Summer 2006

Estimating Familial Correlations Using a Kotz Type Density

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A Dissertation Submitted to the Faculty of
Old Dominion University in Partial Fulfillment of the
Requirement for the Degree of

DOCTOR OF PHILOSOPHY

COMPUTATIONAL AND APPLIED MATHEMATICS

OLD DOMINION UNIVERSITY
August 2005

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ABSTRACT

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Two useful familial correlations often used to study the resemblance between the family members are the sib-sib correlation (ρ_{ss}) and the mom-sib or parent-sib correlation (ρ_{ps}). Since their introduction early in the last century by Galton, Fisher and others, many improved estimators of these correlations have been suggested in the literature. Several moment based estimators as well as the maximum likelihood estimators under the assumption of multivariate normality have been extensively studied and compared by various authors. However, the performance of these estimators when the data are not from multivariate normal distribution is poor. In this dissertation, we provide alternative estimates of ρ_{ss} and ρ_{ps} by minimizing the objective function,

$$n \log |\boldsymbol{\Sigma}| + \sum_{i=1}^{n} [(\mathbf{x}_i - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{x}_i - \boldsymbol{\mu})]^{\frac{1}{2}},$$

where $\boldsymbol{\Sigma}$ is a positive definite matrix with an appropriate structure involving ρ_{ss} and ρ_{ps}. Using extensive simulations from different multivariate distributions and using the bias, the mean squared error, and Pitman probability of nearness we have established that the alternative estimators are better than the existing estimators in most situations. The problems of testing of hypothesis about ρ_{ss} and ρ_{ps} and those of testing the equality of two sib-sib correlations and two mom-sib correlations are also considered. Alternative tests using Srivastava's well known estimators of sib-sib and mom-sib correlations and their asymptotic variances are proposed and compared using simulations. The proposed tests have better estimated sizes and powers than the likelihood based tests when data are from a multivariate normal distribution. Proposed methods are illustrated on Galton's famous classical data set on statures of families. These data are important, in that, the original note book on which these data were recorded by Galton in 1886 has been recently discovered and digitalized.

ACKNOWLEDGMENTS

I would like to extend my sincere appreciation to the following people who have made this work possible.

First and foremost, I thank my principal advisor Dr. Dayanand Naik to whom I am deeply indebted for his continuous supervision and guidance throughout all the stages of my research which constituted this dissertation. Thank you for your patience and understanding and thank you for sharing your tremendous intellect with me.

I am sincerely grateful to the honorable professors who have served in my committee; Drs. Rao Chaganty, Larry Lee and Edward Markowski. I thank you for your time, advice and constructive comments.

I would also like to express my sincere gratitude to the department of Mathematics and Statistics at Old Dominion University represented by the chairman, Dr. J. Mark Dorrepaal, and the secretary Mrs. Barbara Jeffrey. I thank Dr. Dorrepaal for his cooperation and understanding which facilitated my study and work. To Mrs. Jeffrey I say: Thank you so much Barbara for all the support and help you have provided me and for the care and concern you have always expressed, especially during the work on my dissertation. I also extend my warm thanks to secretary Gayle for her help and cooperation.

A warm note of well-deserved thanks goes to Dr. J. Wallace Van Orden, Arturo Tejada Ruiz, Bonney Berck, Deepak Mav, Yiaho Deng. I thank you all for your help.

Last but not least, I would like to express my gratefulness to my husband Majdi for his unwavering support and remarkable patience through it all. Thank you for the priceless gift of time that you have provided me. It is with loving gratitude that I dedicate this dissertation to you and to our beloved son, Saloha.

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CHAPTER I

INTRODUCTION

An interest in the study of familial correlations has been around for a long time, since Galton (1888). Galton worked with data on human stature of family members. In an attempt to study the relationships between various measurements on parents and their children, Galton and then Pearson (1896) considered different correlation measures, such as, sib-sib (or child to child) and parent-sib (or mom-sib) correlations. Numerous attempts have been made since then to provide better estimates of these correlations. Even after more than one hundred years, the study of these correlations have not ceased to exist. The problems of studying the familial correlations such as sibling-sibling (sib-sib) and parent-sibling (mom-sib) correlations are still important is clear from the vast amount of literature that as is being produced. The literature mainly deals with estimation of sib-sib (or intraclass) correlation (ρ_{ss}) and mom-sib (or interclass) correlation (ρ_{ms} or ρ_{ps}) under equal and unequal family sizes. Several articles also have considered the problems of testing of hypothesis about these parameters. There are also some articles discussing testing problems involving equality of these correlations from two or more populations.

Pearson (1896) proposed estimating sib-sib correlation by computing the sample product moment of every possible pair of observations from siblings. Fisher (1925) proposed a sib-sib correlation based on the analysis of variance (ANOVA) and this was appropriate for the balanced case where sib-ships are identical in size. Fisher's ANOVA estimator was generalized by Fieller and Smith (1951) to accommodate unbalanced sibship sizes. Smith (1957) proposed a further improvement by associating a weight with each component of the between sum of squares, yielding the uniform ANOVA (ρ_u) estimator in the case where weights are equal. Donner and Koval (1980) derived the maximum likelihood estimator of the sib-sib correlation for the unbalanced case. Using simulations from a multivariate normal distribution they showed their MLE outperforms the ANOVA intraclass correlation coefficient for extreme values of sib-sib correlation.

Although many measures of correlation among siblings have been proposed, each one has its own deficiency. For example, Pearson's pairwise intraclass correlation

The journal model for this thesis is *Biometrics* coefficient weights a sibship of size 10 by 45 times as much as a sibship of size 2, although it does not provide 45 times as much information. The dependence on balance limits the usefulness of Fisher's estimator, since families vary in size. While Fieller and Smith's estimator addresses both Pearson's and Fisher's problems, providing only 5 times as much weight in this instance, their estimator is inefficient for small sib-sib correlations. Although Smith's estimator, ρ_u, addresses several weaknesses of previous estimators, it also possesses Fieller and Smith's inefficiency for small sib-sib correlations. While Donner and Koval's ML estimator outperforms the ANOVA and Pearson's correlation measures for the unbalanced case, there is no closed form for this measure. Keen (1993) showed that the product-moment estimators and ANOVA estimators have similar efficiencies, but again noted that efficiencies are cumbersome to calculate without closed forms in the unbalanced case, even with the aid of numerical methods.

Srivastava (1993) suggested an estimator which is an efficient combination of two non-iterative estimators proposed by Smith (1957) and showed that this combination estimator has better efficiency than either one of them.

Estimation of the mom-sib correlation has been of interest since early 1950's. Kempthorne and Tandan (1953) used a linear model to estimate this interclass correlation under the assumptions that $\sigma_s^2 = \sigma_p^2$, that is, the parent and sibling variances are the same and ρ_{ss}, the intraclass correlation, is given. Since these assumptions are unnecessarily restrictive, another measure called the pairwise estimator, was introduced. The pairwise estimator of mom-sib correlation (r_p) is computed by pairing values for each sibling in the family with the parent's value. While the pairwise estimator has intuitive merit, it violates the required assumption that the data are independent.

Other estimators used in lieu of the pairwise estimator are the sib-mean estimator, where the mean value for all siblings is paired with the parent's value, and the random sib estimator, where a single sibling from the family is chosen randomly and this sibling's value is paired with the parent's value. Rosner, Donner and Hennekens (1977) proposed the ensemble estimator, based on the random sib estimator, and compared these three measures to the pairwise estimator. This ensemble estimator computes an expected mom-sib correlation over all possible random mom-sib pairings as described in the random sib estimator. Rosner, et al. (1977) determined that the ensemble and pairwise estimators were superior to both the sib-mean and random sib estimators based on smallest MSE criteria. And when ρ_{ps} is small (<=0.1), the pairwise estimator outperforms the ensemble estimator and when ρ_{ps} is large (>=0.5), the ensemble estimator outperforms the pairwise estimator.

The linear model approach was re-examined by Mak and Ng (1981) following Rosner's (1979) development of MLEs for the balanced and unbalanced cases. Mak and Ng's approach simplified derivation of the MLEs by using a linear model, such that given the mother's value, the mom-sib correlation can be determined and tested. Their approach assumes, as in Kempthorne (1953), that only the mother's score is random, whereas Rosner's approach allows both mom and sib scores to be random. Srivastava (1984) gives an estimate of mom-sib correlation for the unbalanced case that has a similar bias, but smaller asymptotic variance compared to the ensemble estimator, as noted in Velu and Rao (1990). Srivastava and Katapa (1986) provided the asymptotic variance of the Srivastava estimator.

Since the introduction of Srivastava's estimator, variants of the traditional interclass correlations have been proposed, but in each case compared to Srivastava's estimator and the ensemble estimator. Two comparisons by Srivastava and Keen (1988) and Eliasziw and Donner (1990) determine that Srivastava's estimator is uniformly more efficient than the ensemble estimator, but the magnitude is relatively small. Thus, both estimators perform similarly and choice of approach lies in the small increase in efficiency by using the Srivastava estimator.

Although not directly applicable to familial data, in the same spirit, Khattree and Naik (1994) considered the problem of estimating interclass correlation under a circular covariance matrix and this work was later expanded by Hartley (1997) and Hartley and Naik (2001).

Testing theory for the intraclass and interclass correlations was discussed in the late 1970s and early 1980s. In 1984, around the same time when Srivastava proposed his estimator (but before Srivastava discussed testing), Donner and Bull (1984) compared four methods for testing that the mom-sib correlation is zero. The four tests they considered were the likelihood ratio test (LRT), a test based on the large sample variance of the maximum likelihood estimator (MLE), an adjusted pairwise test, and a test (Z_p) based on the large sample variance of the pairwise estimator. Z_p uses the ratio of the pairwise estimator to its large sample standard error for testing. They found that under certain conditions, including that the data are from a normal distribution and family size is around 25, Z_p has size and power comparable to the LRT, especially for the most common moderate-to-small values of the mom-sib correlation.

Once the ensemble and Srivastava's estimators were introduced, comparison of testing procedures using these was done. For example, Konishi (1985) proposed two tests based on pairwise and ensemble estimators and via simulation from the normal distribution compared these estimators with the LRT. His simulation results showed that the LRT was most efficient. Velu and Rao (1990) studied testing procedures using the mean-sib correlation, the ensemble estimator, and Srivastava's estimator for small sample situations. They derived the exact null distribution of Srivastava's estimator.

The problem of testing the equality of sib-sib correlations for two populations was considered by Donner and Bull (1983) when family sizes within a population and between populations were the same, and by Khatri, Pukkila, and Rao (1989) when family sizes in the two populations were different. These authors derived and studied the performance of the likelihood ratio test. For the problem of testing equality of several correlations, Konishi and Gupta (1989) have suggested a modified likelihood ratio test and a test based on Fisher's z-transformation. Paul and Barnwal (1990) suggested a $C(\alpha)$ test, and Haung and Sinha (1993) derived the optimum invariant test, assuming the family sizes within populations are the same, but different for different populations.

Young and Bhandary (1998) and Bhandary and Alam (2000) respectively considered the problems of testing the equality of two and three correlation coefficients when the family sizes are unequal. They used Srivastava (1984)'s estimator of intraclass correlation and proposed the approximate likelihood ratio test and compared its performance with two other asymptotic tests based on normal distribution. They also made the assumption that the variances for different populations are the same.

It appears that there is not much work done for testing the equality of two parent-sib correlations.

So far in the literature, all the comparisons, whether it is of different estimators of intraclass or interclass correlations or of testing procedures for testing hypothesis about these correlations, is performed mostly via simulation experiments where the data are generated from multivariate normal distribution. The performance of these methods is unclear when the data may be from other symmetric but heavy tailed distributions. The main objective of this thesis is to investigate the performances of various procedures under non-normal distributions, such as a Kotz type distribution and multivariate T distribution. The T distribution has been used in the context of repeated measures study by Lange et al. (1989).

The Kotz type distribution, the probability density function (*pdf*) of which is provided below, has fatter tail regions than that of multivariate normal distribution.

$$f(\mathbf{x}, \boldsymbol{\mu}, \Sigma) = c \mid \Sigma \mid^{-\frac{1}{2}} \exp\ \{-[(\mathbf{x}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})]^{\frac{1}{2}}\},$$

where $\boldsymbol{\mu} \in \Re^p$, Σ is a positive definite matrix and $c = \frac{\Gamma(\frac{p}{2})}{2\pi^{\frac{p}{2}}\Gamma(p)}$.

See Plungpongpun (2003) and the references therein for properties and other details about this distribution. The contour plots of the pdf of this and normal distribution for the same set of parameters are provided in Figures I and 2 for comparison. Notice that the Kotz type density has a much narrower peak as compared to normal. Hence it is clear that the area covered in the tail regions of Kotz type density is larger than that under a normal density.

The Kotz distribution we have considered here was used in the context of multivariate analysis of variance and discriminant analysis by Plungpongpun (2003) and Naik and Plungpongpun (2004). The motivation behind using this distribution was two fold. First, by extensive simulation study it was determined in Plungpongpun (2003) that the tests for multivariate normality against other symmetric heavy tailed alternatives are not powerful and hence one cannot easily guaranty, based on these tests, that the data in hand are multivariate normal. Secondly, this particular Kotz type distribution has the property that the MLE of the location parameter is the generalized spatial median, which is a robust (against outliers) estimate of location parameter. It is expected that the estimators of scale parameters are also robust.

Hence in this thesis, we have adopted this density for estimation of scale matrix, in particular the intra and interclass correlations. Taking this Kotz type distribution as a model for fitting the familial data, we derive the maximum likelihood estimates and asymptotic tests based on the maximum likelihood theory. Our interest then points to a comparison of these estimators and tests with those derived using the maximum likelihood theory for multivariate normal distribution.

While it is true that the normal distribution based maximum likelihood estimators may be asymptotically biased under the assumption that the true model is a Kotz type distribution and vice versa, it is not uncommon in statistics to compare such estimators using the mean squared error and other criteria, such as, bias and Pitman nearness probability. In the statistical literature, in fact, there is a whole topic named "biased estimation" under which comparison of biased and unbiased estimators is routinely done. Of course one would not use the variance of the estimators for comparison here. For example, in the standard linear regression theory, the least squares estimators (which are unbiased) are routinely compared with the ridge regression estimators (which are by construction biased).

In this thesis we investigate the problem of estimation of the sib-sib correlation in Chapter 2 and that of mom-sib correlation in Chapter 3. Using simulation experiments and data from normal, Kotz type and T distributions, we compare different estimates by comparing their bias, mean squared error, and Pitman nearness probability.

Chapter 4 deals with four different testing of hypothesis problems. Testing of sib-sib correlation equal to zero is considered first. We use two sets (one under multivariate normal distribution and another under Kotz type distribution) of three well known asymptotic likelihood theory based tests, namely the likelihood ratio test (LRT), the Wald's test, and Rao's score test. These tests along with a test based on Srivastava's estimator are compared using extensive simulations. The data are simulated from different distributions. The simulation estimate of the sizes of the tests and powers are used for comparison. The problem of testing mom-sib correlation is zero is considered next and a similar type of comparison study is performed.

Next, the problems of testing the equality of two sib-sib correlations and testing the equality of two mom-sib correlations are considered. By adopting Srivastava's estimate we have provided estimates of common correlations under each of the null hypothesis. Alternative tests based on Srivastava's estimate and its asymptotic variance are proposed and compared with the likelihood based tests.

Recently, Hanley (2004) worked with family data on human stature obtained directly from Galton's note books (cf. Galton, 1886, 1889). The data consists of heights of 205 families with the number of children ranging from 1 to 15. For more details on this data set and on how to obtain it, see Hanely (2004), or visit *http : //www.epi.mcgill.ca/hanely/galton*. At the end of each chapter we illustrate our procedures on Galton's data.

All the computations and simulations are performed using SAS software. To obtain the programs electronically, send a request by e-mail to *dnaik@odu.edu*.

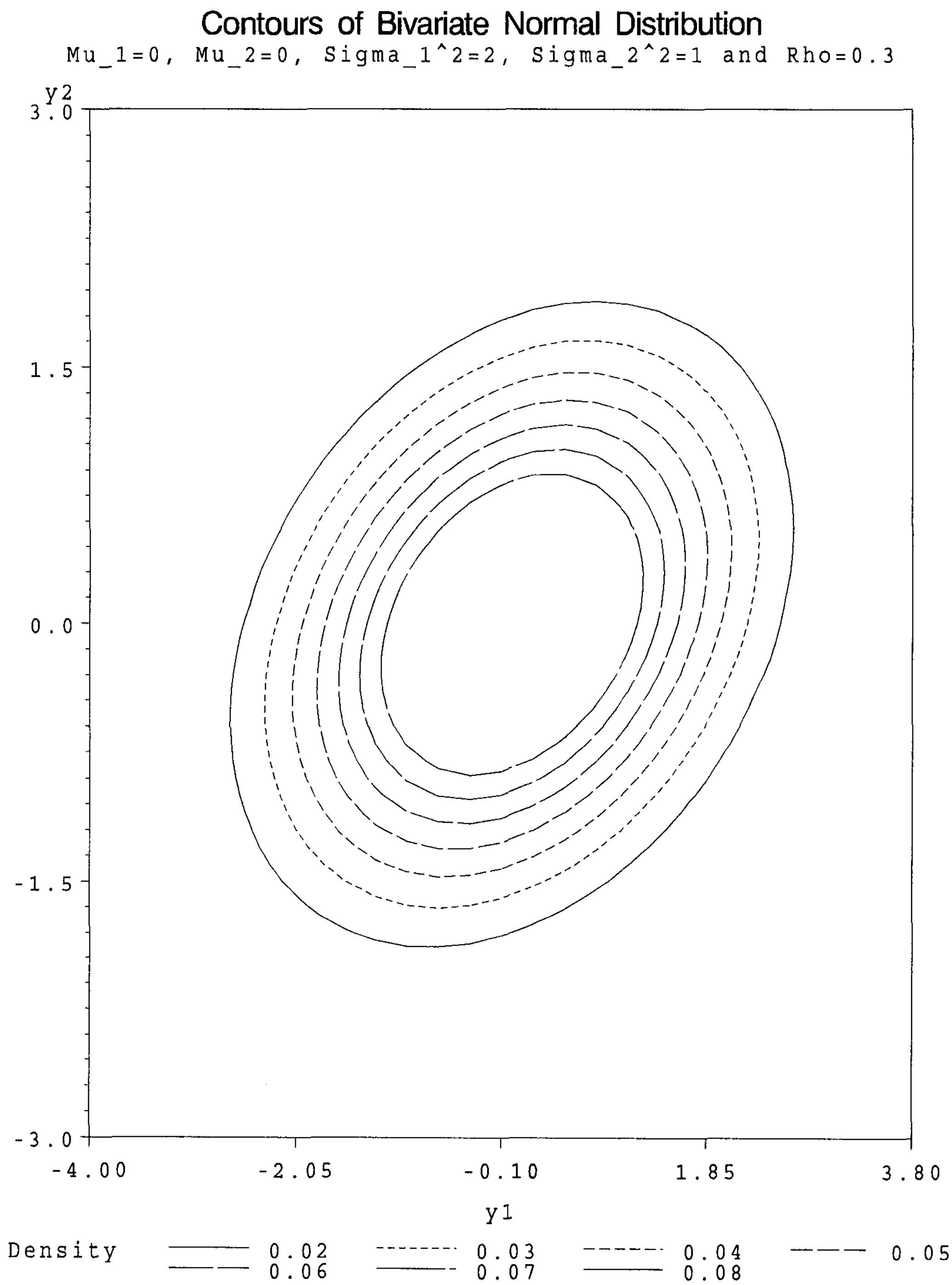


Figure 1. Contours of Bivariate normal distribution.

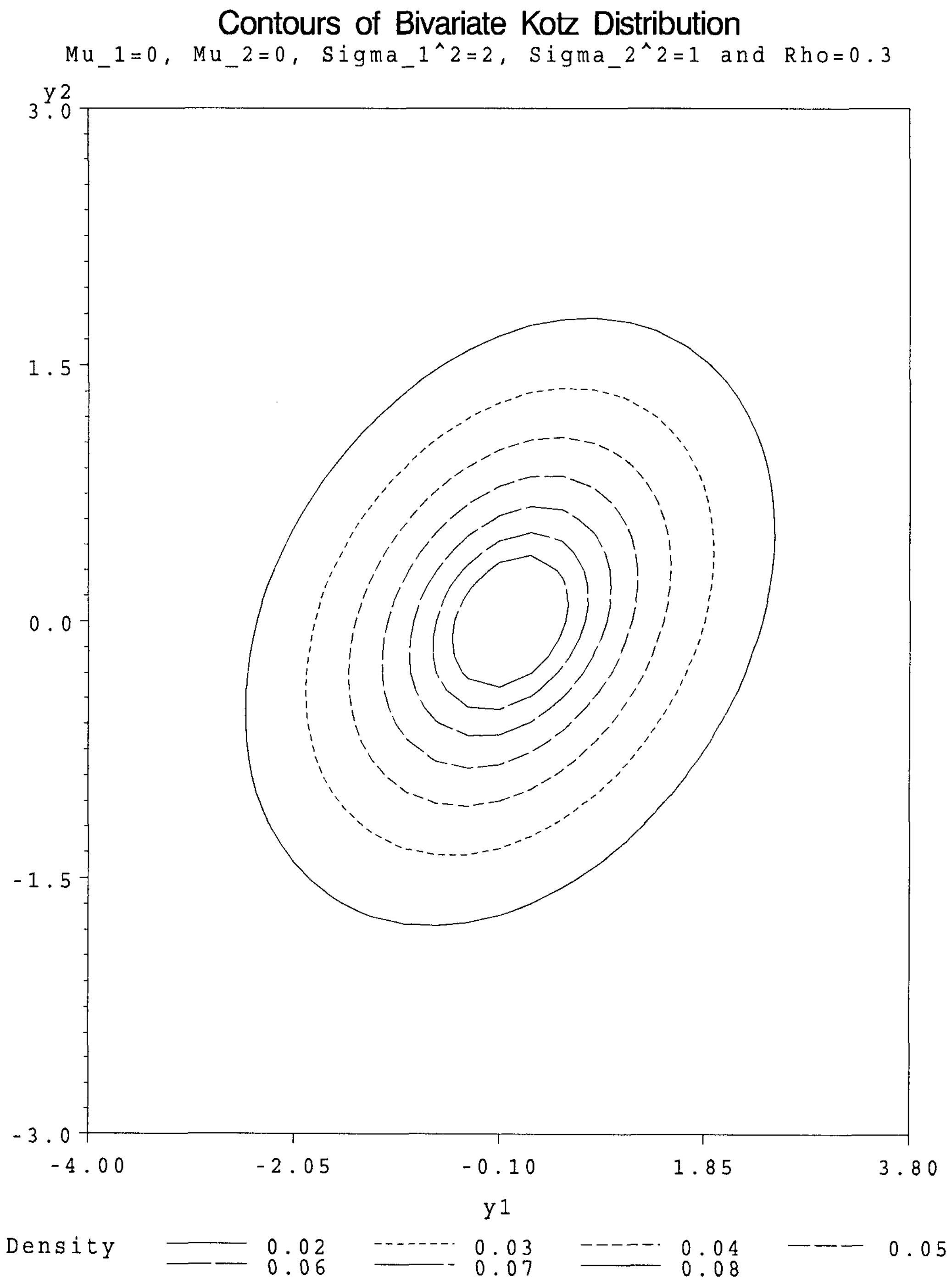


Figure 2. Contours of Kotz type distribution.

CHAPTER II

ESTIMATION OF SIB-SIB CORRELATIONS

II.1 Introduction

To measure the degree of resemblance between family members with respect to a specified characteristic, sib-sib correlation and parent-sib correlation are used. Numerous methods have been proposed in the literature to estimate the sib-sib or intraclass correlation (ρ_{ss}) since its introduction by Galton (1888). Pearson (1896) suggested estimating it by computing the product moment over every possible pair of observations from siblings. Fisher (1925) suggested using the analysis of variance (ANOVA) method for the balanced data where the sibship sizes are all equal. Fisher's estimator was generalized by Fieller and Smith (1951) to accommodate unbalanced sibship sizes. Smith (1957) proposed a further improvement by associating a weight with each component of the between sum of squares.

Suppose x_{ij}, $j = 1, ..., m_i$; $i = 1, ..., n$ is an observation on the jth child of the ith family. Let the vector of observations on the ith family be $\mathbf{x}_i' = (x_{i1}, ..., x_{im_i})$. Then the sib-sib correlation coefficient ρ_{ss} is the correlation coefficient between any x_{ij} and $x_{ij'}$ for $j \neq j'$.

The variance covariance matrix of $\mathbf{x}_i$ then can be written as

$$\begin{aligned} cov(\mathbf{x}_i) = \mathbf{\Sigma}_i &= \sigma^2[(1-\rho_{ss})\mathbf{I}_{m_i} + \rho_{ss}\mathbf{J}_{m_i}] \\ &= \sigma^2 \begin{pmatrix} 1 & \rho_{ss} & \cdots & \rho_{ss} \\ \rho_{ss} & 1 & \cdots & \rho_{ss} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{ss} & \rho_{ss} & \cdots & 1 \end{pmatrix} = \sigma^2 \mathbf{V}_i(\rho_{ss}), \end{aligned}$$

where $\sigma^2 = var(x_{ij})$, and $\mathbf{I}_{m_i}$ is an identity matrix of order m_i and $\mathbf{J}_{m_i}$ is the $m_i \times m_i$ matrix of all ones.

The uniform ANOVA estimator ($\hat{\rho}_{ss,u}$) of Smith (1957) in the case where weights are equal is given by

$$\hat{\rho}_u = \frac{\sum\limits_{i=1}^{n}(\bar{x}_i - \bar{x}_u)^2 - \frac{(n-1)(1-\bar{a})}{N-n}\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m_i}(x_{ij}-\bar{x}_i)^2}{\sum\limits_{i=1}^{n}(\bar{x}_i - \bar{x}_u)^2 - \frac{(n-1)\bar{a}}{N-n}\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m_i}(x_{ij}-\bar{x}_i)^2}, \tag{II.1}$$

where $\bar{x}_u = n^{-1}\sum_{i=1}^{n}\bar{x}_i$, $a_i = 1 - m_i^{-1}$, $\bar{a} = n^{-1}\sum_{i=1}^{n}a_i$ and $N = \sum_{i=1}^{n}m_i$.

Smith additionally proposed a generalized weighted ANOVA estimator, $\hat{\rho}_w$, which can be rewritten with weights $p_i = m_i(m_i - 1)$ as

$$\hat{\rho}_w = \frac{\sum_{i=1}^{n} p_i(\bar{x}_i - \bar{x}_p)^2 - \frac{p_c}{N-n}\sum_{i=1}^{n}\sum_{j=1}^{m_i}(x_{ij}-\bar{x}_i)^2}{\sum_{i=1}^{n} p_i(\bar{x}_i - \bar{x}_p)^2 - \frac{p_c(p_0-1)}{N-n}\sum_{i=1}^{n}\sum_{j=1}^{m_i}(x_{ij}-\bar{x}_i)^2}, \tag{II.2}$$

where $\bar{x}_p = \sum_{i=1}^{n}\bar{x}_i p_i/P$, $p_c = \sum_{i=1}^{n}(p_i - p_i^2/P)/m_i$, $p_0 = (P - \sum_{i=1}^{n}p_i^2/P)/p_c$, and $P = \sum_{i=1}^{n}p_i$.

Donner and Koval (1980) derived the maximum likelihood estimator (MLE_N) of ρ_{ss} and showed, when data are simulated from a multivariate normal distribution, that the MLE_N outperformed the ANOVA intraclass correlation coefficient for extreme values of sib-sib correlation. Additionally, for the unbalanced case, the MLE_N outperformed Pearson's correlation for all but the zero correlation scenario. Srivastava (1993) proposed an improved estimator of ρ_{ss} based on an efficient combination of $\hat{\rho}_u$ and $\hat{\rho}_w$ as

$$\hat{\rho}_S = \frac{\hat{\rho}_w}{1 + \hat{\rho}_w - \hat{\rho}_u}. \tag{II.3}$$

This estimator shows a great increase in asymptotic relative efficiency over either $\hat{\rho}_u$ or $\hat{\rho}_w$. While it is clear why MLE_N, being asymptotically the most efficient estimator, performs better than the other estimators under the multivariate normal data, its performance when data are not normal is not clear.

In this chapter we provide an alternative estimator for estimating ρ_{ss} and assess the performance of the new estimator against the others via a simulation experiment. For our simulation we generate data from multivariate normal and other symmetrical multivariate distributions, namely multivariate T and Kotz type.

II.2 An Alternative Approach: Balanced Case

First let us consider the case when all families have the same number of children. Suppose $\mathbf{x}_i$, $i = 1, ..., n$ is a vector of observations on m children in the ith family such that $E(\mathbf{x}_i) = \boldsymbol{\mu} = \mu\mathbf{1}$, $var(\mathbf{x}_i) = \boldsymbol{\Sigma} = \sigma^2[(1-\rho_{ss})\mathbf{I}_m + \rho_{ss}\mathbf{J}_m] = \sigma^2\mathbf{V}(\rho_{ss})$.

Note that the determinant and inverse of $\boldsymbol{\Sigma}$ respectively are

$$|\boldsymbol{\Sigma}| = (\sigma^2)^m[(1-\rho_{ss})^{m-1}(1+(m-1)\rho_{ss})] \text{ and}$$
$$\boldsymbol{\Sigma}^{-1} = \frac{1}{\sigma^2(1-\rho_{ss})}[\mathbf{I}_m - \frac{\rho_{ss}}{(1+(m-1)\rho_{ss})}\mathbf{J}_m].$$

In the new approach we propose to estimate ρ_{ss} by minimizing the objective function

$$F(\mu, \sigma^2, \rho_{ss}) = \log |\Sigma|^{\frac{n}{2}} + \sum_{i=1}^{n} \sqrt{(\mathbf{x}_i - \boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x}_i - \boldsymbol{\mu})}, \tag{II.4}$$

simultaneously with respect to ρ_{ss}, μ and σ^2.

Let $v = 1 + (m-1)\rho_{ss}^2$ and $w = 1 + (m-1)\rho_{ss}$.

This process leads to solving the following three equations

$$\frac{\partial F}{\partial \mu} = \sum_{i=1}^{n} \frac{\mathbf{1}'\Sigma^{-1}(\mathbf{x}_i - \mu\mathbf{1})}{\sqrt{(\mathbf{x}_i - \mu\mathbf{1})'\Sigma^{-1}(\mathbf{x}_i - \mu\mathbf{1})}} = 0,$$

$$\frac{\partial F}{\partial \sigma^2} = \frac{-mn}{2\sigma^2} + \frac{1}{2(\sigma^2)^{3/2}} \sum_{i=1}^{n} \sqrt{(\mathbf{x}_i - \mu\mathbf{1})'(\mathbf{V}(\rho_{ss}))^{-1}(\mathbf{x}_i - \mu\mathbf{1})} = 0,$$

$$\frac{\partial F}{\partial \rho_{ss}} = \frac{mn(m-1)\rho_{ss}}{2(1-\rho_{ss})w} - \frac{1}{2\sigma^2(1-\rho_{ss})^2} \sum_{i=1}^{n} \frac{(\mathbf{x}_i - \mu\mathbf{1})'(\mathbf{I}_m - \frac{v}{w^2}\mathbf{J}_m)(\mathbf{x}_i - \mu\mathbf{1})}{\sqrt{(\mathbf{x}_i - \mu\mathbf{1})'\Sigma^{-1}(\mathbf{x}_i - \mu\mathbf{1})}} = 0, \tag{II.5}$$

simultaneously with respect to μ, σ^2 and ρ_{ss}. Note the second equation can be written more compactly as

$$\hat{\sigma} = \frac{1}{mn} \sum_{i=1}^{n} \sqrt{(\mathbf{x}_i - \mu\mathbf{1})'(\mathbf{V}(\rho_{ss}))^{-1}(\mathbf{x}_i - \mu\mathbf{1})}.$$

There is no closed form solutions to these equations and hence one needs to solve these iteratively and numerically. Alternatively, using software one can directly numerically minimize the objective function (II.4) w.r.t. μ, σ^2, and ρ_{ss}. We have used SAS/IML procedure's dual Quasi Newton Method (NLPQN) routine for obtaining the estimates. The optimization gives unique estimates in the feasible regions under the above covariance structure. We observe that these estimators are also the maximum likelihood estimators of μ, σ^2, and ρ_{ss} when $\mathbf{x}_i, i = 1, ..., n$ is a random sample from a Kotz type distribution where the probability density function of an $m \times 1$ random vector $\mathbf{x}$ is given by

$$f(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}) = c|\boldsymbol{\Sigma}|^{-\frac{1}{2}} \exp\ \{-[(\mathbf{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})]^{\frac{1}{2}}\},\ \boldsymbol{\mu} \in \Re^m,\ \boldsymbol{\Sigma}\ p.\ d., \tag{II.6}$$

where $c = \frac{\Gamma(\frac{m}{2})}{2\pi^{\frac{m}{2}}\Gamma(m)}$, $\boldsymbol{\mu} = \mu\mathbf{1}$, and $\boldsymbol{\Sigma} = \sigma^2\mathbf{V}(\rho_{ss})$. The asymptotic distribution of these estimators under Kotz type distribution can be studied by computing the Fisher information matrix.

The following theorem is useful for computing the elements of the Fisher information matrix. See Mitchell (1989) and Lange, Little, and Taylor (1989) for details and proofs.

