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Secrecy Rates and Optimal Power Allocation for Full-Duplex Decode-and-Forward Relay Wire-Tap Channels

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ABSTRACT This paper investigates the secrecy rates and optimal power allocation schemes for a decode-and-forward wiretap relay channel where the transmission from a source to a destination is aided by a relay operating in a full-duplex (FD) mode under practical residual self-interference. By first considering static channels, we address the non-convex optimal power allocation problems between the source and relay nodes under individual and joint power constraints to establish closed-form solutions. An asymptotic analysis is then given to provide important insights on the derived power allocation solutions. Specifically, by using the method of dominant balance, it is demonstrated that full power at the relay is only optimal when the power at the relay is sufficiently smaller when compared with that of the source. When the power at the relay is larger than the power at the source, the power consumed at the relay saturates to a constant for an effective control of self-interference. The analysis is also helpful to demonstrate that the secrecy capacity of the FD system is twice as much as that of the half-duplex system. The extension to fast fading channels with channel state information being available at the receivers but not the transmitters is also studied. To this end, we first establish a closed-form expression of the ergodic secrecy rate using simple exponential integrals for a given power allocation scheme. The results also show that with optimal power allocation schemes, FD can significantly improve the secrecy rate in fast fading environments.

INDEX TERMS Decode-and-forward, full-duplex relaying, residual self-interference, physical-layer security, optimal power allocation.

I. INTRODUCTION

Cooperative relaying has been considered as an effective method to enhance the transmission security in wireless networks under the context of Physical Layer (PHY) [2]. Under this framework, most of the works consider the use of a trusted relay operating in half-duplex (HD) mode, i.e., the relay cannot transmit and receive at the same time in a single channel. Given the recent development of several encouraging full-duplex (FD) radio front-ends [3]–[6], the capability of a FD relay in transmitting and receiving simultaneously to further enhance the secrecy has also gained attention in the literature. For instance, FD relaying has been exploited in [7] and [8] to send jamming signals to the eavesdropper while forwarding information to the destination. However, one of the main drawbacks of these studies is the assumption of significant suppression of self-interference in FD operation. In a realistic FD operation, residual self-interference needs to be explicitly taken into account.

Besides the capability of receiving data and sending jamming signals as in [7] and [8], a trusted FD relay can also send and receive data coherently. In this line of research, the work in [9] and [10] has examined the secrecy capacity and the respective optimal power allocation schemes for a relay wire-tap channel using an amplify-and-forward (AF) FD relay. Interestingly, it was shown in [10] that while FD relaying significantly outperforms HD relaying, full power allocation (PA) at the FD relay might not always be needed.

In this work, we extend the results in [9] and [10] by investigating the optimal power allocation schemes at the source and relay nodes for a FD relay wiretap
channel where the relay is assumed to operate in decode-and-forward (DF) mode. Both static, and ergodic fading channels are considered. While AF and DF relaying strategies have been considered as the two most popular relaying schemes [11], there exist fundamental differences between these two relaying strategies [12]. Therefore, the extension from AF in [9] and [10] to DF in this work under the framework of PHY security and FD relaying is certainly not trivial and it poses different challenges. The main contributions of the paper are summarized as follows:

- By first focusing on the static channels, we establish the optimal power allocation schemes under both individual and joint power constraints for the considered relay wiretap channel. We adopt the practical self-interference model established from experimental results in [13] to take into account the effect of residual self-interference in DF operation. While the optimization problems are non-convex, we show that the closed-form solutions can still be established.

- The second contribution of the paper lies in the asymptotic analysis on the derived optimal power allocation scheme to shed important insights on the obtained solutions. Specifically, using the method of dominant balance, it is shown that full power at the relay is needed only when the power at relay is sufficiently smaller when compared to that of the source. Furthermore, when the power at the relay is larger than the power at the source, the power consumed at the relay saturates to a constant for an effective control of the self-interference. The analysis is also helpful to demonstrate the superiority of FD DF relaying over HD DF relaying as well as FD AF relaying.

- By further considering ergodic fading, we derive closed-form solutions of the ergodic secrecy rates in terms of the well-known exponential integral functions by calculating the expectation of an exponentially distributed random variable. Numerical results show that the closed-form solutions can be used to accurately calculate the ergodic secrecy rates without the need of lengthy Monte Carlo simulations. The closed-form solutions allow us to obtain the optimal power allocation scheme to further improve the secrecy rates in fast fading environments.

