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Kumudu Gamage
Old Dominion University

Yan Peng
Old Dominion University

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A spatially and temporally second order method for solving parabolic interface problems

Kumudu Gamage(kgamage@odu.edu) and Yan Peng (ypeng@odu.edu)
Department of Mathematics and Statistics, Old Dominion University, Norfolk, VA

Introduction

Parabolic interface problems have many applications in physics and biology, such as hyperthermia treatment of cancer, underground water flow, and food engineering.

Goal: Obtain a numerical methodology for solving 2D parabolic interface problems, which results in second-order accuracy in both space and time for both solutions and the solution's gradient.

We studied the following interface problem:

$$u_t = (\beta u_x)_x + (\beta u_y)_y - f(x,y,t), \qquad (x,y) \in \Omega \backslash \Gamma$$
$$[u] = w(:,t), [\beta u_n] = v(:,t),$$

with specified boundary and initial conditions.

Crank-Nicolson scheme and direct Immersed Interface Method (IIM) for solving parabolic interface problems

For simplicity, we consider the interface problems with piecewise constant coefficients. For any grid point (x_i, y_j), we can write the Crank-Nicolson scheme as,

$$\frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{\Delta t \beta_{i,j}} = \frac{1}{2}\left(\delta_x U_{i,j}^{n+1} + \left(C_{i,j}^{x}\right)^{n+1} + \delta_y U_{i,j}^{n+1} + \left(C_{i,j}^{y}\right)^{n+1} - \frac{f_{i,j}^{n+1}}{\beta_{i,j}} + \delta_x U_{i,j}^{n} + \left(C_{i,j}^{x}\right)^{n} + \delta_y U_{i,j}^{n} + \left(C_{i,j}^{y}\right)^{n}\right),$$

where, $\delta_x U_{i,j}^n = \frac{U_{i+1,j}^n - 2U_{i,j}^n + U_{i-1,j}^n}{h^2}$ and correction term at an irregular grid point (See Fig.1.) is given by,

$$(C_{i,j}^x)^n = \sum_{m}^{9} \alpha_m U_{i+i_m, j+j_m}^n + \alpha_c^x .$$

where, α_m and α_c^x depend on the position of (x_i, y_j) and β. Moreover, α_m is time-independent while α_c^x is time-dependent.

Now, the linear system of the Crank-Nicolson scheme can be written as,

$$AU^{n+1} = BU^n + \text{F},$$

where, U^n is the numerical solution. A and B are the coefficient matrix, and F is a vector constructed from the source term, boundary conditions and correction terms.

Note: A and B are only needed to be found at once. For piecewise constant β, A and B are nine-point banded matrices for irregular grid points and five-diagonal for regular points.

Numerical Examples

Consider the following equation with the interface Γ: $x^2 + y^2 = 0.25$ on the domain $\Omega = [-1,1] \times [-1,1]$,

$$u_t = (\beta u_x)_x + (\beta u_y)_y - f(x,y,t),$$

The source term f is defined as,

$$f(x,y,t) = \begin{cases} 0 & if \quad (x,y) \in \Omega^- \\ e^{-t}(x^2 - y^2) & if \quad (x,y) \in \Omega^+ \end{cases}$$

Jump conditions are given by,

$$[u] = e^{-t}(x^2 - y^2), \quad [\beta u_n] = 4\beta^+ e^{-t}(x^2 - y^2).$$

Boundary conditions are obtained from the exact solution of,

$$u(x,y,t) = \begin{cases} 0 & if \quad (x,y) \in \Omega^- \\ e^{-t}(x^2 - y^2) & if \quad (x,y) \in \Omega^+ \end{cases}$$