Theorem 1 *Suppose the pdf of* $\mathbf{x}$ *is given by*

$$c_m|\Sigma|^{-\frac{1}{2}}g((\mathbf{x}-\boldsymbol{\delta})'\Sigma^{-1}(\mathbf{x}-\boldsymbol{\delta}))$$

for an appropriate constant c_m *and nonnegative function* $g(.)$, $\mathbf{x} \in R^m$, $\boldsymbol{\delta} \in R^m$, *positive definite (p.d.) matrix* Σ. *Then* $\mathbf{y} = \Sigma^{-\frac{1}{2}}(\mathbf{x}-\boldsymbol{\delta})$ *(i.e.* $\mathbf{x} = \boldsymbol{\delta} + \Sigma^{\frac{1}{2}}\mathbf{y}$*) has the pdf* $c_m\ g(\mathbf{y}'\mathbf{y})$ *and further* $t = r^2 = \mathbf{y}'\mathbf{y}$ *has the probability density function*

$$c_m\frac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})}t^{\frac{m}{2}-1}g(t), t > 0.$$

Suppose $u = \dfrac{g'(r^2)}{g(r^2)}(=\dfrac{g'(t)}{g(t)})$, *then we have the following results:*

(a) $E(r^2u) = -\frac{m}{2}\ \{= E(Tu); T = r^2\}$

(b) $E\left(y'A\ y\ u^k\right) = \frac{1}{m}tr(A)E\left(r^2\ u^k\right)$

(c) $E\left(y'A\ y\ y'B\ y\ u^k\right) = \frac{1}{m(m+2)}\left\{2tr(AB) + tr(A)tr(B)\right\} E\left(r^4\ u^k\right)$

For a given function $g(t)$, c_m, u, *and pdf of* r^2 *are completely specified and hence (b) and (c) can be completely evaluated.*

For the Kotz type distribution,

$$g(t) = e^{-\sqrt{t}},\ t = (\mathbf{x}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu}),$$
$$u = \frac{g'(t)}{g(t)},\ g'(t) = \frac{d}{dt}g(t).$$

Let $\mathbf{y} = (\mathbf{x}_i - \boldsymbol{\mu})$. Then $f(\mu, \sigma^2, \rho_{ss}|\mathbf{y}) = c_m\,|\Sigma|^{-\frac{1}{2}}\,g(t)$ and

$$\begin{aligned}\log f &= \log\ c_m + \frac{1}{2}\log\ \left|\Sigma^{-1}\right| + \log\ g(t)\\ &= \log\ c_m - \frac{m}{2}\log\sigma^2 - \frac{(m-1)}{2}\log(1-\rho_{ss}) - \frac{\log w}{2} + \log g(t).\end{aligned}$$

We have $\dfrac{\partial \log f}{\partial \sigma^2} = -\dfrac{m}{2\sigma^2} + \dfrac{g'(t)}{g(t)}\dfrac{\partial t}{\partial \sigma^2}$, but

$$\frac{\partial t}{\partial \sigma^2} = -\frac{\mathbf{y}'(\mathbf{I}-\frac{\rho_{ss}}{w}\mathbf{J})\mathbf{y}}{\sigma^4(1-\rho_{ss})} = -\frac{t}{\sigma^2}.\ \text{Hence}$$

$$\frac{\partial \log f}{\partial \sigma^2} = \frac{-m}{2\sigma^2} - \frac{g'(t)t}{\sigma^2 g(t)}\ \text{and}$$

$$(\frac{\partial \log f}{\partial \sigma^2})^2 = \frac{m^2}{4\sigma^4} + \frac{u^2t^2}{\sigma^4} + \frac{mut}{\sigma^4} = \frac{m^2+t}{4\sigma^4} + \frac{mtu}{\sigma^4}.$$

Using the expected values formula in Theorem 1 we get

$$E(\frac{\partial \log f}{\partial \sigma^2})^2 = \frac{m}{4\sigma^4}.$$

Next,

$$\frac{\partial \log f}{\partial \rho_{ss}} = \frac{(m-1)}{2}\left[\frac{1}{(1-\rho_{ss})} - \frac{1}{w}\right] + \frac{g'(t)}{g(t)}\frac{\partial t}{\partial \rho_{ss}},$$

$$\frac{\partial t}{\partial \rho_{ss}} = \frac{\mathbf{y}'(\mathbf{I} - \frac{v}{w^2}\mathbf{J})\mathbf{y}}{\sigma^2(1-\rho_{ss})}.$$

$$\frac{\partial \log f}{\partial \rho_{ss}} = \frac{m(m-1)\rho_{ss}}{2(1-\rho_{ss})w} + \frac{u\mathbf{y}'\mathbf{A}\mathbf{y}}{\sigma^2(1-\rho_{ss})^2}, \quad \mathbf{A} = \mathbf{I} - \frac{v}{w^2}\mathbf{J} \text{ and}$$

$$(\frac{\partial \log f}{\partial \rho_{ss}})^2 = \frac{m^2(m-1)^2\rho_{ss}^2}{4(1-\rho_{ss})^2w^2} + \frac{m(m-1)\rho_{ss}u\mathbf{y}'\mathbf{A}\mathbf{y}}{\sigma^2 w(1-\rho_{ss})^3} + \frac{(u\mathbf{y}'\mathbf{A}\mathbf{y})^2}{\sigma^4(1-\rho_{ss})^4}.$$

Then we can show that

$$E(\frac{\partial \log f}{\partial \rho_{ss}})^2 = \frac{m(m-1)((m+2)v+m)}{4(m+2)(1-\rho_{ss})^2w^2}.$$

Also,

$$\frac{\partial \log f}{\partial \sigma^2}\frac{\partial \log f}{\partial \rho_{ss}} = -\frac{(m+2ut)}{2\sigma^2}[\frac{m(m-1)\rho_{ss}}{2(1-\rho_{ss})w} + \frac{u\mathbf{y}'\mathbf{A}\mathbf{y}}{\sigma^2(1-\rho_{ss})^2}]. \text{ Then}$$

$$E\left(\frac{\partial \log f}{\partial \sigma^2}\frac{\partial \log f}{\partial \rho_{ss}}\right) = -E\frac{(m+2ut)}{2\sigma^2}\frac{m(m-1)\rho_{ss}}{2(1-\rho_{ss})w} - E\frac{(m+2ut)u\mathbf{y}'\mathbf{A}\mathbf{y}}{2\sigma^4(1-\rho_{ss})^2}$$

$$= -\frac{[mE(u\mathbf{y}'\mathbf{A}\mathbf{y}) + 2Eu^2t\mathbf{y}'\mathbf{A}\mathbf{y}]}{2\sigma^4(1-\rho_{ss})^2}, \text{ but}$$

$$E\,(u^2t\mathbf{y}'\mathbf{A}\mathbf{y}) = \left\{\frac{2tr(\boldsymbol{\Sigma}^{\frac{1}{2}}\mathbf{A}\boldsymbol{\Sigma}^{\frac{1}{2}} + mtr(\boldsymbol{\Sigma}^{\frac{1}{2}}\mathbf{A}\boldsymbol{\Sigma}^{\frac{1}{2}})}{m(m+2)}\right\} E\,(t^2u^2)$$

$$= \frac{m(m+1)}{4m}tr(B) = \frac{m(m+1)\sigma^2}{4}(1-\frac{v}{w}). \text{ Thus}$$

$$E\left(\frac{\partial \log f}{\partial \sigma^2}\frac{\partial \log f}{\partial \rho_{ss}}\right) = -\frac{m(m-1)\rho_{ss}}{4(1-\rho_{ss})w\sigma^2}.$$

Finally,

$$\frac{\partial \log f}{\partial \mu} = \frac{g'(t)}{g(t)}\frac{\partial t}{\partial \mu} = -2\frac{g'(t)}{g(t)}(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{y}) \text{ and}$$

$$(\frac{\partial \log f}{\partial \mu})^2 = 4u^2(\mathbf{y}'\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{J}\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{y}). \text{ Then we get } E(\frac{\partial \log f}{\partial \mu})^2 = \frac{1}{w\sigma^2},$$

$$E(\frac{\partial \log f}{\partial \mu}\frac{\partial \log f}{\partial \sigma^2}) = 0, \text{ and } E(\frac{\partial \log f}{\partial \mu}.\frac{\partial \log f}{\partial \rho_{ss}}) = 0.$$

Thus the information matrix for the Kotz type distribution can be written as

$$\mathcal{I}_K = \begin{pmatrix} \dfrac{1}{\sigma^2 w} & 0 & 0 \\ 0 & \dfrac{m}{4\sigma^4} & -\dfrac{m(m-1)\rho_{ss}}{4\sigma^2(1-\rho_{ss})w} \\ 0 & -\dfrac{m(m-1)\rho_{ss}}{4\sigma^2(1-\rho_{ss})w} & \dfrac{m(m-1)[(m+2)v+m]}{4(m+2)w^2(1-\rho_{ss})^2} \end{pmatrix}.$$

Theorem 2 *Suppose* $\mathbf{x}$ *has probability density function as in (II.6) and* $\mathbf{x}_1, ..., \mathbf{x}_n$ *is a random sample from this distribution. Suppose the maximum likelihood estimator of* μ, σ^2 *and* ρ_{ss} *obtained by solving the equations in (II.5) are* $\hat{\mu}, \hat{\sigma}^2$ *and* $\hat{\rho}_{ss}$. *Then* $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N_3(0, \mathcal{I}_K^{-1})$, *where* $\theta = (\mu, \sigma^2, \rho_{ss})'$ *and* $\hat{\theta} = (\hat{\mu}, \hat{\sigma}^2, \hat{\rho}_{ss})'$ *is the MLE of* θ.

A similar result for multivariate normal distribution is given below.

Theorem 3 *(Donner and Koval, 1980) Suppose* $\mathbf{x}_i, i = 1, ..., n,$ *is a random sample from multivariate normal distribution with mean* $\mu\mathbf{1}$, *and variance covariance matrix* $\mathbf{\Sigma} = \sigma^2\mathbf{V}(\rho_{ss})$ *and* $\tilde{\theta} = (\tilde{\mu}, \tilde{\sigma}^2, \tilde{\rho}_{ss})'$ *is the MLE of* θ. *Then* $\sqrt{n}(\tilde{\theta} - \theta) \xrightarrow{d} N_3(0, \mathcal{I}_N^{-1})$,

$$\text{where } \mathcal{I}_N = \begin{pmatrix} \dfrac{m}{\sigma^2 w} & 0 & 0 \\ 0 & \dfrac{m}{2\sigma^4} & \dfrac{m(m-1)\rho_{ss}}{2\sigma^2(1-\rho_{ss})w} \\ 0 & \dfrac{m(m-1)\rho_{ss}}{2\sigma^2(1-\rho_{ss})w} & \dfrac{m(m-1)v}{2w^2(1-\rho_{ss})^2} \end{pmatrix}.$$

The Kotz type distribution has fatter tail regions than that of multivariate normal distribution and hence can be an alternative model to the multivariate normal distribution. In the following we will simulate data from multivariate normal distribution, multivariate T distribution with degrees of freedom=5, and Kotz type distribution, and in each case we compute the bias, the relative efficiency (RE), and the Pitman nearness (PN) probabilities for $\hat{\rho}$ and $\tilde{\rho}$.

II.3 Simulation Study

In the following we provide algorithms for simulating random samples from each of the three multivariate distributions mentioned above.

II.3.1 Simulating from multivariate normal distribution

(a) Generate m independent standard normal random variables $z_1, ..., z_m$ and let $\mathbf{z} = (z_1, ..., z_m)'$.

(b) Suppose $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma} = \boldsymbol{\Gamma}'\boldsymbol{\Gamma}$ are given. Then $\mathbf{x} = \boldsymbol{\Gamma}'\mathbf{z} + \boldsymbol{\mu}$. Then $\mathbf{x}$ has a multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and variance covariance matrix $\boldsymbol{\Sigma}$. See Khattree and Naik (1999).

(c) Repeat the above steps n times to obtain a sample of size n.

II.3.2 Simulating from multivariate T distribution

(a) Generate m independent standard normal random variables $z_1, ..., z_m$ and let $\mathbf{z} = (z_1, ..., z_m)'$.

(b) Suppose $\boldsymbol{\Sigma} = \boldsymbol{\Gamma}'\boldsymbol{\Gamma}$ is given. Then $\mathbf{x} = \boldsymbol{\Gamma}'\mathbf{z}$ has a multivariate normal distribution with mean vector $\mathbf{0}$ and variance covariance matrix $\boldsymbol{\Sigma}$.

(c) Generate a gamma random variate V with a shape parameter $v/2$ and scale parameter 1, (in our case $v = 5$).

(d) Let $\chi^2 = 2V$. Then χ^2 is distributed as a chi-square random variable with v degrees of freedom.

(e) Let $\mathbf{t} = \sqrt{\frac{v}{\chi^2}}\mathbf{x} + \boldsymbol{\mu}$. Then $\mathbf{t}$ has a multivariate T distribution with parameters $\boldsymbol{\mu}, \boldsymbol{\Sigma}$, and v. See Lange et al. (1989).

(f) Repeat the above steps n times to obtain a sample of size n.

II.3.3 Simulating from Kotz type distribution

The following algorithm to generate a random sample from an m-variate Kotz type distribution (II.6) is given in Naik and Plungpongpun (2004). Here we provide only an outline.

(a) Simulate $\mathbf{y}' = (y_1, ..., y_m)$ having the density

$$f(\mathbf{y}) = c \ \exp\{-\sqrt{\mathbf{y}'\mathbf{y}}\},$$

where $-\infty < y_i < \infty$, and $c = \frac{\Gamma(\frac{m}{2})}{2\pi^{\frac{m}{2}}\Gamma(m)}$. Note that $f(\mathbf{y})$ is the standardized version of Kotz type distribution given in (II.6) and also $E(\mathbf{y}) = \mathbf{0}$ and $Var(\mathbf{y}) = (m+1)\mathbf{I}$.

(b) Obtain $\mathbf{x}' = (x_1, ..., x_m)$ having the distribution as in (II.6) by making the transformation $\mathbf{x} = \mathbf{\Gamma}'\mathbf{y} + \boldsymbol{\mu}$, where $\boldsymbol{\mu}' = (\mu_1, ..., \mu_m)$ and $\mathbf{\Sigma} = \mathbf{\Gamma}'\mathbf{\Gamma}$.

The simulation of $\mathbf{y}$ is achieved by using the polar coordinate transformation,

$$\begin{aligned} y_1 &= R\cos\theta_1 \\ y_2 &= R\sin\theta_1\cos\theta_2 \\ &\vdots \\ y_{m-1} &= R\sin\theta_1\sin\theta_2\cdots\sin\theta_{m-2}\cos\theta_{m-1} \\ y_m &= R\sin\theta_1\sin\theta_2\cdots\sin\theta_{m-2}\sin\theta_{m-1}, \end{aligned}$$

where $R = \sqrt{\mathbf{y}'\mathbf{y}}$, $\theta_j \in [\,0, \pi)$ for $1 \le j \le m-2$ and $\theta_{m-1} \in [\,0, 2\pi)$. The Jacobian of the transformation is $R^{m-1}\prod_{j=1}^{m-2}\sin^{m-j-1}(\theta_j)$.

To simulate $\theta \sim g(\theta)$, we use the bisection method which is one of the popular numerical inversion algorithms and is described below. See Devroye (1986) for details.

Algorithm: Find an initial interval $[a, b]$ to which the solution belongs.

REPEAT

$\theta \leftarrow \frac{(a+b)}{2}$

IF $G(\theta) \le U$ THEN $a \leftarrow \theta$

ELSE $b \leftarrow \theta$

UNTIL $b - a \le 2\delta$

RETURN θ

Here $\delta > 0$ is a small number.

Our simulation study includes generating data from each one of these three distributions and comparing the MLE's based on Kotz type and normal distribution. For all three simulation experiments, the parameters used include the total number of families $n = 10, 20, 30, 50, 100$ to cover small, medium, and large sample sizes, the family sizes, $m = 3, 4, 5, 6$, sib-sib correlations $\rho_{ss} = 0.1, 0.3, 0.5, 0.7, 0.9$, $\sigma^2 = 1$, and $\mu = 0$. For each set of these parameters, 10,000 data sets were simulated. The three criteria used for comparison are simulation estimates of (i) Bias (ii) Relative efficiency and (iii) Pitman Nearness (PN) probability.

Bias is computed as $\bar{\hat{\rho}}_{ss} - \rho_{ss}$, where $\bar{\hat{\rho}}$ is the average of the 10,000 estimates of ρ_{ss}, one for each simulation, and ρ_{ss} is the value that is used in the simulation.

Relative efficiency is computed as $RE(\hat{\rho}_{ss_1}, \hat{\rho}_{ss_2}) = \frac{MSE(\hat{\rho}_{ss_2})}{MSE(\hat{\rho}_{ss_1})}$, where $MSE(\hat{\rho}_{ssj}) = \frac{1}{10{,}000}\sum_{i=1}^{10000}(\hat{\rho}_{ssj_i} - \rho_{ss})^2$, $\hat{\rho}_{ssj_i}$ being the estimate of ρ_{ss} for the ith simulated data set and $j = 1, 2$. We say that $\hat{\rho}_{ss_1}$ is better estimator of ρ_{ss} than $\hat{\rho}_{ss_2}$, if $RE > 1$.

Pitman Nearness (PN) Probability is computed as $PN = P\{|\hat{\rho}_{ss_1} - \rho_{ss}| < |\hat{\rho}_{ss_2} - \rho_{ss}|\} = \frac{1}{10{,}000}\#\{\left|\hat{\rho}_{ss1_i} - \rho_{ss}\right| < \left|\hat{\rho}_{ss2_i} - \rho_{ss}\right|\}$, where $\hat{\rho}_{ss1_i}$ and $\hat{\rho}_{ss2_i}$ are the estimates of ρ_{ss} for the ith simulated data set. We say that $\hat{\rho}_{ss_1}$ is a better estimator than $\hat{\rho}_{ss_2}$ for ρ_{ss} if $PN > 0.5$.

II.4 Results and Remarks

Various results are provided in Figures 3 - 17 given at the end of the chapter and a summary of the results are provided below.

- In general we observe that the MLE's underestimate the sib-sib correlation ρ_{ss}.
- The family size (m) seems not to affect the bias.
- When data are simulated from a multivariate normal distribution, the MLE_K are slightly different from the MLE_N and for extreme values of ρ_{ss} (0.1 and 0.9) or when (n) increases, they both have the same bias. On the other hand MLE_N has noticeably higher bias than the MLE_K when the simulation is from a Kotz type distribution (Figures 3-11).
- When data are simulated from a multivariate normal distribution, the relative efficiency (RE) of MLE_K as relative to MLE_N is almost 1 for large m $(= 5, 6)$ and small n $(= 10)$ and RE is close to 1.1 when n is large and m is small $(= 3, 4)$. These values indicate that the MLE_K is at least as efficient as MLE_N when sampling is from multivariate normal distribution. However, when the sampling is from Kotz type distribution, the MLE_N is not as efficient as MLE_K, this is more so for large sample sizes and small m values (Figures 12-14).

- We note from Figures 15-17 that MLE_K is better than MLE_N for all m, n and for different values of ρ_{ss} with high PN values. However, MLE_N is slightly better than MLE_K for small n and also for large n, when $m = 3$ and $\rho_{ss} = 0.3$.

In Summary, these estimators perform as expected in relation to one another with regard to bias, relative efficiency, and PN probability. MLE_K outperforms MLE_N when data come from Kotz type distribution and vice versa. However, the magnitudes of the differences in these criteria are much greater when using MLE_N. When data are simulated from the multivariate T distribution, the estimators under Kotz type distribution are superior for all values of m, n and ρ_{ss}. The results are provided in the Figures 9 - 11, 14, and 17. Clearly we can see that the alternative estimator proposed here is more efficient.

II.5 Alternative Approach: Unbalanced Case

The proposed estimators under unbalanced case can be similarly obtained by minimizing the objective function

$$\begin{aligned} F(\mu, \sigma^2, \rho_{ss}) &= \frac{N}{2}\log\sigma^2 + \frac{(N-n)}{2}\log(1-\rho_{ss}) + \\ &\sum_{i=1}^{n}\frac{\log(w_i)}{2} + \sum_{i=1}^{n}\sqrt{\frac{(\mathbf{x}_i-\boldsymbol{\mu})'(\mathbf{I}_{m_i}-\frac{\rho_{ss}}{w_i}\mathbf{J}_{m_i})(\mathbf{x}_i-\boldsymbol{\mu})}{\sigma^2(1-\rho_{ss})}} \end{aligned} \tag{II.7}$$

simultaneously w.r.t. μ, σ^2, and ρ_{ss}, where $w_i = 1 + (m_i - 1)\rho_{ss}$, $i = 1, 2, ..., n$ and $N = \sum_{i=1}^{n} m_i$. The solutions $\hat{\mu}, \hat{\sigma}^2$, and $\hat{\rho}_{ss}$ are the maximum likelihood estimators when the distribution is Kotz type.

The Fisher's Information matrix for a given $\mathbf{x}_{i(m_i\times 1)}$ is given by

$$\mathcal{I}_{Ki} = \begin{pmatrix} \frac{1}{\sigma^2}\frac{1}{w_i} & 0 & 0 \\ 0 & \frac{1}{4\sigma^4}m_i & \frac{\rho_{ss}}{4\sigma^2(1-\rho_{ss})}\left\{\frac{m_i(m_i-1)}{w_i}\right\} \\ 0 & \frac{\rho_{ss}}{4\sigma^2(1-\rho_{ss})}\left\{\frac{m_i(m_i-1)}{w_i}\right\} & \frac{1}{4(1-\rho_{ss})^2}\frac{m_i(m_i-1)[(m_i+2)v_i+m_i]}{(m_i+2)w_i^2} \end{pmatrix} \tag{II.8}$$

For large n, $\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_{K_i}$ will converge to $\mathcal{I}_K$,

$$\text{where } \mathcal{I}_K = \begin{pmatrix} \frac{1}{\sigma^2}a_{11} & 0 & 0 \\ 0 & \frac{1}{4\sigma^4}a_{22} & \frac{\rho_{ss}}{4\sigma^2(1-\rho_{ss})}a_{23} \\ 0 & \frac{\rho_{ss}}{4\sigma^2(1-\rho_{ss})}a_{23} & \frac{1}{4(1-\rho_{ss})^2}a_{33} \end{pmatrix},$$

with $a_{11} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{1}{w_i}$, $a_{22} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} m_i$,

$a_{23} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{m_i(m_i-1)}{w_i}$, $a_{33} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{m_i(m_i-1)[(m_i+2)v_i+m_i]}{(m_i+2)w_i^2}$.

If $\hat{\theta} = (\hat{\mu}, \hat{\sigma}^2, \hat{\rho}_{ss})'$ is the MLE of $\theta = (\mu, \sigma^2, \rho_{ss})'$ then $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N_3(0, \mathcal{I}_K^{-1})$.

Similarly, for normal distribution we have $\sqrt{n}(\widetilde{\theta} - \theta) \xrightarrow{d} N_3(0, \mathcal{I}_N^{-1})$, where $\widetilde{\theta}$ is the MLE of θ under normal distribution and

$$\mathcal{I}_N = \begin{pmatrix} \frac{1}{\sigma^2} b_{11} & 0 & 0 \\ 0 & \frac{1}{2\sigma^4} b_{22} & \frac{\rho_{ss}}{2\sigma^2(1-\rho_{ss})} b_{23} \\ 0 & \frac{\rho_{ss}}{2\sigma^2(1-\rho_{ss})} b_{23} & \frac{1}{2(1-\rho_{ss})^2} b_{33} \end{pmatrix},$$

where $w_i = 1 + (m_i - 1)\rho_{ss}$, $v_i = 1 + (m_i - 1)\rho_{ss}^2$,

$b_{11} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{m_i}{w_i}$, $b_{22} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} m_i$,

$b_{23} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{m_i(m_i-1)}{w_i}$, $b_{33} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{m_i(m_i-1)v_i}{w_i^2}$.

In practice we can calculate the asymptotic variance of $\hat{\theta}$ as:

$var(\hat{\theta}) \approx \left[\sum_{i=1}^{n} \mathcal{I}_{Ki}\right]^{-1}$, where $\mathcal{I}_{Ki}$ is given in (II.8). Similarly the asymptotic variance of $\tilde{\theta}$ is calculated.

II.5.1 Simulating data with unequal family sizes

In the unbalanced case we follow the same procedure as that for the balanced case in for generating data from the three types of distribution, namely, multivariate normal, multivariate T, and Kotz type. Along with MLE_N and MLE_K, we consider a non-iterative estimator given in (II.3) due to Srivastava (1993). Only this non-iterative estimator is considered because it has been shown in Srivastava (1993) that it is more efficient compared to the other commonly used non iterative estimators.

In the unbalanced case, one extra problem we face is the determination of m_i, the family sizes. For this we used a procedure due to Brass (1958) who suggested that the negative binomial distribution truncated below 1 fits the observed distribution of sibship sizes very well in a variety of human populations for appropriate choice of parameters n and P. The distribution has the probability mass function given by

$$P_r(m = r) = \frac{(n + r - 1)!Q^{-n}(\frac{P}{Q})^r}{(n - 1)!r!(1 - Q^{-n})}, \ Q = 1 + P, \ r = 1, 2, ...$$

Since recent census data notes that family sizes are small, (for example, see *http://www.census.gov/population/socdemo/hh-fam/cps2002/tabAVG1.pdf)*, for our simulations we used the negative binomial distribution truncated above by 6, that is $1 \leqslant m \leqslant 6$.

We include the total number of families $n = 50$, and 100 and sib-sib correlation coefficient, $\rho_{ss} = 0, 0.1, 0.3, 0.5, 0.7$, and 0.8. Also to avoid certain convergence problems we encountered during simulations, we simulated data from multivariate T distribution with df=3 instead of 5. As before we will compare the bias, RE and PN probabilities for MLE_N, MLE_K, and Srivastava's estimator.

II.6 Results and Remarks

- Increasing the sample size didn't affect the bias for the three estimators. When the sampling is from multivariate normal distribution, MLE_K has slightly higher bias compared to MLE_N and Srivastava's estimator, especially for moderate to high values of ρ_{ss}. When data are simulated from Kotz type distribution, there is a major difference between MLE_K and the other estimators, compared to the differences that existed when data were from normal distribution. When the simulation is from multivariate T distribution, the MLE_K has the highest bias, but the magnitude of the bias is smaller than 0.05 (Figures 18 - 20).

- When data are simulated from multivariate normal distribution, the relative efficiency (RE) of MLE_K relative to MLE_N is close to 1 as ρ_{ss} increases and when $n = 5$ (Figure 21). When data are simulated from Kotz type distribution, the RE value of MLE_N relative to MLE_K is at least 1.5 for $n = 50$ and for moderate to large values of ρ_{ss} (≥ 0.5). Also we can see that the relative efficiency of Srivastava's estimator relative to MLE_K is at least 2.5 for all values of ρ_{ss} and all values of n (Figure 22).

- When the simulation is from multivariate T, the efficiency of MLE_K is considerably higher than the efficiency of MLE_N and Srivastava's estimator (Figure 23).

- By the PN probability, when the simulation is from Kotz type distribution, the MLE_N is almost as good as MLE_K when $\rho_{ss} = 0, 0.1$. But for the other values, MLE_K is better in all the cases and the PN value is more than 0.7 for large ρ_{ss}. MLE_K is more efficient than Srivastava's estimator for all n and all values of ρ_{ss} (Figure 24). Figure 25 shows that when data are simulated from multivariate normal distribution, then MLE_N is better, but the PN values do not exceed 0.7.

- When the simulation is from multivariate T distribution, clearly MLE_K is better than both MLE_N and Srivastava's estimator for all values of ρ_{ss} and all n, (Figure 26).

In summary, here we have considered two multivariate heavy-tailed distributions which have fatter tail regions than that of multivariate normal distribution and studied the estimation of sib-sib correlation. We find the estimators of ρ_{ss} by maximizing the log-likelihood function of Kotz type distribution and normal distribution and using other non-iterative methods. We have provided a simulation algorithm for generating samples from Kotz type distribution with unequal family sizes. Next, we performed a simulation experiment to compare the ML estimators of the sib-sib correlation by using three measures, Bias, RE and Pitman Nearness probability under multivariate normal, multivariate T and Kotz type samples. Based on all three criteria and using the results provided in previous subsections we conclude that these estimators perform as expected in relation to one another with regard to Bias, RE and PN probability. MLE_N and Srivastava's estimator outperform the MLE_K when data come from a normal distribution, and MLE_K outperforms all other estimators when data come from the Kotz-type distribution. However, the magnitudes of the differences in these criteria are much greater if MLE_N is used, when the parent distribution is heavy tailed. This implies that the greatest loss occurs if normal estimates are used for non-normal cases.

II.7 Analysis of Galton's Data

Recently, Hanley (2004) worked with family data on human stature obtained directly from Galton's note books (cf. Galton, 1886, 1889). The data consists of heights of 205 families with the number of children ranging from 1 to 15. Over all, there were 962 children, 486 of them were sons and the remaining 476 were daughters. However, only 934 children had numerical values. The remaining children scores were described as "tall", "tallish", "short", etc.. Some other children's hights were described as "deformed" and "idiotic". For more details on this data set see Hanely (2004).

We will use these data to illustrate our procedures discussed earlier for estimating the sib-sib correlations. The correlations we have considered include the son-son, daughter-daughter, and child child correlations. For the son-son correlation, we will consider data on only sons, and for the daughter-daughter correlation, we will consider data on only daughters and for the child-child correlation we will consider all the children. For each case we will use the normal and Kotz likelihood based methods to compute the maximum likelihood estimates. We will also use Srivastava's estimator as an example of non-iterative moments based method. In each case, we assume that the expected value of sib score is the same and the variance of the sib score is the same for all the sibs and families. The standard errors of these estimates will also be provided.

- Out of the 205 families, only 168 families have girls and for these families, the pairs: (the number of daughters, the number of families having those many daughters) are (1, 56), (2, 39), (3, 33), (4, 20), (5, 9), (6, 5), (7, 3), (8, 2), and (9, 1). That is, there are 56 families with one daughter, 39 families with two daughters and so on. Our interest is to estimate ρ_{dd} the correlation between the daughters. The maximum likelihood estimates of ρ_{ss} with their standard errors are provided in the Table below.

- Out of 205 families, only 173 families have boys and for these families, the pairs: (the number of sons, the number of families having those many sons) are (1, 40), (2, 49), (3, 40), (4, 7), (5, 8), (6, 6), (7, 2), and (10, 1). That is, there are 40 families with one son, 49 families with two sons and so on.

  If ρ_{ss} is the correlation between the sons then the estimators and the standard errors of these estimators are provided in the Table below.

- Out of 205 families there are 197 families with at least one child. The pairs (the number of children, the number of families having those many children) are (1, 32), (2, 20), (3, 22), (4, 29), (5, 27), (6, 20), (7, 16), (8, 16), (9, 7), (10, 4), (11, 3), and (15, 1).

The results are summarized in the Table below.

Method	$\hat{\rho}_{dd}$	$\hat{\rho}_{ss}$	$\hat{\rho}_{cc}$
MLE_N	0.420 (0.0545)	0.366 (0.0532)	0.175 (0.0341)
MLE_K	0.533 (0.0554)	0.476 (0.0575)	0.207 (0.0382)
Srivastava	0.439 (0.0552)	0.374 (0.0546)	0.17 (0.0356)

Remarks and conclusions:

- We notice that the strongest correlation exists between daughters and daughters.
- The child-child correlation estimates are smaller than the son-son or daughter-daughter correlations. This may be because of the assumption that the daughters and the sons have common mean and common variances and also the assumptions that the correlation is common.
- We notice that all the methods have provided almost identical standard errors for each category of sib-sib correlation estimates. However, the estimators are some what different for different methods.

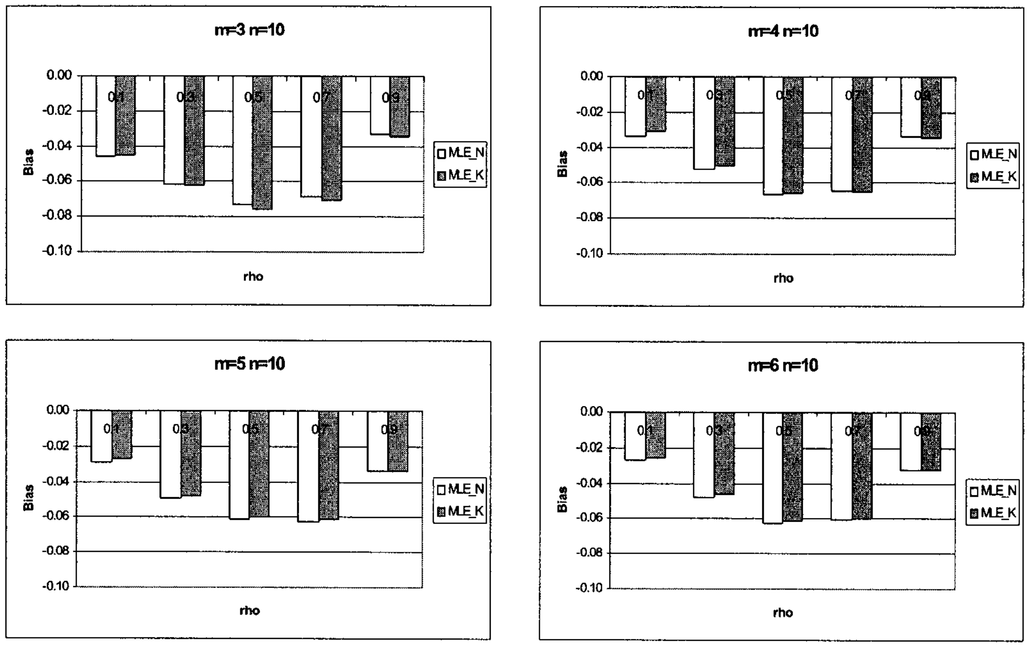


Figure 3. Bias comparison of MLE_N with MLE_K when simulation is from normal for n=10.

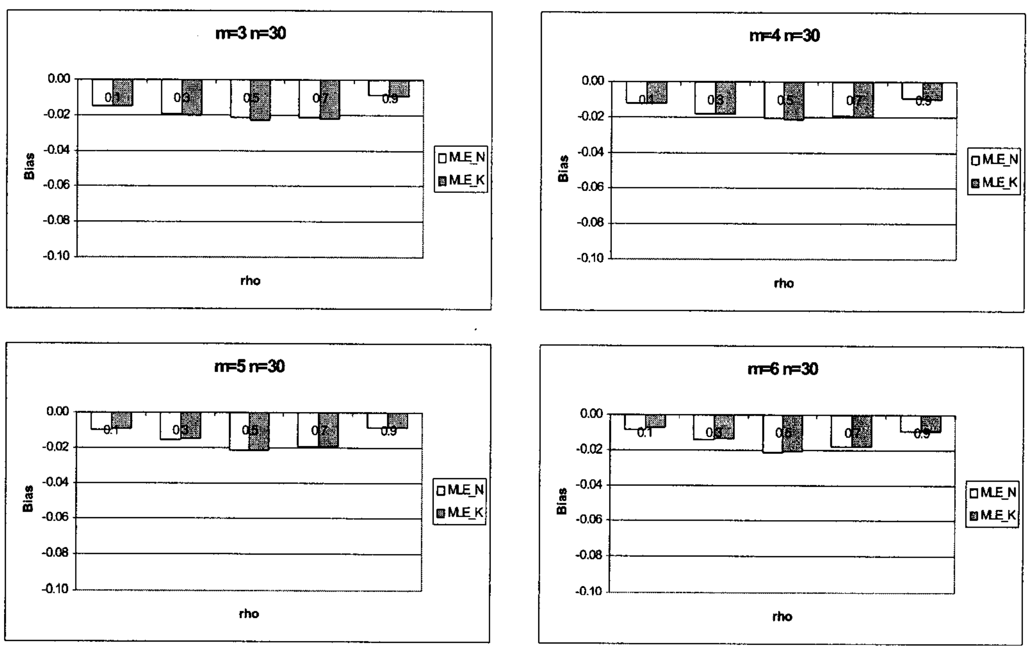


Figure 4. Bias comparison of MLE_N with MLE_K when simulation is from Normal for n=30.