The remainder of the paper is organized as follows. Section II describes the considered relay channels and formulates the secrecy rates. The optimal power allocation and asymptotic analysis for the static channels are given in Section III. In Section IV, the results are extended to ergodic fading channels. Numerical results are then presented in Section V to confirm the analysis. Finally, Section VI concludes the paper.

II. SYSTEM MODEL AND SECRECY RATES

The considered relay wiretap model is shown in Fig. 1. In this model, we have a HD source node $S$, a HD destination $D$, a HD eavesdropper $E$, and the transmission from the source to the destination is aided by a DF relay $R$ that uses DF. At a given time $i$, $S$ transmits the signal $x_i$ to the relay $R$. At the relay, the received signal is given by

$$ r_i = \sqrt{P_i} h_1 x_i + n_{r,i} + v_i. $$

In (1), $P_i$ is a constant associated with the power transmitted at $S$ and $h_1$ is the channel gain for the S-R link. In addition, $n_{r,i}$ is the zero-mean circularly Gaussian noise at $R$, which is denoted as $n_{r,i} \sim CN(0, N_r)$, while $v_i$ is the residual self-interference resulting from the FD operation at the relay. While receiving $x_i$, the relay can decode the signal it received at time $i-1$ before extracting the data and retransmitting this information to the destination. Let $\hat{x}_{i-1}$ be the signal forwarded by $R$ at time $i$. The signals received at $D$ and overheard at $E$ can then be written respectively as:

$$ y_{d,i} = \sqrt{P_i} h_3 \hat{x}_{i-1} + n_{d,i}, $$

$$ y_{e,i} = \sqrt{P_i} h_4 \hat{x}_{i-1} + n_{e,i}. $$

Here, $P_i$ is a constant related to the power consumed at $R$. Furthermore, $h_3$ and $h_4$ are channel gains of the $R$-$D$ and $R$-$E$ links, respectively, and $n_{d,i} \sim CN(0, N_d)$ and $n_{e,i} \sim CN(0, N_e)$ are the thermal noises at $D$ and $E$.

A. STATIC CHANNELS

For static channels, that the gains $h = [h_1, h_3, h_4]$ change very slowly and can be considered as constant. Furthermore, as in [14]–[18], these channel gains are assumed to be available at all nodes, including $E$. This assumption corresponds to the case where the eavesdropper is a lower-level user belonging to the same legitimate network. Without loss of generality, we also assume that the thermal noise levels are the same at all nodes, i.e., $N_r = N_e = N_d = N_0$. Now, let $\mathbb{E}[|x|^2] = q_1$ and $\mathbb{E}[|\hat{x}^2|] = q_2$, i.e., $S$ and $R$ use an average power $q_1 P_i$ and $q_2 P_i$, respectively. Additionally, we adopt the model in [13] and [19] so that the residual self-interference is given by $v_i \sim CN(0, \lambda)$ and the variance $V = \beta(q_2 P_i)^{2\lambda}$, where $\beta$ and $\lambda(0 \leq \lambda \leq 1)$ are constants that depend on the self-interference cancellation techniques [13], [19].

In DF relaying, the achievable rate at the destination is bounded by the capacities of the two links: from the source to the relay and from the relay to the destination [20], [21], so that a reliable decoding at $R$ and $D$ can be guaranteed. As such, the maximum end-to-end achievable
rates at $D$ and $E$, respectively, can be written as:

$$I_d = \log \left(1 + \min \left(\frac{q_1 y_1}{1 + q_2^2 y_2}, q_2 y_3\right)\right),$$

$$I_e = \log \left(1 + \min \left(\frac{q_1 y_1}{1 + q_2^2 y_2}, q_2 y_4\right)\right),$$

(4)

where $y_1 = \alpha_1 P_s / N_0$, $y_2 = \beta P_r / N_0$, $y_3 = \alpha_3 P_r / N_0$, $y_4 = \alpha_1 P_s / N_0$ and $y_d = |h|^2$. It is important to note from (4) that the $(q_2 P_r)^2$ term represents how the self-interference power scales as a function of the relay power $q_2 P_r$ [13]. The achievable secrecy rate for a given power allocation $q_1$ and $q_2$ can be then given from (4) by

$$R_s = [I_d - I_e]^+,$$

(5)

where $[x]^+ = \max(0, x)$.

