Spontaneous Symmetry Breaking and Goldstone Theorem

Emilia Szymańska
University of Wrocław

SPONTANEOUS SYMMETRY BREAKING AND GOLDSTONE THEOREM

Emilia Szymańska[1]
Mentor: Akaki Rusetsky[2]
[1]Faculty of Physics and Astronomy, University of Wrocław, pl. Maxa Borna 9, 50-204 Wrocław, Poland
[2]HISKP, University of Bonn, Nussallee 14-16, 53715 Bonn, Germany

Abstract:

We discuss the concept of spontaneous symmetry breaking and illustrate it with a general example. We consider Wigner-Weyl and Nambu-Goldstone realisations of symmetry in the quantum theory. Next, we state Goldstone's theorem and sketch its proof. We discuss why quantum chromodynamics is not realised in the Wigner-Weyl mode. We also consider different order parameters of spontaneous chiral symmetry breaking.

Key words: symmetry breaking, Goldstone bosons

1. INTRODUCTION: SYMMETRIES IN QUANTUM THEORIES

In a quantum theory we define a physical symmetry as a probability amplitude preserving map in the Hilbert space of the theory. Explicitly speaking, it is an injection acting on two arbitrary states in the Hilbert space $\mathscr{H}$:

$$|\alpha\rangle \longrightarrow |\alpha'\rangle\,, |\beta\rangle \longrightarrow |\beta'\rangle \tag{1}$$

which satisfies

$$|\langle\alpha|\beta\rangle| = |\langle\alpha'|\beta'\rangle|\,. \tag{2}$$

By Wigner's theorem these transformations are implemented by either unitary or antiunitary operators. Unitary operators implement continuous symmetries (they are "continuously connected" with identity - $\mathbb{1}$ is unitary), while discrete symmetries can be implemented by either anti- or unitary operators.

By Noether's theorem, the conserved charges $\mathcal{Q}^a$ associated with continuous symmetries are the generators of infinitesimal transformations of quantum fields:

$$[\mathcal{Q}^a, H] = 0, \tag{3}$$

where H is the Hamiltonian of the theory.

The symmetry group generated by the operators $\mathcal{Q}^a$ is implemented in $\mathscr{H}$ by a set of unitary operators $\mathscr{U}(\alpha)$, with α^a labelling the transformation ($a = 1, \ldots, \dim \mathfrak{g}$, with $\mathfrak{g}$ being the algebra associated with the generators). The operators $\mathscr{U}(\alpha)$ can be written as

$$\mathscr{U}(\alpha) = e^{i\alpha^a \mathcal{Q}^a}. \tag{4}$$

2. WIGNER-WEYL MODE VS NAMBU-GOLDSTONE MODE

There are two different ways in which a symmetry group can be realised in a quantum theory, depending on the way its elements act on the ground state of the theory. Let us first consider the case:

$$\mathscr{U}(\alpha)|0\rangle = e^{i\alpha^a \mathcal{Q}^a}|0\rangle = |0\rangle \qquad \text{and} \qquad \mathcal{Q}^a|0\rangle = 0, \tag{5}$$

then the symmetry is manifest: vacuum shares the symmetry of the theory. This realisation of the symmetry is called **Wigner-Weyl mode**.

Let us now consider the case:

$$\mathscr{U}(\alpha)|0\rangle = e^{i\alpha^a \mathcal{Q}^a}|0\rangle \neq |0\rangle \qquad \text{and} \qquad \mathcal{Q}^a|0\rangle \neq 0. \tag{6}$$

Using our interpretation of the $\mathcal{Q}^a$ we can conclude that "the vacuum $|0\rangle$ is charged". We know that (3) holds, hence $\mathcal{Q}^a|0\rangle$ is degenerate with $|0\rangle$. It turns out we have a set of degenerate vacua. This means that the symmetry is spontaneously broken. This realisation of symmetry is called **Nambu-Goldstone mode**.

Let us now construct the states

$$|\pi^a(\vec{p})\rangle = \int d^3\vec{x}\; e^{i\vec{p}\cdot\vec{x}} j_0^a(\vec{x})\,|0\rangle\,, \tag{7}$$

where $j_0^a(x)$ is the conserved current associated with $\mathcal{Q}^a$. Let us call the vacuum energy E_0. These states have then the energy $E_0 + E(\vec{p})$. It is easy to check that $E(\vec{p})|_{\vec{p}=0} = 0$. Hence the state $|\pi^a(\vec{p})\rangle$ contains massless particles that emerge in result of the symmetry breaking. These particles are called Goldstone bosons.