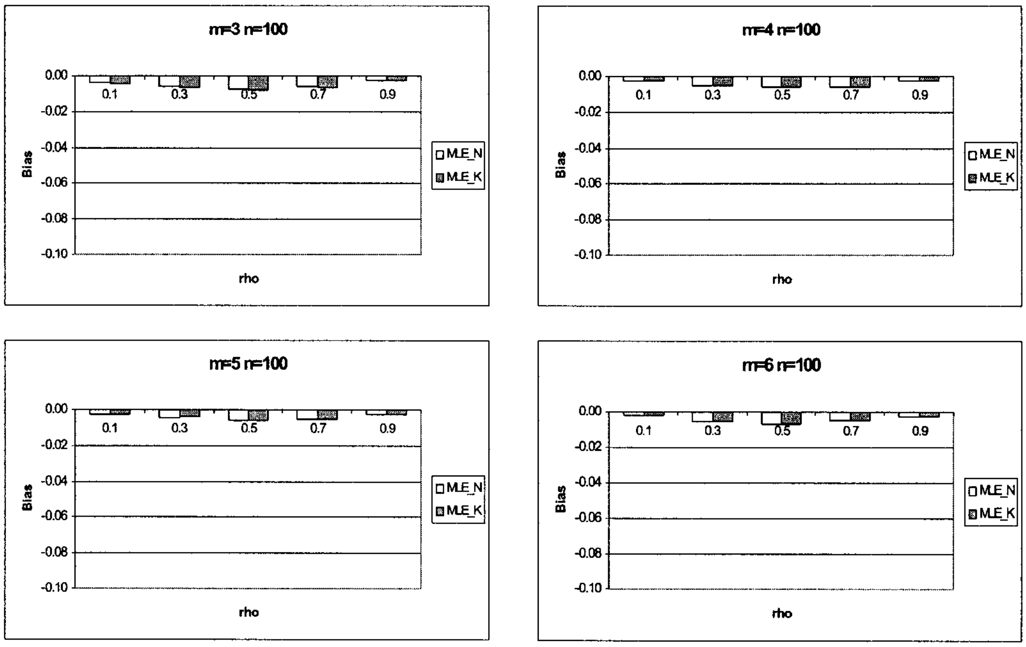


Figure 5. Bias comparison of MLE_N with MLE_K when simulation is from normal for n=100.

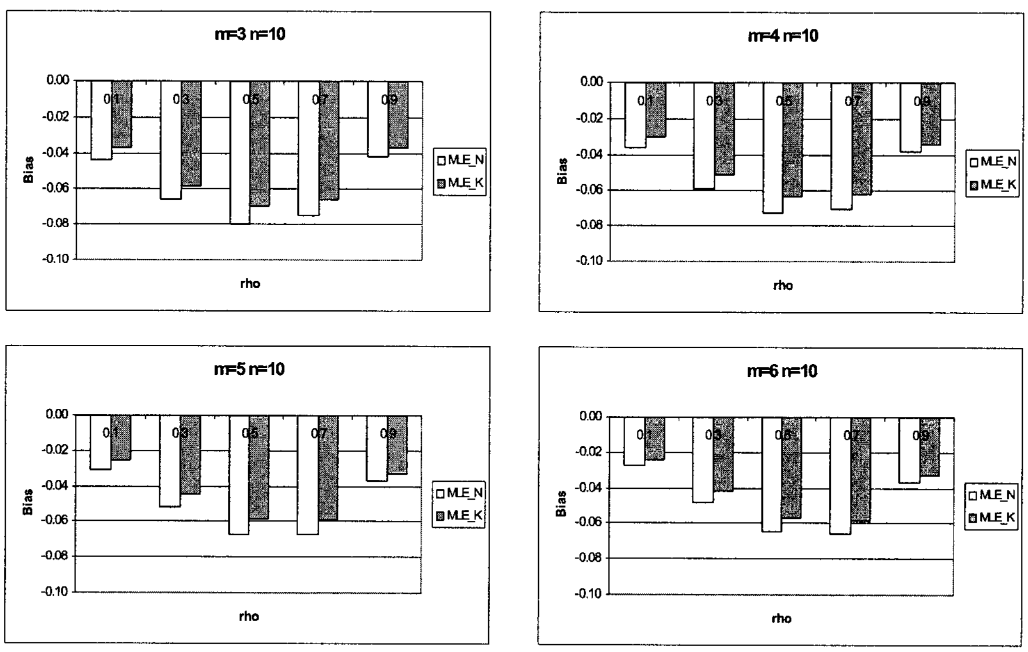


Figure 6. Bias comparison of MLE_N with MLE_K when simulation is from Kotz type distribution for n=10.

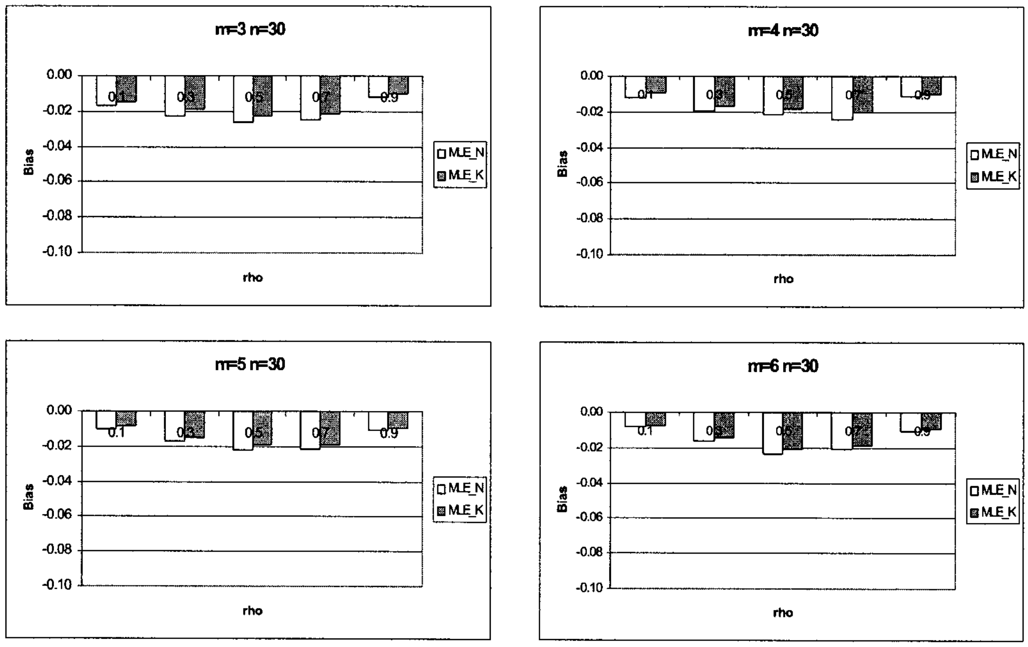


Figure 7. Bias comparison of MLE_N with MLE_K when simulation is from Kotz type distribution for n=30.

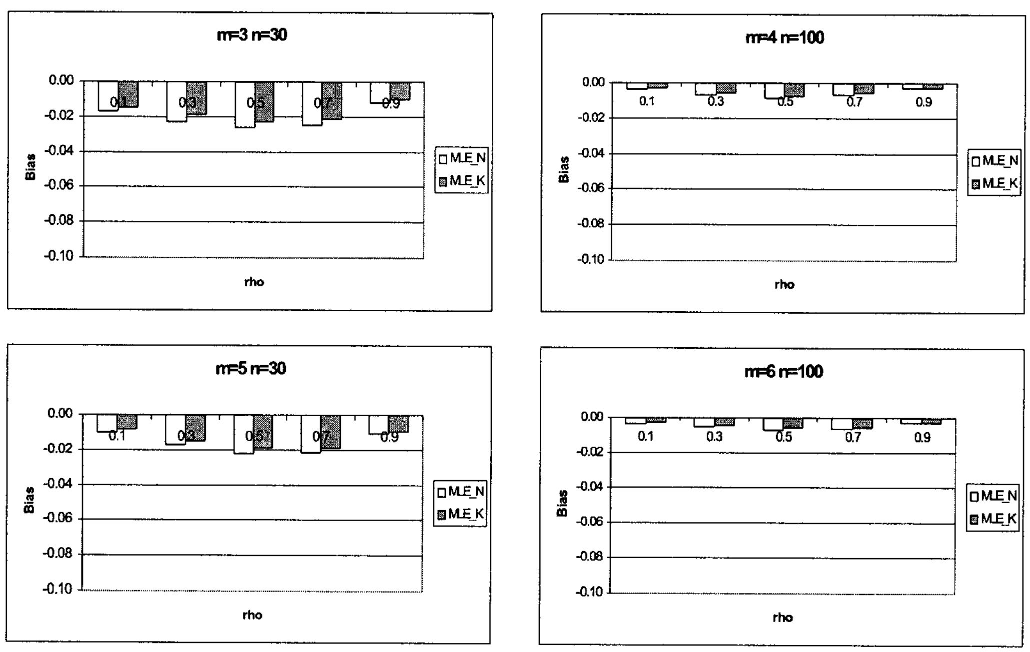


Figure 8. Bias comparison of MLE_N with MLE_K when simulation is from Kotz type distribution for n=100.

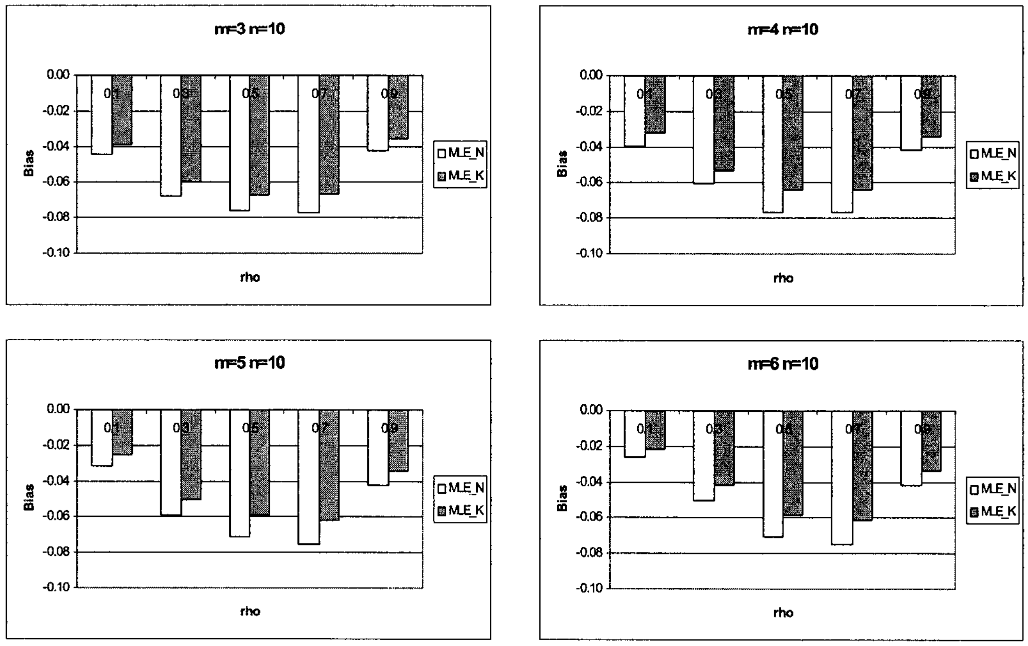


Figure 9. Bias comparison of MLE_N with MLE_K when simulation is from T distribution for n=10.

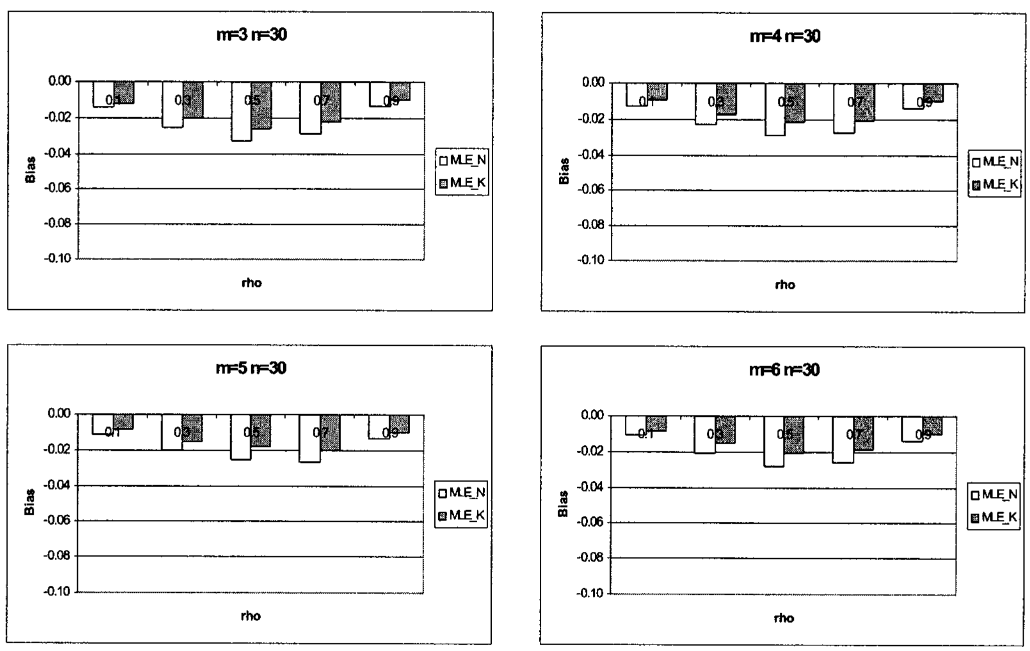


Figure 10. Bias comparison of MLE_N with MLE_K when simulation is from T distribution for n=30.

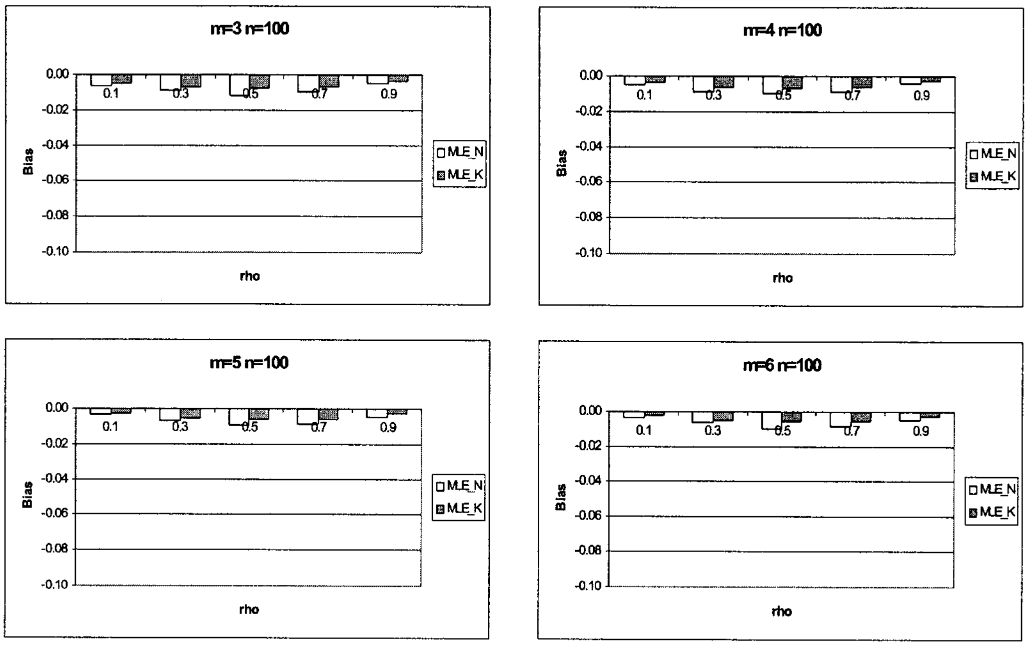


Figure 11. Bias comparison of MLE_N with MLE_K when simulation is from T distribution for n=100.

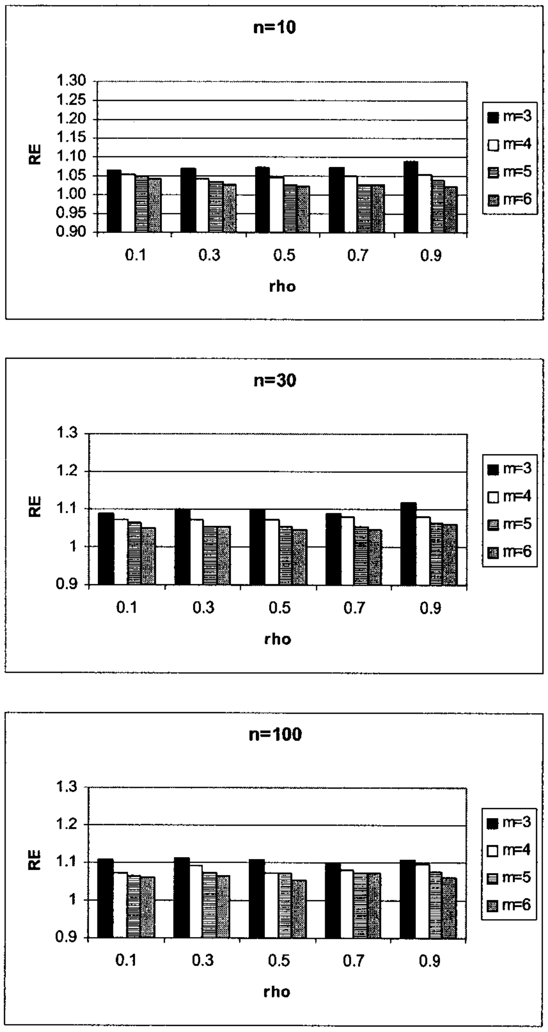


Figure 12. RE comparison of MLE_N with MLE_K when simulation is from normal distribution.

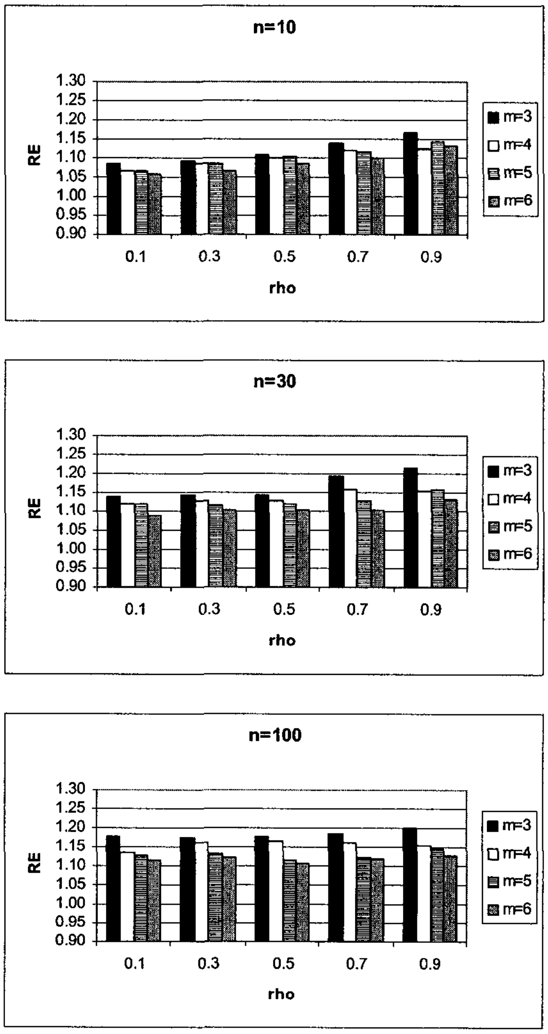


Figure 13. RE comparison of MLE_K with MLE_N when simulation is from Kotz type distribution.

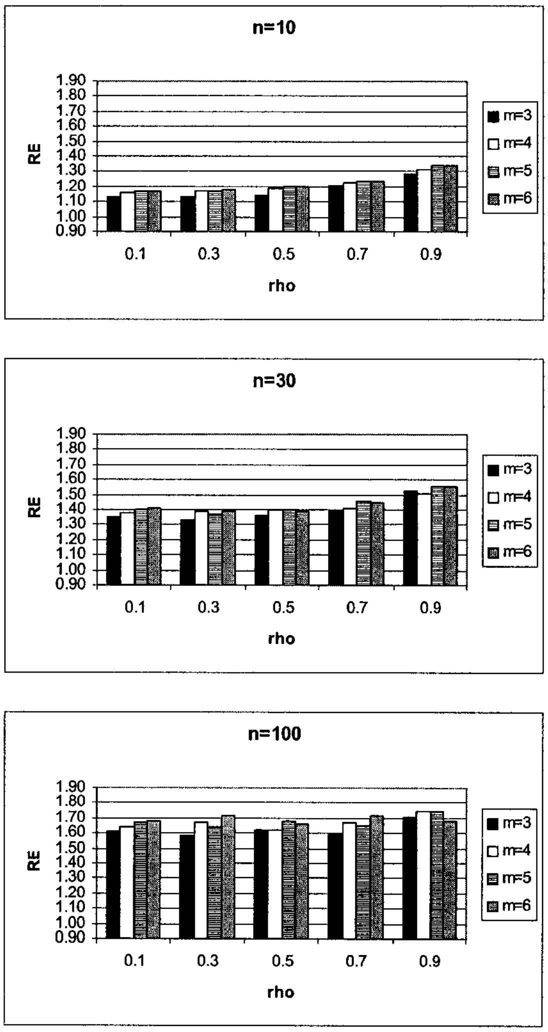


Figure 14. RE comparison of MLE_K with MLE_N when simulation is from T distribution, with df=5.

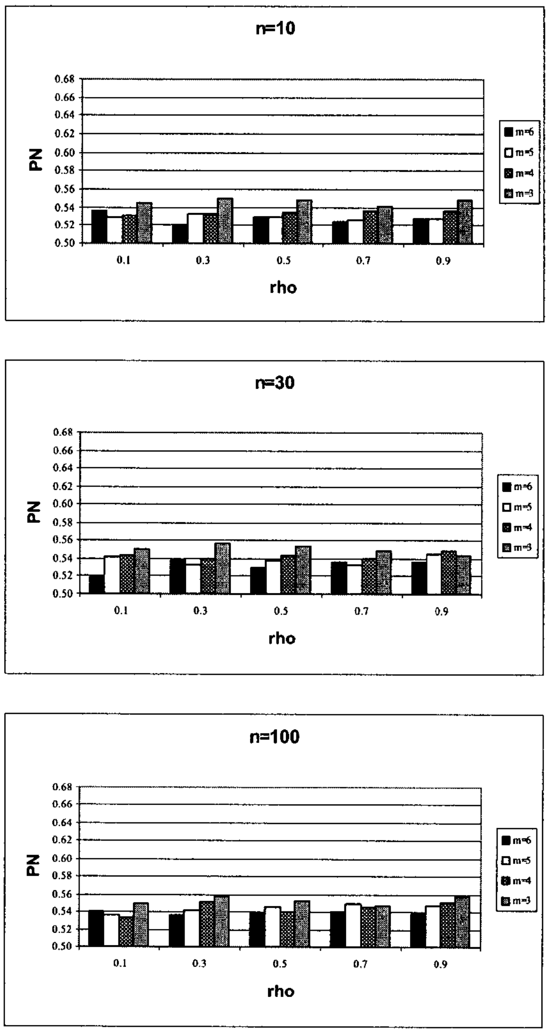


Figure 15. PN comparison of MLE_N with MLE_K when simulation is from normal distribution.

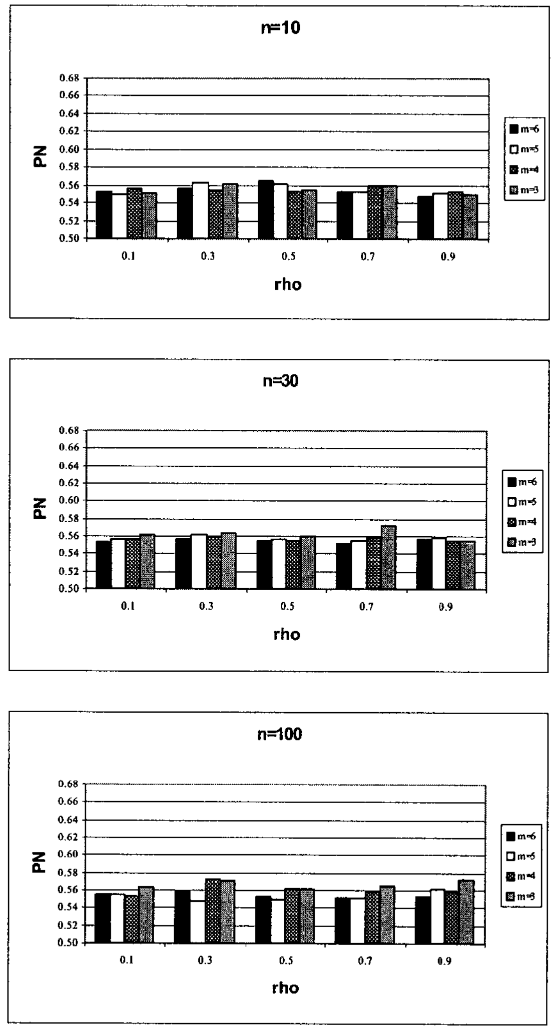


Figure 16. PN comparison of MLE_K with MLE_N when simulation is from Kotz type distribution.

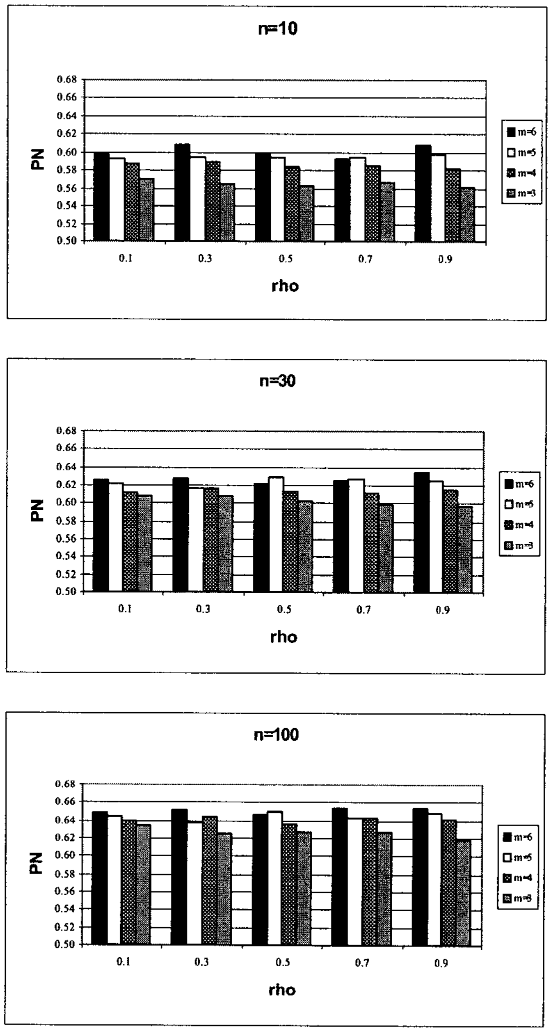


Figure 17. PN comparison of MLE_K with MLE_N when simulation is from T distribution, with df=5.

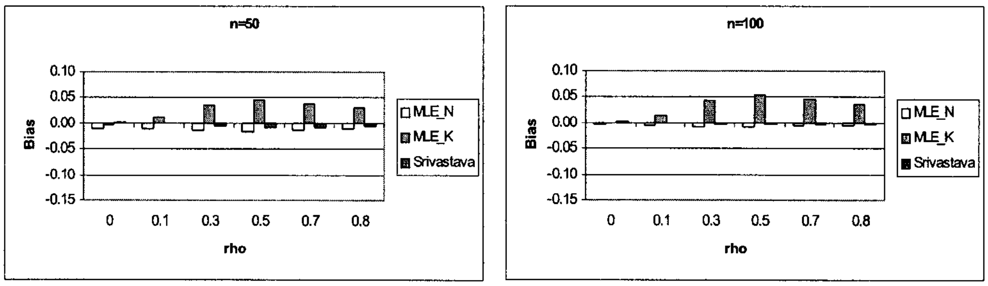


Figure 18. Bias comparison of MLE_N with MLE_K in the unbalanced case when simulation is from normal distribution.

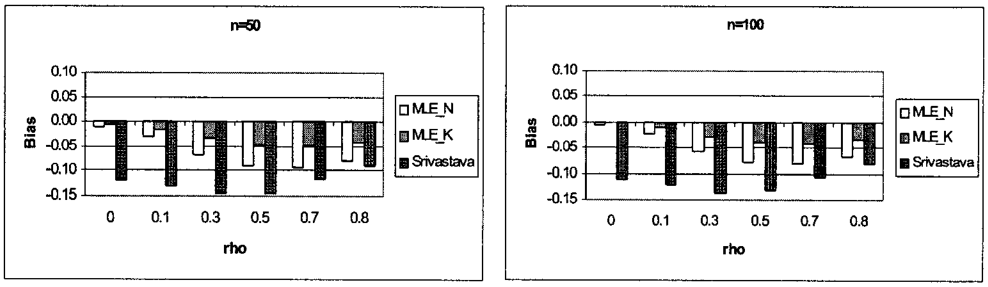


Figure 19. Bias comparison of MLE_N with MLE_K in the unbalanced case when simulation is from Kotz type distribution.

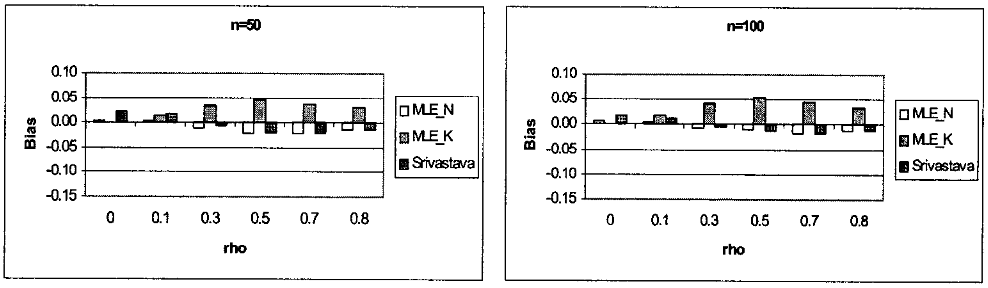


Figure 20. Bias comparison of MLE_N with MLE_K in the unbalanced case when simulation is from T distribution, with df=3.

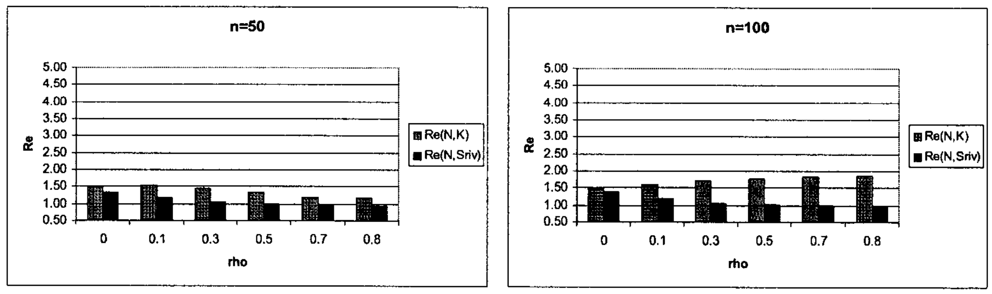


Figure 21. RE comparison of MLE_N with MLE_K in the unbalanced case when simulation is from normal distribution.

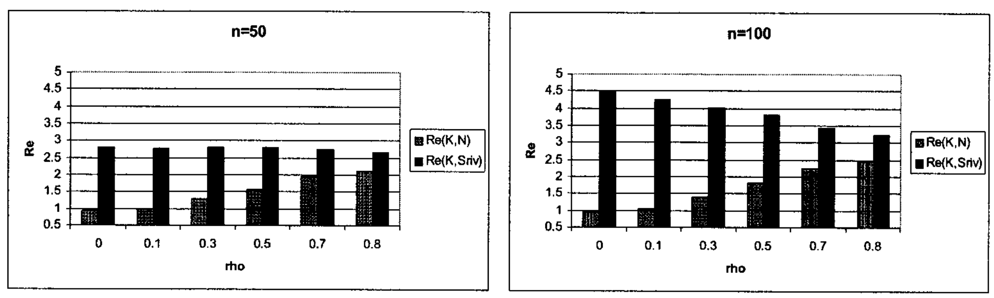


Figure 22. RE comparison of MLE_K with MLE_N in the unbalanced case when simulation is from Kotz type distribution.

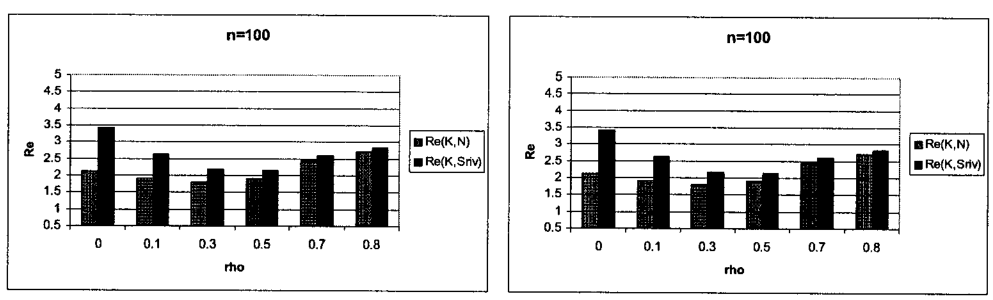


Figure 23. RE comparisons in the unbalanced case when simulation is from T distribution, with df=3.

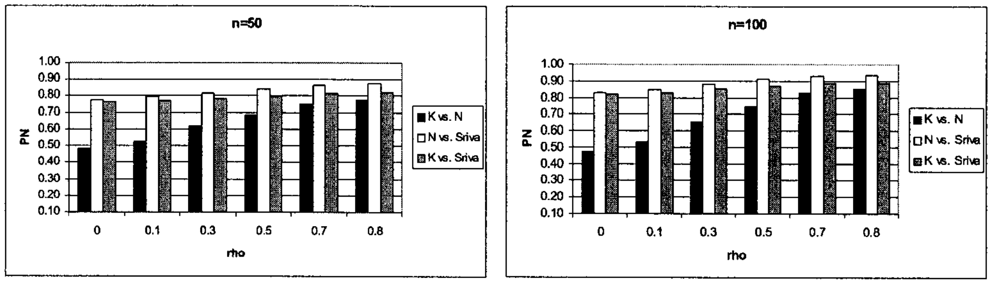


Figure 24. PN comparisons in the unbalanced case when simulation is from Kotz type distribution.