### B. ERGODIC FADING CHANNELS

For ergodic fading, it is assumed that all the channel gains change independently from frame to frame. Specifically, in (1), the channel gain $h_1$ is replaced by a fading gain $h_1^{(i)}$, which is a complex Gaussian random variable with variance $\phi_1$, i.e., $h_1^{(i)} \sim CN(0, \phi_1)$. In a similar manner, the channel gains $h_3$ and $h_4$ in (2) become $h_3^{(i)}$ and $h_4^{(i)}$, respectively, and they are the complex Gaussian variables with variances $\phi_3$ and $\phi_4$, respectively, i.e., $h_3^{(i)} \sim CN(0, \phi_3)$, $h_4^{(i)} \sim CN(0, \phi_4)$. This means that $y_1$, $y_3$ and $y_4$ are exponentially distributed. Furthermore, we assume that the channel state information (CSI) is available at the receivers but not the transmitters, i.e., the destination $D$ and eavesdropper $E$ have full knowledge of channel gains $h_i = [h_1^{(i)}, h_3^{(i)}, h_4^{(i)}]$, while $R$ can estimate $h_1^{(i)}$ perfectly. Note that different from the static case, the parameters $y_1$, $y_3$, $y_4$ depend on the instantaneous channels, i.e., $y_1 = |h_1^{(i-1)}|^2 P_s / N_0$, $y_3 = |h_3^{(i)}|^2 P_r / N_0$, and $y_4 = |h_4^{(i)}|^2 P_r / N_0$, while we still have $y_2 = \beta P_r^2 / N_0$. The ergodic secrecy rate for DF relaying can then be obtained by averaging the difference between the two instantaneous rates over three channel gains as follows:

$$R_s = [E[I_d - I_e]]^+, \quad (6)$$

where the expectation $E[\cdot]$ is over $\gamma = [y_1, y_3, y_4]$, and $I_d$ and $I_e$ are given by (4).

In the following, by first focusing on the static channels, we shall derive the optimal power allocation at the source and relay $\{q_1, q_2\}$ that maximizes $R_s$ in (5) before extending the results to fading channels.

### III. STATIC CHANNELS: OPTIMAL SOURCE AND RELAY POWER ALLOCATION

In this section, we shall consider both individual and joint power constraints under static channels. Recall that under this static setting, the channel gains are assumed to be time-invariant so that all nodes have full CSI. For individual constraints, it is assumed that $0 \leq q_1 \leq q_1^*$ and $0 \leq q_2 \leq q_2^*$ so that the power constraints at the source and the relay are $q_1 P_s$ and $q_2 P_r$, respectively. Under these individual constraints, the optimization problem of interest is to find the set $\{q_1, q_2\}$ to maximize $R_s$, which is written as:

$$\max_{q_1, q_2} R_s \quad \text{s.t.} \quad 0 \leq q_1 \leq q_1^*, \quad 0 \leq q_2 \leq q_2^* \quad \text{(indiv.)} \quad (7)$$

In the case of a joint power constraint, the total power budget at both source and relay is $q_1 P_s$ and thus $P_s = P_r = P_r^*$. The optimization problem can then be expressed as:

$$\max_{q_1, q_2} R_s \quad \text{s.t.} \quad q_1 \geq 0, \quad q_2 \geq 0, \quad q_1 + q_2 \leq q^*_1 \quad \text{(joint)} \quad (8)$$

It should be noted that the total power constraint is a useful criterion to compare different transmission schemes. In addition, a network with such a constraint can serve as a system benchmark to any system under individual power constraints.

Before proceeding further with the solutions to (7) and (8), let us first examine the positivity of the secrecy rate in (5). From (4) and (5), it is clear that $R_s = 0$ when $y_1 \geq y_4$, or equivalently, $\alpha_3 \leq \alpha_4$. It means that a necessary condition for the system to achieve a positive secrecy rate is that the relay-destination channel must be stronger than the relay-eavesdropper channel. As such, hereafter, we only need to focus on the condition that $y_3 > y_4$. Now, denote $y_1 = q_1 y_1 / (1 + q_2^2 y_2)$, $y_d = q_2 y_3$ and $y_e = q_2 y_4$. Note that since $y_3 > y_4$, we have $y_d > y_e$. Therefore, if $y_r \leq y_e$, $R_s = 0$. As result, we can only achieve a positive $R_s$ when $y_r \leq \min (y_d, y_e)$ and the secrecy rate in (5) can then be rewritten as:

$$R_s = \log \left(\frac{1 + \min (y_r, y_e)}{1 + y_e}\right). \quad (9)$$

To this end, the optimization problem to maximize $R_s$ in (9) under individual power constraints is solved first, before we extend the results to the case with a joint power constraint.

