Algebraic Tunnelling

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Abstract

We study the quantum phenomenon of tunnelling in the framework of algebraic quantum theory, motivated by the tunnelling aspects of false vacuum decay. We see that resolvent C*-algebra, proposed relatively recently by Buchholz and Grundling rather than Weyl algebra provides an appropriate framework for treating the dynamics of non-free quantum mechanical system as an algebraic automorphism. At the end, we propose to investigate false vacuum decay in algebraic quantum field theoretic setting in terms of the two-point correlation function which gives us the tunneling probability, with the corresponding C*-algebraic construction.

1 INTRODUCTION

Let us consider a scalar relativistic quantum field theory given by the Lagrangian density

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi - U(\phi), \tag{1.1}$$

where $U(\phi)$ as a potential function of ϕ consists of local minima ϕ_A and ϕ_B, with $U(\phi_A) > U(\phi_B)$ (see Callan and Coleman (1977); Coleman (1977) and references cited there for initial steps taken into the study of such theory and Steingasser (2022) for recent investigation with interesting implications). The quantum state of the field at ϕ_A is unstable and one expects the field to decay to the state corresponding to ϕ_B, via the tunnelling process. This is what we refer to as the false vacuum decay, where false vacuum refers to the meta-stable quantum state at ϕ_A. In this brief report, we would like to take first step towards formulating such processes in the algebraic framework postulated in Haag (2012); Haag and Kastler (1964) for relativistic QFT, preceded by the work of Segal (1947) for quantum mechanics. To do so, we consider a simple quantum mechanical picture of tunnelling in system with non-trivial potential in Hamiltonian.

Let us give a brief overview of the report. We start by investigating the tunnelling phenomenon at the level of quantum mechanics in section [2]. We see that the tunnelling dynamics for particles can be seen as an automorphism of the so called resolvent C*-algebra defined in Buchholz and Grundling (2008) rather than the Weyl algebra. This in essence shows us that the construction of appropriate C*-algebra is a non-trivial problem already at the level of quantum mechanics. Thus, for algebraic field theoretic analysis of such phenomenon some care is needed in construction of the appropriate algebra and corresponding states (expectation functional). We conclude the report with some further points of investigation and possible constructive ideas in section [3].

2 TUNNELLING DYNAMICS IN ALGEBRA

We want to construct algebra for the quantum mechanical system exhibiting phenomenon of tunnelling. We refer to Bratteli and Robinson (2012); Petz (1990) for mathematical definitions and detailed discussion of concepts invoked in the following discussion. We follow Dybalski (2017) closely for the following presentation .

2.1 Weyl C*-algebras and its limitation

Let us consider abstract Weyl operators $\{W(z), z \in \mathbb{C}^n\}$, and take $z = u + iv$, $w = x + iy$. Consider operators $A = i(uP + vQ)$ and $B = i(xP + yQ)$ where Q, P satisfy the canonical commutation relation (set $\hbar = 1$)

$$[Q, P] = i\,\mathbf{1}, \quad [Q, Q] = 0, \quad [P, P] = 0. \tag{2.1}$$

Commutator of operators A and B gives $[A, B] = i(uy - vx)$. On the other hand, imaginary part of canonical product is $\Im\langle z, w\rangle = uy - vx$, where $\langle z, w\rangle = \sum_k \bar{z}_k w_k$ is the canonical scalar product on $\mathbb{C}^n$. Defining exponentiation of these operators as $W(z) := e^A$, $W(w) := e^B$ we get using the BCH formula,

$$W(z)W(w) = e^A e^B = e^{A+B+\frac{1}{2}[A,B]} = e^{\frac{1}{2}[A,B]} e^{A+B} = e^{\frac{i}{2}\Im\langle z,w\rangle} W(z + w). \tag{2.2}$$