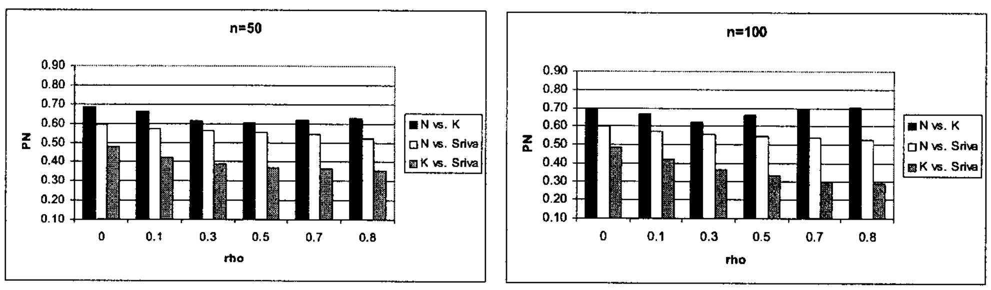


Figure 25. PN comparisons in the unbalanced case when simulation is from normal distribution.

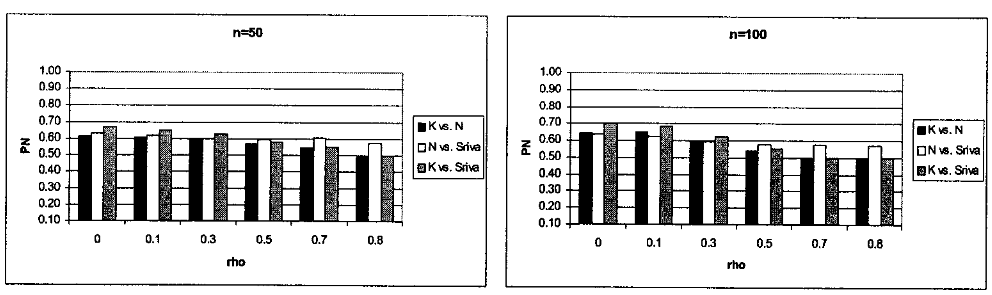


Figure 26. PN comparisons in the unbalanced case when simulation is from T distribution, with df=3.

CHAPTER III

ESTIMATION OF MOM-SIB CORRELATIONS

III.1 Introduction

From the early 1950's parent-sibling (mom-sib) or interclass correlation has been of interest. Sibling-sibling (sib-sib) correlation or intraclass correlation (ρ_{ss}) was already in use by this time, and methods for estimating mom-sib correlation were being developed. Kempthorne and Tandan (1953) used a linear model to estimate this interclass correlation (ρ_{ps}), though made the assumptions that the variance of the parent (σ_p^2) and that of children (σ_s^2) populations were equal (that is, $\sigma_s^2 = \sigma_p^2$) and ρ_{ss} is given. Since these assumptions are unnecessarily restrictive, the pairwise estimator, was computed in the spirit of pearson's correlation coefficient. The pairwise estimator of mom-sib correlation (r_p) is computed by pairing values for each sibling in the family with the parent's value. While the pairwise estimator has an intuitive merit, it violates the required assumption that the data are independent. To over come these problems various estimators are introduced in the literature.

Other estimators used in lieu of the pairwise estimator are the sib-mean estimator, where the mean value of all siblings is paired with the parent's value, and the random sib estimator, where a single sibling from the family is chosen randomly and this sibling's value is paired with the parent's value. Rosner, Donner and Hennekens (1977) proposed the ensemble estimator ($\hat{\rho}_{ps,E}$), based on the random sib estimator. This estimator computes an expected mom-sib correlation over all possible random mom-sib pairings as described in the random sib estimator. Rosner, et al. (1977) compared these three estimators to the pairwise estimator and determined that the ensemble and pairwise estimators are superior to both the sib-mean and random sib estimators based on smallest MSE criteria. And when ρ_{ss} is small, that is, ≤ 0.1, the pairwise estimator outperforms the ensemble estimator and when ρ_{ss} is large, that is, ≥ 0.5, the ensemble estimator outperforms the pairwise estimator.

The linear model approach was re-examined by Mak and Ng (1981) following Rosner's (1979) development of MLEs for the balanced and unbalanced cases. Rosner assumed the scores followed a multivariate normal distribution and derived the MLEs. However, the algorithm used by Rosner is complicated. Mak and Ng's approach simplified derivation of the MLEs by using a linear model, such that given the mother's score, the mom-sib correlation can be determined and tested. Their approach assumes, as Kempthorne and Tandon (1953), that only the mother's score is random, whereas Rosner's approach allows both mom and sib scores to be random.

Srivastava (1984) gives an estimator of ρ_{ps} that has a similar bias, but smaller asymptotic variance compared to the ensemble estimator, as noted in Velu and Rao (1990). Srivastava and Katapa (1986) provided the asymptotic variance of the Srivastava estimator.

Since the introduction of Srivastava's estimator, variants of the traditional interclass correlations have been proposed, and are compared to Srivastava's estimator and the ensemble estimator. Two comparisons by Srivastava and Keen (1988) and Eliasziw and Donner (1990) determine that Srivastava's estimator is uniformly more efficient than the ensemble estimator, but the magnitude of the difference is relatively small.

In this chapter we provide an alternative method of estimating parent-sib correlation, by minimization the negative log of of Kotz type probability density based likelihood function, as in the previous case where sib-sib correlation was estimated.

Assume that we have a sample of measurements from n families and let $(x_i, y_{i1}, y_{i2}, ..., y_{im_i}) = (x_i, \mathbf{y}_i), i = 1, 2, ..., n$, be the measurements from the ith family, where x_i is the measurement of the mom (in general, the parent's measurement) and $y_{i1}, y_{i2}, ..., y_{im_i}$ are the measurements on her m_i siblings. It is assumed that the families are independently distributed with $(m_i + 1) \times 1$ mean vector $\boldsymbol{\mu}_i = (\mu_p, \mu_s, ..., \mu_s)'$ and $(m_i + 1) \times (m_i + 1)$ variance covariance matrix $\boldsymbol{\Sigma}_i$ given by $\begin{pmatrix} \sigma_p^2 & \boldsymbol{\sigma}'_{ps_i} \\ \boldsymbol{\sigma}_{ps_i} & \boldsymbol{\Sigma}_{ss_i} \end{pmatrix}$, where $\boldsymbol{\sigma}_{ps_i} = \rho_{ps}\sigma_p\sigma_s\mathbf{1}_{m_i}$. Also, σ_p^2 is variance of mom's score, σ_s^2 is variance of sib's score, and $\boldsymbol{\Sigma}_{ss_i} = \sigma_s^2 \left\{ (1 - \rho_{ss}) \mathbf{I}_{m_i} + \rho_{ss}\mathbf{1}_{m_i}\mathbf{1}'_{m_i} \right\}$. Recall that ρ_{ps} is the mom-sib correlation and ρ_{ss} is the sib-sib correlation.

Necessary and sufficient conditions for $\boldsymbol{\Sigma}_i$ to be positive definite for all m_i are, $\rho_{ps}^2 \leq \rho_{ss}$ and $0 \leq \rho_{ss} \leq 1$ (Rosner, et al., 1977). Note that

$$\begin{aligned}
\boldsymbol{\Sigma}_i^{-1} &= \begin{pmatrix} \dfrac{1 + (m_i - 1)\rho_{ss}}{\sigma_p^2 g_i} & \dfrac{-\rho_{ps}}{\sigma_p\sigma_s g_i}\mathbf{1}' \\ \dfrac{-\rho_{ps}}{\sigma_p\sigma_s g_i}\mathbf{1} & \dfrac{1}{\sigma_s^2(1 - \rho_{ss})}\{\mathbf{I}_{m_i} - \frac{\rho_{ss} - \rho_{ps}^2}{g_i}\mathbf{J}_{m_i}\} \end{pmatrix} \text{ and} \\
|\boldsymbol{\Sigma}_i| &= \sigma_p^2\sigma_s^{2m_i}(1 - \rho_{ss})^{m_i - 1} g_i,
\end{aligned}$$

where $g_i = 1 + (m_i - 1)\rho_{ss} - m_i\rho_{ps}^2$.

Then the Srivastava's estimator is given by

$$\hat{\rho}_{ps,s} = \frac{\sum_{i=1}^{n}(x_i - \bar{x}_m)(\bar{y}_{is} - \bar{y}_s)}{\left[\sum_{i=1}^{n}(x_i - \bar{x}_m)^2\right]^{1/2}\left[N^*\sum_{i=1}^{n}\sum_{j=1}^{m_i}(y_{ij} - \bar{y}_{is})^2 + \sum_{i=1}^{n}(\bar{y}_{is} - \bar{y}_s)^2\right]^{1/2}}, \tag{III.1}$$

where

$\bar{x}_m = \frac{1}{n}\sum_{i=1}^{n} x_i$, $\bar{y}_{is} = \frac{1}{m_i}\sum_{i=1}^{n} y_{ij}$, $\bar{y}_s = \frac{1}{n}\sum_{i=1}^{n}\bar{y}_{is}$, $N^* = (1 - \overline{m}_h^{-1})(n-1)/\sum_{i=1}^{n}(m_i - 1)$, and $\overline{m}_h^{-1} = \left\{(1/n)\sum_{i=1}^{n}(1/m_i)\right\}^{-1}$.

The asymptotic variance of $\hat{\rho}_{ps,s}$ is

$$AV(\hat{\rho}_{ps,s}) = \frac{1}{n}[\rho_{ps}^4 + \rho_{ps}^2\{\frac{1}{2}c^2 - 2\lambda - \frac{1}{2}\} + \lambda], \tag{III.2}$$

where $c^2 = 1 - 2(1-\rho_{ss})(1 - \overline{m}_h^{-1}) + (1-\rho_{ss})^2\left[\frac{1}{n}\sum_{i=1}^{n}\left\{1 - \frac{1}{m_i}\right\}^2 + \frac{(1-\overline{m}_h^{-1})^2}{(\overline{m}-1)}\right]$,
$\lambda = 1 - (1-\rho_{ss})(1 - \overline{m}_h^{-1})$, and $\overline{m} = (1/n)\sum_{i=1}^{n} m_i$.

III.2 An Alternative Approach: Balanced Case

As in the case of sib-sib correlation, we provide an alternative method for estimating ρ_{ps}, the parent-sib correlation. Given the objective function, when $m_i = m$ for all $i = 1, ..., n$, as

$$\begin{aligned} F(\mu_p, \mu_s, \sigma_p, \sigma_s, \rho_{ps}, \rho_{ss}) &= n\log\sigma_p + N\log\sigma_s + \frac{n}{2}(m-1)\log(1-\rho_{ss}) \\ &\quad + \frac{n}{2}\log g + \sum_{i=1}^{n}[(\mathbf{x}_i - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}_i - \boldsymbol{\mu})]^{\frac{1}{2}}, \end{aligned} \tag{III.3}$$

where $g = 1 + (m-1)\rho_{ss} - m\rho_{ps}^2$, our procedure involves minimizing F simultaneously with respect to μ_p, μ_s, σ_p, σ_s, ρ_{ps}, and ρ_{ss}. Let $\mathbf{v}_1 = (1, 0, ..., 0)'$ and $\mathbf{v}_2 = (0, 1, ..., 1)'$. This process leads to solving the following six equations simultaneously

$$
\begin{aligned}
\frac{\partial F}{\partial \mu_p} &= \frac{1}{2}\sum_{i=1}^{n}\frac{\mathbf{v}_1'\Sigma^{-1}(\mathbf{x}_i-\boldsymbol{\mu})+(\mathbf{x}_i-\boldsymbol{\mu})'\Sigma^{-1}\mathbf{v}_1}{\sqrt{(\mathbf{x}_i-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x}_i-\boldsymbol{\mu})}}=0,\\
\frac{\partial F}{\partial \mu_s} &= \frac{1}{2}\sum_{i=1}^{n}\frac{\mathbf{v}_2'\Sigma^{-1}(\mathbf{x}_i-\boldsymbol{\mu})+(\mathbf{x}_i-\boldsymbol{\mu})'\Sigma^{-1}\mathbf{v}_2}{\sqrt{(\mathbf{x}_i-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x}_i-\boldsymbol{\mu})}}=0,\\
\frac{\partial F}{\partial \sigma_p} &= -\frac{n}{\sigma_p}-\frac{1}{2}\sum_{i=1}^{n}\frac{(\mathbf{x}_i-\boldsymbol{\mu})'\frac{\partial}{\partial\sigma_p}\Sigma^{-1}(\mathbf{x}_i-\boldsymbol{\mu})}{\sqrt{(\mathbf{x}_i-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x}_i-\boldsymbol{\mu})}}=0,\\
\frac{\partial F}{\partial \sigma_s} &= -\frac{nm}{\sigma_s}-\frac{1}{2}\sum_{i=1}^{n}\frac{(\mathbf{x}_i-\boldsymbol{\mu})'\frac{\partial}{\partial\sigma_s}\Sigma^{-1}(\mathbf{x}_i-\boldsymbol{\mu})}{\sqrt{(\mathbf{x}_i-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x}_i-\boldsymbol{\mu})}}=0,\\
\frac{\partial F}{\partial \rho_{ps}} &= -\frac{nm\rho_{ps}}{w}-\frac{1}{2}\sum_{i=1}^{n}\frac{(\mathbf{x}_i-\boldsymbol{\mu})'\frac{\partial}{\partial\rho_{ps}}\Sigma^{-1}(\mathbf{x}_i-\boldsymbol{\mu})}{\sqrt{(\mathbf{x}_i-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x}_i-\boldsymbol{\mu})}}=0,\\
\frac{\partial F}{\partial \rho_{ss}} &= -\frac{n(m-1)}{2}\left[\frac{m\left(\rho_{ss}-\rho_{ps}^2\right)}{(1-\rho_{ss})\,w}\right]-\frac{1}{2}\sum_{i=1}^{n}\frac{(\mathbf{x}_i-\boldsymbol{\mu})'\frac{\partial}{\partial\rho_{ss}}\Sigma^{-1}(\mathbf{x}_i-\boldsymbol{\mu})}{\sqrt{(\mathbf{x}_i-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x}_i-\boldsymbol{\mu})}}=0.
\end{aligned}
$$

None of the above partials have an explicit solution. However an iterative approach can be adopted using SAS/IML procedure. We used a dual Quasi Newton Method (NLPQN) routine for optimizing the objective function to get the estimators. The optimization gives unique estimates in the feasible regions under the above covariance structures. The only difficulty with the suggested algorithm (NLPQN) is that it fails to converge when ρ_{ps}^2 is close to ρ_{ss}. We will report the results from only those iterations where the convergence was achieved.

We observe that these estimators are also the maximum likelihood estimators of μ_p, μ_s, σ_p, σ_s, ρ_{ps}, and ρ_{ss} when maximizing the log-likelihood function of Kotz type distribution with respect to μ_p, μ_s, σ_p, σ_s, ρ_{ps}, and ρ_{ss}.

For finding the asymptotic distribution of the maximum likelihood estimators under the assumed Kotz type distribution, we determine the Fisher information matrix. We will apply **Theorem 1** to compute the various elements of this matrix. Suppose $\mathcal{I}_{ps,K}=\begin{pmatrix} I_{11,K} & I_{12,K}\\ I_{21,K} & I_{22,K}\end{pmatrix}$. The $I_{12,K}=I_{21,K}'=\mathbf{0}$,

$I_{11,K}=\begin{pmatrix} \frac{w}{(m+1)\sigma_p^2 g} & \frac{-2m\rho_{ps}}{(m+1)\sigma_p\sigma_s g}\\ \frac{-2m\rho_{ps}}{(m+1)\sigma_p\sigma_s g} & \frac{m}{(m+1)\sigma_s^2 g}\end{pmatrix}$, and the symmetric matrix

$$I_{22,K} = \begin{pmatrix} \frac{g(m+1)+(m+2)w}{(m+3)\sigma_p^2 g} & \frac{-m}{\sigma_p\sigma_s}\left\{\frac{w+2\rho_{ps}^2}{(m+3)g}\right\} & \frac{-m(m+1)\rho_{ps}}{(m+3)\sigma_p g} & \frac{-m(m-1)(\rho_{ps}^2-\rho_{ss})}{2(m+3)g(1-\rho_{ss})\sigma_p} \\ & \frac{m[(m+4)g+(m+2)\rho_{ps}^2]}{(m+3)\sigma_s^2 g} & \frac{-2m\rho_{ps}}{(m+3)\sigma_s g} & \frac{m(m-1)(m+4)(\rho_{ps}^2-\rho_{ss})}{2(m+3)g\sigma_s(1-\rho_{ss})} \\ & & \gamma_1 & \gamma_2 \\ & symm & & \gamma_3 \end{pmatrix}$$

where $w = (1 + (m-1)\rho_{ss})$,

$\gamma_1 = \frac{m(m+2)w+m^2(m+1)\rho_{ps}^2}{(m+3)g^2}$,

$\gamma_2 = \frac{m(m-1)\rho_{ps}[\,m(\rho_{ps}^2-\rho_{ss})-2(m+2)(1-\rho_{ss})\,]}{2(m+3)g^2(1-\rho_{ss})}$,

$\gamma_3 = \frac{-m(m-1)(\rho_{ps}^2-\rho_{ss})[\,(m+4)g+m(1-\rho_{ps}^2)+2(m+2)(\rho_{ps}^2-1)(1-\rho_{ss})\,]}{4(m+3)g^2(1-\rho_{ss})^2}$.

If the distribution of $(x_i, \mathbf{y}_i)$ is assumed to be Kotz type distribution as in equation (II.6) then as in **Theorem 2**, the asymptotic distribution of the MLE's $\hat{\theta}_{6\times 1} = (\hat{\mu}_p, \hat{\mu}_s, \hat{\sigma}_p, \hat{\sigma}_s, \hat{\rho}_{ps}, \hat{\rho}_{ss})'$ is $\sqrt{n}(\hat{\theta} - \theta) \underrightarrow{d} N_6(0, \mathcal{I}_{ps,K}^{-1})$.

Similarly, if the distribution of $(x_i, \mathbf{y}_i)$ is assumed to be multivariate normal then as in **Theorem 3**, the asymptotic distribution of MLE's $\widetilde{\theta} = (\widetilde{\mu}_p, \widetilde{\mu}_s, \widetilde{\sigma}_p, \widetilde{\sigma}_s, \widetilde{\rho}_{ps}, \widetilde{\rho}_{ss})'$ is $\sqrt{n}(\widetilde{\theta} - \theta) \underrightarrow{d} N_6(0, \mathcal{I}_{ps,N}^{-1})$, where $\mathcal{I}_{ps,N} = \begin{pmatrix} I_{11,N} & I_{12,N} \\ I_{21,N} & I_{22,N} \end{pmatrix}$ with $I_{12,N} = I'_{21,N} = \mathbf{0}$,

$$I_{11,N} = \begin{pmatrix} \frac{w}{\sigma_p^2 g} & \frac{-m\rho_{ps}}{\sigma_p\sigma_s g} \\ \frac{-m\rho_{ps}}{\sigma_p\sigma_s g} & \frac{m}{\sigma_s^2 g} \end{pmatrix}, \text{ and}$$

$$I_{22,N} = \begin{pmatrix} \frac{w+g}{\sigma_p^2 g} & \frac{-m\rho_{ps}^2}{\sigma_p\sigma_s g} & \frac{-m\rho_{ps}}{\sigma_p g} & 0 \\ & \frac{2mg+m\rho_{ps}^2}{\sigma_s^2 g} & \frac{-m\rho_{ps}}{\sigma_s g} & \frac{m(m-1)(\rho_{ps}^2-\rho_{ss})}{g\sigma_s(1-\rho_{ss})} \\ & & \frac{m(w+m\rho_{ps}^2)}{g^2} & \frac{-m(m-1)\rho_{ps}}{g^2} \\ & symm & & \frac{(m-1)[\,(m-1)(1-\rho_{ss})^2+g^2\,]}{2g^2(1-\rho_{ss})^2} \end{pmatrix}.$$

In the following we will simulate data from multivariate normal distribution, multivariate T distribution, with degrees of freedom = 3 and Kotz type distribution, and in each case we will compare the bias, Mean squared error (MSE) and the Pitman Nearness (PN) probability, for balanced and unbalanced cases. Our simulation study includes generating data from each of these three distributions and comparing the MLE's based on Kotz type and normal distribution as well as Srivastava's non-iterative estimator provided in (3.1). For all simulations, the parameters used are the total number of families n = 10 and 50 to cover small and large sample sizes, family sizes, $m = 2, 3, 4$, and 5 and different combinations of mom-sib (ρ_{ps}) and sib-sib (ρ_{ss}) correlations. For these, we used the values 0.1, 0.3, 0.5, and 0.7 with the restrictions $\rho_{ps}^2 \leq \rho_{ss}$ and $0 \leq \rho_{ss} \leq 1$. Also we used $\sigma_m^2 = 2$, $\sigma_s^2 = 1$, $\mu_s = 0$, and $\mu_p = 0$. For each set of these parameters, 10,000 data sets were simulated. The three criteria used for comparison are simulation estimates of bias, MSE, and PN values.

III.3 Results and Remarks

Various results based on the simulation are provided in Tables III.1 - III.48 at the end of the chapter. However, a summary of the conclusion is given below.

- Tables III.1 - III.8 show that when data are simulated from multivariate normal distribution, the three estimators $\hat{\rho}_{ps,K}$, $\hat{\rho}_{ps,N}$ and $\hat{\rho}_{ps,S}$ have relatively large bias when ρ_{ps} and ρ_{ss} are both ≥ 0.3 and $n = 10$. However, the bias decreases as n increases to 50. Srivastava's estimator has higher bias for smaller values of ρ_{ss} that is, when $\rho_{ss} = 0.1$. We also notice that the bias of $\hat{\rho}_{ps,K}$ is either smaller than the bias of $\hat{\rho}_{ps,N}$ or very close to it in magnitude when $n = 50$ and $m \geq 4$.

- As for the MSE, we notice that $\hat{\rho}_{ps,N}$ has the smallest mean squared error as expected, except when $n = 10$ and for extreme values of (ρ_{ps}, ρ_{ss}), i.e. (0.1, 0.1) and (0.7, 0.7). At these values, the Srivastava's estimator had the smallest MSE. Furthermore, for large n (= 50) the Srivastava's estimator and the $\hat{\rho}_{ps,N}$ have almost the same MSE. However the MSE value for $\hat{\rho}_{ps,K}$ is comparable to the MSE value for $\hat{\rho}_{ps,N}$ and $\hat{\rho}_{ps,S}$. If we consider the magnitude of the differences between the MSE's of $\hat{\rho}_{ps,N}$ and $\hat{\rho}_{ps,K}$, we find the differences to be negligible when $n = 50$. More specifically, the differences are shown below:

  - For $n = 10$ and for
    $m = 2$ the differences run from 0.0023 to 0.0058;
    $m = 3$ the differences run from 0.0016 to 0.0051;
    $m = 4$ the differences run from 0.0011 to 0.0043;
    $m = 5$ the differences run from 0.0009 to 0.0035.

  - For $n = 50$ and for
    $m = 2$ the differences run from 0.0003 to 0.0018;
    $m = 3$ the differences run from 0.0003 to 0.0018;

$m = 4$ the differences run from 0.0003 to 0.001;

$m = 5$ the differences run from 0.0001 to 0.0009.

As can be seen easily the differences decrease as m increases, and the highest difference occurs when $\rho_{ps} = 0.1$ and $\rho_{ss} = 0.7$, whereas the smallest difference occurs when $\rho_{ps} = 0.5$ and $\rho_{ss} = 0.3$.

- When data are simulated from Kotz type distribution, it can be seen from Tables III.9 - III.16 that $\hat{\rho}_{ps,K}$ and $\hat{\rho}_{ps,N}$ have a relatively smaller bias when $\rho_{ss} = 0.1$ as compared to that of the Srivastava's estimator and this is more so when n is small $(= 10)$. Furthermore, $\hat{\rho}_{ps,K}$ has the lowest MSE in almost all the cases except when n is small and $m = 2, 3, and 4$. The Srivastava's estimator has the lowest MSE when $(\rho_{ps}, \rho_{ss}) = (0.1, 0.1)$. We notice also that the performance of $\hat{\rho}_{ps,S}$ and $\hat{\rho}_{ps,N}$ is the same especially for large n $(= 50)$. The differences between the $\hat{\rho}_{ps,K}$ and $\hat{\rho}_{ps,N}$ in their MSE values are provided below:

  - For $n = 10$ and for

    $m = 2$ the differences run from 0.0032 to 0.0067;

    $m = 3$ the differences run from 0.0014 to 0.0053;

    $m = 4$ the differences run from 0.0014 to 0.0036;

    $m = 5$ the differences run from 0.001 to 0.0042.

  - For $n = 50$ and for

    $m = 2$ the differences run from 0.0009 to 0.0034;

    $m = 3$ the differences run from 0.0006 to 0.0024;

    $m = 4$ the differences run from 0.0005 to 0.0021;

    $m = 5$ the differences run from 0.0005 to 0.0016.

- When the data are simulated from multivariate T distribution with degrees of freedom $= 3$, $\hat{\rho}_{ps,K}$ has the least bias for all m and n except for the case when n and m are small $(n = 10, m = 3)$, and for lower value of ρ_{ss} $(= 0.1)$. When $n = 50$ we find that $\hat{\rho}_{ps,N}$ and $\hat{\rho}_{ps,S}$ have almost the same bias and most of the time these are at least twice as much as the bias of $\hat{\rho}_{ps,K}$. Refer to Tables III.17 - III.24.

- Next we note that $\hat{\rho}_{ps,K}$ always has the smallest MSE value, and for large n this value is almost half the value of the MSE for either the $\hat{\rho}_{ps,S}$ or $\hat{\rho}_{ps,N}$. Also, the MSE values for $\hat{\rho}_{ps,S}$ and $\hat{\rho}_{ps,N}$ are almost equal. Having n to be large didn't reduce the differences between the MSE's of the estimators, but for sure it reduces the MSE value of each estimator.

- Tables III.25 - III.32 show that when data are simulated from normal distribution, the PN probability of $\hat{\rho}_{ps,N}$ relative to $\hat{\rho}_{ps,K}$ is greater than 0.5, while the PN probability of $\hat{\rho}_{ps,K}$ relative to $\hat{\rho}_{ps,S}$ is less than 0.5, but this last value increases to 0.5 as ρ_{ps} and ρ_{ss} increase. Also, we notice that $\hat{\rho}_{ps,K}$ is better than $\hat{\rho}_{ps,S}$ for small n $(= 10)$, small m $(= 2,3)$, and $(\rho_{ps},\ \rho_{ss}) = (0.5, 0.3)$, and for $m = 4,5$ and $\rho_{ps} = 0.3$ and 0.5. When samples are from Kotz type distribution, we see from Tables III.33 - III.40 that the PN probability of $\hat{\rho}_{ps,N}$ relative to $\hat{\rho}_{ps,K}$ is less than 0.5, while the PN probability of $\hat{\rho}_{ps,K}$ relative to $\hat{\rho}_{ps,S}$ is greater than 0.5. Also, the PN probability of $\hat{\rho}_{ps,N}$ relative to $\hat{\rho}_{ps,S}$ is ≥ 0.5. We also notice that even when data are from multivariate T distribution, the results were similar to what we found earlier, in that the estimates of the Kotz type distribution are more efficient than their competitors. Tables III.41 - III.48 show this clearly.

III.4 Alternative Approach: Unbalanced Case

The proposed estimators under the unbalanced case can be similarly obtained by minimizing

$$\begin{aligned} F(\mu_p, \mu_s, \sigma_p, \sigma_s, \rho_{ps}, \rho_{ss}) &= n \log \sigma_p + \sum_{i=1}^{n} (m_i \log \sigma_s + \frac{1}{2}(m_i - 1) \log(1 - \rho_{ss})) \\ &\quad + \sum_{i=1}^{n} \frac{1}{2} \log g_i + \sum_{i=1}^{n} [(\mathbf{x}_i - \boldsymbol{\mu}_i)' \boldsymbol{\Sigma}_i^{-1} (\mathbf{x}_i - \boldsymbol{\mu}_i)]^{\frac{1}{2}} \qquad \text{(III.4)} \end{aligned}$$

with respect to $\mu_p, \mu_s, \sigma_p, \sigma_s, \rho_{ps}$, and ρ_{ss}, where $g_i = 1 + (m_i - 1)\rho_{ss} - m_i \rho_{ps}^2$, $i = 1, 2, ..., n$. The solutions $\hat{\mu}_p, \hat{\mu}_s, \hat{\sigma}_p, \hat{\sigma}_s, \hat{\rho}_{ps}$, and $\hat{\rho}_{ss}$ are the maximum likelihood estimators when the underlying distribution is Kotz type. The Fisher information matrix for a given $(x_i, \mathbf{y}_i)$ is given by

$\mathcal{I}_{ps,Ki} =$

$$\begin{pmatrix} \frac{1}{\sigma_p^2}c_{11_i} & \frac{-2\rho_{ps}}{\sigma_p\sigma_s}c_{12_i} & 0 & 0 & 0 & 0 \\ & \frac{1}{\sigma_s^2}c_{22_i} & 0 & 0 & 0 & 0 \\ & & \frac{1}{\sigma_p^2}c_{33_i} & \frac{-1}{\sigma_p\sigma_s}c_{34_i} & \frac{-\rho_{ps}}{\sigma_p}c_{35_i} & \frac{-(\rho_{ps}^2-\rho_{ss})}{2(1-\rho_{ss})\sigma_p}c_{36_i} \\ & & & \frac{1}{\sigma_s^2}c_{44_i} & \frac{-2\rho_{ps}}{\sigma_s}c_{45_i} & \frac{(\rho_{ps}^2-\rho_{ss})}{2(1-\rho_{ss})\sigma_s}c_{46_i} \\ & symm & & & c_{55_i} & \frac{1}{2(1-\rho_{ss})}c_{56_i} \\ & & & & & -\frac{(\rho_{ps}^2-\rho_{ss})}{4(1-\rho_{ss})^2}c_{66_i} \end{pmatrix} \quad \text{(III.5)}$$

where $w_i = 1 + (m_i - 1)\rho_{ss}$, $g_i = 1 + 1 + (m_i - 1)\rho_{ss} - m_i\rho_{ps}^2$,

$c_{11_i} = \frac{w_i}{(m_i+1)g_i}$, $c_{12_i} = \frac{m_i}{(m_i+1)g_i}$, $c_{22_i} = \frac{m_i}{(m_i+1)g_i}$,

$c_{33_i} = \frac{g_i(m_i+1)+(m_i+2)w_i}{(m_i+3)g_i}$, $c_{34_i} = \frac{(w_i+2\rho_{ps}^2)m_i}{(m_i+3)g_i}$, $c_{35_i} = \frac{m_i(m_i+1)}{(m_i+3)g_i}$,

$c_{36_i} = \frac{m_i(m_i-1)}{(m_i+3)g_i}$, $c_{44_i} = \frac{m_i[(m_i+4)g_i+(m_i+2)\rho_{ps}^2]}{(m_i+3)g_i}$,

$c_{45_i} = \frac{m_i}{(m_i+3)g_i}$, $c_{46_i} = \frac{m_i(m_i-1)(m_i+4)}{(m_i+3)g_i}$,

$c_{55_i} = \frac{m_i(m_i+2)w_i+m_i^2(m_i+1)\rho_{ps}^2}{(m_i+3)g_i^2}$,

$c_{56_i} = \frac{m_i(m_i-1)[m_i(\rho_{ps}^2-\rho_{ss})-2(m_i+2)(1-\rho_{ss})]}{(m_i+3)g_i^2}$,

$c_{66_i} = \frac{m_i(m_i-1)[\ (m_i+4)g_i+m_i(1-\rho_{ps}^2)+2(m_i+2)(\rho_{ps}^2-1)(1-\rho_{ss})\ ]}{(m_i+3)g_i^2}$.

For large n, $\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_{ps,Ki}$ will converges to $\mathcal{I}_{ps,K}$, where

$$\mathcal{I}_{ps,K} = \begin{pmatrix} \frac{1}{\sigma_p^2}c_{11} & \frac{-2\rho_{ps}}{\sigma_p\sigma_s}c_{12} & 0 & 0 & 0 & 0 \\ & \frac{1}{\sigma_s^2}c_{22} & 0 & 0 & 0 & 0 \\ & & \frac{1}{\sigma_p^2}c_{33} & \frac{-1}{\sigma_p\sigma_s}c_{34} & \frac{-\rho_{ps}}{\sigma_p}c_{35} & \frac{-(\rho_{ps}^2-\rho_{ss})}{2(1-\rho_{ss})\sigma_p}c_{36} \\ & & & \frac{1}{\sigma_s^2}c_{44} & \frac{-2\rho_{ps}}{\sigma_s}c_{45} & \frac{(\rho_{ps}^2-\rho_{ss})}{2(1-\rho_{ss})\sigma_s}c_{46} \\ & symm & & & c_{55} & \frac{1}{2(1-\rho_{ss})}c_{56} \\ & & & & & -\frac{(\rho_{ps}^2-\rho_{ss})}{4(1-\rho_{ss})^2}c_{66} \end{pmatrix}$$

where $c_{11} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}c_{11_i}$, $c_{12} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}c_{12_i}$, $c_{22} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}c_{22_i}$,

$c_{33} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}c_{33_i}$, $c_{34} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}c_{34_i}$, $c_{35} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}c_{35_i}$,

$c_{36} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} c_{36_i}$, $c_{44} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} c_{44_i}$, $c_{45} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} c_{45_i}$,

$c_{46} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} c_{46_i}$, $c_{55} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} c_{55_i}$, $c_{56} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} c_{56_i}$,

$c_{66} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} c_{66_i}$.

If $\hat{\theta} = (\hat{\mu}_p, \hat{\mu}_s, \hat{\sigma}_p^2, \hat{\sigma}_s^2, \hat{\rho}_{ps}, \hat{\rho}_{ss})'$ is the MLE of $\theta = (\mu_p, \mu_s, \sigma_p^2, \sigma_s^2, \rho_{ps}, \rho_{ss})'$ then $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N_6(0, \mathcal{I}_{ps,K}^{-1})$.

Similarly, for normal distribution we have $\sqrt{n}(\tilde{\theta} - \theta) \xrightarrow{d} N_6(0, \mathcal{I}_{ps,N}^{-1})$, where $\tilde{\theta}$ is the MLE of θ under normal distribution

$$\mathcal{I}_{ps,N} = \begin{pmatrix} \frac{1}{\sigma_p^2}d_{11} & -\frac{\rho_{ps}}{\sigma_p\sigma_s}d_{12} & 0 & 0 & 0 & 0 \\ & \frac{1}{\sigma_s^2}d_{22} & 0 & 0 & 0 & 0 \\ & & \frac{1}{\sigma_p^2}d_{33} & -\frac{\rho_{ps}^2}{\sigma_p\sigma_s}d_{34} & -\frac{\rho_{ps}}{\sigma_p}d_{35} & 0 \\ & & & \frac{1}{\sigma_s^2}d_{44} & \frac{-\rho_{ps}}{\sigma_s}d_{45} & \frac{(\rho_{ps}^2-\rho_{ss})}{\sigma_s(1-\rho_{ss})}d_{46} \\ & symm & & & d_{55} & -\rho_{ps}d_{56} \\ & & & & & \frac{1}{2(1-\rho_{ss})^2}d_{66} \end{pmatrix}.$$

Here $d_{11} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{w_i}{g_i}$, $d_{12} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{m_i}{g_i}$, $d_{22} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{m_i}{g_i}$,

$d_{33} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{w_i+g_i}{g_i}$, $d_{34} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{m_i}{g_i}$,

$d_{35} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{m_i}{g_i}$, $d_{44} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{2m_ig_i+m_i\rho_{ps}^2}{g_i}$,

$d_{45} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{m_i}{g_i}$, $d_{55} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{mi(1+(mi-1)pss+mPps2)}{gi2}$,

$d_{46} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{m_i(m_i-1)}{g_i}$, $d_{56} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{m_i(m_i-1)}{g_i^2}$,

$d_{66} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{(m_i-1)[\,(m_i-1)(1-\rho_{ss})^2+g_i^2]}{g_i^2}$.