### A. INDIVIDUAL POWER CONSTRAINTS

Under the individual power constraints, the optimization problem in (7) becomes:

$$\max_{q_1, q_2} \log \left(\frac{1 + \min (y_r, y_d)}{1 + y_e}\right) \quad \text{s.t.} \quad \begin{cases} y_r \leq \min (y_d, y_e) & \leq 0 \leq q_1 \leq q_1^* \\ 0 \leq q_2 \leq q_2^* \end{cases} \quad (10)$$

Let $t = \min (y_r, y_d)$, the max-min optimization problem (10) can be restated as

$$\max_{q_1, q_2, t} \log \left(\frac{1 + t}{1 + y_e}\right) \quad \text{s.t.} \quad \begin{cases} y_r \geq t & y_d \geq t \\ y_e \leq t & 0 \leq q_1 \leq q_1^* \\ 0 \leq q_2 \leq q_2^* \end{cases} \quad (11)$$

It is not hard to see that the above optimization is nonconvex. Thus, it is difficult to find its globally optimal solution. However, it is observed in (11) that it is better to increase $q_1$ as high as possible, which may lead to an increase in $t$. Thus, $q_1^* = q_1$ is an optimal solution. Intuitively, it does not decrease the secrecy rate if the source transmits at its maximum power for
the maximum source-relay rate, although the source may use more power than needed. To find the optimal power allocation at the relay, let us examine the following two cases:

1) CASE 1 \( (q^{t+1}_r + q_r \gamma_3 \leq q_s \gamma_1) \)

Set \( q^*_t = q_t \) at its optimal value, we have

\[
q^{t+1}_r + q_r \gamma_3 + q_s \gamma_1 \leq q^{t+1}_r + q_r \gamma_3 + q_r \gamma_3 \leq q_s \gamma_1, \quad \forall q_2 \leq q_t.
\]

Thus, \( \gamma_d \leq \gamma_r \). The optimization (10) becomes

\[
\max_{q_2} \log(1 + q_2 \gamma_3) - \log(1 + q_2 \gamma_4) \quad \text{s.t.} \quad 0 \leq q_2 \leq q_r,
\]

Since the objective function is increasing in \( \gamma_3 > \gamma_4 \), the optimal transmit power at the relay is \( q^*_2 = q_t \). We note that the optimal power allocation at the source, \( q^*_1 \), can be set anywhere in the range \( \left[ q^{t+1}_r + q_r \gamma_3, q_t \right] \).

2) CASE 2 \( (q^{t+1}_r + q_r \gamma_3 > q_s \gamma_1) \)

Denote \( f(x) \triangleq x^{t+1} \gamma_2 \gamma_3 + \gamma(x_1 + \gamma_3) - q_t \gamma_3 \), which is an increasing function in \( x \). Note that \( f(0) < 0 \) and \( f(q_r) > 0 \). Thus, \( f(x) = 0 \) has a unique solution, denoted as \( q_1 \), which satisfies \( 0 \leq q_1 \leq q_r \). We examine two sub-cases: \( 0 \leq q_2 \leq q_1 \) and \( q_1 < q_2 \leq q_r \).

i) \( 0 \leq q_2 \leq q_1 \): In this case \( \gamma_d \leq \gamma_r \) and the optimization (10) becomes

\[
\max_{q_2} \log(1 + q_2 \gamma_3) - \log(1 + q_2 \gamma_4) \quad \text{s.t.} \quad 0 \leq q_2 \leq q_1,
\]

which attains the optimal solution at \( q^*_2 = q_1 \) as the above expression is increasing with \( q_2 \) when \( \gamma_3 > \gamma_4 \).

ii) \( q_1 < q_2 \leq q_r \): In this case \( \gamma_r < \gamma_d \) and the optimization (10) becomes

\[
\max_{q_2} \log \left( 1 + \frac{q_2 \gamma_1}{1 + q_2 \gamma_2} \right) - \log(1 + q_2 \gamma_4) \quad \text{s.t.} \quad q_1 \leq q_2 \leq q_r.
\]

It can be easily verified through derivative methods that the objective function in (15) is now a decreasing function in \( q_2 \). Thus, \( q^*_2 = q_r \).

Therefore, the optimal power allocation for both sub-cases is \( q^*_t = q_t \) and \( q^*_2 = q_r \).

