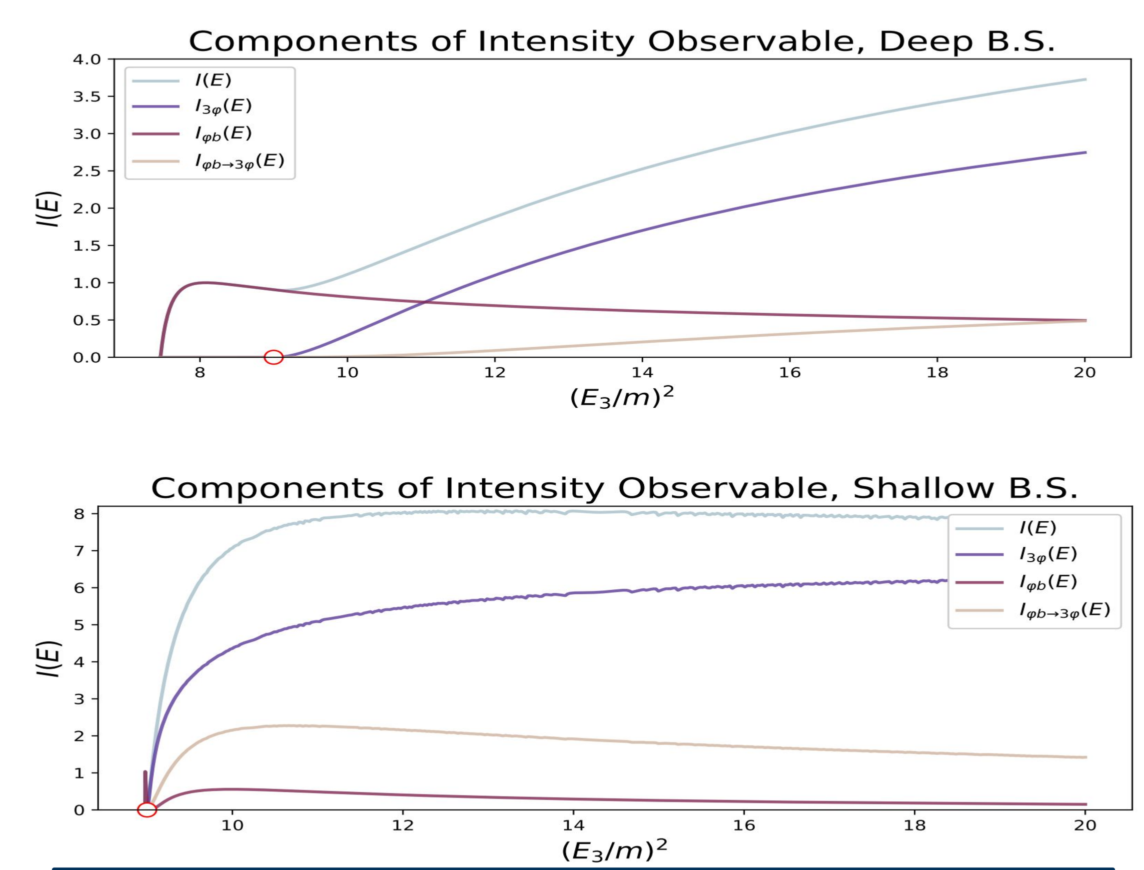


Figure 4: (a) Components of ***I(E)*** for ***ma=2***. (b) Components of ***I(E)*** for ***ma=16***. The red circle indicates the three-body energetic threshold.

Combined Bound State and Resonance Channel

Shown in Figure 5 is the intensity distribution for the combined two-particle amplitude. For low three-particle energies, the behavior closely resembles that of the previous case with a deeply bound state. However, once sufficient energy exists in the two-particle subsystems to interact with the resonance region, a large resonance peak emerges in the total intensity distribution. In addition, another bump in the distribution emerges in the region near $12 < (E_3/m)^2 < 13$ (indicated by a red arrow) which is not directly due to a resonance or bound state. Instead, this bump in the intensity distribution is primarily due to the underlying exchange interactions between the bound state and three-particle scattering (and vice-versa).