In practice we can calculate the asymptotic variance of $\hat{\theta}$ as:

$$var(\hat{\theta}) \approx \left[\sum_{i=1}^{n} \mathcal{I}_{ps,Ki}\right]^{-1},$$

where $\mathcal{I}_{ps,Ki}$ is given in III.5. Similarly the asymptotic variance of $\tilde{\theta}$ is calculated.

III.4.1 Simulating data with unequal family sizes

In the unbalanced case we follow the same procedure as that for the balanced case for generating the data from the three types of distributions, namely, multivariate normal, multivariate T with 3 degrees of freedom and Kotz type distribution. Here we also have to simulate the family sizes (m_i). As in the previous chapter, we used the procedure due to Brass (1958), that is, the negative binomial distribution truncated below by 1 and truncated above by 5, that is, $1 \leqslant m_i \leqslant 5$. As before we will compare the bias, the MSE, and the PN probability values for MLE$_N$, MLE$_K$, and Srivastava's estimator.

We include the total number of families, $n = 50$, and 100, and different combinations of mom-sib (ρ_{ps}) and sib-sib (ρ_{ss}) correlations taking the values $0.1, 0.3, 0.5$, and 0.7, but with the restrictions, $\rho_{ps}^2 \leq \rho_{ss}$ and $0 \leq \rho_{ss} \leq 1$ (these are necessary and sufficient condition for $\mathbf{\Sigma}_i$ to be positive definite for all m_i).

III.5 Results and Remarks

Results are summarized in Tables III.49 - III.60 provided at the end of the chapter. A summary of the results is provided below.

- Tables III.49 - III.50 show that when the simulation is from multivariate normal, $\hat{\rho}_{ps,N}$ has the smallest bias for small ρ_{ss} (=0.1, 0.3). However, interestingly $\hat{\rho}_{ps,K}$ has the the smallest bias when n=50 and for moderate to large values of $\rho_{ss}(= 0.5, 0.7)$ and also when n=100, ρ_{ps} small ($= 0.1$) and $\rho_{ss} = 0.5$ and 0.7. In general, $\hat{\rho}_{ps,N}$ and $\hat{\rho}_{ps,S}$ have relatively smaller bias, especially for n=100.

- In general, $\hat{\rho}_{ps,N}$ has the smallest mean squared error. For moderate to large values of ρ_{ss} (ρ_{ss} = 0.5, 0.7), the MSE of the $\hat{\rho}_{ps,S}$ is close to that of $\hat{\rho}_{ps,N}$. However, it is notable that the MSE of $\hat{\rho}_{ps,K}$ is comparable with that for $\hat{\rho}_{ps,N}$ and $\hat{\rho}_{ps,S}$, especially for large values of ρ_{ps} ($= 0.7$) when n=100.

- Tables III.51 - III.52 show that when simulation is from Kotz type distribution, $\hat{\rho}_{ps,K}$ has the smallest bias, except for the case when n=100 and $\rho_{ps} = 0.1$ in which case $\hat{\rho}_{ps,S}$ has the smallest bias. Moreover, the bias for the Srivastava's estimator ($\hat{\rho}_{ps,S}$) is comparable with the bias of $\hat{\rho}_{ps,K}$, when ρ_{ps} is small ($= 0.1$) and n=100. The MSE of $\hat{\rho}_{ps,K}$ is the smallest in general. However, for $\rho_{ss} = 0.7$ the MSE of $\hat{\rho}_{ps,N}$ and $\hat{\rho}_{ps,S}$, are roughly the same.

- When simulation is from multivariate T distribution with degrees of freedom=3, $\hat{\rho}_{ps,N}$ and $\hat{\rho}_{ps,S}$ have the same bias and it is smaller than the bias of $\hat{\rho}_{ps,K}$ for ρ_{ss} =0.1, $n = 50$, and for $\rho_{ss} = 0.1$ and 0.3 when $n = 100$. For the remaining cases the magnitude of the bias of the $\hat{\rho}_{ps,K}$ is not only the smallest but it is negligible compared to the other estimators for $\rho_{ss} = 0.5$ and 0.7 and when $n = 50$ and 100. The same conclusion holds when $\rho_{ss} = 0.7$, (Tables III.53 - III.54). The MSE values for both $\hat{\rho}_{ps,N}$ and $\hat{\rho}_{ps,S}$ is not only larger than that for the $\hat{\rho}_{ps,K}$ but it is at least twice as large as the MSE for $\hat{\rho}_{ps,K}$.

- For data from multivariate normal we see from Tables III.55 - III.56 that the PN probability of $\hat{\rho}_{ps,N}$ relative to $\hat{\rho}_{ps,K}$ is greater than 0.5, while the PN probability of $\hat{\rho}_{ps,K}$ relative to $\hat{\rho}_{ps,S}$ is less than 0.5. When we simulate from Kotz type distribution Tables III.57 - III.58 show that the PN probability of $\hat{\rho}_{ps,K}$ relative to $\hat{\rho}_{ps,N}$ or relative to $\hat{\rho}_{ps,S}$ is greater than .5. Also, notice that the $\hat{\rho}_{ps,K}$ improves significantly as ρ_{ps} and ρ_{ss} increase. And when simulation is from multivariate T with 3 degrees of freedom, we find from Tables III.59 - III.60 that the PN of $\hat{\rho}_{ps,K}$ relative to $\hat{\rho}_{ps,N}$ or relative to $\hat{\rho}_{ps,S}$ is at least .6, when $n = 50$ and increases up to 0.7 when n=100.

In summary, we considered two multivariate heavy-tailed distributions which have fatter tail regions than that of multivariate normal distribution and studied the performance of estimators of mom-sib correlation. We find the estimator $\hat{\rho}_{ps}$ by maximizing the log-likelihood function of Kotz type distribution and compared it with the estimator based on normal distribution and an estimator based non-iterative method. We have provided a simulation algorithm for generating samples from Kotz type distribution with unequal family sizes. Next, we performed a simulation experiment to compare the ML estimators of the mom-sib correlation by using three measures, namely, bias, MSE and Pitman Nearness probability under multivariate normal, multivariate T and Kotz type samples. Based on all the three criteria and using the results provided in previous subsections we conclude that these estimators perform as expected in relation to one another with regard to bias, MSE and PN probability. The estimator $\hat{\rho}_{ps,N}$ and $\hat{\rho}_{ps,S}$ outperform $\hat{\rho}_{ps,K}$ when data come from a normal distribution, and $\hat{\rho}_{ps,K}$ outperforms all other estimators when data come from the Kotz type distribution. However, the magnitudes of the differences in these criteria values are much greater when $\hat{\rho}_{ps,N}$ is used when the distribution is heavy tailed. This implies that the greatest loss occurs if normal estimates are used for non normal cases.

III.6 Analysis of Galton's Data

We use Galton's data for illustration of the methods that we have used in this chapter. We will consider the problem of computing the mom-daughter (ρ_{md}), mom-son (ρ_{ms}), father-daughter (ρ_{fd}), and father-son (ρ_{fs}) correlations using all the three methods described. The estimates and their asymptotic standard errors are provided below

We notice that all the estimators have very similar standard errors for each category of parent-sib correlation estimates. The strongest correlation exists between father-children (daughters or sons) compared to mother-children correlation.

Method	ρ_{md}	ρ_{ms}	ρ_{fd}	ρ_{fs}
MLE_N	.321 (.0574)	.316 (.0539)	.461 (.0490)	.417 (.0490)
MLE_K	.385 (.0616)	.326 (.0608)	.475 (.0548)	.448 (0.0539)
Srivastava	.337 (.0583)	.309 (.0565)	.466 (.0510)	.428 (.0500)

Table III.1

Bias and MSE ($\times 10^4$) of interclass correlation estimators for different values of ρ_{ss} and ρ_{ms} (n=10, m=2), based on 10,000 simulations from normal distribution

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	-16	691	-52	716	-32	795	-37	904
	$\hat{\rho}_{ps,K}$	-24	743	-61	772	-32	850	-23	962
	$\hat{\rho}_{ps,S}$	-62	602	-58	694	-26	805	-30	918
0.3	$\hat{\rho}_{ps,N}$	25	530	-139	613	-160	680	-152	758
	$\hat{\rho}_{ps,K}$	21	568	-139	660	-159	732	-162	810
	$\hat{\rho}_{ps,S}$	-133	494	-159	607	-137	690	-130	769
0.5	$\hat{\rho}_{ps,N}$			-176	376	-268	467	-251	557
	$\hat{\rho}_{ps,K}$			-176	405	-284	503	-269	601
	$\hat{\rho}_{ps,S}$			-222	404	-233	477	-213	562
0.7	$\hat{\rho}_{ps,N}$					-308	214	-303	282
	$\hat{\rho}_{ps,K}$					-314	230	-312	305
	$\hat{\rho}_{ps,S}$					-261	230	-248	281

Table III.2

Bias and MSE ($\times 10^4$) of interclass correlation estimators for different values of ρ_{ss} and ρ_{ms} (n=10, m=3), based on 10,000 simulations from normal distribution

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	-33	474	-67	548	-98	684	-94	834
	$\hat{\rho}_{ps,K}$	-27	506	-67	588	-108	731	-90	882
	$\hat{\rho}_{ps,S}$	-66	425	-57	556	-86	705	-85	852
0.3	$\hat{\rho}_{ps,N}$	-8	346	-167	453	-214	583	-238	728
	$\hat{\rho}_{ps,K}$	-6	368	-156	489	-214	619	-249	779
	$\hat{\rho}_{ps,S}$	-128	349	-136	467	-168	598	-207	742
0.5	$\hat{\rho}_{ps,N}$			-251	274	-311	395	-300	523
	$\hat{\rho}_{ps,K}$			-244	293	-303	421	-298	555
	$\hat{\rho}_{ps,S}$			-206	299	-235	403	-247	529
0.7	$\hat{\rho}_{ps,N}$					-332	180	-332	256
	$\hat{\rho}_{ps,K}$					-324	188	-335	272
	$\hat{\rho}_{ps,S}$					-223	180	-256	252

Table III.3

Bias and MSE (×10^4) of interclass correlation estimators for different values of ρ_{ss} and ρ_{ms} (n=10, m=4), based on 10,000 simulations from normal distribution

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	-55	368	-94	484	-77	631	-64	789
	$\hat{\rho}_{ps,K}$	-42	388	-86	512	-78	664	-61	832
	$\hat{\rho}_{ps,S}$	-71	343	-74	501	-59	656	-52	809
0.3	$\hat{\rho}_{ps,N}$	-26	253	-206	400	-232	537	-186	674
	$\hat{\rho}_{ps,K}$	-14	272	-194	422	-229	563	-178	711
	$\hat{\rho}_{ps,S}$	-98	272	-148	416	-177	554	-151	689
0.5	$\hat{\rho}_{ps,N}$			-279	233	-342	368	-325	494
	$\hat{\rho}_{ps,K}$			-255	244	-328	386	-312	517
	$\hat{\rho}_{ps,S}$			-185	249	-250	373	-266	500
0.7	$\hat{\rho}_{ps,N}$					-370	166	-343	239
	$\hat{\rho}_{ps,K}$					-344	170	-335	251
	$\hat{\rho}_{ps,S}$					-238	160	-257	234

Table III.4

Bias and MSE (×10^4) of interclass correlation estimators for different values of ρ_{ss} and ρ_{ms} (n=10, m=5), based on 10,000 simulations from normal distribution

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	-64	306	-76	442	-94	608	-75	768
	$\hat{\rho}_{ps,K}$	-58	322	-69	468	-96	639	-70	801
	$\hat{\rho}_{ps,S}$	-74	295	-53	464	-74	633	-63	789
0.3	$\hat{\rho}_{ps,N}$	-83	202	-219	359	-222	517	-232	690
	$\hat{\rho}_{ps,K}$	-69	215	-205	379	-214	540	-234	725
	$\hat{\rho}_{ps,S}$	-123	227	-149	375	-164	535	-195	706
0.5	$\hat{\rho}_{ps,N}$			-323	219	-332	345	-282	478
	$\hat{\rho}_{ps,K}$			-300	228	-319	359	-279	499
	$\hat{\rho}_{ps,S}$			-203	228	-233	350	-219	484
0.7	$\hat{\rho}_{ps,N}$					-387	151	-361	230
	$\hat{\rho}_{ps,K}$					-363	154	-351	240
	$\hat{\rho}_{ps,S}$					-242	142	-269	225

Table III.5

Bias and MSE ($\times 10^4$) of interclass correlation estimators for different values of ρ_{ss} and ρ_{ms} (n=50, m=2), based on 10,000 simulations from normal distribution

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	-7	115	3	130	-13	146	-26	169
	$\hat{\rho}_{ps,K}$	1	127	-1	142	-15	161	-26	187
	$\hat{\rho}_{ps,S}$	-21	112	6	130	-10	147	-24	170
0.3	$\hat{\rho}_{ps,N}$	16	86	-47	106	-43	123	-32	145
	$\hat{\rho}_{ps,K}$	26	94	-43	116	-48	136	-37	159
	$\hat{\rho}_{ps,S}$	-28	89	-39	106	-36	123	-28	146
0.5	$\hat{\rho}_{ps,N}$			-47	67	-65	81	-34	95
	$\hat{\rho}_{ps,K}$			-49	74	-69	89	-38	104
	$\hat{\rho}_{ps,S}$			-33	68	-52	81	-26	95
0.7	$\hat{\rho}_{ps,N}$					-53	36	-56	41
	$\hat{\rho}_{ps,K}$					-57	40	-62	46
	$\hat{\rho}_{ps,S}$					-35	36	-46	41

Table III.6

Bias and MSE ($\times 10^4$) of interclass correlation estimators for different values of ρ_{ss} and ρ_{ms} (n=50, m=3), based on 10,000 simulations from normal distribution

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	-12	80	-22	102	-13	131	-3	157
	$\hat{\rho}_{ps,K}$	-10	87	-22	111	-15	142	-5	170
	$\hat{\rho}_{ps,S}$	-16	80	-18	103	-10	132	-1	158
0.3	$\hat{\rho}_{ps,N}$	-10	59	-43	86	-39	107	-51	133
	$\hat{\rho}_{ps,K}$	-3	64	-44	93	-36	116	-55	145
	$\hat{\rho}_{ps,S}$	-22	62	-29	86	-29	107	-45	134
0.5	$\hat{\rho}_{ps,N}$			-64	52	-53	70	-54	88
	$\hat{\rho}_{ps,K}$			-64	56	-53	75	-58	95
	$\hat{\rho}_{ps,S}$			-41	52	-36	70	-44	88
0.7	$\hat{\rho}_{ps,N}$					-72	28	-52	37
	$\hat{\rho}_{ps,K}$					-73	30	-55	40
	$\hat{\rho}_{ps,S}$					-48	27	-38	37

Table III.7

Bias and MSE ($\times 10^4$) of interclass correlation estimators for different values of ρ_{ss} and ρ_{ms} (n=50, m=4), based on 10,000 simulations from normal distribution

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	-7	63	-3	94	-7	120	-14	151
	$\hat{\rho}_{ps,K}$	-4	68	-2	100	-5	130	-12	161
	$\hat{\rho}_{ps,S}$	-5	64	3	95	-3	121	-12	152
0.3	$\hat{\rho}_{ps,N}$	-28	48	-34	74	-52	100	-33	126
	$\hat{\rho}_{ps,K}$	-23	51	-35	79	-51	106	-33	136
	$\hat{\rho}_{ps,S}$	-22	50	-18	75	-41	100	-26	127
0.5	$\hat{\rho}_{ps,N}$			-61	44	-42	62	-60	85
	$\hat{\rho}_{ps,K}$			-57	47	-40	67	-59	91
	$\hat{\rho}_{ps,S}$			-35	44	-24	62	-49	85
0.7	$\hat{\rho}_{ps,N}$					-72	25	-66	35
	$\hat{\rho}_{ps,K}$					-70	27	-66	38
	$\hat{\rho}_{ps,S}$					-45	25	-49	35

Table III.8

Bias and MSE ($\times 10^4$) of interclass correlation estimators for different values of ρ_{ss} and ρ_{ms} (n=50, m=5), based on 10,000 simulations from normal distribution

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	-4	54	-20	88	-8	116	-11	154
	$\hat{\rho}_{ps,K}$	-2	57	-15	94	-13	123	-16	163
	$\hat{\rho}_{ps,S}$	1	54	-14	89	-4	117	-9	154
0.3	$\hat{\rho}_{ps,N}$	-41	41	-34	67	-45	95	-46	125
	$\hat{\rho}_{ps,K}$	-40	43	-31	71	-44	102	-47	132
	$\hat{\rho}_{ps,S}$	-26	42	-17	67	-33	96	-38	125
0.5	$\hat{\rho}_{ps,N}$			-61	40	-80	61	-55	82
	$\hat{\rho}_{ps,K}$			-59	43	-78	64	-60	88
	$\hat{\rho}_{ps,S}$			-33	40	-60	60	-43	83
0.7	$\hat{\rho}_{ps,N}$					-68	22	-68	34
	$\hat{\rho}_{ps,K}$					-65	23	-65	35
	$\hat{\rho}_{ps,S}$					-40	22	-51	33

Table III.9

Bias and MSE (×10^4)of interclass correlation estimators for different values of ρ_{ss} and ρ_{ms} (n=10, m=2), based on 10,000 simulations from Kotz type distribution

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	20	872	-65	904	-84	1006	-37	1110
	$\hat{\rho}_{ps,K}$	33	818	-69	844	-71	940	-43	1043
	$\hat{\rho}_{ps,S}$	-44	747	-78	863	-83	1009	-30	1126
0.3	$\hat{\rho}_{ps,N}$	1	692	-181	774	-208	872	-145	977
	$\hat{\rho}_{ps,K}$	-4	641	-162	719	-198	824	-129	913
	$\hat{\rho}_{ps,S}$	-201	631	-225	759	-196	880	-126	989
0.5	$\hat{\rho}_{ps,N}$			-177	482	-281	600	-346	725
	$\hat{\rho}_{ps,K}$			-154	447	-262	563	-321	683
	$\hat{\rho}_{ps,S}$			-265	514	-266	614	-313	733
0.7	$\hat{\rho}_{ps,N}$					-314	275	-345	375
	$\hat{\rho}_{ps,K}$					-291	258	-308	341
	$\hat{\rho}_{ps,S}$					-291	299	-296	376

Table III.10

Bias and MSE (×10^4)of interclass correlation estimators for different values of ρ_{ss} and ρ_{ms} (n=10, m=3), based on 10,000 simulations from Kotz type distribution

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	13	595	-61	681	-143	852	-68	994
	$\hat{\rho}_{ps,K}$	20	555	-55	646	-129	814	-66	941
	$\hat{\rho}_{ps,S}$	-38	530	-55	683	-130	876	-58	1017
0.3	$\hat{\rho}_{ps,N}$	23	436	-200	569	-203	700	-253	864
	$\hat{\rho}_{ps,K}$	20	408	-170	535	-184	665	-229	820
	$\hat{\rho}_{ps,S}$	-119	431	-184	583	-163	720	-223	881
0.5	$\hat{\rho}_{ps,N}$			-258	343	-344	471	-339	628
	$\hat{\rho}_{ps,K}$			-223	328	-297	446	-292	589
	$\hat{\rho}_{ps,S}$			-240	375	-275	483	-286	636
0.7	$\hat{\rho}_{ps,N}$					-406	223	-423	334
	$\hat{\rho}_{ps,K}$					-350	206	-378	311
	$\hat{\rho}_{ps,S}$					-307	228	-347	330

Table III.11

Bias and MSE ($\times 10^4$) of interclass correlation estimators for different values of ρ_{ss} and ρ_{ms} (n=10, m=4), based on 10,000 simulations from Kotz type distribution

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	-5	443	-75	572	-131	734	-60	924
	$\hat{\rho}_{ps,K}$	3	419	-72	544	-119	702	-58	888
	$\hat{\rho}_{ps,S}$	-36	409	-57	591	-114	761	-48	948
0.3	$\hat{\rho}_{ps,N}$	-31	313	-232	481	-270	638	-272	814
	$\hat{\rho}_{ps,K}$	-15	297	-200	455	-237	614	-254	773
	$\hat{\rho}_{ps,S}$	-131	331	-184	499	-218	659	-238	832
0.5	$\hat{\rho}_{ps,N}$			-360	295	-378	445	-375	585
	$\hat{\rho}_{ps,K}$			-318	281	-332	419	-342	553
	$\hat{\rho}_{ps,S}$			-283	315	-290	453	-317	592
0.7	$\hat{\rho}_{ps,N}$					-419	200	-390	293
	$\hat{\rho}_{ps,K}$					-359	184	-345	274
	$\hat{\rho}_{ps,S}$					-290	194	-304	287

Table III.12

Bias and MSE ($\times 10^4$) of interclass correlation estimators for different values of ρ_{ss} and ρ_{ms} (n=10, m=5), based on 10,000 simulations from Kotz type distribution

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	-25	362	-71	509	-84	697	-81	907
	$\hat{\rho}_{ps,K}$	-14	344	-58	488	-66	678	-69	869
	$\hat{\rho}_{ps,S}$	-43	348	-50	532	-66	726	-69	932
0.3	$\hat{\rho}_{ps,N}$	-80	240	-231	412	-263	596	-250	774
	$\hat{\rho}_{ps,K}$	-58	230	-204	396	-231	571	-227	732
	$\hat{\rho}_{ps,S}$	-130	265	-166	430	-206	617	-213	792
0.5	$\hat{\rho}_{ps,N}$			-366	258	-374	399	-313	567
	$\hat{\rho}_{ps,K}$			-326	249	-333	380	-271	535
	$\hat{\rho}_{ps,S}$			-260	272	-275	404	-250	575
0.7	$\hat{\rho}_{ps,N}$					-422	172	-375	265
	$\hat{\rho}_{ps,K}$					-368	161	-332	246
	$\hat{\rho}_{ps,S}$					-279	162	-282	259

Table III.13

Bias and MSE ($\times 10^4$) of interclass correlation estimators for different values of ρ_{ss} and ρ_{ms} (n=50, m=2), based on 10,000 simulations from Kotz type distribution

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	15	160	-39	186	-10	213	6	240
	$\hat{\rho}_{ps,K}$	8	138	-26	159	-6	184	5	206
	$\hat{\rho}_{ps,S}$	-9	154	-38	186	-8	214	8	241
0.3	$\hat{\rho}_{ps,N}$	38	126	-28	153	-40	177	-41	205
	$\hat{\rho}_{ps,K}$	33	108	-36	133	-32	152	-30	179
	$\hat{\rho}_{ps,S}$	-30	129	-24	155	-33	178	-36	205
0.5	$\hat{\rho}_{ps,N}$			-68	96	-54	117	-81	136
	$\hat{\rho}_{ps,K}$			-57	83	-45	101	-65	117
	$\hat{\rho}_{ps,S}$			-61	99	-42	117	-73	137
0.7	$\hat{\rho}_{ps,N}$					-94	52	-67	61
	$\hat{\rho}_{ps,K}$					-78	45	-59	52
	$\hat{\rho}_{ps,S}$					-77	52	-56	60

Table III.14

Bias and MSE ($\times 10^4$) of interclass correlation estimators for different values of ρ_{ss} and ρ_{ms} (n=50, m=3), based on 10,000 simulations from Kotz type distribution

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	-21	106	-23	144	-22	177	-17	210
	$\hat{\rho}_{ps,K}$	-21	94	-15	128	-14	155	-20	186
	$\hat{\rho}_{ps,S}$	-31	105	-19	145	-19	178	-15	211
0.3	$\hat{\rho}_{ps,N}$	15	80	-32	114	-54	143	-50	175
	$\hat{\rho}_{ps,K}$	13	71	-25	100	-45	125	-46	156
	$\hat{\rho}_{ps,S}$	-14	85	-19	115	-44	143	-44	175
0.5	$\hat{\rho}_{ps,N}$			-81	71	-49	94	-49	117
	$\hat{\rho}_{ps,K}$			-68	62	-37	82	-45	104
	$\hat{\rho}_{ps,S}$			-59	71	-32	94	-39	117
0.7	$\hat{\rho}_{ps,N}$					-73	38	-70	51
	$\hat{\rho}_{ps,K}$					-63	33	-58	45
	$\hat{\rho}_{ps,S}$					-49	37	-55	50

Table III.15

Bias and MSE ($\times 10^4$) of interclass correlation estimators for different values of ρ_{ss} and ρ_{ms} (n=50, m=4), based on 10,000 simulations from Kotz type distribution

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	-3	85	-24	122	-17	162	-46	199
	$\hat{\rho}_{ps,K}$	-2	76	-13	109	-12	144	-43	178
	$\hat{\rho}_{ps,S}$	-4	85	-19	124	-14	163	-44	200
0.3	$\hat{\rho}_{ps,N}$	-27	59	-60	97	-51	133	-40	163
	$\hat{\rho}_{ps,K}$	-25	53	-49	87	-41	119	-30	146
	$\hat{\rho}_{ps,S}$	-30	62	-45	98	-40	134	-33	164
0.5	$\hat{\rho}_{ps,N}$			-81	59	-68	80	-66	108
	$\hat{\rho}_{ps,K}$			-65	52	-63	72	-52	96
	$\hat{\rho}_{ps,S}$			-55	59	-49	80	-54	108
0.7	$\hat{\rho}_{ps,N}$					-85	33	-79	46
	$\hat{\rho}_{ps,K}$					-73	29	-64	41
	$\hat{\rho}_{ps,S}$					-59	32	-63	46

Table III.16

Bias and MSE ($\times 10^4$) of interclass correlation estimators for different values of ρ_{ss} and ρ_{ms} (n=50, m=5), based on 10,000 simulations from Kotz type distribution

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	-31	69	-8	105	-36	144	-31	188
	$\hat{\rho}_{ps,K}$	-25	63	-6	95	-32	131	-25	172
	$\hat{\rho}_{ps,S}$	-27	70	-3	106	32	145	-28	189
0.3	$\hat{\rho}_{ps,N}$	-30	50	-42	85	-24	122	-43	154
	$\hat{\rho}_{ps,K}$	-24	45	-33	77	-19	111	-38	140
	$\hat{\rho}_{ps,S}$	-20	52	-25	86	-12	123	-36	155
0.5	$\hat{\rho}_{ps,N}$			-63	50	-71	75	-77	100
	$\hat{\rho}_{ps,K}$			-55	45	-61	67	-71	91
	$\hat{\rho}_{ps,S}$			-36	50	-50	75	-65	100
0.7	$\hat{\rho}_{ps,N}$					-76	28	-82	44
	$\hat{\rho}_{ps,K}$					-64	26	-73	39
	$\hat{\rho}_{ps,S}$					-47	28	-65	43

Table III.17

Bias and MSE ($\times 10^4$) of interclass correlation estimators for different values of ρ_{ss} and ρ_{ms} (n=10, m=2), based on 10,000 simulations from T distribution, when df=3

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	35	1194	-16	1258	-162	1374	-126	1479
	$\hat{\rho}_{ps,K}$	46	933	-25	1010	-137	1109	-115	1197
	$\hat{\rho}_{ps,S}$	-47	1006	-44	1183	-163	1361	-121	1490
0.3	$\hat{\rho}_{ps,N}$	-17	968	-105	1043	-301	1194	-207	1309
	$\hat{\rho}_{ps,K}$	-46	741	-84	822	-261	964	-166	1051
	$\hat{\rho}_{ps,S}$	-207	865	-182	1002	-314	1188	-193	1318
0.5	$\hat{\rho}_{ps,N}$			-214	706	-339	831	-387	992
	$\hat{\rho}_{ps,K}$			-178	539	-270	649	-310	778
	$\hat{\rho}_{ps,S}$			-353	735	-359	846	-366	1003
0.7	$\hat{\rho}_{ps,N}$					-380	397	-472	564
	$\hat{\rho}_{ps,K}$					-315	297	-371	418
	$\hat{\rho}_{ps,S}$					-409	440	-445	575

Table III.18

Bias and MSE ($\times 10^4$) of interclass correlation estimators for different values of ρ_{ss} and ρ_{ms} (n=10, m=3), based on 10,000 simulations from T distribution, when df=3

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	-14	850	-115	983	-111	1141	-69	1340
	$\hat{\rho}_{ps,K}$	19	649	-99	772	-78	905	-47	1071
	$\hat{\rho}_{ps,S}$	-84	734	-118	964	-105	1160	-59	1366
0.3	$\hat{\rho}_{ps,N}$	-11	647	-277	822	-317	1002	-314	1231
	$\hat{\rho}_{ps,K}$	-29	482	-215	634	-249	784	-215	974
	$\hat{\rho}_{ps,S}$	-190	618	-292	824	-289	1018	-284	1251
0.5	$\hat{\rho}_{ps,N}$			-310	510	-443	717	-402	888
	$\hat{\rho}_{ps,K}$			-225	384	-337	548	-302	690
	$\hat{\rho}_{ps,S}$			-341	552	-397	735	-355	899
0.7	$\hat{\rho}_{ps,N}$					-467	326	-519	489
	$\hat{\rho}_{ps,K}$					-356	243	-381	355
	$\hat{\rho}_{ps,S}$					-408	352	-451	490

Table III.19

Bias and MSE ($\times 10^4$) of interclass correlation estimators for different values of ρ_{ss} and ρ_{ms} (n=10, m=4), based on 10,000 simulations from T distribution, when df=3

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	14	663	-120	842	-94	1041	-102	1296
	$\hat{\rho}_{ps,K}$	10	499	-99	639	-74	825	-72	1028
	$\hat{\rho}_{ps,S}$	-35	597	-117	850	-80	1072	-92	1327
0.3	$\hat{\rho}_{ps,N}$	-36	494	-322	701	-302	905	-356	1163
	$\hat{\rho}_{ps,K}$	-13	358	-254	534	-220	697	-265	910
	$\hat{\rho}_{ps,S}$	-191	497	-297	715	-256	929	-324	1188
0.5	$\hat{\rho}_{ps,N}$			-366	430	-517	661	-453	892
	$\hat{\rho}_{ps,K}$			-272	322	-390	498	-339	671
	$\hat{\rho}_{ps,S}$			-330	465	-448	678	-398	906
0.7	$\hat{\rho}_{ps,N}$							-544	472
	$\hat{\rho}_{ps,K}$							-389	331
	$\hat{\rho}_{ps,S}$							-461	4680

Table III.20

Bias and MSE ($\times 10^4$) of interclass correlation estimators for different values of ρ_{ss} and ρ_{ms} (n=10, m=5), based on 10,000 simulations from T distribution, when df=3

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	-44	571	-92	760	-119	1024	-73	1274
	$\hat{\rho}_{ps,K}$	-21	413	-81	578	-99	796	-60	1002
	$\hat{\rho}_{ps,S}$	-79	529	-76	781	-102	1060	-61	1308
0.3	$\hat{\rho}_{ps,N}$	-90	389	-280	615	-368	889	-318	1137
	$\hat{\rho}_{ps,K}$	-60	287	-220	469	-268	688	-233	877
	$\hat{\rho}_{ps,S}$	-205	411	-238	637	-317	918	-282	1165
0.5	$\hat{\rho}_{ps,N}$			-391	372	-508	636	-486	839
	$\hat{\rho}_{ps,K}$			-288	277	-378	473	-359	624
	$\hat{\rho}_{ps,S}$			-326	402	-419	649	-425	851
0.7	$\hat{\rho}_{ps,N}$							-551	461
	$\hat{\rho}_{ps,K}$							-385	311
	$\hat{\rho}_{ps,S}$							-461	455

Table III.21

Bias and MSE ($\times 10^4$) of interclass correlation estimators for different values of ρ_{ss} and ρ_{ms} (n=50, m=2), based on 10,000 simulations from T distribution, when df=3

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	52	426	20	460	-21	495	-26	563
	$\hat{\rho}_{ps,K}$	38	183	8	206	-7	229	-17	260
	$\hat{\rho}_{ps,S}$	9	392	13	447	-22	492	-25	564
0.3	$\hat{\rho}_{ps,N}$	44	323	-62	382	-119	455	-107	484
	$\hat{\rho}_{ps,K}$	39	136	-37	172	-53	201	-56	231
	$\hat{\rho}_{ps,S}$	-92	315	-86	377	-120	454	-105	485
0.5	$\hat{\rho}_{ps,N}$			-75	242	-130	314	-156	350
	$\hat{\rho}_{ps,K}$			-72	108	-63	136	-78	152
	$\hat{\rho}_{ps,S}$			-119	255	-130	317	-150	349
0.7	$\hat{\rho}_{ps,N}$					-153	133	-171	176
	$\hat{\rho}_{ps,K}$					-81	57	-85	70
	$\hat{\rho}_{ps,S}$					-152	139	-164	178

Table III.22

Bias and MSE ($\times 10^4$) of interclass correlation estimators for different values of ρ_{ss} and ρ_{ms} (n=50, m=3), based on 10,000 simulations from T distribution, when df=3

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	19	303	-36	367	-38	444	-17	529
	$\hat{\rho}_{ps,K}$	3	129	-5	163	-23	202	-11	241
	$\hat{\rho}_{ps,S}$	-19	288	-38	365	-36	446	-15	531
0.3	$\hat{\rho}_{ps,N}$	-37	232	-67	295	-105	395	-95	453
	$\hat{\rho}_{ps,K}$	11	97	-45	131	-45	170	-50	204
	$\hat{\rho}_{ps,S}$	-67	237	-68	298	-96	396	-89	454
0.5	$\hat{\rho}_{ps,N}$			-117	181	-172	262	-159	309
	$\hat{\rho}_{ps,K}$			-69	81	-78	111	-75	131
	$\hat{\rho}_{ps,S}$			-118	188	-160	263	-150	310
0.7	$\hat{\rho}_{ps,N}$					-165	108	-188	158
	$\hat{\rho}_{ps,K}$					-80	45	-88	60
	$\hat{\rho}_{ps,S}$					-148	111	-174	158

Table III.23

Bias and MSE ($\times 10^4$) of interclass correlation estimators for different values of ρ_{ss} and ρ_{ms} (n=50, m=4), based on 10,000 simulations from T distribution, when df=3