By combining the results in Cases 1 and 2, the optimal power allocation scheme under individual power constraints can finally be expressed as follows:

\[
\begin{align*}
\begin{cases}
q^*_t = q_t, \\
q^*_2 = q_r,
\end{cases}
\end{align*}
\]

when \( q^{t+1}_r + q_r \gamma_3 + q_r \gamma_3 < q_s \gamma_1 \) \hspace{1cm} (16)

It can be seen from the final solution given in (16) that under individual power constraints, full-power at the relay is not always optimal.

B. JOINT POWER CONSTRAINT

We restate the problem (8) as follows:

\[
\max_{q_1,q_2} \log \left( \frac{1 + t}{1 + \gamma_e} \right) \quad \text{s.t.} \quad \begin{cases}
\gamma_r \geq t \\
\gamma_d \geq t \\
\gamma_e \leq t \\
q_1 + q_2 \leq q_t.
\end{cases}
\]

Similar to the arguments following problem (11), we can always find an optimal solution \((q^*_1, q^*_2)\) such that \( q^*_1 + q^*_2 = q_t \). By introducing an auxiliary variable \( \hat{q}_r \leq q_r \), the optimization is recast as

\[
\max_{q_1,q_2,t} \log \left( \frac{1 + t}{1 + \gamma_e} \right) \quad \text{s.t.} \quad \begin{cases}
\gamma_r \geq t \\
\gamma_d \geq t \\
\gamma_e \leq t \\
q_1 = q_t - \hat{q}_r \\
q_2 \leq \hat{q}_r.
\end{cases}
\]

Denote \( \hat{q}_1 \) as a solution to equation \( f(x) = x^{t+1} \gamma_3 + x(\gamma_1 + \gamma_3) - q_1 \gamma_3 = 0 \). Note that \( f(x) \) is an increasing function with \( f(0) < 0 \) and \( f(q_r) > 0 \). Thus, \( f(x) = 0 \) has a unique solution in \( \hat{q}_1 \) and \( 0 < \hat{q}_1 < q_t \). By introducing an auxiliary variables \( \hat{q}_1 \leq q_t \), we consider two cases: i)

1) \( \hat{q}_r \leq \hat{q}_1 \): In this case \( \hat{q}^{t+1}_r + \hat{q}_r \gamma_3 \leq (q_t - \hat{q}_r) \gamma_1 \), which is similar to Case 1 for individual power constraints and the solution is given as \( q^*_2 = \hat{q}_r \) and \( q^*_1 = (q^{t+1}_r + \hat{q}_r \gamma_3, q_t - \hat{q}_r) \).

2) \( \hat{q}_1 \leq \hat{q}_r \leq q_t \): In this case \( \hat{q}^{t+1}_r + \hat{q}_r \gamma_3 \geq (q_t - \hat{q}_r) \gamma_1 \), which is similar to Case 2 for individual power constraints and the solution is given as \( q^*_2 = \hat{q}_r \) and \( q^*_1 = q_t - \hat{q}_1 \).

Using simple calculations to compare the objective function values obtained from the above two cases, it is straightforward to check that the solution of the second case results in the maximum secrecy rate. Thus, the optimal power allocation scheme under the joint power constraints is \( q^*_2 = \hat{q}_r \) and \( q^*_1 = q_t - \hat{q}_1 \).

C. ASYMPOTIC ANALYSIS AND COMPARISON TO HD RELAYING

Given the solutions of the optimization problem under individual and joint power constraints, this section provides an asymptotic analysis to shed light on the derived solutions in different high power regions. As we will show later, the analysis is helpful to make a direct comparison with traditional HD relaying. Note that for the HD system, the secrecy rates can be obtained by setting \( \gamma_2 = 0 \) in (4) and pre-multiplying \( R_t \) by a factor of 1/2. Also, for a fair comparison, the same average power constraints between HD and FD are used, i.e., \( q_{s,HD} = 2q_s \), \( q_r = 2q_r \), and \( q_{t,HD} = 2q_t \) for the
HD system. In the following, we will examine different high power regions at the source and relay.