A bilinear form $\sigma : X \times X \to \mathbb{R}$ on a real linear space X is said to be symplectic form if $\sigma(z, w) = -\sigma(w, z) \quad \forall z, w \in X$, and non-degenerate if $\sigma(z, w) = 0,\ \forall w \in X \implies z = 0$. The pair (X, σ) is referred to as the symplectic space. Generally, if $\mathcal{H}$ is a complex Hilbert space then $\sigma(z, w) = \Im\langle z, w\rangle$ is a non-degenerate symplectic form on the real linear space $\mathcal{H}$. A unital *-algebra is a quadruple $(\mathfrak{A}, \times, *, \mathbf{1})$ of vector space $\mathfrak{A}$, composition map $\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ following distributive laws, involution operation $*$ such that $\forall A \in \mathfrak{A}$ there exists $A^* \in \mathfrak{A},\ A^{**} = A$ and $\forall A, B \in \mathfrak{A}$ there holds $(AB)^* = B^*A^*, (zA + wB)^* = \bar{z}A + \bar{w}B$ and an unique identity element $\mathbf{1}$. If in addition, we equip $*$-algebra with norm $||.||$ such that it is complete with respect to the norm induced topology and follows C*-norm condition $||A^{**}|| = A$, we get unital abstract C*-algebra denoted by $\mathcal{A}$.

Definition 2.1. *The Weyl algebra over a symplectic space (X, σ) denoted by $\mathcal{W}(X, \sigma)$, is defined to be a unital $*$-algebra generated by abstract symbols $\{W(z) \mid z \in X = \mathbb{C}^n\}$ modulo the relations*

$$W(z)W(w) = e^{\frac{i}{2}\sigma(z,w)} W(z + w), \quad W(-z) - W(z)^* = 0. \tag{2.3}$$

Furthermore, properties of identity $W(0) = \mathbf{1}$, unitarity $W(z)^*W(z) = \mathbf{1}$, and linear combinations: $\sum_z \alpha_z W(z) \sum_w \beta_w W(w) = \sum_{z,w} \alpha_z \beta_w\ e^{\frac{i}{2}\sigma(z,w)} W(z + w)$ are satisfied for the Weyl algebra.

The Weyl algebra $\mathcal{W}(X, \sigma)$ admits a unique C*-norm, so that its norm completion yields Weyl C*-algebra denoted here by $\mathcal{A}(X, \sigma)$. However, in the situation we consider here of non-trivial potential, Weyl algebra proves to be limited in its application. The difficulty is crystallized in the following theorem:

Theorem 2.2 (Fannes and Verbeure (1974))**.** *Consider Hamiltonian of the form $H = \frac{P^2}{2m} + V(Q)$, where $V(Q) \in L^1(\mathbb{R}) \cup L^\infty(\mathbb{R})$. Let the time translation be implemented by unitary operator $U(t) = e^{itH}$ whose action on the Weyl algebra ought to be automorphic in nature. However, automorphism of Weyl algebra under time-translation implies triviality of potential, i.e. $U(t)\pi(A)U(t)^{-1} \in \pi(\mathcal{W}),\ \forall A \in \mathcal{W},\ \forall t \in \mathbb{R} \implies V = 0$.*

Thus, the situation of quantum mechanical tunnelling cannot be realized as the automorphism of Weyl algbera. Interestingly for us, we have an alternative in the form of relatively recent proposal by Buchholz and Grundling (2008). In this approach, one rather considers element of the form

$$R(\lambda, z) = \frac{1}{i\lambda\mathbf{1} - uP - vQ} \tag{2.4}$$

where $\lambda \in \mathbb{R} \setminus 0$ and subsequently the C*-algebra generated by these elements and relations between them. Let us recall that resolvent of $A \in \mathcal{A}$ is $(\lambda\mathbf{1} - A)^{-1}$ for $\lambda \in \mathbb{C}$. Here, we expect that the process of quantum tunneliing can treated as a C*-system $(\mathcal{R}, \{\alpha_t\})$, with dynamics being implemented as automorphism $\alpha_t : \mathcal{R} \to \mathcal{R}$ over resolvent C*-algebra $\mathcal{R}$, $\forall t \in \mathbb{R}$. Let us see how $\mathcal{R}$ can be constructed in the following sub-section closely adapted from Buchholz and Grundling (2008), where further details and proof of the statements can be found which are not reproduced here for the sake of brevity.

2.2 Construction of resolvent algebra

Definition 2.3. *For the symplectic space (X, σ), we may define the pre-resolvent algebra $\mathcal{R}_0$ to be the unital $*$-algebra generated by the elements $\{R(\lambda, z) \mid \lambda \in \mathbb{R}\backslash 0, z \in X\}$ modulo the relations exhibiting linearity of the map $(u, v) \mapsto uP + vQ$, algebraic properties of the resolvent of self-adjoint operator, and canonical commutation relation respectively:*