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	7	235	-22	324	-55	422	-50	516
	$\hat{\rho}_{ps,K}$	-4	96	-9	142	-32	188	-22	234
	$\hat{\rho}_{ps,S}$	-16	226	-19	324	-52	424	-48	518
0.3	$\hat{\rho}_{ps,N}$	-1	174	-122	273	-131	353	-142	471
	$\hat{\rho}_{ps,K}$	-7	73	-58	115	-52	154	-66	202
	$\hat{\rho}_{ps,S}$	-66	183	-113	274	-121	355	-135	473
0.5	$\hat{\rho}_{ps,N}$			-135	157	-172	242	-192	302
	$\hat{\rho}_{ps,K}$			-60	67	-74	98	-91	130
	$\hat{\rho}_{ps,S}$			-120	161	-155	242	-181	304
0.7	$\hat{\rho}_{ps,N}$					-184	96	-170	148
	$\hat{\rho}_{ps,K}$					-74	38	-72	56
	$\hat{\rho}_{ps,S}$					-160	96	-154	147

Table III.24

Bias and MSE ($\times 10^4$) of interclass correlation estimators for different values of ρ_{ss} and ρ_{ms} (n=50, m=5), based on 10,000 simulations from T distribution, when df=3

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	-16	207	-54	300	-39	415	-44	510
	$\hat{\rho}_{ps,K}$	-9	83	-18	130	-12	180	-29	223
	$\hat{\rho}_{ps,S}$	-32	202	-49	302	-34	418	-42	513
0.3	$\hat{\rho}_{ps,N}$	-26	144	-105	249	-106	329	-106	432
	$\hat{\rho}_{ps,K}$	-21	61	-34	104	-43	145	-55	188
	$\hat{\rho}_{ps,S}$	-66	153	-92	252	-93	332	-99	434
0.5	$\hat{\rho}_{ps,N}$			-142	150	-174	227	-171	307
	$\hat{\rho}_{ps,K}$			-67	61	-70	93	-71	129
	$\hat{\rho}_{ps,S}$			-121	152	-154	228	-158	307
0.7	$\hat{\rho}_{ps,N}$					-188	88	-193	139
	$\hat{\rho}_{ps,K}$					-75	34	-84	53
	$\hat{\rho}_{ps,S}$					-161	87	-176	138

Table III.25

PN comparison of interclass correlation estimators for different values of ρ_{ss}, ρ_{ms} and (n=10, m=2), based on 10,000 simulations from normal distribution

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimate	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,K}$	0.5504	0.545	0.5492	0.5511
	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,S}$	0.5766	0.7206	0.8284	0.8733
	$\hat{\rho}_{ps,K}$vs$\hat{\rho}_{ps,S}$	0.3878	0.4405	0.4599	0.4626
0.3	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,K}$	0.5481	0.5477	0.5568	0.5454
	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,S}$	0.5922	0.6315	0.6598	0.6723
	$\hat{\rho}_{ps,K}$vs$\hat{\rho}_{ps,S}$	0.4483	0.4664	0.4677	0.4695
0.5	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,K}$		0.54	0.5433	0.5484
	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,S}$		0.6372	0.58	0.5654
	$\hat{\rho}_{ps,K}$vs$\hat{\rho}_{ps,S}$		0.5272	0.4887	0.469
0.7	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,K}$			0.5329	0.5496
	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,S}$			0.563	0.5246
	$\hat{\rho}_{ps,K}$vs$\hat{\rho}_{ps,S}$			0.5082	0.4702

Table III.26

PN comparison of interclass correlation estimators for different values of ρ_{ss}, ρ_{ms} and (n=10, m=3), based on 10,000 simulations from normal distribution

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimate	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,K}$	0.5521	0.5521	0.5509	0.5474
	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,S}$	0.5951	0.772	0.8518	0.8736
	$\hat{\rho}_{ps,K}$vs$\hat{\rho}_{ps,S}$	0.4006	0.4559	0.4723	0.4711
0.3	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,K}$	0.5492	0.5479	0.5415	0.5483
	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,S}$	0.6226	0.6357	0.6368	0.6621
	$\hat{\rho}_{ps,K}$vs$\hat{\rho}_{ps,S}$	0.4877	0.4924	0.4892	0.476
0.5	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,K}$		0.5466	0.5485	0.5386
	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,S}$		0.5929	0.5345	0.5555
	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,K}$		0.5262	0.4828	0.4857
0.7	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,K}$			0.5466	0.5335
	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,S}$			0.4975	0.5083
	$\hat{\rho}_{ps,N}$vs $\hat{\rho}_{ps,k}$			0.47	0.489

Table III.27

PN comparison of interclass correlation estimators for different values of ρ_{ss}, ρ_{ms} and (n=10, m=4), based on 10,000 simulations from normal distribution

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimate	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,K}$	0.5422	0.5405	0.5409	0.5437
	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,S}$	0.6329	0.7944	0.8508	0.8717
	$\hat{\rho}_{ps,K}$vs$\hat{\rho}_{ps,S}$	0.436	0.492	0.4954	0.4827
0.3	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,K}$	0.5554	0.5442	0.5407	0.5415
	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,S}$	0.6563	0.6069	0.6331	0.6596
	$\hat{\rho}_{ps,K}$vs$\hat{\rho}_{ps,S}$	0.5305	0.5049	0.4956	0.4852
0.5	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,K}$		0.5438	0.5357	0.5374
	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,S}$		0.5424	0.5113	0.5433
	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,K}$		0.5123	0.4937	0.4913
0.7	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,K}$			0.53	0.5444
	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,S}$			0.4631	0.5005
	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,K}$			0.4673	0.4848

Table III.28

PN comparison of interclass correlation estimators for different values of ρ_{ss}, ρ_{ms} and (n=10, m=5), based on 10,000 simulations from normal distribution

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimate	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,K}$	0.5407	0.5559	0.5362	0.534
	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,S}$	0.6468	0.8106	0.85	0.8703
	$\hat{\rho}_{ps,K}$vs$\hat{\rho}_{ps,S}$	0.4528	0.4995	0.506	0.4938
0.3	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,K}$	0.5587	0.542	0.5471	0.5481
	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,S}$	0.6694	0.5944	0.6204	0.6585
	$\hat{\rho}_{ps,K}$vs$\hat{\rho}_{ps,S}$	0.563	0.5171	0.5003	0.4846
0.5	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,K}$		0.531	0.536	0.5332
	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,S}$		0.5078	0.5131	0.5518
	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,K}$		0.5101	0.4981	0.4934
0.7	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,K}$			0.5312	0.538
	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,S}$			0.4521	0.4922
	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,K}$			0.4736	0.4842

Table III.29

PN comparison of interclass correlation estimators for different values of ρ_{ss}, ρ_{ms} and (n=50, m=2), based on 10,000 simulations from normal distribution

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimate	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,K}$	0.5582	0.5515	0.5521	0.5502
	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,S}$	0.6077	0.686	0.7142	0.7281
	$\hat{\rho}_{ps,K}$vs$\hat{\rho}_{ps,S}$	0.4305	0.4533	0.4516	0.4512
0.3	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,K}$	0.5512	0.5497	0.5545	0.5456
	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,S}$	0.5914	0.5132	0.5128	0.5158
	$\hat{\rho}_{ps,K}$vs$\hat{\rho}_{ps,S}$	0.4706	0.4563	0.4484	0.4553
0.5	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,K}$		0.5432	0.5371	0.5562
	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,S}$		0.5115	0.4966	0.5136
	$\hat{\rho}_{ps,K}$vs$\hat{\rho}_{ps,S}$		0.4636	0.4643	0.4458
0.7	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,K}$			0.5485	0.5512
	$\hat{\rho}_{ps,N}$vs$\hat{\rho}_{ps,S}$			0.5079	0.5025
	$\hat{\rho}_{ps,K}$vs$\hat{\rho}_{ps,S}$			0.4555	0.45

Table III.30

PN comparison of interclass correlation estimators for different values of ρ_{ss}, ρ_{ms} and (n=50, m=3), based on 10,000 simulations from normal distribution

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimate	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,N}$ *vs* $\hat{\rho}_{ps,K}$	0.5446	0.5455	0.5528	0.55
	$\hat{\rho}_{ps,N}$ *vs* $\hat{\rho}_{ps,S}$	0.6149	0.6613	0.6975	0.7159
	$\hat{\rho}_{ps,K}$ *vs* $\hat{\rho}_{ps,S}$	0.4539	0.4601	0.4518	0.4519
0.3	$\hat{\rho}_{ps,N}$ *vs* $\hat{\rho}_{ps,K}$	0.5443	0.5433	0.5558	0.5538
	$\hat{\rho}_{ps,N}$ *vs* $\hat{\rho}_{ps,S}$	0.5862	0.4969	0.5082	0.508
	$\hat{\rho}_{ps,K}$ *vs* $\hat{\rho}_{ps,S}$	0.4888	0.4629	0.4503	0.4474
0.5	$\hat{\rho}_{ps,N}$ *vs* $\hat{\rho}_{ps,K}$		0.5528	0.5515	0.5529
	$\hat{\rho}_{ps,N}$ *vs* $\hat{\rho}_{ps,S}$		0.4889	0.5024	0.5083
	$\hat{\rho}_{ps,K}$ *vs* $\hat{\rho}_{ps,S}$		0.4552	0.4546	0.4509
0.7	$\hat{\rho}_{ps,N}$ *vs* $\hat{\rho}_{ps,K}$			0.5445	0.5414
	$\hat{\rho}_{ps,N}$ *vs* $\hat{\rho}_{ps,S}$			0.4807	0.506
	$\hat{\rho}_{ps,K}$ *vs* $\hat{\rho}_{ps,S}$			0.4574	0.4631

Table III.31

PN comparison of interclass correlation estimators for different values of ρ_{ss}, ρ_{ms} and (n=50, m=4), based on 10,000 simulations from normal distribution

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimator	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,N}$ *vs* $\hat{\rho}_{ps,K}$	0.5517	0.5378	0.5543	0.5424
	$\hat{\rho}_{ps,N}$ *vs* $\hat{\rho}_{ps,S}$	0.61	0.6599	0.6906	0.7073
	$\hat{\rho}_{ps,K}$ *vs* $\hat{\rho}_{ps,S}$	0.4562	0.4701	0.4509	0.4611
0.3	$\hat{\rho}_{ps,N}$ *vs* $\hat{\rho}_{ps,K}$	0.5449	0.542	0.5386	0.5506
	$\hat{\rho}_{ps,N}$ *vs* $\hat{\rho}_{ps,S}$	0.5561	0.4994	0.5029	0.5172
	$\hat{\rho}_{ps,K}$ *vs* $\hat{\rho}_{ps,S}$	0.4859	0.462	0.4634	0.4525
0.5	$\hat{\rho}_{ps,N}$ *vs* $\hat{\rho}_{ps,K}$		0.5443	0.543	0.5436
	$\hat{\rho}_{ps,N}$ *vs* $\hat{\rho}_{ps,S}$		0.4917	0.5103	0.508
	$\hat{\rho}_{ps,K}$ *vs* $\hat{\rho}_{ps,S}$		0.4624	0.4665	0.4627
0.7	$\hat{\rho}_{ps,N}$ *vs* $\hat{\rho}_{ps,K}$			0.5434	0.5483
	$\hat{\rho}_{ps,N}$ *vs* $\hat{\rho}_{ps,S}$			0.4865	0.495
	$\hat{\rho}_{ps,K}$ *vs* $\hat{\rho}_{ps,S}$			0.458	0.4554

Table III.32

PN comparison of interclass correlation estimators for different values of ρ_{ss}, ρ_{ms} and (n=50, m=5), based on 10,000 simulations from normal distribution

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimator	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,N}$ *vs* $\hat{\rho}_{ps,K}$	0.5425	0.5447	0.5382	0.5432
	$\hat{\rho}_{ps,N}$ *vs* $\hat{\rho}_{ps,S}$	0.5952	0.6436	0.6825	0.7094
	$\hat{\rho}_{ps,K}$ *vs* $\hat{\rho}_{ps,S}$	0.4667	0.464	0.4655	0.4616
0.3	$\hat{\rho}_{ps,N}$ *vs* $\hat{\rho}_{ps,K}$	0.5352	0.5431	0.5427	0.5387
	$\hat{\rho}_{ps,N}$ *vs* $\hat{\rho}_{ps,S}$	0.5279	0.5005	0.4985	0.5064
	$\hat{\rho}_{ps,K}$ *vs* $\hat{\rho}_{ps,S}$	0.4853	0.4667	0.4611	0.4655
0.5	$\hat{\rho}_{ps,N}$ *vs* $\hat{\rho}_{ps,K}$		0.5397	0.5357	0.5397
	$\hat{\rho}_{ps,N}$ *vs* $\hat{\rho}_{ps,S}$		0.4858	0.4869	0.5092
	$\hat{\rho}_{ps,K}$ *vs* $\hat{\rho}_{ps,S}$		0.4643	0.4676	0.4637
0.7	$\hat{\rho}_{ps,N}$ *vs* $\hat{\rho}_{ps,K}$			0.5473	0.5402
	$\hat{\rho}_{ps,N}$ *vs* $\hat{\rho}_{ps,S}$			0.4819	0.4947
	$\hat{\rho}_{ps,K}$ *vs* $\hat{\rho}_{ps,S}$			0.4576	0.4644

Table III.33

PN comparison of interclass correlation estimators for different values of ρ_{ss}, ρ_{ms} and (n=10, m=2), based on 10,000 simulations from Kotz type distribution

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
K	Estimate	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.544	0.543	0.535	0.542
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.5593	0.7053	0.8279	0.8832
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.4519	0.5087	0.5354	0.5518
0.3	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.548	0.542	0.535	0.538
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.5841	0.6353	0.6878	0.7157
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.5087	0.5434	0.548	0.552
0.5	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$		0.5341	0.5272	0.5324
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$		0.6471	0.6083	0.5968
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$		0.584	0.5649	0.5482
0.7	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$			0.5179	0.5308
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$			0.5872	0.5451
0	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$			0.5674	0.5559

Table III.34

PN comparison of interclass correlation estimators for different values of ρ_{ss}, ρ_{ms} and (n=10, m=3), based on 10,000 simulations from Kotz type distribution

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimate	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.5482	0.538	0.5231	0.5347
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.5949	0.7705	0.859	0.8881
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.4737	0.536	0.5443	0.5515
0.3	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.5353	0.529	0.5377	0.528
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.6219	0.6466	0.6666	0.6867
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.5465	0.5598	0.5652	0.5467
0.5	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$		0.5189	0.5286	0.5311
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$		0.622	0.5546	0.5034
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$		0.5946	0.5595	0.5513
0.7	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$			0.5362	0.5242
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$			0.5004	0.5113
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$			0.5524	0.5469

Table III.35

PN comparison of interclass correlation estimators for different values of ρ_{ss}, ρ_{ms} and (n=10, m=4), based on 10,000 simulations from Kotz type distribution

ρ_{ps}/ρ_{ss}	Estimate	0.1 PN	0.3 PN	0.5 PN	0.7 PN
0.1	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.5418	0.5307	0.5244	0.5207
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.6231	0.793	0.8598	0.8871
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.4829	0.5494	0.555	0.544
0.3	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.5338	0.5309	0.5197	0.5279
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.6576	0.6351	0.6497	0.6856
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.5904	0.577	0.549	0.5518
0.5	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$		0.5223	0.5325	0.5319
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$		0.5547	0.528	0.5521
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$		0.5831	0.5655	0.5568
0.7	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$			0.5426	0.5289
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$			0.4708	0.5032
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$			0.541	0.5488

Table III.36

PN comparison of interclass correlation estimators for different values of ρ_{ss}, ρ_{ms} and (n=10, m=5), based on 10,000 simulations from Kotz type distribution

ρ_{ps}/ρ_{ss}	Estimate	0.1 PN	0.3 PN	0.5 PN	0.7 PN
0.1	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.5408	0.5313	0.5176	0.5253
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.6531	0.8126	0.8583	0.8787
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.5088	0.561	0.5516	0.5467
0.3	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.5224	0.5224	0.5233	0.5364
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.6721	0.611	0.6492	0.6811
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.6132	0.5676	0.56	0.5602
0.5	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$		0.5153	0.5216	0.5309
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$		0.5158	0.5114	0.5609
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$		0.5606	0.5479	0.5601
0.7	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$			0.5299	0.5274
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$			0.4493	0.5044
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$			0.5294	0.5548

Table III.37

PN comparison of interclass correlation estimators for different values of ρ_{ss}, ρ_{ms} and (n=50, m=2), based on 10,000 simulations from Kotz type distribution

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimate	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.5627	0.5697	0.5634	0.5649
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.6103	0.7218	0.7551	0.7704
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.5434	0.5693	0.5657	0.5658
0.3	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.5607	0.5607	0.5662	0.5619
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.6026	0.5406	0.5274	0.5359
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.5686	0.5675	0.5695	0.5633
0.5	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$		0.5538	0.5624	0.5619
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$		0.5399	0.5107	0.5023
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$		0.565	0.5669	0.5636
0.7	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$			0.5596	0.5553
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$			0.4869	0.5042
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$			0.5645	0.5581

Table III.38

PN comparison of interclass correlation estimators for different values of ρ_{ss}, ρ_{ms} and (n=50, m=3), based on 10,000 simulations from Kotz type distribution

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimate	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.5531	0.5453	0.5594	0.5554
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.6292	0.7009	0.7338	0.7489
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.5428	0.55	0.5631	0.5571
0.3	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.5477	0.5611	0.556	0.5495
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.6038	0.5158	0.5153	0.5228
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.5812	0.5646	0.5592	0.5523
0.5	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$		0.5564	0.5507	0.5557
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$		0.4857	0.5107	0.5149
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$		0.5587	0.557	0.5601
0.7	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$			0.5587	0.5553
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$			0.4919	0.5047
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$			0.5612	0.5543

Table III.39

PN comparison of interclass correlation estimators for different values of ρ_{ss}, ρ_{ms} and (n=50, m=4), based on 10,000 simulations from Kotz type distribution

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimate	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.5495	0.5537	0.5585	0.5578
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.6416	0.6917	0.7206	0.7482
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.5516	0.5594	0.5623	0.5598
0.3	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.5339	0.556	0.551	0.5511
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.5798	0.4907	0.5119	0.5227
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.5703	0.5618	0.5534	0.5555
0.5	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$		0.5547	0.549	0.5539
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$		0.487	0.5009	0.5085
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$		0.5594	0.5556	0.5584
0.7	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$			0.5591	0.5605
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$			0.4874	0.4959
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$			0.5571	0.5635

Table III.40

PN comparison of interclass correlation estimators for different values of ρ_{ss}, ρ_{ms} and (n=50, m=5), based on 10,000 simulations from Kotz type distribution

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimate	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.5402	0.5523	0.5478	0.5451
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.6207	0.6708	0.7083	0.7411
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.5481	0.5587	0.5525	0.5477
0.3	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.543	0.5429	0.5499	0.5503
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.5417	0.4994	0.517	0.5236
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.5681	0.5481	0.5529	0.5521
0.5	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$		0.5495	0.5495	0.5464
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$		0.4893	0.4929	0.5057
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$		0.5553	0.5521	0.5498
0.7	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$			0.5489	0.5551
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$			0.481	0.49
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$			0.5486	0.5571

Table III.41

PN comparison of interclass correlation estimators for different values of ρ_{ss}, ρ_{ms} and (n=10, m=2), based on 10,000 simulations from T distribution, with df=3

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimate	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.6427	0.6294	0.623	0.6201
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.5532	0.7068	0.8136	0.8831
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.5204	0.5768	0.6063	0.6251
0.3	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.628	0.6188	0.6078	0.6195
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.5842	0.6575	0.7001	0.7434
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.5714	0.5993	0.6174	0.6227
0.5	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$		0.5783	0.5959	0.6041
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$		0.6663	0.6463	0.6312
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$		0.6431	0.6373	0.6238
0.7	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$			0.6127	0.6128
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$			0.6206	0.5699
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$			0.6471	0.6251

Table III.42

PN comparison of interclass correlation estimators for different values of ρ_{ss}, ρ_{ms} and (n=10, m=3), based on 10,000 simulations from T distribution, with df=3

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimate	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.6421	0.6407	0.6124	0.6231
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.5781	0.7494	0.8491	0.8946
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.5343	0.606	0.6286	0.6314
0.3	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.6347	0.6174	0.6212	0.6256
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.6309	0.6669	0.6987	0.7269
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.6119	0.6352	0.6336	0.6393
0.5	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$		0.5898	0.6083	0.6288
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$		0.6591	0.6023	0.6097
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$		0.6629	0.6339	0.6283
0.7	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$			0.3873	0.6354
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$			0.5369	0.5253
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$			0.6424	0.6283

Table III.43

PN comparison of interclass correlation estimators for different values of ρ_{ss}, ρ_{ms} and (n=10, m=4), based on 10,000 simulations from T distribution, with df=3

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimate	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.6421	0.6407	0.6124	0.6231
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.5986	0.7882	0.8649	0.8944
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.5566	0.6315	0.6302	0.6375
0.3	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.6347	0.6174	0.6212	0.6256
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.6575	0.6625	0.6873	0.7271
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.6548	0.6429	0.6464	0.6403
0.5	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$		0.5898	0.6083	0.6288
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$		0.6028	0.5654	0.6077
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$		0.6601	0.6389	0.6474
0.7	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$				0.6354
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$				0.5097
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$				0.6526

Table III.44

PN comparison of interclass correlation estimators for different values of ρ_{ss}, ρ_{ms} and (n=10, m=5), based on 10,000 simulations from T distribution, with df=3

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimate	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.6575	0.6318	0.6241	0.6253
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.6249	0.8105	0.87	0.9008
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.5883	0.636	0.6497	0.6448
0.3	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.6182	0.6081	0.6109	0.621
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.6795	0.6464	0.6865	0.7248
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.673	0.643	0.6368	0.6381
0.5	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$		0.6045	0.6164	0.62
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$		0.5626	0.5481	0.5927
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$		0.6626	0.6424	0.6424
0.7	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$				0.6294
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$				0.5028
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$				0.643

Table III.45

PN comparison of interclass correlation estimators for different values of ρ_{ss}, ρ_{ms} and (n=50, m=2), based on 10,000 simulations from T distribution, with df=3

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimate	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.7083	0.6956	0.6976	0.7041
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.6194	0.7361	0.8046	0.8241
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.6736	0.6902	0.6975	0.7054
0.3	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.6996	0.6901	0.6927	0.6914
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.6034	0.5868	0.5842	0.6035
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.6884	0.6925	0.6934	0.6919
0.5	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$		0.6812	0.6874	0.6895
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$		0.5865	0.5376	0.5294
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$		0.702	0.6928	0.6918
0.7	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$			0.6951	0.6965
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$			0.5172	0.5154
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$			0.7011	0.6992

Table III.46

PN comparison of interclass correlation estimators for different values of ρ_{ss}, ρ_{ms} and (n=50, m=3), based on 10,000 simulations from T distribution, with df=3

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimate	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.7026	0.6987	0.6981	0.7011
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.645	0.7589	0.8023	0.8212
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.6817	0.6967	0.6981	0.7018
0.3	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.6906	0.696	0.6979	0.7044
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.6325	0.5683	0.5695	0.5933
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.7059	0.7029	0.7005	0.7058
0.5	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$		0.6843	0.6945	0.6959
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$		0.5233	0.5099	0.523
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$		0.6965	0.6977	0.6979
0.7	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$			0.7027	0.7057
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$			0.4838	0.4956
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$			0.7023	0.704

Table III.47

PN comparison of interclass correlation estimators for different values of ρ_{ss}, ρ_{ms} and (n=50, m=4), based on 10,000 simulations from T distribution, with df=3

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimate	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.7101	0.6995	0.7003	0.7004
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.6552	0.7609	0.7978	0.8245
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.6961	0.7002	0.703	0.7018
0.3	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.6759	0.6975	0.6974	0.7034
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.6366	0.5347	0.5563	0.5839
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.7109	0.7035	0.7	0.7037
0.5	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$		0.699	0.7069	0.6976
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$		0.4974	0.4992	0.5151
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$		0.7047	0.7101	0.7
0.7	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$			0.71	0.7124
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$			0.4596	0.5009
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$			0.7069	0.7113

Table III.48

PN comparison of interclass correlation estimators for different values of ρ_{ss}, ρ_{ms} and (n=50, m=5), based on 10,000 simulations from T distribution, with df=3

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimate	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.7089	0.6999	0.7016	0.7108
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.6627	0.7605	0.7959	0.8147
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.7011	0.7021	0.7047	0.7118
0.3	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.6768	0.6998	0.6971	0.7105
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.6115	0.5353	0.5485	0.5829
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.7191	0.7031	0.7012	0.7127
0.5	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$		0.7107	0.7126	0.7007
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$		0.4803	0.494	0.5121
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$		0.7137	0.7137	0.7023
0.7	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$			0.719	0.6982
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$			0.4448	0.4889
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$			0.7174	0.7019

Table III.49

Bias and MSE ($\times 10^4$) of ρ_{ps} for the unbalanced case when n=50, based on 10,000 simulations from normal distribution

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	8	107	-4	130	-22	152	-22	172
	$\hat{\rho}_{ps,K}$	54	130	33	157	9	179	-12	200
	$\hat{\rho}_{ps,S}$	-14	122	-10	139	-30	154	-34	170
0.3	$\hat{\rho}_{ps,N}$			-43	105	-50	127	-27	145
	$\hat{\rho}_{ps,K}$			74	126	43	148	29	167
	$\hat{\rho}_{ps,S}$			-63	114	-79	130	-62	144
0.5	$\hat{\rho}_{ps,N}$					-53	84	-65	97
	$\hat{\rho}_{ps,K}$					97	97	23	112
	$\hat{\rho}_{ps,S}$					-95	90	-121	98

Table III.50

Bias and MSE ($\times 10^4$) of ρ_{ps} for the unbalanced case when n=100, based on 10,000 simulations from normal distribution

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	5	47	-7	60	-22	72	3	84
	$\hat{\rho}_{ps,K}$	43	57	27	71	3	83	14	96
	$\hat{\rho}_{ps,S}$	-1	55	-8	64	-26	73	-1	84
0.3	$\hat{\rho}_{ps,N}$	0	35	-15	48	-25	59	-34	70
	$\hat{\rho}_{ps,K}$	105	42	83	57	52	68	11	81
	$\hat{\rho}_{ps,S}$	-17	45	-22	53	-32	61	-46	71
0.5	$\hat{\rho}_{ps,N}$			-23	29	-29	39	-23	46
	$\hat{\rho}_{ps,K}$			126	34	93	45	54	53
	$\hat{\rho}_{ps,S}$			-26	33	-42	41	-42	47
0.7	$\hat{\rho}_{ps,N}$					-30	15	-19	20
	$\hat{\rho}_{ps,K}$					112	17	83	22
	$\hat{\rho}_{ps,S}$					-44	17	-44	21

Table III.51

Bias and MSE ($\times 10^4$) of ρ_{ps} for the unbalanced case when n=50, based on 10,000 simulations from Kotz type distribution

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	-57	140	-104	169	-67	206	-21	234
	$\hat{\rho}_{ps,K}$	3	134	-41	158	-25	189	-3	212
	$\hat{\rho}_{ps,S}$	-24	154	-60	180	-33	212	-9	234
0.3	$\hat{\rho}_{ps,N}$			-203	141	-168	173	-121	204
	$\hat{\rho}_{ps,K}$			-32	127	-34	155	-26	179
	$\hat{\rho}_{ps,S}$			-68	146	-70	174	-78	202
0.5	$\hat{\rho}_{ps,N}$					-289	124	-190	143
	$\hat{\rho}_{ps,K}$					-56	101	-36	120
	$\hat{\rho}_{ps,S}$					-125	116	-119	137
0.7	$\hat{\rho}_{ps,N}$							-265	71
	$\hat{\rho}_{ps,K}$							-61	52
	$\hat{\rho}_{ps,S}$							-173	63

Table III.52

Bias and MSE ($\times 10^4$) of ρ_{ps} for the unbalanced case when n=100, based on 10,000 simulations from Kotz type distribution

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	-37	61	-58	81	-50	99	-18	117
	$\hat{\rho}_{ps,K}$	13	57	-13	73	-19	88	5	102
	$\hat{\rho}_{ps,S}$	5	68	-15	86	-20	102	0	119
0.3	$\hat{\rho}_{ps,N}$	-141	48	-148	67	-129	84	-119	101
	$\hat{\rho}_{ps,K}$	1	43	16	58	-26	73	-41	85
	$\hat{\rho}_{ps,S}$	-18	55	-28	69	-33	84	-63	100
0.5	$\hat{\rho}_{ps,N}$			-221	46	-194	56	-143	68
	$\hat{\rho}_{ps,K}$			-18	36	-22	45	-23	55
	$\hat{\rho}_{ps,S}$			-30	43	-43	53	-52	66
0.7	$\hat{\rho}_{ps,N}$					-239	29	-196	34
	$\hat{\rho}_{ps,K}$					-38	19	-35	24
	$\hat{\rho}_{ps,S}$					-69	23	-78	29

Table III.53

Bias and MSE ($\times 10^4$) of ρ_{ps} for the unbalanced case when n=50, based on 10,000 simulations from T distribution, with df=3

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	20	410	-25	456	-19	522	-97	583
	$\hat{\rho}_{ps,K}$	74	199	13	225	13	263	-44	292
	$\hat{\rho}_{ps,S}$	-19	434	-35	470	-20	523	-105	576
0.3	$\hat{\rho}_{ps,N}$			-79	379	-159	448	-128	508
	$\hat{\rho}_{ps,K}$			74	188	10	214	1	242
	$\hat{\rho}_{ps,S}$			-116	399	-178	452	-150	500
0.5	$\hat{\rho}_{ps,N}$					-158	312	-176	362
	$\hat{\rho}_{ps,K}$					75	138	12	159
	$\hat{\rho}_{ps,S}$					-188	320	-214	357

Table III.54

Bias and MSE ($\times 10^4$) of ρ_{ps} for the unbalanced case when n=100, based on 10,000 simulations from T distribution, df=3

ρ_{ps}/ρ_{ss}		0.1		0.3		0.5		0.7	
	Estimate	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.1	$\hat{\rho}_{ps,N}$	-14	252	-18	295	-36	342	-30	397
	$\hat{\rho}_{ps,K}$	38	86	23	106	10	124	7	144
	$\hat{\rho}_{ps,S}$	-45	271	-19	306	-40	343	-33	396
0.3	$\hat{\rho}_{ps,N}$			-62	243	-111	297	-109	346
	$\hat{\rho}_{ps,K}$			73	85	29	104	21	120
	$\hat{\rho}_{ps,S}$			-72	257	-115	301	-115	344
0.5	$\hat{\rho}_{ps,N}$			-98	146	-114	199	-92	229
	$\hat{\rho}_{ps,K}$			102	50	83	65	47	79
	$\hat{\rho}_{ps,S}$			-115	166	-124	204	-102	229
0.7	$\hat{\rho}_{ps,N}$					-106	90	-308	41
	$\hat{\rho}_{ps,K}$					105	26	-214	31
	$\hat{\rho}_{ps,S}$					-127	99	-239	36

Table III.55
PN comparison of ρ_{ps} for the unbalanced case when n=50, based on 10,000 simulations from normal distribution

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimate	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,K}$	0.5885	0.5950	0.5831	0.5692
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.5565	0.5318	0.5106	0.4914
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.4752	0.4386	0.4177	0.4191
0.3	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,K}$		0.5983	0.5831	0.5791
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$		0.5418	0.5284	0.4995
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$		0.4463	0.4300	0.4194
0.5	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,K}$			0.5851	0.5871
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$			0.5284	0.5081
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$			0.4326	0.4168

Table III.56
PN comparison of ρ_{ps} for the unbalanced case when n=100, based on 10,000 simulations from normal distribution

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimate	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,K}$	0.5960	0.5914	0.5793	0.5646
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.5648	0.5456	0.5229	0.5057
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.4893	0.4442	0.4313	0.4305
0.3	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,K}$	0.5264	0.5845	0.5825	0.5730
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.5804	0.5492	0.5252	0.5114
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.5133	0.4578	0.4327	0.4255
0.5	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,K}$		0.5928	0.5843	0.5802
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$		0.5559	0.5239	0.5086
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$		0.4553	0.4353	0.4211
0.7	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,K}$			0.5832	0.5836
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$			0.5480	0.5271
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$			0.4544	0.4302

Table III.57

PN comparison of ρ_{ps} for the unbalanced case when n=50, based on 10,000 simulations from Kotz type distribution

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimate	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.5140	0.5166	0.5280	0.5400
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.5436	0.5440	0.5295	0.4996
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.5531	0.5517	0.5426	0.5401
0.3	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$		0.5248	0.5353	0.5050
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$		0.5225	0.5171	0.4894
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$		0.5506	0.5462	0.5406
0.5	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$			0.5499	0.5516
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$			0.4884	0.4800
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$			0.5468	0.5461
0.7	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$				0.5646
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$				0.4611
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$				0.5475

Table III.58

PN comparison of ρ_{ps} for the unbalanced case when n=100, based on 10,000 simulations from Kotz type distribution

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimate	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.5260	0.5379	0.5450	0.5530
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.5450	0.5451	0.539	0.54150
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.5660	0.5710	0.5680	0.5670
0.3	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.5407	0.5394	0.5411	0.5567
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.5433	0.5276	0.5143	0.5112
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.5800	0.5678	0.5578	0.5730
0.5	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$		0.5625	0.5505	0.5564
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$		0.4914	0.4843	0.4840
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$		0.5653	0.5510	0.5637
0.7	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$			0.6057	0.5707
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$			0.4424	0.4527
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$			0.5651	0.5596

Table III.59

PN comparison of ρ_{ps} for the unbalanced case when n=50, based on 10,000 simulations from T distribution, with df=3

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimate	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.6719	0.6742	0.6741	0.6685
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.5384	.5191	0.4997	0.4699
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.6756	.6812	0.6720	0.6675
0.3	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$		0.6550	0.6703	0.6773
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$		0.5376	0.4996	0.4739
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$		0.6789	0.6706	0.6739
0.5	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$			0.6548	0.6714
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$			0.5237	0.4830
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$			0.6630	0.6656

Table III.60

PN comparison of ρ_{ps} for the unbalanced case when n=100, based on 10,000 simulations from T distribution, with df=3

ρ_{ps}/ρ_{ss}		0.1	0.3	0.5	0.7
	Estimate	PN	PN	PN	PN
0.1	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.7114	0.7016	0.7033	0.7164
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$	0.5396	0.5288	.5026	0.4886
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.7161	0.7136	0.7065	0.7187
0.3	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$	0.6830	0.7039	0.6901	0.7118
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S})$	0.5929	0.5371	0.5532	0.4779
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$	0.7222	0.7222	0.7106	0.7125
0.5	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$		0.6901	0.7012	0.7012
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$		0.5532	0.5160	0.4943
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$		0.7106	0.7066	0.7014
0.7	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,N}$			0.6886	0.5889
	$\hat{\rho}_{ps,N}$ vs $\hat{\rho}_{ps,S}$			0.5182	0.4337
	$\hat{\rho}_{ps,K}$ vs $\hat{\rho}_{ps,S}$			0.6900	0.5593

CHAPTER IV

TESTING SIB-SIB AND MOM-SIB CORRELATIONS

IV.1 Introduction

Hypothesis testing is an important part of statistical inference. In this chapter we consider problems of hypothesis testing for ρ_{ss}, the sib-sib or intraclass correlation coefficient and ρ_{ps}, the mom-sib or interclass correlation coefficient, when familial data with unequal (unbalanced) number of children per family are available. If a known form of the distribution, like normal, can be assumed for the data then the likelihood based tests and the asymptotic distribution of the test statistic under the null hypothesis, can be adopted for testing. The three famous tests, under this approach are the likelihood ratio test, Wald test, and Rao's score test. The test statistics under all approaches have the same asymptotic distribution, but none is found to be uniformly better than the other. Hence all three tests are generally considered and efforts are made to determine the best test for the particular testing problem in hand.