1) INDIVIDUAL POWER CONSTRAINTS

a) LARGE $P_s/N_0$

Assume that $P_s$ is sufficiently higher than $P_r$. By choosing $q_1^t = q_s$, when $\lambda = 0$, we have $q_2^t = \min(\frac{q_2}{\gamma(1+\rho)}, q_s)$. The secrecy capacity $C_t$ therefore approaches $\log(\frac{q_2}{\gamma(1+\rho)})$. On the other hand, if $\frac{q_2}{\gamma(1+\rho)} > q_s$, then $q_2^t = q_s$, and the secrecy capacity goes to $\log(\frac{q_2}{\gamma(1+\rho)})$. When $0 < \lambda \leq 1$, applying the method of dominant balance to the equation $q_2^{t+1} = \gamma q_2 + q_2^{t+1} - q \gamma q_1$, we have $O(q_2^{t+1}) = O(q_s^{t+1})$. As a result, $q_2^{t+1} = O(q_s^{t+1})$, and the secrecy capacity $C_t$ approaches $\log(\frac{q_2}{\gamma(1+\rho)})$, or equivalently, $C_t = \log(\frac{q_2}{\gamma(1+\rho)}) = q_s$. When $0 < \lambda \leq 1$, applying the method of dominant balance to $q_2^{t+1} = \gamma q_2 + q_2^{t+1} - q \gamma q_1$, we have $\frac{q_2}{\gamma(1+\rho)} = O(q_s^{t+1})$. Thus $q_2^{t+1} = O(q_s^{t+1})$. Note that for the HD mode, one obtains $q_2^{t+1} = q_t^t = 0$, which is the highest secrecy capacity $C_t$ approaches $1/2 \log(\frac{q_2}{\gamma(1+\rho)})$. Therefore, HD relaying only achieves half of the secrecy capacity of FD relaying.

IV. FADING CHANNELS: ERGODIC SECRECY RATES AND POWER ALLOCATION

In the previous sections, we have focused on channels that change slowly and can be considered as constant for the duration of the transmission. We now turn our attention to the fading channel. It can be observed from (6) that the average secrecy rates involve triple-integrals. A consequence, calculating them with high accuracy is very cumbersome. Monte Carlo simulations can be used as an alternative to estimate these rates. However, it is a time consuming process and does not give an insight on the behavior of the secrecy rate. In the following, we propose a simple method to establish $R_s$ in (6) in closed-form. To this end, the following lemma first states an important result related to the exponential integral.

Lemma 1: Consider an exponentially distributed random variable $\omega$ with mean $\phi$. Let $J(x) = \exp(x)E_1(x)$, with $E_1(x)$ being the well-known the exponential integral given as:

$$E_1(x) = \int_x^{\infty} \frac{e^{-u}}{u} du = -\left(\gamma + \ln(x) + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n! n}\right),$$

and $\gamma$ being the Euler number. For any positive $a_0$, we have:

$$E_0[\ln(a_0 + \omega)] = J(a_0/\phi).$$

Proof: By factoring $a_0$, the expectation in (19) is expressed as:

$$E_0[\ln(a_0 + \omega)] = \ln(a_0) + E_0[\ln(1 + \omega/a_0)],$$

The above expression is similar to the rate achieved in a single-input single-output system with instantaneous SNR $\omega/a_0$, and average SNR $\phi/a_0$ in [22, eq. (15.26)].

Given the results in Lemma 1, we are now ready to find the closed-form expression of the rate formula in (6), which is described in the following.

The average secrecy rate given in (6) can be re-written as:

$$R_s = E_{h_1, h_3} \left[ \log \left( 1 + \min \left( \frac{q_1}{1 + q_2}, q_2 \gamma \right) \right) \right] - E_{h_1, h_3} \left[ \log \left( 1 + \min \left( \frac{q_1}{1 + q_2}, q_2 \gamma \right) \right) \right].$$

The above expectation is similar to the rate achieved in a single-input single-output system with instantaneous SNR $\omega/a_0$, and average SNR $\phi/a_0$ in [22, eq. (15.26)].

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$$R_s = E_{h_1, h_3} \left[ \log \left( 1 + \min \left( \frac{q_1}{1 + q_2}, q_2 \gamma \right) \right) \right] - E_{h_1, h_3} \left[ \log \left( 1 + \min \left( \frac{q_1}{1 + q_2}, q_2 \gamma \right) \right) \right].$$

