A Spatially and Temporally Second Order Method for Solving Parabolic Interface Problems

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A spatially and temporally second order method for solving parabolic interface problems

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Introduction
Parabolic interface problems have many applications in physics and biology, such as hyperthermia treatment of cancer, underground water flow, and food engineering.

Goal: Obtain a numerical methodology for solving 2D parabolic interface problems, which results in second-order accuracy in both space and time for both solutions and the solution’s gradient.

We studied the following interface problem:

\[ u_t = \beta u_{xx} + \beta u_{xy} - f(x,y,t), \quad (x,y) \in \Omega \cap \Gamma \]

\[ u = w(x,t), \quad [u] = \psi(x,t), \]

with specified boundary and initial conditions.

Crank-Nicolson scheme and direct Immersed Interface Method (IIM) for solving parabolic interface problems

For simplicity, we consider the interface problems with piecewise constant coefficients. For any grid point \((x_i,y_j)\), we can write the Crank-Nicolson scheme as,

\[ \frac{U^{n+1}_{i,j} - U^n_{i,j}}{\Delta \beta} = \frac{1}{2} \left( \delta_x U^m_{i,j} + (C^x_{i,j})^{n+1} + \delta_y U^n_{i,j} + (C^y_{i,j})^{n+1} - \frac{f^m_{i,j}}{\beta} \right) + \delta_x U^n_{i,j} + (C^x_{i,j})^{n+1} + \delta_y U^n_{i,j} + (C^y_{i,j})^{n+1}, \]

where, \(\delta_x U^n_{i,j} = \frac{U^n_{i+1,j} - 2U^n_{i,j} + U^n_{i-1,j}}{\Delta x^2}\) and correction term at an irregular grid point (See Fig.1) is given by,

\[ (C^x_{i,j}) = \sum_{m=0}^{9} a_m U^n_{M_{i,j}+\alpha_{m}}, \]

where, \(a_m\) and \(a_{m}^x\) depend on the position of \((x_i,y_j)\) and \(\beta\). Moreover, \(a_m\) is time-independent while \(a_{m}^x\) is time-dependent.

Now, the linear system of the Crank-Nicolson scheme can be written as,

\[ A U^{n+1} = B U^n + F, \]

where, \(U^n\) is the numerical solution, \(A\) and \(B\) are the coefficient matrix, and \(F\) is a vector constructed from the source term, boundary conditions and correction terms.

Note: \(A\) and \(B\) are only needed to be found at once. For piecewise constant \(\beta\), \(A\) and \(B\) are nine-point banded matrices for irregular grid points and five-diagonal for regular points.

Numerical Examples
Consider the following equation with the interface \(\Gamma: x^2 + y^2 = 0.25\) on the domain \(\Omega = [-1.1] \times [-1.1],\)

\[ u_t = (\beta u_{xx})^1 + (\beta u_{xy})^1 - f(x,y,t), \]

The source term \(f\) is defined as,

\[ f(x,y,t) = \begin{cases} 0 & \text{if } (x,y) \in \Omega^- \\ e^{-t(x^2 - y^2)} & \text{if } (x,y) \in \Omega^+ \end{cases} \]

Jump conditions are given by,

\[ [u] = e^{-t(x^2 - y^2)}, \quad [\beta u_{x}] = 4\beta^+ e^{-t(x^2 - y^2)}. \]

Boundary conditions are obtained from the exact solution of,

\[ u(x,y,0) = \begin{cases} 0 & \text{if } (x,y) \in \Omega^- \\ e^{-t(x^2 - y^2)} & \text{if } (x,y) \in \Omega^+ \end{cases} \]

Numerical and error distribution

Fig. 1. Illustration of regular, irregular, and control points in the domain \(\Omega\) with the interface \(\Gamma\).

Grid refinement analysis
A grid refinement analysis with \(\beta^+ = 1\) and \(\beta^+ = 10000\) at final time \(T = 1\).

<table>
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<th>N</th>
<th>(E(u))</th>
<th>r</th>
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Summary and discussion
We present an algorithm for solving two-dimensional parabolic interface problems. Here we use the Crank-Nicolson scheme together with direct IIM to solve parabolic interface problems. The resulting method is second-order accurate in both space and time for both solution and gradients.

Temporal discretization analysis

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(a) Plot of numerical solution, (b) error with \(\beta^+ = 1\) and \(\beta^+ = 1000\).