In this chapter, the sections that follow we discuss various testing problems for the sib-sib and mom-sib correlations. For example, in the next section we consider hypothesis testing problems for sib-sib correlations, in Section 4.3 we consider testing of hypothesis for mom-sib correlation. In the sections 4.4 and 4.5 we consider testing the equality of two sib-sib and two mom-sib correlations respectively.

The testing procedures are developed under three different scenarios: (i) using normal likelihood function, (ii) using Kotz likelihood function and (iii) using Srivastava's non-iterative estimators. The first two cases above will provide us three tests each, namely, the LRT, Wald and Score tests. Further, at least one test can be constructed using Srivastava's estimator. Thus for each testing problem we have at least seven tests. In this chapter, we undertake an extensive simulation study hoping to identify the best test for each testing problem considered.

IV.2 Testing for ρ_{ss}, the Sib-Sib Correlation

Suppose $\mathbf{x}_i = (x_{i1}, ..., x_{im_i})'$ is the vector of observations on the ith family, where x_{ij}, $j = 1, ..., m_i$; $i = 1, ..., n$ is an observation on the jth child of the ith family. Let

$E(x_{ij}) = \mu$, $var(x_{ij}) = \sigma^2$, and $corr(x_{ij}, x_{ij'}) = \rho_{ss}$ for $j \neq j'$.

In this section, we consider the problem of testing $H_0 : \rho_{ss} = 0$ vs. $H_1 : \rho_{ss} \neq 0$. Estimation of ρ_{ss} has been already considered in Chapter 2. Let $\theta = (\mu, \sigma^2, \rho_{ss})'$ be the vector of the parameters and $L(\theta)$ be the likelihood function of θ. Suppose $\hat{\theta} = (\hat{\mu}, \hat{\sigma}^2, \hat{\rho}_{ss})'$ is the vector of maximum likelihood estimators obtained by maximizing $L(\theta)$ w.r.t. μ, σ^2, and ρ_{ss}, and $\hat{\theta}_0 = (\hat{\mu}_0, \hat{\sigma}_0^2, \hat{\rho}_{ss0})'$ is the vector of the MLE's computed under H_0.

For balanced data, that is when the number of children is the same for every family, assuming normality of the scores, Fisher's ANOVA F-statistic can be used for testing H_0. However, in general, for any likelihood function $L(\theta)$, we have the following procedures.

(a) Likelihood ratio test (LRT)

The log likelihood ratio for testing H_0 vs. H_1 is given by

$$\Lambda = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})}.$$

Then by the asymptotic theory (see Serfling, 1980) we have $LRT = 2\log L(\hat{\theta}) - 2\log L(\hat{\theta}_0) \xrightarrow{d} \chi_1^2$. We would reject H_0 if $LRT > \chi^2_{\alpha,1}$, where $\chi^2_{\alpha,1}$ is the αth upper tail cut off point of the chi-square distribution with 1 degrees of freedom.

(b) Wald's test

Suppose $\mathcal{I}(\theta) = E\left[\left(\frac{\partial \log L(\theta)}{\partial \theta}\right)\left(\frac{\partial \log L(\theta)}{\partial \theta}\right)'\right]$ is the Fisher information matrix of θ. If $\hat{\rho}_{ss}$ is the MLE of the sib-sib correlation (ρ_{ss}) and $\mathcal{I}(\hat{\theta})$ is the information matrix evaluated at the MLE, $\hat{\theta}$, then the Wald test statistic for testing H_0 vs. H_1 is

$$W = \left(\frac{\hat{\rho}_{ss}}{\sqrt{\frac{\mathcal{I}^{-1}_{(3,3)}}{n}}}\right)^2,$$

where $\mathcal{I}^{-1}_{(3,3)}$ is the 3rd diagonal element of $\left(\mathcal{I}(\hat{\theta})\right)^{-1}$.

From the asymptotic distribution of MLE's it is clear that $W \xrightarrow{d} \chi_1^2$ as $n \longrightarrow \infty$ (Serfling, 1980). Hence we reject H_0 if $W > \chi^2_{\alpha,1}$.

(c) Rao's score test

Suppose $S(\theta) = \frac{\partial \log L(\theta)}{\partial \theta}$ is the score vector and $\mathcal{I}(\theta)$ is the information matrix. Suppose $S(\hat{\theta}_0)$ and $\mathcal{I}(\hat{\theta}_0)$ are the score and information marices evaluated at $\theta = \hat{\theta}_0$, the MLE under null hypothesis. Then the score test statistic is:

$$R = S\left(\hat{\theta}_0\right)' (\mathcal{I}(\hat{\theta}_0))^{-1} S\left(\hat{\theta}_0\right).$$

The asymptotic distribution of R is χ^2 with 1 degrees of freedom (Serfling, 1980). Then reject H_0 if $R > \chi^2_{\alpha,1}$.

(d) A test based on Srivastava's estimator

We can also suggest a test based on Srivastava's combination estimator which was given earlier in (2.3), and its asymptotic variance (Srivastava, 1993) given by

$$AV(\hat{\rho}_{ss,S}) = 2(1-\rho_{ss})^2 \cdot \left\{ \begin{array}{c} \frac{(1-\rho_{ss})^2}{b_A^2} tr\{(\mathbf{AD}_{\eta^2})^2\} + \frac{\rho_s^2}{b_B^2} tr\{(\mathbf{BD}_{\eta^2})^2\} + \frac{2\rho_{ss}(1-\rho_{ss})}{b_A b_B} tr(\mathbf{AD}_{\eta^2}\mathbf{BD}_{\eta^2}) + \\ \frac{\{b_B - a_B(1-\rho_{ss}) - (a_A \frac{b_B}{b_A} - a_B)(1-\rho_{ss})^2\}^2}{b_B^2(N-n)} \end{array} \right\},$$

where $\mathbf{D}_{\eta^2} = diag(\eta_1^2, \ldots\ldots, \eta_n^2)$, $\eta_i^2 = 1-(1-\rho_s)a_i$, $a_i = 1 - m_i^{-1}$, $\mathbf{A} = \mathbf{D}_{\eta^2} - N^{-1}\omega\omega'$, $\omega = (m_1, \ldots, m_n)'$, $\mathbf{B} = \mathbf{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n'$, $a_A = b_A - (n-1)$, $b_A = N - N^{-1}\sum_{i=1}^{n} m_i^2$, $N = \sum_{i=1}^{n} m_i$, $a_B = (n-1)n^{-1}\sum_{i=1}^{n}(1 - m_i^{-1})$, $b_B = (n-1)$. Then the test statistic and its asymptotic distribution under $H_0 : \rho_{ss} = 0$ vs. $H_1 : \rho_{ss} \neq 0$ is given by:

$$T_s = \frac{(\hat{\rho}_{ss,S})^2}{AV(\hat{\rho}_0)} \xrightarrow{d} \chi_1^2,$$

where $AV(\hat{\rho}_0)$ is the asymptotic variance evaluated at $\rho_{ss} = 0$, the null hypothesis value.

IV.2.1 Performance of the tests: a simulation study

In this section, we adopt the following strategy for comparing various tests for testing $H_0 : \rho_{ss} = 0$ vs. $H_1 : \rho_{ss} \neq 0$. First, we consider the three likelihood based tests under normal distribution and the test based on Srivastava, and compare their performance (using estimated size and power of the tests) when data are simulated from normal distribution. Then we compare these tests when data are simulated from a non-normal distribution, namely, the Kotz type distribution. As will be observed later using the simulation results, neither the normal likelihood based tests nor Srivastava test achieve the assumed level of the test. Hence, to find a better test, we consider the three likelihood tests based on Kotz type distribution and compare the performance of all the seven tests when data are simulated from normal, Kotz type, and T distributions.

Thus, our interest is to first compare the three normal distribution based likelihood tests with the non-iterative test based on Srivastava's estimator as described in (d). To assess the performance of these four tests, we conduct a simulation study. Familial data on $n = 50$ and 100 families with unequal family sizes ranging from 1 to 6 children per family are considered. When $n = 50$ using truncated negative binomial distribution, as described earlier in Chapters 2 and 3, we determine n_i, the number of families with i children, $i = 1, 2, .., 6$. We have used $n_1 = 14$, $n_2 = 11$, $n_3 = 15$, $n_4 = 5$, $n_5 = 4$ and $n_6 = 1$. Similarly, when $n = 100$ we take $n_1 = 27$, $n_2 = 23$, $n_3 = 23$, $n_4 = 19$, $n_5 = 6$ and $n_6 = 2$. Ten thousand data sets for each set of n and m are generated when $\rho_{ss} = 0$ and the test statistics are calculated. For all the simulation runs we used $\mu = 0$ and $\sigma^2 = 1$. The simulation estimate of size of the test, when the assumed level is $\alpha = 0.05$, is computed as $\frac{1}{10{,}000}\#\{T_i \geqslant \chi^2_{\alpha,1}\}$, where T_i is the value of the particular test statistic in use for the ith simulation, $i = 1, ..., 10,000$.

For larger sample size, that is $n = 100$, we perform a study to compare the powers of these tests. Power of each test is evaluated as $\frac{1}{10{,}000}\#\{T_i \geqslant \chi^2_{\alpha,1}\}$ when the data are generated under the alternative hypotheses with the values of $\rho_{ss} = 0.1$, 0.2 and 0.25. As we will see later, all the tests already have high power for $\rho_{ss} = 0.25$. Hence there was no need to compute the power at larger values of ρ_{ss}. Results (sizes and power of the tests) are provided in Tables IV.1 - IV.13. The simulation estimates of sizes and powers of the normal distribution likelihood based tests are denoted by LRT_N, $Wald_N$ and $Score_N$, and these values for the non-iterative procedure is denoted by $Moment$. For convenience, we have used these same symbols for refereing to the tests as well. For example, the notation LRT_N is used to represent both the likelihood ratio test based on normal likelihood and the estimated size (or power) of that test. We notice from Table IV.1 that $Score_N$ and $Moment$ values are less than the nominal level ($\alpha = 0.05$). Although LRT_N values are higher than the nominal level, they do improve when $n = 100$. Notice that $Wald_N$ values are significantly higher than 0.05 which indicates that $Wald_N$ test is not a good test for testing $H_0 : \rho_{ss} = 0$ even for the sample sizes as large as $n = 100$. Hence we dropped $Wald_N$ from our power computations. All three tests, namely the LRT_N, $Score_N$ and $Moment$ tests, have very high power (Table IV.2). Based on the size and power of these tests we will recommend the $Score_N$ test as the best, if data are from normal distribution. Moment test also performs comparably well.

Next, we want to study the performance of these three tests if they were applied on a non-normal set of data, such as from a Kotz type distribution. From Table IV.3 we see that these tests have significantly higher estimated sizes than the nominal level of 0.05.

Next, we consider the likelihood based tests constructed from Kotz type distribution. We will denote the simulation estimates of sizes and powers of the Kotz type likelihood based tests by LRT_K, $Wald_K$ and $Score_K$. Table IV.4 summarizes these results. We note that the LRT_K and $Score_K$ are slightly higher than the nominal level, but have quite high power.

Finally, we study the performance of all the likelihood based tests, that is, those based on normal and Kotz type distribution, and the non-iterative test, when data under investigation are from multivariate T with different degrees of freedom. Results are provided in Tables IV.6 - IV.11. We observe that, for different degrees of freedoms the $Score_K$ values are the only values that are ≤ 0.05, and they have very high power. (Table IV.6 - IV.11). One last thing we want to do is to study the performance of Kotz type likelihood based tests when data are from normal distribution. Tables IV.12 and IV.13 provide these results. We note that LRT_K has higher size than α when $n = 50$ and this size improves as n increases to 100, while the $Score_K$ has nominal level that is significantly smaller than 0.05 for any value of n, with high power.

IV.2.2 Recommendations

We do not recommend using Wald test for testing $H_0 : \rho_{ss} = 0$. If it is known to us that the data under consideration are from normal distribution then we highly recommend using $Score_N$ or the non-iterative test based on Srivastava's estimator. But if this can not be guaranteed then we highly recommend using $Score_K$. Based on its performance under different distributions and for large or small samples, we believe that it is the best test to use.

IV.3 Testing for ρ_{ps}, the Mom-Sib Correlation

Suppose $(x_i, y_{i1}, y_{i2}, ..., y_{im_i})$ is the data vector on the ith family where x_i is the observation made on a parent and $y_{i1}, y_{i2}, ..., y_{im_i}$ are those made on m_i children in that family. Suppose $E(x_i) = \mu_p$, $E(y_{ij}) = \mu_s$, $var(x_i) = \sigma_p^2$, $var(y_{ij}) = \sigma_s^2$, $corr(y_{ij}, y_{ij'}) = \rho_{ss}$, and $corr(x_i, y_{ij}) = \rho_{ps}$. The main parameter of interest in this section is ρ_{ps}. We have studied estimation of ρ_{ps} in the previous chapter. Here we want to test $H_0 : \rho_{ps} = 0$ vs. $H_1 : \rho_{ps} \neq 0$.

Let $\theta = (\mu_p, \mu_s, \sigma_p, \sigma_s, \rho_{ps}, \rho_{ss})'$ be the vector of parameters and $L(\theta)$ be the likelihood function of θ given familial data on n families. Let $\hat{\theta}$ be the MLE of θ obtained by maximizing $L(\theta)$ with respect to θ and $\hat{\theta}_0$ be the MLE of θ obtained by maximizing $L(\theta)$ with respect to θ under the null hypothesis. Let $S(\theta)$ be the score vector and $\mathcal{I}(\theta)$ be the Fisher information matrix. Then the three asymptotic tests based on the likelihood theory are the likelihood ratio test (*LRT*), *Wald* test and *Rao's score* test. We use these three tests each under the normal distribution and the Kotz distribution. Thus we have six tests for testing $H_0 : \rho_{ps} = 0$ vs. $H_1 : \rho_{ps} \neq 0$.

Testing for mom-sib correlation was discussed first by Donner and Bull (1984). They considered the likelihood ratio test (LRT), a test based on the large sample variance of the maximum likelihood estimator (MLE), an adjusted pairwise test, and a test (Z_p) based on the large sample variance of the pairwise estimator, which uses the ratio of the pairwise estimator to its large sample standard error for testing, that mom-sib correlation is zero. They found that under certain conditions, including that the data is from a normal distribution and family size is around 25, Z_p has size and power comparable to the LRT, especially for the most common moderate-to-small values of the mom-sib correlation.

Velu and Rao (1990) studied testing procedures using the mean-sib correlation, the ensemble estimator, and Srivastava's estimator for small sample situations. They derived the exact distribution for Srivastava's estimator because it has smaller asymptotic variance, and gave sizes of the test based on the asymptotic variance of Srivastava's estimator. Since then there have been no discussions regarding testing for mom-sib correlations using either the ensemble or Srivastava's estimator.

Along with the likelihood based tests discussed above we propose to consider a test based on Srivastava's estimator for testing H_0. A brief description of these tests are provided next.

(a) Likelihood ratio test (LRT)

By the asymptotic theory we have $LRT = 2\log L(\hat{\theta}) - 2\log L(\hat{\theta}_0) \xrightarrow{d} \chi_1^2$. Then we reject H_0 if $LRT > \chi^2_{\alpha,1}$, where $\chi^2_{\alpha,1}$ is the α upper tail cut off point of the chi-square distribution with 1 degrees of freedom.

(b) Wald's test

Let $\hat{\rho}_{ps}$ be the MLE of the mom-sib correlation ρ_{ps} and $\mathcal{I}(\hat{\theta})$ be the information matrix evaluated at $\hat{\theta}$. Then by the large sample theory,

$$W = \left(\frac{\hat{\rho}_{ps}}{\sqrt{\frac{\mathcal{I}^{-1}_{(5,5)}}{n}}} \right)^2 \xrightarrow{d} \chi_1^2,$$

where $\mathcal{I}^{-1}_{(5,5)}$ is the 5th diagonal element of the inverse of $\mathcal{I}(\hat{\theta})$. We reject H_0 if $W > \chi^2_{\alpha,1}$.

(c) Rao's score test

Let $\theta = (\mu_p, \mu_s, \sigma_p, \sigma_s, \rho_{ps}, \rho_{ss})'$ be the vector of parameters and $\hat{\theta}_0 = (\hat{\mu}_{po}, \hat{\mu}_{so}, \hat{\sigma}_{po}, \hat{\sigma}_{so}, \hat{\rho}_{pso}, \hat{\rho}_{sso})'$ be the vector of the MLE's under H_0. Then the Score test statistic is:

$$R = S\left(\hat{\theta}_0\right)' (\mathcal{I}(\hat{\theta}_0))^{-1} S\left(\hat{\theta}_0\right) \xrightarrow{d} \chi_1^2,$$

where $S\left(\hat{\theta}_0\right)$ and $\mathcal{I}(\hat{\theta}_0)$ are the score function and information matrix evaluated at $\hat{\theta}_0$. Then we reject H_0 if $R > \chi^2_{\alpha,1}$.

(d) A test based on Srivastava's estimator

Srivastava (1984) introduced his remarkable estimator for the mom-sib correlation $\hat{\rho}_{ps,S}$ which is given in (III.1), and its asymptotic variance derived by Srivastava and Katapa (1986) is given in (III.2). Then $\hat{\rho}_{ps,S} \approx N(\rho_{ps,S}, AV(\hat{\rho}_{ps,S}))$. Hence a test can be proposed as follows:

Reject H_0 if $\dfrac{(\hat{\rho}_{ps,S})^2}{AV(\hat{\rho}_{ps,S})_0} > \chi^2_{\alpha,1}$, where $AV(\hat{\rho}_{ps,S})_0$ is the asymptotic variance with the nuisance parameter ρ_{ss} replaced by $\hat{\rho}_{ss0}$, the estimator of ρ_{ss} under H_0.

As in the sib-sib case, using simulation we will evaluate the performance of various tests.

Familial data on $n = 50$ and 100 families with unequal family sizes ranging from 1 to 5 children per family are generated. When $n = 50$, using truncated negative binomial distribution, as described earlier in Chapters 2 and 3, we determine n_i, the number of families with i children, $i = 1, 2, .., 5$. We have used $n_1 = 17$, $n_2 = 15$, $n_3 = 9$, $n_4 = 6$, and $n_5 = 3$. Similarly, when n=100 we took $n_1 = 21$, $n_2 = 32$, $n_3 = 29$, $n_4 = 11$, and $n_5 = 7$. Ten thousand data sets for each set of n and m, when $\rho_{ps} = 0$ and $\rho_{ps} = 0.2$ and 0.5 are generated and the test statistics are calculated. In the simulations we used $\mu_p = 0, \mu_s = 0$, $\sigma_p^2 = 1$, $\sigma_s^2 = 1$. Then the simulation estimate of the size of the test when the assumed level of the test is $\alpha = 0.05$ is computed as $\frac{1}{10,000}\#\{T_i \geq \chi^2_{\alpha,1}\}$, where T_i is the value of the particular test statistic in use, for the ith simulation, $i = 1, ..., 10,000$. Notice that we computed the size of the test using two different values of ρ_{ss}. Based on the results we claim that for large n the effect of the nuisance parameter (ρ_{ss}) seems to be small. To further assess these tests we computed the powers. However, the powers are computed for only those tests whose sizes were closer to the nominal size. Power of each test is calculated as $\frac{1}{10,000}\#\{T_i \geq \chi^2_{\alpha,1}\}$ when the data are generated under the alternative hypotheses with the values of $\rho_{ps} = 0.1, 0.2$ and 0.25 and using $\rho_{ss} = 0.2$. Results (sizes and power of the tests) are provided in (Tables IV.14 - IV.24) for different values of n, m and for a variety of simulations. The estimated sizes and powers are denoted in tables using the same symbols described in the sib-sib case.

IV.3.1 Performance of the tests

The estimated sizes and powers of the tests, when data are simulated from normal distribution are provided in Tables IV.14 and IV.15. Note from the values in Table IV.14 that $Score_N$ and $Moment$ values are very close to 0.05 (nominal level). LRT_N and $Wald_N$ are slightly higher than 0.05. The powers for $Score_N$ and $Moment$ tests are provided in Table IV.15. The values in the table show clearly that both tests have at least 90% power. Based on the size and power of these tests, we recommend $Score_N$ for testing $H_0 : \rho_{ps} = 0$.

When data are simulated from Kotz type distribution, it is clear from the values in Table IV.16 that the normal likelihood based tests and the $Moment$ test have significantly larger size compared to $\alpha = 0.05$. However, among the tests based on

Kotz type likelihood, we find that the $Score_K$ values are always less than or equal 0.05 and LRT_K values are slightly higher than 0.05 (Table IV.17). Table IV.18 provides the power of LRT_K and $Score_K$. These tests have high power, and both of them achieved at least 97% power even for small values of ρ_{ps}.

Next we would like to study the performance of these two groups of tests, namely the *Moment* test and the three likelihood tests based on normal and the three likelihood tests based on Kotz type, when data are generated from multivariate T with different degrees of freedom, that is, df=5 and df=10 (due to convergence problem we couldn't perform the test when df=3). The values in Tables IV.19 - IV.22 show that regardless of the degrees of freedom of the T distribution, the tests based on normal distribution along with the non-iterative test do not perform well. On the other hand the tests based on Kotz type perform fairly well. For example, the $Score_K$ values are always less than or equal to 0.05 with very high power. The LRT_K also performs well. Tables IV.20 and IV.22 provide the power of the LRT_K and the $Score_K$. Finally, for normal distribution we find that LRT_K and $Score_K$ values are ≤ 0.05 and have very high power.

IV.3.2 Recommendations

If it is known to us that the data under consideration are from normal distribution then we highly recommend using $Score_N$ or the non-iterative test based on Srivastava's estimator. But if this can not be guaranteed, we highly recommend using $Score_K$ based on its performance under different distributions and the fact that the test doesn't depend on nuisance parameters or on how large or small the sample size is.

IV.4 Testing the Equality of Sib-Sib Correlations for Two Independent Populations

Suppose there are two independent populations (or groups) and data on children of the families randomly selected from these populations are available. Suppose in the ith population there are n_i families and the number of children in families are allowed to be different. We denote the number of children in the jth family from the ith population by m_{ij}.

Suppose x_{ijk}, $k = 1, ..., m_{ij}$; $j = 1, ..., n_i$; $i = 1, 2$ are the observations on the

kth child of the jth family belonging to the ith population. We assume for the ith population that $E(x_{ijk}) = \mu_i$, $var(x_{ijk}) = \sigma_i^2$, and the sib-sib or intraclass correlation, $corr(x_{ijk}, x_{ijk'}) = \rho_{ssi}$ for $k \neq k'$. For every i, we have $-\infty < \mu_i < \infty$, $\sigma_i^2 > 0$, and $(-\frac{1}{\max_{1\le j\le n_i}(m_{ij})-1}) < \rho_{ssi} < 1$. Let the vector of observations on the jth family from ith group be $\mathbf{x}_{ij} = (x_{ij1}, ..., x_{ijm_{ij}})'$. Then $E(\mathbf{x}_{ij}) = \boldsymbol{\mu}_{ij} = \mu_i \mathbf{1}_{m_{ij}}$, where $\mathbf{1}_m$ is an $m \times 1$ vector of all ones, and the variance covariance matrix of $\mathbf{x}_{ij}$ is

$$
\begin{aligned}
var(\mathbf{x}_{ij}) = \boldsymbol{\Sigma}_{ij} &= \sigma_i^2[(1-\rho_{ssi})\mathbf{I}_{m_{ij}} + \rho_{ssi}\mathbf{J}_{m_{ij}}] \\
&= \sigma_i^2 \mathbf{V}_{ij}(\rho_{ssi}) = \sigma_i^2 \begin{pmatrix} 1 & \rho_{ssi} & \cdots & \rho_{ssi} \\ \rho_{ssi} & 1 & \cdots & \rho_{ssi} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{ssi} & \rho_{ssi} & \cdots & 1 \end{pmatrix},
\end{aligned}
$$

where $\mathbf{I}_m$ is an identity matrix of order m and $\mathbf{J}_m$ is the $m \times m$ matrix of all ones.

Note that the determinant and inverse of $\boldsymbol{\Sigma}_{ij}$ respectively are

$$|\boldsymbol{\Sigma}_{ij}| = (\sigma_i^2)^{m_{ij}}[(1-\rho_{ssi})^{m_{ij}-1}(1+(m_{ij}-1)\rho_{ssi})]$$

and

$$\boldsymbol{\Sigma}_{ij}^{-1} = \frac{1}{\sigma_i^2(1-\rho_{ssi})}[\mathbf{I}_{m_{ij}} - \frac{\rho_{ssi}}{1+(m_{ij}-1)\rho_{ssi}}\mathbf{J}_{m_{ij}}].$$

In this section, we consider the problem of testing equality of two sib-sib correlations, namely testing $H_0 : \rho_{ss_1} = \rho_{ss_2} = \rho_{ss}$ *vs.* $H_1 : \rho_{ss_1} \neq \rho_{ss_2}$.

Note that the common values of ρ_{ss} under H_0 is a nuisance paremeter and an interest is also to estimate it.

The problem of testing equality of two intraclass correlations is considered by many authors. Donner and Bull (1983) considered this problem when family sizes within a population and between populations were the same. Khatri, Pukkila, and Rao (1989) considered this when family sizes in the two populations were different. These authors derived and studied the performance of the likelihood ratio test. For the problem of testing equality of several correlations, Konishi and Gupta (1989) have suggested a modified likelihood ratio test and a test based on Fisher's z-transformation, Paul and Barnwal (1990) suggested a $C(\alpha)$ test, and Haung and Sinha (1993) derived the optimum invariant test, assuming the family sizes within population are the same, but different for different populations.

Young and Bhandary (1998) and Bhandary and Alam (2000) respectively considered the problems of testing the equality of two and three correlation coefficients when the family sizes are unequal. They used Srivastava (1984)'s estimator of intraclass correlation and proposed the approximate likelihood ratio test and compared its performance with two other asymptotic tests based on normal distribution. They also made the assumption that the variances for different populations are the same.

We consider the problem of testing $H_0 : \rho_{ss_1} = \rho_{ss_2} = \rho_{ss}$ $vs.$ $H_1 : \rho_{ss_1} \neq \rho_{ss_2}$, using tests based on the likelihood theory. Let $\theta = (\mu_1, \sigma_1^2, \rho_{ss_1}, \mu_2, \sigma_2^2, \rho_{ss_2})'$ be the vector of the parameters and $L(\theta)$ be the likelihood function of θ. Let $\hat{\theta}$ be the maximum likelihood estimator (MLE) of θ which is obtained by maximizing $L(\theta)$ and $\hat{\theta}_0$ be the MLE obtained by maximizing $L(\theta)$ under the null hypothesis $H_0 : \rho_{ss_1} = \rho_{ss_2} = \rho_{ss}$. Note that $\theta_0 = (\mu_1, \sigma_1^2, \rho_{ss}, \mu_2, \sigma_2^2, \rho_{ss})'$ which is the same as θ but under the null hypothesis. We will assume that the two samples are drawn from multivariate normal, T, and Kotz type distributions when the family sizes are unequal and the variances are different for different populations with possibly different means but the same intraclass correlation coefficient, ρ_{ss}. The hypothesis of a common ρ_{ss} maybe be tested through the application of likelihood based tests for normal distribution as well as the likelihood based tests for Kotz type distributions. We also propose using two non-iterative tests based on Srivastava's combination estimator and its asymptotic variance.

The the likelihood ratio test statistic for testing H_0 $vs.$ H_1 is given by

(a) Likelihood ratio test (LRT)

$$\Lambda = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})}.$$

Then by the asymptotic theory (see Serfling, 1980), we have $LRT = 2\log L(\hat{\theta}) - 2\log L(\hat{\theta}_0) \xrightarrow{d} \chi_1^2$. We would reject H_0 if $LRT > \chi^2_{\alpha,1}$, where $\chi^2_{\alpha,1}$ is the αth upper tail cut off point of the chi-square distribution with 1 degrees of freedom.

(b) Rao's score test

Let $S(\theta) = \partial logL(\theta)/\partial\theta$ be the 6×1 vector of the score function and $\mathcal{I}(\theta) = E[(\partial logL(\theta)/\partial\theta)(\partial logL(\theta)/\partial\theta)']$ be the 6×6 Fisher information matrix. Suppose $S(\hat{\theta}_0)$ and $\mathcal{I}(\hat{\theta}_0)$ are the score and information matrices at $\theta = \hat{\theta}_0$, the MLE under null hypothesis. Then the score test statistic is

$$R = S\left(\hat{\theta}_0\right)' (\mathcal{I}(\hat{\theta}_0))^{-1} S\left(\hat{\theta}_0\right).$$

The asymptotic distribution of R is χ^2 with 1 degrees of freedom (Serfling, 1980). Then reject H_0 if $R > \chi^2_{\alpha,1}$.

Note that $S(\theta) = (S_1(\theta), S_2(\theta))'$, where

$$S_i(\theta) = (\partial logL(\theta)/\partial\mu_i, \partial logL(\theta)/\partial\sigma_i^2, \partial logL(\theta)/\partial\rho_i)'$$

Next, we note that the Fisher information matrix $\mathcal{I}(\theta)$ is a block diagonal matrix containing two blocks of 3×3 matrices. Then $\mathcal{I}(\theta)^{-1}$, the inverse of the Fisher information matrix, will also be block diagonal. The ith block, $\mathcal{I}_i$, of the Fisher information matrix (for the normal distribution, as an example) is the following:

$$\mathcal{I}_i = \begin{pmatrix} \sum_{j=1}^{n_i} \frac{m_{ij}}{\sigma_i^2(1+(m_{ij}-1)\rho_i)} & 0 & 0 \\ 0 & \sum_{j=1}^{n_i} \frac{m_{ij}}{2\sigma_i^4} & \sum_{j=1}^{n_i} \frac{\rho_i m_{ij}(m_{ij}-1)}{2\sigma_i^2(1-\rho_i)(1+(m_{ij}-1)\rho_i)} \\ 0 & \sum_{j=1}^{n_i} \frac{\rho_i m_{ij}(m_{ij}-1)}{2\sigma_i^2(1-\rho_i)(1+(m_{ij}-1)\rho_i)} & \sum_{j=1}^{n_i} \frac{m_{ij}(m_{ij}-1)(1+(m_{ij}-1)\rho_i^2)}{2(1-\rho_i)^2(1+(m_{ij}-1)\rho_i)^2} \end{pmatrix}.$$

(c) Wald's test

Suppose $\mathcal{I}(\theta)$ is the fisher information matrix of θ. If $\hat{\rho}_{ss_1}$ and $\hat{\rho}_{ss_2}$ are the MLEs of the sib-sib correlations ρ_{ss_1} and ρ_{ss_2} and $\mathcal{I}(\theta)$ is the information matrix evaluated at the MLE, $\hat{\theta} = (\hat{\mu}_1, \hat{\sigma}_1^2, \hat{\rho}_{ss_1}, \hat{\mu}_2, \hat{\sigma}_2^2, \hat{\rho}_{ss_2})'$, then the Wald test statistic for testing H_0 vs. H_1 is:

$$W = \left(\frac{\hat{\rho}_{ss_1} - \hat{\rho}_{ss_2}}{\sqrt{\frac{\mathcal{I}^{-1}_{(3,3)}}{n_1} + \frac{\mathcal{I}^{-1}_{(6,6)}}{n_2}}} \right)^2,$$

where $\mathcal{I}^{-1}_{(3,3)}$ and $\mathcal{I}^{-1}_{(6,6)}$ are the 3rd and the 6th diagonal elements of $(\mathcal{I}(\theta))^{-1}$ respectively. From the asymptotic distribution of MLEs it is clear that $W \xrightarrow{d} \chi^2_1$ as $n \longrightarrow \infty$ (Serfling, 1980). Hence we reject H_0 if $W > \chi^2_{\alpha,1}$.

(d) Non-iterative tests

The two non-iterative tests we propose are based on Srivastava's combined estimator of the intraclass correlation coefficient and its asymptotic variance. The two tests differ in the way we estimate the standard error of the difference between the intraclass correlation coefficients.

The uniform ANOVA and the generalized weighted ANOVA estimators of ρ_{ssi} due to Smith's (1957) can be written as follows

$$\hat{\rho}_{ui} = 1 - \frac{b_{ui}\sum_{j=1}^{n_i}\sum_{k=1}^{m_{ij}}(x_{ijk}-\bar{x}_{ij})^2}{(m_i - n_i)\sum_{j=1}^{n_i}(\bar{x}_{ij}-\bar{x}_{wi})^2 + a_{ui}\sum_{j=1}^{n_i}\sum_{k=1}^{m_{ij}}(x_{ijk}-\bar{x}_{ij})^2}, \tag{IV.1}$$

where $a_{ui} = (n_i - 1)n_i^{-1}\sum_{j=1}^{n_i}(1 - m_{ij}^{-1})$, $b_{ui} = (n_i - 1)$.

$$\hat{\rho}_{wi} = 1 - \frac{b_{wi}\sum_{j=1}^{n_i}\sum_{k=1}^{m_{ij}}(x_{ijk}-\bar{x}_{ij})^2}{(m_i - n_i)\sum_{j=1}^{n_i}m_{ij}(\bar{x}_{ij}-\bar{x}_{wi})^2 + a_{wi}\sum_{j=1}^{n_i}\sum_{k=1}^{m_{ij}}(x_{ijk}-\bar{x}_{ij})^2}, \tag{IV.2}$$

where $a_{wi} = b_{wi} - (n_i - 1)$, $b_{wi} = m_i - m_i^{-1}\sum_{j=1}^{n_i}m_{ij}^2$, $m_i = \sum_{j=1}^{n_i}m_{ij}$, $\bar{x}_{wi} = m_i^{-1}\sum_{j=1}^{n_i}m_{ij}\bar{x}_{ij}$, $\bar{x}_{ij} = m_{ij}^{-1}\sum_{k=1}^{m_{ij}}x_{ijk}$,

Then Srivastava's combination estimator of ρ_{ssi} (Srivastava, 1993) is defined as

$$\hat{\rho}_{Si} = \frac{\hat{\rho}_{wi}}{1 + \hat{\rho}_{wi} - \hat{\rho}_{ui}}. \tag{IV.3}$$

The estimation of the common intraclass correlation, ρ_{ss}, using Srivastava's combination estimator (IV.3) can be done as follows. For $i = 1, 2$, suppose we write $\hat{\rho}_{wi} = 1 - \frac{w_{1i}}{w_{2i}}$, where w_{1i} and w_{2i} respectively the numerator and denominator expressions on the right hand side of (IV.1) and similarly $\hat{\rho}_{ui} = 1 - \frac{u_{1i}}{u_{2i}}$, where u_{1i} and u_{2i} respectively the numerator and denominator expressions on the right hand side of (IV.2).

$$\text{Let } \hat{\rho}_w = 1 - \frac{1}{2}\sum_{i=1}^{2}\frac{w_{1i}}{w_{2i}} \text{ and } \hat{\rho}_u = 1 - \frac{1}{2}\sum_{i=1}^{2}\frac{u_{1i}}{u_{2i}}.$$

$$\text{Then we suggest } \hat{\rho}_S = \frac{\hat{\rho}_w}{1 + \hat{\rho}_w - \hat{\rho}_u},$$

as an estimator of the common intraclass correlation. Now using Srivastava (1993)'s expression for the asymptotic variance of his combined estimator we provide the following two results.