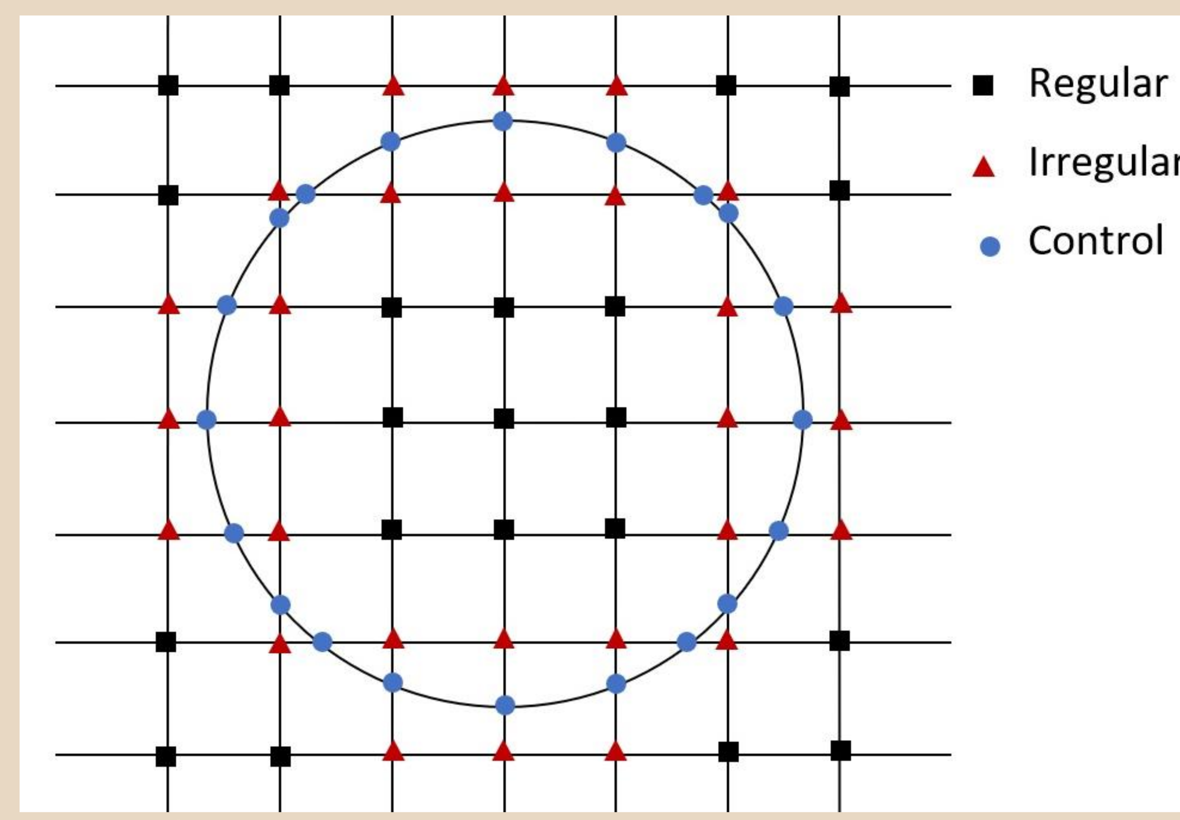


Fig.1. Illustration of regular, irregular, and control points in the domain Ω with the interface Γ

Grid refinement analysis

A grid refinement analysis with $\beta^+ = 1$ and $\beta^+ = 10000$ at final time $T = 1$