1. $R(\lambda, 0) = \frac{1}{i\lambda\mathbf{1}}$,
2. $vR(v\lambda, vz) = R(\lambda, z)$,
3. $R(\lambda, z)\, R(\mu, w) = R(\lambda+\mu, z+w)\, \{R(\lambda, z)+R(\mu, w)+i\, \sigma(z, w)\, R(\lambda, z)^2\, R(\mu, w)\}$, $\lambda+\mu \neq 0$,
4. $R(\lambda, z)^* = R(-\lambda, z)$,
5. $R(\lambda, z) - R(\mu, w) = i\, (\mu - \lambda)\, R(\lambda, z)\, R(\mu, w)$,
6. $[R(\lambda, z), R(\mu, w)] = i\, \sigma(z, w)\, R(\lambda, z)\, R(\mu, w)^2\, R(\lambda, z)$

where, $\lambda, \mu, \nu \in \mathbb{R} \setminus 0$ and $z, w \in X$.

For a symplectic space (X, σ) we have the associated pre-resolvent algebra $\mathcal{R}_0(X, \sigma)$. Consider positive linear functionals $\omega : \mathcal{A} \to \mathbb{C}$, i.e. satisfying $\omega(zA + yB) = z\omega(A) + y\omega(B)$ and $\omega(A^*A) \geq 0\, \forall z, y \in \mathbb{C}$, $\forall A, B \in \mathcal{A}$. We can construct the Gelfand Naimark Segal (GNS) representation of ω which is cyclic *-representation into the space of bounded Hilbert space operators, $\pi_\omega : \mathcal{R}_0 \to B(\mathcal{H})$. However, note that ω is not yet what we call an algebraic state. Although we can perform the GNS construction, we get non-degenerate representations from such non-state positive linear functionals, which is not a set. We further need to define a set $\mathcal{S}$ which consists of positive linear functionals which also satisfies $\omega(\mathbf{1}) = 1$, i.e. set of algebraic states. To establish connection of algebraic state with more generally utilized notion of state, let us consider a density matrix belonging to the class of bounded operators acting on a Hilbert space $\rho \in \mathcal{B}(\mathcal{H})$. Then, if we define for all $A \in \mathcal{A}$, $\omega(A) := Tr(\rho A)$ it can be seen that all the properties for ω to be an algebraic state is fulfilled. GNS construction with states $\omega \in \mathcal{S}$ yields bounded representations constituting a set in its totality. In fact, we will see next that representations induced by set of algebraic states are uniformly bounded.

Let us consider representation $\pi_0 : R_0 \to B(\mathcal{H})$. Then, it can be shown that norm of resolvent element in representation space is bounded with $||\pi_0(R(\lambda, z))|| \leq \lambda^{-1}$. This means that for any representation π of $R_0(X, \sigma)$ we have a positive bound $c_r \geq 0$ from above, i.e. $\pi(r) \leq c_r\ \forall r \in R_0(X, \sigma)$. This leaves a room to create direct sum over infinite number of representations while maintaining the boundedness. This is used to create universal representation.

The universal representation $\pi_u : \mathcal{R}_0 \to B(\mathcal{H}_u)$ is given by the relation,$\pi_u(A) := \bigoplus\{\pi_\omega(A) \text{ for } \omega \in \mathcal{S}\}$. We can define a enveloping C*-norm on $\mathcal{R}_0$ by $||A||_u := ||\pi_u(A)|| = \sup_{\omega\in\mathcal{S}} ||\pi_\omega(A)|| = \sup_{\omega\in\mathcal{S}} \omega(A^*A)^{1/2}$. Let us denote kernel with respect to the C*-norm by Ker $||.||_u := \{A \in \mathcal{R}_0;\ ||A||_u = 0\}$. Finally, we are in the position to define the resolvent C*-algebra.

Definition 2.4. *The resolvent algebra $\mathcal{R}(X, \sigma)$ is defined to be the C*-algebra generated by universal representation of $\mathcal{R}_0$, $\mathcal{R}_0 \setminus$ Ker $||.||_u$ completed with respect to the enveloping norm $||.||_u$.*

A representation $(\mathcal{H}, \pi)$ of $\mathcal{R}$ is said to be regular if there exists self-adjoint operators P_i, Q_i such that

$$\pi(R(\lambda, z)) = (i\lambda\mathbf{1} - uP - vQ)^{-1}. \tag{2.5}$$

An example of this is the Schrödinger representation which can be constructed as follows: consider Schödinger representation $(\mathcal{H}_S, \pi_S)$ of Weyl algebra $\mathcal{W}$. Because it is regular there exists P_i, Q_j as self-adjoint operators on $\mathcal{H}_S = L^2(\mathbb{R}^n)$. Thus, we can define $\pi_S(R(\lambda, z)) = (i\lambda\mathbf{1} - uP - vQ)^{-1}$. In addition, it is also irreducible. As is stated in (Dybalski, 2017, Prop. 1.41), any irreducible regular representation of $\mathcal{R}$ is infact unitarily equivalent to the Schrödinger representation, due to Stone-von