The existence of a non-vanishing vacuum expectation value of the commutator $\langle 0| [\mathcal{Q}^a, \phi(x)] |0\rangle \neq 0$ for a field in a continuous symmetry yields massless particle(s) in the theory (the number depends on the number of the "broken generators"), see e.g. the discussion in (Meissner, 2002). We can prove the contraposition of the theorem:

$$P_\mu P^\mu \geq \varepsilon > 0 \Rightarrow \langle 0| [\mathcal{Q}^a, \phi(x)] |0\rangle = 0, \tag{8}$$

where ε is the mass gap (Coleman et al. 2018). Since

$$i \langle 0| [\mathcal{Q}^a, \phi(x)] |0\rangle = i \int d^3x \langle 0| [j_0^a(\vec{x}, t), \phi(y)] |0\rangle , \tag{9}$$

it is sufficient to show that $\langle 0| [j_\mu(x), \phi(y)] |0\rangle = 0$.

Proof. Consider $\langle 0| j_\mu(x)\phi(y) |0\rangle$. We can write a Källén-Lehmann-like spectral decomposition for this matrix element:

$$\langle 0| j_\mu(x)\phi(y) |0\rangle = \int d^4p\, \sigma(p^2)\theta(p_0)\, p_\mu e^{ip\cdot(x-y)}. \tag{10}$$

The vacua do not contribute, as by Lorentz invariance $\langle 0| j_\mu(x) |0\rangle = 0$ (there are no Lorentz covariant vectors in the theory).

Next, we differentiate our result. Using the current conservation we find

$$\langle 0| \partial^\mu j_\mu(x)\phi(y) |0\rangle = \int d^4p\, \sigma(p^2)\theta(p_0)\, p_\mu p^\mu e^{ip\cdot(x-y)} = 0. \tag{11}$$

Hence, $p^2\sigma(p^2) = 0$. Now, by assumption, $p^2 \geq \varepsilon > 0$, so we can safely divide by p^2 to obtain $\sigma(p^2) = 0$. This means that

$$\langle 0| j_\mu(x)\phi(y) |0\rangle = 0. \tag{12}$$

Similar reasoning provides us with $\langle 0| \phi(y) j_\mu(x) |0\rangle = 0$. We can finally write

$$\langle 0| [j_\mu(x), \phi(y)] |0\rangle = 0. \tag{13}$$

□

It is worthwhile to note that after the symmetry is broken by the choice of a possible vacuum all other possible vacua become inaccessible in the infinite volume limit. In ordinary quantum mechanics (where the number of degrees of freedom is finite) tunnelling between different vacua is possible, so the true ground state is their symmetric combination. On the other hand, in quantum field theory (where the number of degrees of freedom is infinite) switching from one vacuum into another would induce the change of the vacuum everywhere in space. The probability of such switching is vanishing. To make the above arguments more explicit, let us consider a set of spin-$\frac{1}{2}$ magnets with nearest neighbour interactions (Álvarez-Gaumé, et al., 2011). Our space is now a lattice with spacing a and lattice vectors $\vec{x} = (n_1 a, n_2 a, n_3 a)$. At each lattice site $\vec{x}$, there is a spin-$\frac{1}{2}$ degree of freedom $\vec{s} = (\frac{1}{2}\sigma_1, \frac{1}{2}\sigma_2, \frac{1}{2}\sigma_3)$ with σ_i being Pauli matrices. The Hamiltonian is defined as

$$H = -J \sum_{\langle \vec{x}, \vec{x}' \rangle} \vec{s}(\vec{x}) \cdot \vec{s}(\vec{x}'), \qquad J > 0, \tag{14}$$

where $\langle \vec{x}, \vec{x}' \rangle$ indicates that we're summing over the nearest neighbours on the lattice. For each lattice site we have a 2-dimensional Hilbert space. We can take its basis to be the two $s_3(\vec{x})$ eigenstates $\{|\vec{x};\uparrow\rangle, |\vec{x};\downarrow\rangle\}$. The state corresponding to the spin aligned along direction $\hat{r}$ at the site $\vec{x}$ is

$$\hat{r} \cdot \vec{s}(\vec{x}) |\vec{x};\hat{r}\rangle = \frac{1}{2} |\vec{x};\hat{r}\rangle \tag{15}$$