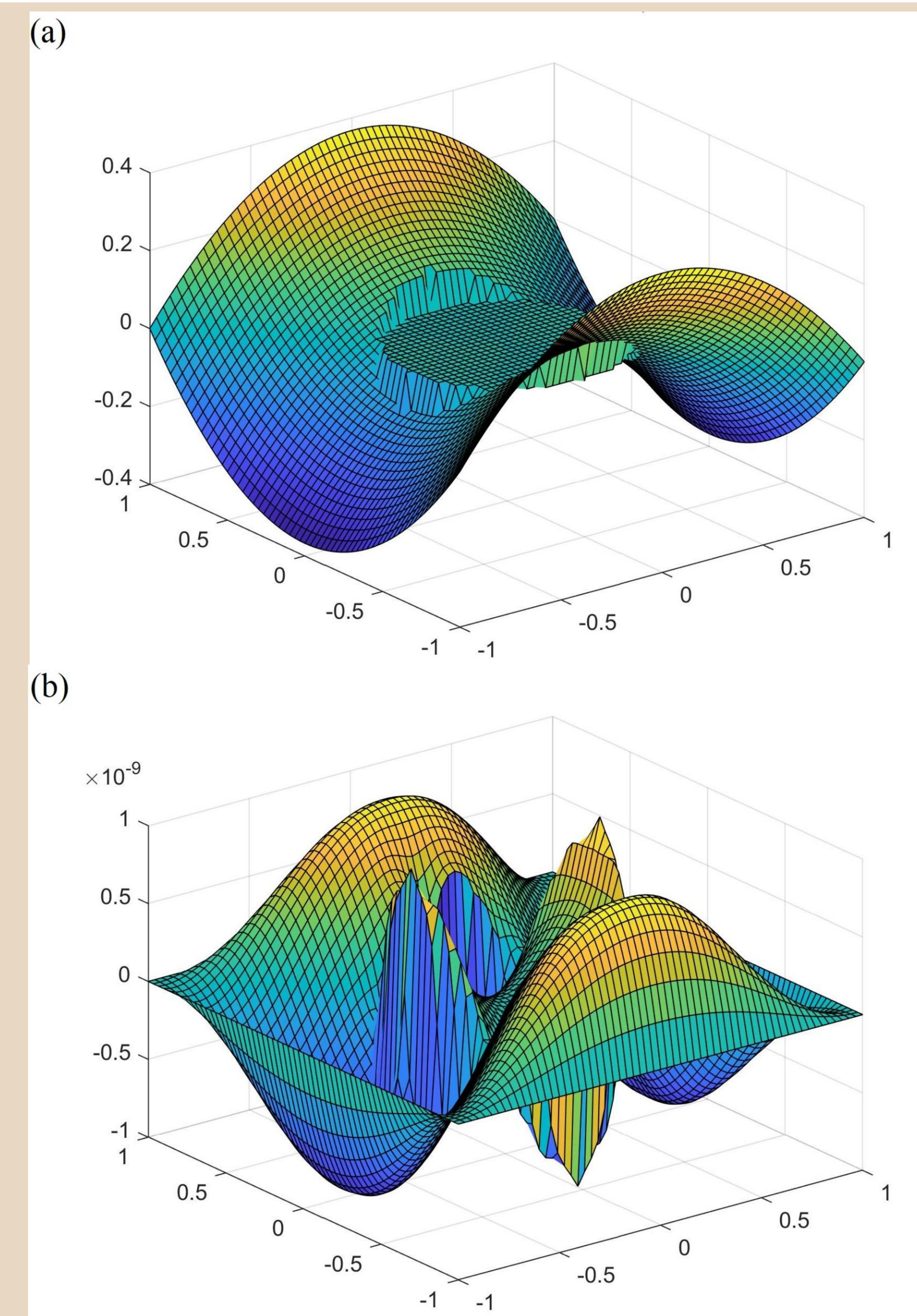
N	$E(u)$	r	$E(u_n)$	r	$E(u_\eta)$	r
20	9.80E-10		7.19E-09		6.51E-09	
40	2.66E-10	1.88	2.99E-09	1.27	3.51E-09	0.89
80	6.83E-11	1.96	1.13E-09	1.41	1.33E-09	1.40
160	1.59E-11	2.10	3.49E-10	1.69	4.96E-10	1.42
360	3.11E-12	2.35	8.06E-11	2.12	1.87E-10	1.41

Temporal discretization analysis

Temporal discretization with $\beta^+ = 1$ and $\beta^+ = 10000$ at $T = 2$

Δt	$E(u)$	r	$E(u_n)$	r	$E(u_\eta)$	r
1/40	6.83E-11		1.13E-09		1.33E-09	
1/80	2.14E-11	1.68	4.43E-10	1.35	4.96E-10	1.42
1/160	3.24E-12	2.72	6.41E-11	2.79	9.87E-11	2.33
1/320	8.10E-13	2.00	3.95E-12	4.02	6.09E-12	4.02

Numerical solution and error distribution



(a) Plot of numerical solution, (b) error with $\beta^+ = 1$ and $\beta^+ = 1000$

Summary and discussion

We present an algorithm for solving two-dimensional parabolic interface problems. Here we use the Crank-Nicolson scheme together with direct IIM to solve parabolic interface problems. The resulting method is second-order accurate in both space and time for both solution and gradients.