Neumann uniqueness theorem. For our understanding of the potential well, we may thus pass onto irreducible Schrödinger representation of $\mathcal{R}(X,\sigma)$. The unitary equivalence of various representations of Weyl CCR (see Summers (1998) for further discussion on this point) translates into unitary equivalence of representations of the resolvent algebra. We have following proposition which sums up the discussion nicely:

Proposition 2.5 (Buchholz and Grundling (2008))**.** *For continuous potential $V \in C_0(\mathbb{R})$ on real line, the associated self-adjoint Hamiltonian $H = P^2 + V(Q)$ induces dynamics on $\mathcal{R}(X,\sigma)$. H generates unitary group via $U(t) = e^{iHt}, \forall t \in \mathbb{R}$ such that*

$$U(t)\pi_0(\mathcal{R}(X,\sigma))U(t)^{-1} \in \pi_0(\mathcal{R}(X,\sigma)), \tag{2.6}$$

i.e. we can define automorphic action by the following expression

$$\alpha_t(R) := \pi_0^{-1}(U(t)\pi_0(R)U(t)^{-1}), \quad R \in \mathcal{R}(X,\sigma). \tag{2.7}$$

It is clear from the above proposition that the framework of resolvent C*-algebra is suitable for algebraic treatment of quantum tunnelling. Subsequently, we can pass to the usual Schrödinger representation for system with non-trivial potential, and solve for reflection and transmission coefficients as done in quantum mechanics textbooks. A more general conclusion we would like to draw here is that the quantum mechanical dynamics can be incorporated into algebraic framework using the notion of resolvent C*-algebra, with some exceptions, for e.g. system with Hamiltonian $H = P^2 - Q^2$ (Buchholz & Grundling, 2008, Prop. 6.3).

3 DISCUSSION

We would like to stress on the fact that the program of analysing false vacuum decay in algebraic framework, has not in any sense been completed here. There are several conceptual and technical difficulties of mathematical nature to be resolved on the way to full-fledged algebraic field theoretical analysis. Let us briefly indicate some of them with some speculative comments.

We notice that in the field theoretic description of Hawking radiation as tunneling phenomenon in Moretti and Pinamonti (2011), tunneling probability is given in terms of the two-point correlation function. We can adapt this idea directly into the context of false vacuum decay. However, authors of this paper consider $*$-algebra rather than C*-algebra. We would like to construct C*-algebra for scalar quantum field theory yielded by the Lagrangian density $\mathcal{L} = 1/2\ \partial_\mu\phi\ \partial^\mu\phi - U(\phi)$ so that upon their representation, we could do our analysis at the level of von Neumann algebras. Here, mathematical machinery of Tomita-Takesaki modular theory and several other notions like the type III property, the Reeh-Schlieder property could be utilized towards deeper understanding of the false vacuum decay; see Fewster and Rejzner (2019); Haag (2012) for discussion on these concepts. However, construction of the suitable C*-algebra is no trivial matter, as we witnessed already at the level of quantum mechanics in the discussion of resolvent C*-algebra. In this regard, we think that the approach advocated in Buchholz and Fredenhagen (2020) serves as the correct procedure for our purposes. As pointed out in these papers, the subtle problem of constructing states for the algebra of interacting quantum field theories arises. However, states describing Bose-Einstein condensates were constructed in Brunetti, Fredenhagen, and Pinamonti (2020) using the methods developed in Drago, Hack, and Pinamonti (2016); Fredenhagen and Lindner (2014). Fortunately for us, phenomenon false vacuum decay is mimicked by Bose-Einstein condensates as pointed out in Jenkins et al. (2023) and references cited therein. We think this is a positive news for us and we speculate that construction of states can be achieved for the false vacuum in a similar fashion. This would lead towards complete treatment of false vacuum decay in the framework of C*-algebras. An advantage of such treatment would be the ease of generalization in algebraic framework from the Minkowski spacetime to a more generic curved spacetime, and subsequent studies of false vacuum decay in cosmological models.

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