We can express it in this basis as

$$|\vec{x};\hat{r}\rangle = \cos\left(\frac{\theta}{2}\right)|\vec{x};\uparrow\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|\vec{x};\downarrow\rangle\,. \tag{16}$$

We can easily show that $\langle\vec{x};\hat{r}|\vec{x};\hat{r}'\rangle = \cos(\alpha/2)$, where α is the angle between $\hat{r}$ and $\hat{r}'$. Let us now construct the ground states of our Hamiltonian. Each of them corresponds to all spins in the ferromagnet being aligned along a direction $\hat{r}$. We can then write:

$$|\hat{r}\rangle = \bigotimes_{\vec{x}} |.\vec{x};\hat{r}\rangle \tag{17}$$

The overlap between two different ground states is thus given by

$$\langle\hat{r}|\hat{r}'\rangle = \left[\cos\left(\frac{\alpha}{2}\right)\right]^N, \tag{18}$$

where $N = V/a^3$ is the number of lattice sites. As the number of lattice sites increases, the overlap (18) vanishes (unless $\hat{r}$ and $\hat{r}'$ are parallel): the ground states associated to different directions mix less and less. In large volumes the mixing of the vacua is suppressed enough to approximate the finite volume theory by the theory with Goldstone bosons. In the limit, $V \to \infty$, the vacua become orthogonal and the spontaneous symmetry breaking occurs.

3. AN EXPLICIT EXAMPLE OF SYMMETRY BREAKING

We consider a set of n real scalar fields $\{\phi^i\}$ which we assemble into a vector $\mathbf{\Phi}$. We have an N-parameter group G with elements $g \in G$ which is characterised by real parameters $\alpha_a,\ a = 1,\ldots,N$. The action of the group takes the form:

$$G \ni g : \mathbf{\Phi} \to \mathscr{U}(g)\mathbf{\Phi} = e^{i\alpha^a \mathcal{Q}^a}\mathbf{\Phi}, \tag{19}$$

where $\mathcal{Q}^a$ are the generators of the group. With g near identity:

$$\mathscr{U}(g)\mathbf{\Phi} = \mathbf{\Phi} + i\alpha^a\mathcal{Q}^a\mathbf{\Phi} + \mathcal{O}\left(\alpha^2\right), \tag{20}$$

where $\mathcal{Q}^a$ are Hermitean, so $\partial_\mu\mathbf{\Phi}\cdot\partial^\mu\mathbf{\Phi}$ is invariant under the transformation shown in (19). The number of parameters α^a is equal to the number of generators. Commutators of any two generators of the group must give another generator:

$$\left[\mathcal{Q}^j, \mathcal{Q}^k\right] = i f_{jkl}\mathcal{Q}^l, \tag{21}$$

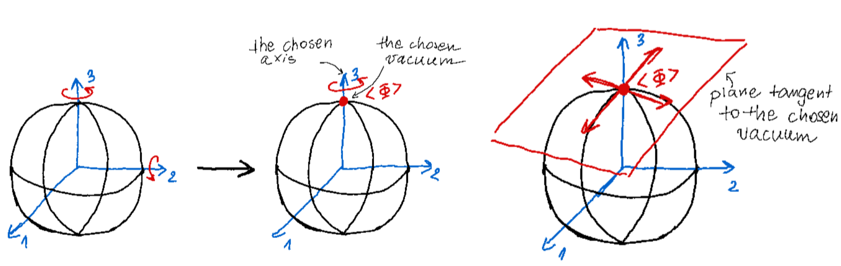
where f_{jkl} are the structure constants of group G. This means that the group generators form a closed Lie algebra.

Let us now consider a general Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\mathbf{\Phi})(\partial^\mu\mathbf{\Phi}) - U(\mathbf{\Phi}), \tag{22}$$

with the potential $U(\mathbf{\Phi})$ that is invariant under the group G. We assume that the potential $U(\mathbf{\Phi})$ is invariant under the group G. We pick one of the potential's minima to be our vacuum denoted as $\langle\mathbf{\Phi}\rangle$, which is not invariant under G. We will consider the case where $\langle\mathbf{\Phi}\rangle$ is invariant only under a subgroup $H \subset G$, which we will call the unbroken group. The remaining generators are the spontaneously broken generators.