The above expectation is similar to the rate achieved in a single-input single-output system with instantaneous SNR $\omega/a_0$, and average SNR $\phi/a_0$ in [22, eq. (15.26)].
It is known that the minimum of two exponential random variables (RV) with the two corresponding expected values \(1/\alpha_1\) and \(1/\alpha_2\) is also an exponential RV with expected value \(1/(\alpha_1 + \alpha_2)\) [23]. Therefore, (21) can be rewritten as:

\[
R_s = E \left[ \log (1 + y_r) \right] - E \left[ \log (1 + y_e) \right],
\]

where

\[
y_r = \min \left( \frac{q_1 r_1}{1 + q_2 r_2}, \frac{q_2 r_4}{1 + q_4 r_2} \right)
\]

and

\[
y_e = \min \left( \frac{q_1 r_1}{1 + q_2 r_2}, \frac{q_2 r_4}{1 + q_4 r_2} \right)
\]

are two exponential RVs with the following expected values, respectively:

\[
\gamma_r = \frac{q_1 \gamma_1 \times q_2 \gamma_3}{q_1 \gamma_1 + \frac{q_4 r_2}{q_4} + q_2 \gamma_3},
\]

\[
\gamma_e = \frac{q_1 \gamma_1 \times q_2 \gamma_4}{q_1 \gamma_1 + \frac{q_4 r_2}{q_4} + q_2 \gamma_4}.
\]

Here, \(\gamma_1 = E[r_1] = P_r \phi_1/N_0\), \(\gamma_2 = E[r_2] = P_r \phi_3/N_0\), and \(\gamma_4 = E[r_4] = P_r \phi_4/N_0\). Thus, from the results in Lemma 1, the expectation in (21) can be obtained in closed-form as follows:

\[
R_s = \frac{1}{\ln(2)} \left[ J \left( \frac{q_1 \gamma_1 + \frac{q_2 \gamma_3}{q_2} + \frac{q_2 \gamma_4}{q_4}}{q_1 \gamma_1 \times q_2 \gamma_4} \right) - J \left( \frac{q_1 \gamma_1 + \frac{q_2 \gamma_3}{q_2} + \frac{q_2 \gamma_4}{q_4}}{q_1 \gamma_1 \times q_2 \gamma_4} \right) \right]^{1/2}.
\]

It can be seen from (23) the ergodic secrecy rate for DF relaying can be easily evaluated. It is because the expression in (23) involves only the well-known exponential integral. In the next section, numerical results are presented to confirm the accuracy of the proposed solution in (23).

Given the above closed-form expression, the power allocation \(q_1\) and \(q_2\) at the source and relay, respectively, can be further optimized to improve the secrecy rate. Since this rate involves exponential integrals, obtaining a rigorous solution of such a non-convex optimization problem is challenging, and it is a subject for future research. However, as an alternative, we can perform an exhaustive search over the two variables \(q_1\) and \(q_2\) under either individual power constraints \(q_1 \leq q_s\) and \(q_2 \leq q_r\), or the joint power constraint \(q_1 + q_2 \leq q_r\), to find the optimal values \(q_1^*\) and \(q_2^*\). Under individual power constraints, via extensive numerical results, we have observed that \(q_1^* = q_s\), while \(q_2^* < q_r\). It means that full-power allocation at the relay is not an optimal solution to maximize the ergodic secrecy rate. This observation shall be confirmed shortly via numerical results.

V. ILLUSTRATIVE EXAMPLES

In this section, numerical results are provided to confirm the proposed power allocation solutions, the asymptotic analysis, as well as the derived closed-from expressions. In all simulations, we consider \(q_s = q_r = q_t = 1\) so that the power constraints are simply given by \(P_s\) and \(P_r\) for the individual setting, and by \(P_t\) for the joint one. In addition and similar to [9], it is assumed that \(\beta = 0.1\).

A. STATIC CHANNELS

For static channels, we consider the case where \(\alpha_1 = 1\), \(\alpha_3 = 2\), and \(\alpha_4 = 1\), which corresponds to a 3dB gain difference between R-D and R-E channels. Besides illustrating the performance of the considered FD DF system, the performance of two benchmark systems are also provided for comparison: 1) HD DF relaying, and 2) FD AF relaying in [10]. For all FD AF systems, we assume that a perfect self-interference cancellation is achieved, i.e., \(\lambda = 0\).

1) INDIVIDUAL POWER CONSTRAINTS

Under individual power constraints, Fig. 2 first shows the secrecy rates versus \(P_s/N_0\) for both optimal and full power allocation \((q_1 = q_s\) and \(q_2 = q_r\)) with \(\lambda\) being either 0, 0.5, or 1. Here, \(P_r/N_0\) is fixed at 5dB. The secrecy rates obtained by the optimal HD DF scheme as well as the optimal FD AF system are also provided. Observe that the secrecy rates in FD systems asymptotically approach \(\log(1 + q_r y_2) = 0.8745\) for both power allocation schemes, which confirms our asymptotic analysis. It can also be observed that the FD system outperforms the HD system. Specifically, \(C_s = \log(1 + q_r y_2) = 0.8745 > C_{s,HD} = 1/2 \log(1 + 2q_r y_2) = 0.4664\). Furthermore, it can be seen that FD DF relaying is significantly better than DF AF relaying.