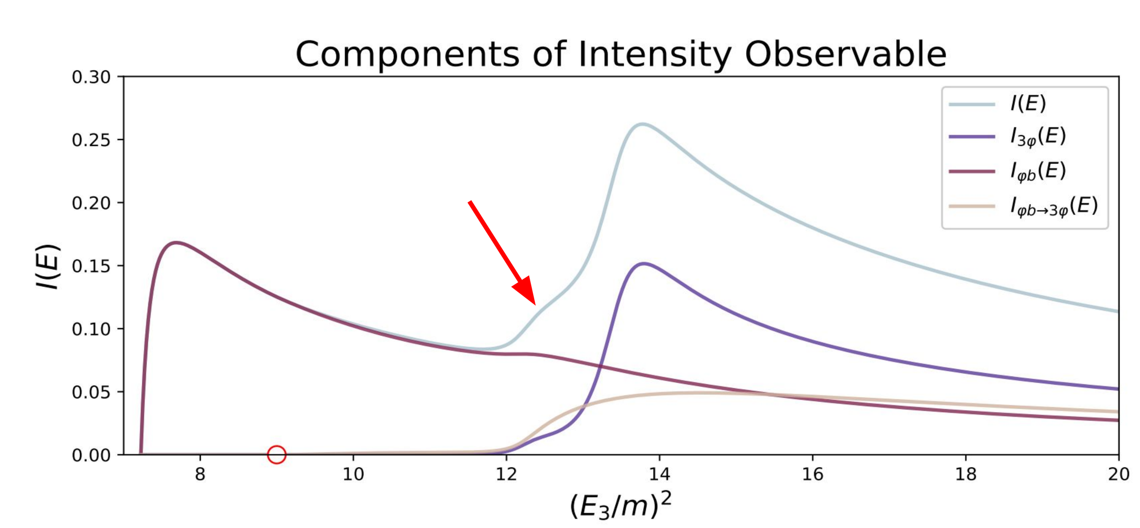


Figure 5: Components of ***I(E)*** for the combined two particle amplitude. The red arrow indicates a 'bump' in the overall intensity profile due to exchange interactions. The red circle indicates the three-body energetic threshold.

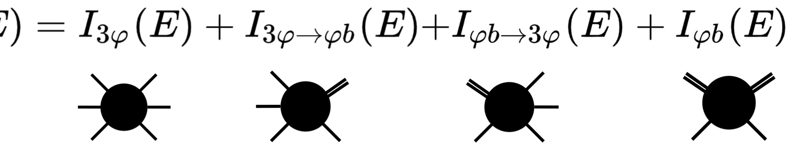
Intensity Observable

In order to conduct a single-variable study of various three-particle scattering amplitudes, each of which are complex, multivariable functions, we derive an observable which is proportional to the three-body scattering amplitude probability density:

$$\text{Prob} \propto \int_0^{(E-m)^2} \frac{k^2\,dk}{(2\pi)^2\omega_k}\rho(k) \int_0^{(E-m)^2} \frac{p^2\,dp}{(2\pi)^2\omega_p}\rho(p)\left|\mathcal{M}_3(p,E,k)\right|^2$$

Which is denoted as an intensity distribution. In the presence of a bound state, unitarity conditions for two-body scattering are used to expand the intensity distribution to include contributions from the pole singularity to give:

$$I(E) = I_{3\varphi}(E) + I_{3\varphi\to\varphi b}(E) + I_{\varphi b\to 3\varphi}(E) + I_{\varphi b}(E)$$



Where the first term represents a scattering event where the initial and final states are three independent particles, the second and third terms represent an exchange between an initial state with three-particles and a final state with a bound state + spectator and vice versa, and the final term represents initial and final states with a bound state + spectator.

This observable is then computed across various three-body energies for each two-body scattering amplitude in order to generate intensity distributions.

References

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Contact

Taylor R. Powell
Email: trpowe1994@gmail.com

Jefferson Lab
Thomas Jefferson National Accelerator Facility

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