For illustration, one could consider a 3-component real scalar field $\mathbf{\Phi} = (\phi_1, \phi_2, \phi_3)$ with the SO(3)-invariant "Mexican hat" potential. $\langle\mathbf{\Phi}\rangle$ would then be a vector of fixed length pointing in an arbitrary direction. We can choose it to point in the 3-direction: H would then be a subgroup containing rotations about the 3-axis, see Fig. 1.



(a) Spontaneous symmetry breaking in SO(3) (b)

Figure 1

The potential is constant on the plane tangent to the chosen vacuum (see Figure (1b)). This is why the particles are massless: they need no energy to move on this plane (Zee, 2010). One can easily see that we reduce our SO(3) symmetry to the SO(2) symmetry, which has one generator instead of three. This means there are two spontaneously broken generators and thus we have two Nambu-Goldstone bosons. It is important to note that the theory is indeed symmetric under the whole SO(3). The physics doesn't depend on any particular choice of vacuum and the different vacua are related by the full SO(3) symmetry.

In the generic case, let us arrange the generators G in the following way:

$$H = \{\mathcal{Q}^1, \mathcal{Q}^2, \ldots, \mathcal{Q}^m\} \tag{23}$$

and

$$G = \{\mathcal{Q}^1, \mathcal{Q}^2, \ldots, \mathcal{Q}^m, \mathcal{Q}^{m+1}, \ldots, \mathcal{Q}^N\}. \tag{24}$$

Generators of G not included in H do not leave $\langle \mathbf{\Phi} \rangle$ unchanged. This means that:

$$\sum_{a=m+1}^{N} \lambda^a \mathcal{Q}^a = 0 \quad \Rightarrow \quad \lambda^a = 0, \quad \forall a = m+1, \ldots, N. \tag{25}$$

No linear combination of spontaneously broken generators acting on the vacuum can give zero. Such a combination would have to be in H. The generators in (25) form an $(N-m)$-dimensional manifold. There are $N-m$ spontaneously broken symmetry generators, thus there are $N-m$ Goldstone bosons in the theory.

Let the potential be a multi-dimensional Mexican hat potential:

$$U(\mathbf{\Phi}) = \frac{1}{4}\lambda(\mathbf{\Phi}\cdot\mathbf{\Phi} - d^2)^2. \tag{26}$$

The symmetry group of the theory is $G = SO(n)$. Its dimension is the number of independent planes in the N-space: $\dim G = \frac{1}{2}n(n-1)$. The ground state satisfies: $\langle \mathbf{\Phi} \rangle \cdot \langle \mathbf{\Phi} \rangle = d^2$. Let us pick our vacuum to be

$$\langle \phi^n \rangle = d, \quad \langle \phi^a \rangle = 0, \quad a < n, \tag{27}$$

and label the remaining fields as $\mathbf{\Phi}_\perp = (\phi^1, \phi^2, \ldots, \phi^{n-1})$.

We define $\mathbf{\Phi'}$ as $\mathbf{\Phi} = \mathbf{\Phi'} + \langle \mathbf{\Phi} \rangle$. The potential has then the form

$$U = \frac{1}{4}\lambda({\phi_n}'\phi_n' + 2d\phi_n' + \mathbf{\Phi'}_\perp \cdot \mathbf{\Phi'}_\perp)^2. \tag{28}$$

We identify the masses of the particles: $m_N^2 = 2d^2\lambda$, $m_\perp^2 = 0$. The subgroup H is $SO(n-1)$ with $\dim H = \frac{1}{2}(n-1)(n-2)$. We explicitly see that there are $n-1$ Goldstone bosons: $\dim G - \dim H = n-1$.

4. SPONTANEOUS BREAKING OF CHIRAL SYMMETRY

The full symmetry of massless quantum chromodynamics is $SU(N_f)_R \otimes SU(N_f)_L \otimes U(1)_V \otimes U(1)_A$, N_f being the number of light flavours. The conserved charges correspond to the vector and axial vector currents:

$$\mathcal{Q}_V^i = \int d^3\vec{x} V_0^i(\vec{x}, t), \quad \mathcal{Q}_A^i = \int d^3\vec{x} A_0^i(\vec{x}, t), \tag{29}$$

where $i = 1, \ldots, N_f^2 - 1$. The conservation of charges means that

$$\left[H, \mathcal{Q}_V^i\right] = \left[H, \mathcal{Q}_A^i\right] = 0. \tag{30}$$

The conserved charges form a closed group structure.