Fig. 3 plots the secrecy rate versus \(P_s/N_0\) for the same systems but now with \(P_s/N_0 = 5dB\). Observe from Fig. 3 that the secrecy rate achieved by FD DF relaying with the optimal power allocation becomes insensitive to \(P_r\)
when $P_r$ increases. With full-power allocation, the secrecy rate of the FD DF system goes to zero. Similar to the previous result, FD DF relaying with optimal power allocation outperforms both HD DF and FD AF relaying.

Finally, under the individual power constraints, Fig. 4 presents the secrecy rates achieved by different systems when $P_s/N_0 = P_r/N_0 = P/N_0$. It can be noticed from Fig. 4 that in FD mode with full power allocation, the secrecy rates are zero. Furthermore, when the optimal power allocation scheme in the FD mode is used, the secrecy rates asymptotically approach $\log(\alpha/3)$. As similar to the previous results, FD DF relaying provides much better performance than those achieved in HD DF and FD AF systems.

2) JOINT POWER CONSTRAINT

Under the case of a joint power constraint, Fig. 5 plots the secrecy rate versus $P_t/N_0$ for the FD systems using the optimal power allocation and the uniform power allocation scheme ($q_1 = q_2 = q_t/2$). The rate achieved by the optimal HD system under the joint power constraint is also provided for comparison. As expected, the secrecy capacity of the FD system approaches $\log(\alpha_3/\alpha_4) \to 1$. The uniform power allocation performs poorly and it does not provide any secrecy. Compared to the HD mode, the FD system is far superior. It is interesting to see that in this case, while FD DF relaying is much better than DF AF relaying in the low power region, the two systems achieve a similar performance when $P_t/N_0$ increases.

B. FADING CHANNELS

Let us now consider the case of fading channels. To confirm the accuracy of the proposed closed-form expression, here we compare the rates obtained by (23) to those obtained by Monte Carlo simulations. In this simulation, we assume that the channel variances $\phi_1 = 2, \phi_3 = 2.5,$ and $\phi_4 = 1.5$.

Fig. 6 plots the secrecy rate versus signal-to-noise ratio $\text{SNR} = P/N_0$ for different values of $\lambda$. In Fig. 6, we assume
that $P_s = P_r = P$ and full power allocation is used at both source and relay nodes, i.e., $q_1 = q_s$ and $q_2 = q_r$. It can be seen from Fig. 6 that the results obtained by the proposed closed-form are identical to the Monte Carlo simulations. This verifies the accuracy of our proposed solution in (23).

Regarding the optimal power allocation, Fig. 7 shows the ergodic secrecy rates versus SNR for two different power allocation schemes under DF relaying: the full power allocation scheme and the optimal allocation scheme obtained by brute-force search. All channel parameters are the same as the ones we considered earlier. It can be observed from Fig. 7 that full power allocation scheme is suboptimal and the secrecy rates approach 0 when $0 < \lambda \leq 1$. When $\lambda = 0$, the rate is positive. However, it is much smaller that the secrecy capacity obtained using the optimal power allocation scheme.

Similar results can also be obtained under the joint power constraint. In particular, Fig. 8 plots the secrecy rate versus $P_t/N_0$ when the optimal power allocation scheme and the uniform power allocation scheme are used. Apparently, a significant gain can be achieved with the optimal solution.

**VI. CONCLUSION**

This paper investigated the optimal power allocation schemes for both static and ergodic fading two-hop relay channels with FD DF relaying under residual self-interference. Individual and joint power constraints were considered. For the static case, while the optimization problems are non-convex, we demonstrated that closed-form solutions can still be obtained. By further exploiting these solutions via an asymptotic analysis using the method of dominant balance, important insights on the derived solutions have been provided to demonstrate the advantage of FD relaying over HD relaying. A comparison between FD DF and FD AF also showed that FD DF relaying outperforms FD AF relaying under both individual and joint power constraints in low power regions. In high power regions, their performances are almost the same under joint power constraints. The extension to fast fading channels has also been studied. Specifically, we derived a closed-form expression of the ergodic secrecy rate using simple exponential integrals for a given power allocation scheme. By further optimizing the power allocation, we demonstrated that FD also greatly enhances the secrecy rate in Rayleigh fading.

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**REFERENCES**


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