Let us now discuss the possible realisations of the symmetry. In Wigner-Weyl mode the vacuum is not charged: $\mathcal{Q}_V^i \left|0\right\rangle = \mathcal{Q}_A^i \left|0\right\rangle = 0$. Let us consider a one-particle state $\left|a\right\rangle$, with $\vec{p} = 0$. This means that $H\left|a\right\rangle = m\left|a\right\rangle$. Thanks to (30) we can see that the states $\mathcal{Q}_V^i \left|a\right\rangle$, $\mathcal{Q}_A^i \left|a\right\rangle$ have the same energy:

$$H\left(\mathcal{Q}_{V/A}^i \left|a\right\rangle\right) = \mathcal{Q}_{V/A}^i \left(H\left|a\right\rangle\right) = m\left(\mathcal{Q}_{V/A}^i \left|a\right\rangle\right). \tag{31}$$

The states $\mathcal{Q}_V^i \left|a\right\rangle = \sum_b c_{ab} \left|b\right\rangle$, $i = 1, \ldots, N_f^2 - 1$, form a multiplet of $SU(N_f)$. The states $\mathcal{Q}_A^i \left|a\right\rangle$ will form a $SU(N_f)$ multiplet as well, but with different parity and in general, corresponding to a different irreducible representation of $SU(N_f)$. Thus we have two multiplets corresponding to the same mass, yet having different parity. This is not observed in nature. QCD is not realised in Wigner-Weyl mode. On the other hand, in Nambu-Goldstone mode we have

$$\mathcal{Q}_V^i \left|0\right\rangle = 0, \mathcal{Q}_A^i \left|0\right\rangle \neq 0, \tag{32}$$

and there is no parity doubling.

Consider now the quantity

$$\left\langle 0\right| A_\mu^k(x) \left|\pi^i(q)\right\rangle = i q_\mu F_\pi e^{-iqx}, \tag{33}$$

where $\left|\pi^i(q)\right\rangle$ is a pion state with momentum q and F_π is the pion decay constant. If the chiral symmetry is spontaneously broken, then $F_\pi \neq 0$ and vice versa. Thus, F_π is an order parameter of spontaneous breaking of chiral symmetry.

Another order parameter is the quark condensate:

$$\left\langle 0\right| \bar{\psi}(0)\psi(0) \left|0\right\rangle = \left\langle 0\right| \left[\bar{\psi}_L(0)\psi_R(0) + \bar{\psi}_R(0)\psi_L(0)\right] \left|0\right\rangle . \tag{34}$$

If $\left\langle 0\right| \bar{\psi}(0)\psi(0) \left|0\right\rangle \neq 0$, then we can be sure that the symmetry is broken, though it can be broken even if $\left\langle 0\right| \bar{\psi}(0)\psi(0) \left|0\right\rangle = 0$.

5. CONCLUSIONS

Symmetries in quantum theories can be realised in two different ways. In Wigner-Weyl mode the symmetry transformation leaves the vacuum invariant. In Nambu-Goldstone mode the vacuum is not invariant under the transformation and we obtain a set of degenerate vacua. In this case the symmetry is spontaneously broken: we have to choose one of the many ground states to be our vacuum. Goldstone theorem states that every generator of the spontaneously broken continuous symmetry corresponds to one massless boson. Spontaneous symmetry breaking manifests itself in the choice of the ground state. The Lagrangian itself retains its fundamental symmetry. It can be also shown that Goldstone bosons have perturbative dynamics at small momenta. Their interactions are vanishing in the limit $p_\mu \to 0$.

Quantum chromodynamics is not realised in Wigner-Weyl mode, since the degenerate multiplets with different parity are not observed experimentally. The unique order parameter of chiral symmetry breaking is the pion decay constant F_π. If its value is nonzero, the symmetry is spontaneously broken and vice versa. Another order parameter is the quark condensate. Its nonzero value is sufficient to determine that the symmetry is spontaneously broken, although spontaneous symmetry breaking can occur even if $\left\langle 0\right| \bar{\psi}(0)\psi(0) \left|0\right\rangle = 0$.

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