For the first, the test statistic and the asymptotic distribution are:

$$TS = \frac{(\hat{\rho}_{ss_1} - \hat{\rho}_{ss_2})^2}{AV_1(\hat{\rho}_S) + AV_2(\hat{\rho}_S)} \xrightarrow{d} \chi_1^2 \; as \; n \longrightarrow \infty.$$

Hence we reject H_0 if $TS > \chi^2_{\alpha,1}$. And the second test is given by:

$$TS^* = \frac{(\hat{\rho}_{ss_1} - \hat{\rho}_{ss_2})^2}{AV_1(\hat{\rho}_{ss_1}) + AV_2(\hat{\rho}_{ss_2})} \xrightarrow{d} \chi_1^2 \; as \; n \longrightarrow \infty.$$

Hence we reject H_0 if $TS^* > \chi^2_{\alpha,1}$.

IV.4.1 Performance of the tests

Next, we compare the performance of the likelihood based tests based on multivariate normal distribution with the non-iterative tests based on Srivastava's combination estimator using simulation study. Familial data on 30 families with unequal family sizes ranging from 2 to 4 children per family are simulated from a multivariate normal distribution. As before we computed the significance levels and power of these five tests. The results are presented in Table IV.25 and Figure 27. It shows that $Score_N$ and TS values are $\leq$ nominal level and do not depend on the nuisance parameter (ρ_{ss}, the common value of intraclass correlation). LRT_N values are slightly higher than 0.05 for all values of ρ_{ss}. On the other hand we can see that $Wald_N$ and TS^* depend on the nuisance parameter, so both of these tests have size of test that is higher than 0.05 when ρ_{ss} is small to moderate, and they both have size that is ≤ 0.05 when ρ_{ss} is large ($= 0.8$). Next, we compare these tests when data are simulated from a Kotz type distribution. It is clear from the values which are provided in Table IV.26 that the normal likelihood based tests and TS and TS^* tests have sizes larger than $\alpha = 0.05$. On the other hand, among the tests based on Kotz type likelihood, the $Score_K$ and the LRT_K tests do not depend on the nuisance parameter. Moreover $Score_K$ has size that is considerably close to 0.05. LRT_K values are always higher than the nominal level, but do not depend on the nuisance parameter (Table IV.27). The power comparison between $Score_K$ and LRT_K is shown in Figure 28 and it is clear from the graph that both tests have similar power. We also notice that the performance of $Wald_K$ is similar to $Wald_N$ when data are from normal distribution. The Wald test, however, depends on the nuisance parameter and hence it is not a reliable test for testing $H_0 : \rho_{ss_1} = \rho_{ss_2} = \rho_{ss}$.

Next we compare all the above mentioned tests under the multivariate T distribution. The results are recorded in Table IV.28. Clearly $Score_K$ has the best size (which is less than or equal 0.05) and has a very high power (see Figure 29). Finally we study the performance of the likelihood based tests based on Kotz type distribution, but when data are simulated from a multivariate normal distribution. We find that $Score_K$ values are always less than or equal to the nominal level. Further the power of $Score_K$ do not differ from the power of $Score_N$, or the power of the TS (Table IV.29 and Figure 30).

IV.4.2 Recommendations

If the sample under study is from a multivariate normal distribution then we recommend using normal based score test or TS test because of their small nominal significance levels and due to the fact that these tests are independent of the nuisance parameter ρ_{ss}. However, if normality cannot be guaranteed then we recommend using the score test based on Kotz likelihood, $Score_K$, since our simulation results show that this test performs well under different distributions, and it doesn't depend on the nuisance parameter ρ_{ss}.

IV.5 Testing the Equality of Mom-Sib Correlations for Two Independent Populations

Suppose there are two independent populations and data on mother and her children are randomly selected. Suppose in the ith population there are n_i families and the number of children in families are allowed to be different. We denote the number of children in the jth family from the ith population by m_{ij}.

Suppose x_{ij}, $j = 1, ..., n_i$; $i = 1, 2$ are the observations on the mom's score from the jth family belonging to the ith population and $\mathbf{y}_{ij} = (y_{ij1,...,}y_{ijm_{ij}})$ is the vector of children scores, such that y_{ijk} is the observation on the kth child of the jth family from the ith population. We assume for the ith population that $E(x_{ij}) = \mu_{pi}$, $var(x_{ij}) = \sigma^2_{pi}$, $E(y_{ijk}) = \mu_{si}$, $var(y_{ijk}) = \sigma^2_{si}$ and the sib-sib or intraclass correlation, $corr(y_{ijk}, y_{ijk'}) = \rho_{ssi}$ and the mom-sib correlation, $corr(x_{ij}, y_{ijk}) = \rho_{psi}$. For every i, we have $-\infty < \mu_{pi}, \mu_{si} < \infty$, σ^2_{pi}, $\sigma^2_{si} > 0$, $\rho^2_{psi} \leq \rho_{ssi}$, and $0 \leq \rho_{ssi} \leq 1$. Recall that the last two conditions are needed to keep the variance covariance matrix positive definite.

Our main interest in this section is testing $H_0 : \rho_{ps_1} = \rho_{ps_2} = \rho_{ps}$ $vs.$ $H_1 : \rho_{ps_1} \neq \rho_{ps_2}$. However, estimating the common mom-sib correlation, ρ_{ps}, becomes an important problem as well.

The hypothesis of a common ρ_{ps} can be tested using likelihood based tests for normal distribution as well as the likelihood based tests for Kotz type distribution. We also proposed two non-iterative tests based on Srivastava's estimator and its asymptotic variance.

Let $\theta = (\mu_{p1}, \mu_{s1}, \sigma_{p1}, \sigma_{s1}, \rho_{ps1}, \rho_{ss1}, \mu_{p2}, \mu_{s2}, \sigma_{p2}, \sigma_{s2}, \rho_{ps2}, \rho_{ss2})'$ be the vector of parameters and $L(\theta)$ be the likelihood function of θ. Let $\hat{\theta}$ be the maximum likelihood estimator (MLE) of θ which is obtained by maximizing $L(\theta)$ and $\hat{\theta}_0$ be the MLE obtained by maximizing $L(\theta)$ under the null hypothesis $H_0 : \rho_{ps_1} = \rho_{ps_2} = \rho_{ps}$. Note that $\theta_0 = (\mu_{p1}, \mu_{s1}, \sigma_{p1}, \sigma_{s1}, \rho_{ps}, \rho_{ss1}, \mu_{p2}, \mu_{s2}, \sigma_{p2}, \sigma_{s2}, \rho_{ps}, \rho_{ss2})'$ which is the same as θ, but under the null hypothesis.

We will assume that the two samples are drawn from multivariate normal, T, and Kotz type distributions, the family sizes are unequal, and the means and variances are different for different populations.

(a) Likelihood ratio test (LRT)

By the asymptotic theory we have $LRT = 2\log L(\hat{\theta}) - 2\log L(\hat{\theta}_0) \xrightarrow{d} \chi^2_1$. We would reject H_0 if $LRT > \chi^2_{\alpha,1}$, where $\chi^2_{\alpha,1}$ is the αth upper tail cut off point of the chi-square distribution with 1 degrees of freedom.

(b) Rao's score test

Let $S(\theta) = \partial logL(\theta)/\partial\theta$ be 12×1 vector of the score function and $\mathcal{I}(\theta) = E[(\partial logL(\theta)/\partial\theta)(\partial logL(\theta)/\partial\theta)']$ be the 12×12 Fisher information matrix. $S(\hat{\theta}_0)$ and $\mathcal{I}(\hat{\theta}_0)$ are these when evaluated at $\theta = \hat{\theta}_0$, the MLE under null hypothesis. Then the score test statistic is

$$R = S\left(\hat{\theta}_0\right)'(\mathcal{I}\ (\hat{\theta}_0))^{-1}S\left(\hat{\theta}_0\right).$$

The asymptotic distribution of R is χ^2 with 1 degrees of freedom. Then reject H_0 if $R > \chi^2_{\alpha,1}$.

Note that $S(\theta) = (S_1(\theta), S_2(\theta))'$, where

$$\begin{aligned} S_i(\theta) &= (\partial logL(\theta)/\partial\mu_{pi}, \partial logL(\theta)/\partial\mu_{si}, \partial logL(\theta)/\partial\sigma_{pi}, \partial logL(\theta)/\partial\sigma_{si} \\ &\quad , \partial logL(\theta)/\partial\rho_{psi}, \partial logL(\theta)/\partial\rho_{ssi})' \end{aligned}$$

Next, we note that Fisher information matrix $\mathcal{I}\ (\hat{\theta})$ is a block diagonal matrix containing two blocks of 6×6 matrices. Then $(\mathcal{I}\ (\hat{\theta}))^{-1}$, the inverse of Fisher information matrix, will also be block diagonal. In the following we provide the expressions for the ith block, $\mathcal{I}_{psi}$, of Fisher information matrix (for normal likelihood, as an example), which can be used in practice and inverse of which can be computed.

$$\mathcal{I}_{psi} = \begin{pmatrix} \frac{1}{\sigma_{p_i}^2} h_{11} & -\frac{\rho_{ps}}{\sigma_p \sigma_s} h_{12} & 0 & 0 & 0 & 0 \\ & \frac{1}{\sigma_{si}^2} h_{22} & 0 & 0 & 0 & 0 \\ & & \frac{1}{\sigma_{pi}^2} h_{33} & -\frac{\rho_{psi}^2}{\sigma_{pi}\sigma_{si}} h_{34} & -\frac{\rho_{ps}}{\sigma_{pi}} h_{35} & 0 \\ & & & \frac{1}{\sigma_{si}^2} h_{44} & \frac{-\rho_{psi}}{\sigma_{si}} h_{45} & \frac{(\rho_{psi}^2 - \rho_{ssi})}{\sigma_{si}(1-\rho_{ssi})} h_{46} \\ & symm & & & h_{55} & -\rho_{psi}\, h_{56} \\ & & & & & \frac{1}{2(1-\rho_{ssi})^2} h_{66} \end{pmatrix}$$

where $w_i = (1 + (m_{ij} - 1)\rho_{ssi})$, and $g_i = 1 + (m_{ij} - 1)\rho_{ssi} - m_{ij}\rho_{psi}^2$.

$$h_{11} = \sum_{j=1}^{ni} \frac{w_i}{g_i},\ h_{12} = \sum_{i=1}^{n} \frac{m_i}{g_i},\ h_{22} = \sum_{j=1}^{ni} \frac{m_{ij}}{g_i},\ h_{33} = \sum_{j=1}^{ni} \frac{w_i + g_i}{g_i},\ h_{34} = \sum_{j=1}^{ni} \frac{m_{ij}}{g_i},$$

$$h_{35} = \sum_{j=1}^{n} \frac{m_{ij}}{g_i},\ h_{44} = \sum_{j=1}^{ni} \frac{2m_{ij}g_i + m_{ij}\rho_{ps}^2}{g_i},\ h_{45} = \sum_{j=1}^{ni} \frac{m_{ij}}{g_i},$$

$$h_{55} = \sum_{j=1}^{ni} \frac{m_{ij}(1+(m_{ij}-1)pss+m_{ij}Pps2)}{gi2},\ h_{46} = \sum_{j=1}^{ni} \frac{m_{ij}(m_{ij}-1)}{g_i},$$

$$h_{56} = \sum_{j=1}^{ni} \frac{m_{ij}(m_{ij}-1)}{g_i^2},\ h_{66} = \sum_{j=1}^{ni} \frac{(m_{ij}-1)[\ (m_{ij}-1)(1-\rho_{ss})^2 + g_i^2]}{g_i^2},$$

(c) Wald's test

Suppose $\mathcal{I}_{ps}(\theta)$ is the Fisher information matrix of θ. If $\hat{\rho}_{ps_1}$ and $\hat{\rho}_{ps_2}$ are the MLEs of the mom-sib correlations ρ_{ps_1} and ρ_{ps_2} and $\mathcal{I}_{ps}(\hat{\theta})$ is the inverse of the information matrix evaluated at the MLE, $\hat{\theta} = (\hat{\mu}_{p1}, \hat{\mu}_{s1}, \hat{\sigma}_{p1}, \hat{\sigma}_{s1}, \hat{\rho}_{ps1}, \hat{\rho}_{ss1}, \hat{\mu}_{p2}, \hat{\mu}_{s2}, \hat{\sigma}_{p2}, \hat{\sigma}_{s2}, \hat{\rho}_{ps2}, \hat{\rho}_{ss2})'$. Then the Wald test statistic for testing H_0 vs. H_1 is:

$$W = \left(\frac{\hat{\rho}_{ps_1} - \hat{\rho}_{ps_2}}{\sqrt{\frac{\mathcal{I}_{ps\,(5,5)}^{-1}}{n_1} + \frac{\mathcal{I}_{ps\,(11,11)}^{-1}}{n_2}}} \right)^2,$$

where $\mathcal{I}_{ps\,(5,5)}^{-1}$ and $\mathcal{I}_{ps\,(11,11)}^{-1}$ are the 5th and the 11th diagonal elements of $(\mathcal{I}_{ps}(\hat{\theta}))^{-1}$. From the asymptotic distribution of MLEs it is clear that $W \xrightarrow{d} \chi_1^2$. Hence we reject H_0 if $W > \chi_{\alpha,1}^2$.

(d) Non-iterative tests

The two non-iterative tests we proposed are based on Srivastava's estimator and its asymptotic variance (see (III.1) and (III.2)). The two tests differ in the way we estimate the standard error of the estimated difference between the interclass correlation coefficient.

The common interclass correlation using Srivastava's estimator can be estimated as

$$\hat{\rho}_{ps,S} = \frac{\sum_{i=1}^{2}(n_i - 1)\hat{\rho}_{ps,S_i}}{\sum_{i=1}^{2}(n_i - 1)}$$

and this is what we suggest as an estimator of the common interclass correlation.

Then the first test is given by,

$$TP = \frac{(\hat{\rho}_{ps,S_1} - \hat{\rho}_{ps,S_2})^2}{AV_1(\hat{\rho}_{ps,S}) + AV_2(\hat{\rho}_{ps,S})} \xrightarrow{d} \chi_1^2 \; as \; n \longrightarrow \infty.$$

Hence we reject H_0 if $TP > \chi^2_{\alpha,1}$. The second test is

$$TP^* = \frac{(\hat{\rho}_{ps,S_1} - \hat{\rho}_{ps,S_2})^2}{AV_1(\hat{\rho}_{ps,S_1}) + AV_2\hat{\rho}_{ps,S_2})} \xrightarrow{d} \chi_1^2 \; as \; n \longrightarrow \infty.$$

Hence we reject H_0 if $TP^* > \chi^2_{\alpha,1}$.

Recall that $AV(.)$ is the asymptotic variance expression for Srivastava's estimator provided in (III.2).

IV.5.1 Performance of the tests

First, we compare the performance of the normal distribution based likelihood tests with the non-iterative tests based on Srivastava's estimator as described in (d). To assess the performance of these 5 tests, we conduct a simulation study. Familial data on n =30 families with unequal family sizes ranging from 2 to 4 children per family are simulated from a multivariate normal distribution. Then the significance levels and power of these five tests are computed. The results are presented in Table IV.30 and Figure 31. Notice that the TP and $Score_N$ values are close to the nominal level 0.05. LRT_N values are moderately higher than 0.05 for all values of ρ_{ps}, but $Wald_N$ and TP^* values are significantly larger than the nominal level. Second, we want to study the performance of these tests under Kotz type distribution. Table IV.31 shows that these tests have significantly higher estimated size compared to 0.05.

Third, we consider the tests based on Kotz type type distribution. Table IV.32 shows that $Score_K$ and LRT_K values are close to the nominal level, but $Wald_K$ values are the highest compared to the other tests. Notice in this table that none of the tests based on Kotz type distribution depend on the nuisance parameter. The values given in Table 32 indicates that the $Score_K$ and LRT_K have the same power.

Fourth, we compare the performance of all the likelihood based tests, that is, those based on normal and Kotz type distributions as well as non-iterative tests, when data are simulated from T distribution. Table IV.33, indicates that $Score_K$ and LRT_K values are higher than the nominal level, but these values are smaller than the nominal levels for the other tests. These two tests also have high power (Table IV.33 and Figure 33).

Finally, we study the performance of the Kotz type likelihood based tests when data are from normal distribution. $Score_K$ and LRT_K values are always less than or equal the nominal level with very high power and the power values do not differ from either the power of $Score_N$ or the power of the TP test (Table IV.34 and Figure 34).

IV.5.2 Recommendations

If we know that the sample under study are from a multivariate normal distribution then we recommend using normal based score test or TP test. However, if this cannot be guaranteed then we recommend using score test based on Kotz type distribution since our simulation results show that this test performs well under different distributions, and it doesn't depend on the nuisance parameter ρ_{ps}.

IV.6 Analysis of Galton's Data

For illustration of our procedures, that is, of testing the equality of two sib-sib and two mom-sib correlations, we divide Galton's data set into two groups. The first group would contain 102 families and the second group would contain the remaining 103 families. From the first group we consider data on only daughters and from the second we consider data on only sons. For the first group, the pairs: (the number of daughters, the number of families with those many daughters) are (1, 25), (2, 21), (3, 12), (4, 10), (5, 5), (6, 4), (7, 1) (8, 1), and (9, 1). That is, there are 25 families with one daughter, 21 families with two daughters and so on. Similarly these pairs for the second group from where only boys are selected are (1, 10), (2, 28), (3, 22), (4, 11), (5, 4), and (6, 4). If ρ_{dd} is the correlation between the daughters from the first group and ρ_{ss} is the correlation between the sons from the second group then our interest is to test the null hypothesis $\rho_{dd} = \rho_{ss}$ $(= \rho)$. The maximum likelihood estimates of ρ_{dd}, $\rho_{ss,}$ and ρ, respectively are, $\hat{\rho}_{dd} = 0.2938$, $\hat{\rho}_{ss} = 0.2023$, and $\hat{\rho} = 0.2489$. Srivastava's estimators are $\hat{\rho}_{dd,S} = 0.3080$, $\hat{\rho}_{ss,S} = 0.2016$, and $\hat{\rho}_{S} = 0.2545$.

The P-values of the tests are provided in the table below. Clearly all tests fail to reject H_0 at $\alpha = 0.05$ significant level.

LRT_N	Wald_N	Score_N	LRT_K	Wald_K	Score_K	TS	TS^*
0.4063	0.4105	0.4004	0.3798	0.3094	0.4457	0.3635	0.3607

Table IV.1

Size of testing $H_0 : \rho_{ss} = 0$ when simulation is from normal

n	m	LRT_N	$Wald_N$	$Score_N$	$Moment$
50	1≤m≤6	0.0687	0.1276	0.0445	0.0454
100		0.0553	0.0855	0.0480	0.0452

Table IV.2

Power of testing $H_0 : \rho_{ss} = 0$ when simulation is from normal, n=100

m	ρ_{ss}	LRT_N	$Score_N$	$Moment$
1≤m≤6	0	0.0553	0.0480	0.0452
	0.1	0.3442	0.3795	0.3129
	0.2	0.8314	0.8452	0.7857
	0.25	0.9479	0.9480	0.9287

Table IV.3

Size of testing $H_0 : \rho_{ss} = 0$ when simulation is from Kotz type distribution

n	m	LRT_N	$Wald_N$	$Score_N$	$Moment$
n=50	1≤m≤6	0.1087	0.1163	0.1576	0.1780
n=100	1≤m≤6	0.1106	0.0861	0.1683	0.371

Table IV.4

Size of testing $H_0 : \rho_{ss} = 0$ when simulation is from Kotz type distribution

n	m	LRT_K	$Wald_K$	$Score_K$
n=50	1≤m≤6	0.065±.001	0.0996±.001	0.059±.001
n=100	1≤m≤6	0.053±.002	0.065±.003	0.063±.001

Table IV.5

Power of testing $H_0 : \rho_{ss} = 0$ when simulation is from Kotz type distribution, n=100

m	ρ_{ss}	LRT_K	$Score_K$
1≤m≤6	0	0.0510	0.061
	0.1	0.2976	0.3659
	0.2	0.7693	0.8129
	0.25	0.9060	0.9258

Table IV.6

Size of testing $H_0 : \rho_{ss} = 0$ when simulation is from T distribution, with df=3

n	m	LRT_N	$Wald_N$	$Score_N$	LRT_K	$Wald_K$	$Score_K$	$Moment$
50	1≤m≤6	0.2700	0.3436	0.2064	0.1307	0.2319	0.047±.003	0.2734
100	1≤m≤6	0.3097	0.3537	0.2758	0.1042	0.1908	0.054±.001	0.3467

Table IV.7

Power of testing $H_0 : \rho_{ss} = 0$ when simulation is from T distribution, with df=3 and n=100

m	ρ_{ss}	$Score_K$
1≤m≤6	0	0.0559
	0.1	0.2759
	0.2	0.6874
	0.25	0.8534
	0.3	0.9421

Table IV.8

Size of testing $H_0 : \rho_{ss} = 0$ when simulation is from T distribution, with df=5

n	LRT_N	$Wald_N$	$Score_N$	LRT_K	$Wald_K$	$Score_K$	$Moment$
50	0.1551	0.2216	0.1173	0.1000±.003	0.1859	0.0300±.001	0.1549
100	0.1561	0.1933	0.1411	0.0663±.002	0.1441	0.033±.001	0.1744

Table IV.9

Power of testing $H_0 : \rho_{ss} = 0$ when simulation is from T distribution, with df=5 and n=100

m	ρ_{ss}	LRT_K	$Score_K$
1≤m≤6	0	0.0663	0.0302
	0.1	0.3109	0.2538
	0.2	0.7767	0.7160
	0.25	0.9114	0.8750
	0.3	0.9736	0.9576

Table IV.10

Size of testing $H_0 : \rho_{ss} = 0$ when simulation is from T distribution, with df=10

n	m	LRT_N	$Wald_N$	$Score_N$	LRT_K	$Wald_K$	$Score_K$	$Moment$
50	1≤m≤6	.102	.163	.071±.002	.085±.002	.174	.024±.002	.082
100	1≤m≤6	.088	.119	.078±.001	.057±.001	.127	.025±.00	.09

Table IV.11

Power of testing $H_0 : \rho_{ss} = 0$ when simulation is from T distribution, with df=10 and n=100

m	ρ_{ss}	LRT_K	$Score_K$
1≤m≤6	0	0.0677	0.0309
	0.1	0.3155	0.2561
	0.2	0.7873	0.7274
	0.25	0.9176	0.8830
	0.3	0.9776	0.9628

Table IV.12

Size of testing $H_0 : \rho_{ss} = 0$ when simulation is from normal distribution

n	m	LRT_N	$Wald_N$	$Score_N$	LRT_K	$Wald_K$	$Score_K$	$Moment$
50	1≤m≤6	0.0687	0.1276	0.0445	0.0760	0.1627	0.0191	0.0454
100		0.0553	0.0855	0.0480	0.0515	0.1170	0.0201	0.0452

Table IV.13

Power of testing $H_0 : \rho_{ss} = 0$ when simulation is from normal distribution, n=100

m	ρ_{ss}	LRT_N	$Score_N$	LRT_K	$Score_K$	$Moment$
1≤m≤6	0	0.0553	0.0480	0.0515	0.0201	0.0452
	0.1	0.3442	0.3795	0.3071	0.2440	0.3129
	0.2	0.8314	0.8452	0.7889	0.7331	0.7857
	0.25	0.9479	0.9480	0.9272	0.8891	0.9287

Table IV.14

Size of testing $H_0 : \rho_{ps} = 0$ when simulation is from normal distribution

n	ρ_{ss}	LRT_N	$Wald_N$	$Score_N$	$Moment$
50	0.2	0.059±.001	0.072±.003	0.052±.001	0.05±.003
50	0.5	0.057±.002	0.068±.003	0.052±.001	0.051±.003
100	0.2	0.053±.003	0.058±.002	0.051±.002	0.051±.002
100	0.5	0.053±.003	0.06±.003	0.051±.003	0.051±.004

Table IV.15

Power of testing $H_0 : \rho_{ps} = 0$ when simulation is from normal distribution

n	ρ_{ms}	$Score_N$	$Moment$
100	0	0.0511	0.0518
100	0.1	0.273	0.253
100	0.2	0.7854	0.7357
100	0.25	0.9348	0.9051

Table IV.16

Size of testing $H_0 : \rho_{ps} = 0$ when simulation is from Kotz type distribution

n	ρ_{ss}	LRT_N	$Wald_N$	$Score_N$	$Moment$
50	0.2	0.115±.004	0.116±.004	0.126±.004	0.086±.005
50	0.5	0.116±.006	0.122±.005	0.118±.007	0.099±.007
100	0.2	0.108±.002	0.102±.003	0.118±.002	0.089±.002
100	0.5	0.111±.001	0.112±.001	0.114±.001	0.099±.002

Table IV.17

Size of testing $H_0 : \rho_{ps} = 0$ when simulation is from Kotz type distribution

n	ρ_{ss}	LRT_K	$Wald_K$	$Score_K$
50	0.2	0.056±.002	0.07±.004	0.051±.003
50	0.5	0.056±.004	0.07±.004	0.052±.003
100	0.2	0.052±.002	0.058±.002	0.049±.002
100	0.5	0.052±.002	0.058±.002	0.05±.002

Table IV.18

Power of testing $H_0 : \rho_{ps} = 0$ when simulation is from Kotz type distribution

n	ρ_{ps}	LRT_K	$Score_K$
100	0	0.055	0.051
100	0.1	0.2397	0.2354
100	0.2	0.7185	0.7092
100	0.25	0.8913	0.8863
100	0.3	0.9750	0.9729

Table IV.19

Size of testing $H_0 : \rho_{ps} = 0$ when simulation is from T distribution, with df=5

n	ρ_{ss}	LRT_N	$Wald_N$	$Score_N$	LRT_K	$Wald_K$	$Score_K$	$Moment$
50	.2	.157	.178	.152	.065±.001	.08±.001	.051±.001	.143
50	.5	.158	.177	.15	.061±.003	.082±.003	.0523±.001	.145
100	.2	.166	.176	.169	.059±.002	.073±.004	.051±.002	.159
100	.5	.17	.18	.17	.059±.004	.072±.005	.053±.004	.166

Table IV.20

Power of testing $H_0 : \rho_{ps} = 0$ when simulation is from T distribution, with df=5

n	ρ_{ms}	$Score_K$
100	0	0.0521
100	0.1	0.2245
100	0.2	0.6699
100	0.25	0.8717
100	0.3	0.9661

Table IV.21

Size of testing $H_0 : \rho_{ps} = 0$ when simulation is from T distribution, with df=10

n	ρ_{ss}	LRT_N	$Wald_N$	$Score_N$	LRT_K	$Wald_K$	$Score_K$	$Moment$
50	.2	.092	.11	.086	.052±.003	.074±.004	.04±.002	.08
50	.5	.093	.107	.086	.053±.001	.071±.001	.045±.002	.083
100	.2	.091	.099	.087	.051±.003	.062±.004	.044±.004	.085
100	.5	.087	.095	.084	.047±.001	.056±.002	.043±.001	.084

Table IV.22

Power of testing $H_0 : \rho_{ps} = 0$ when simulation is from T distribution, with df=10

n	ρ_{ms}	LRT_K	$Score_K$
100	0	0.055	0.048
100	0.1	0.235	0.212
100	0.2	0.713	0.686
100	0.25	0.979	0.974

Table IV.23

Size of testing $H_0 : \rho_{ps} = 0$ when simulation is from normal distribution

n	ρ_{ss}	LRT_N	$Wald_N$	$Score_N$	LRT_K	$Wald_K$	$Score_K$	$Moment$
50	.2	.059	.072	.052	.048±.001	.069±.001	.036	.05
50	.5	.057	.068	.052	.044±.001	.061±.002	.038	.051
100	.2	.053	.058	.051	.042±.00	.053±.001	.035	.051
100	.5	.053	.059	.051	.042±.002	.051±.002	.037	.051

Table IV.24

Power of testing $H_0 : \rho_{ps} = 0$ when simulation is from normal distribution

n	ρ_{ms}	$Score_N$	LRT_K	$Score_K$	$Moment$
100	0	0.0511	0.0427	0.0366	0.0518
100	0.1	0.273	0.231	0.211	0.253
100	0.2	0.7854	0.7256	0.6972	0.7357
100	0.25	0.9348	0.9020	0.8855	0.9051

Table IV.25

Size of testing $H_0 : \rho_{ss1} = \rho_{ss1} = \rho_{ss}$ when simulation is from normal distribution, when $\alpha = 0.05$

ρ_{ss}	LRT_N	$Wald_N$	$Score_N$	TS	TS^*
.2	.055±.003	.07±.003	.049±.003	.047±.003	.059±.003
.5	.057±.001	.063±.002	.053±.003	.049±.002	.058±.001
.8	.059±.002	0.042±.001	.053±.003	.0441±.004	.04±.003

Table IV.26

Size of testing $H_0 : \rho_{ss1} = \rho_{ss1} = \rho_{ss}$ when simulation is from Kotz type distribution, when $\alpha = 0.05$

ρ_{ss}	LRT_N	$Wald_N$	$Score_N$	TS	TS^*
.2	.1111	.1146	.1183	.0996	.1168
.5	.1119	.1122	.1139	.0994	.1121
.8	.1177	.092	.1144	.0972	.0887

Table IV.27

Size of testing $H_0 : \rho_{ss1} = \rho_{ss1} = \rho_{ss}$ when simulation is from Kotz type distribution, when $\alpha = 0.05$

ρ_{ss}	LRT_K	$Wald_K$	$Score_K$
.2	.057±.004	.072±.005	.053±.005
.5	.056±.002	.063±.002	.054±.002
.8	.055±.003	.036±.003	.053±.004

Table IV.28

Size of testing $H_0 : \rho_{ss1} = \rho_{ss1} = \rho_{ss}$ when simulation is from T distribution, when $\alpha = 0.05$

ρ_{ss}	LRT_N	$Wald_N$	$Score_N$	LRT_K	$Wald_K$	$Score_K$	TS	TS^*
.2	.1458	.1667	.1352	.064±.002	.09±.002	.049±.002	.1382	.1575
.5	.1473	.1578	.1394	.06±.002	.068±.002	.053±.003	.1369	.1491
.8	.1476	.1185	.1406	.058±.002	.053±.002	.053±.002	.1257	.1147

Table IV.29
Size of testing $H_0 : \rho_{ss1} = \rho_{ss1} = \rho_{ss}$ when simulation is from normal distribution, with $\alpha = 0.05$

ρ_{ss}	LRT_N	$Wald_N$	$Score_N$	LRT_K	$Wald_K$	$Score_K$	TS	TS^*
.2	.055	.07	.049	.044±.002	.066±.003	.033±.001	.047	.059
.5	.057	.063	.053	.042±.002	.05±.001	.036±.002	.049	.058
.8	.059	.042	.053	.04±.003	.024±.003	.036±.003	.0441	.04

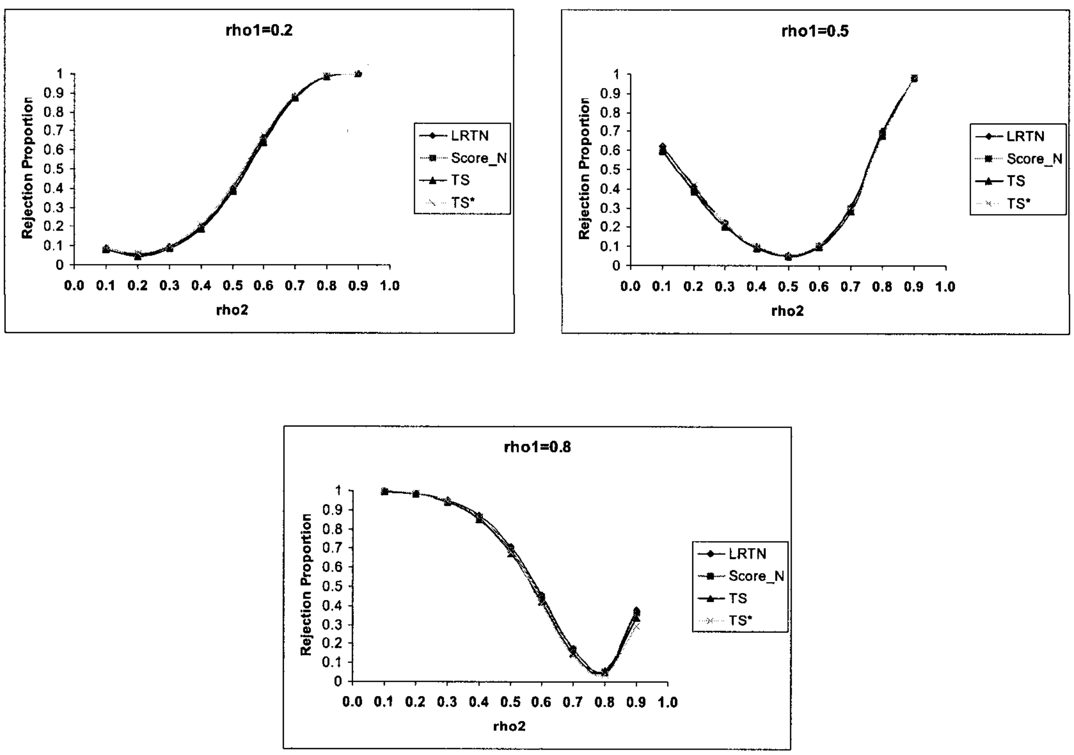


Figure 27. Power estimated with nominal level $\alpha = 0.05$ for testing $H_0 : \rho_{ss1} = \rho_{ss2} = \rho_{ss}$ when simulation is from normal distribution.

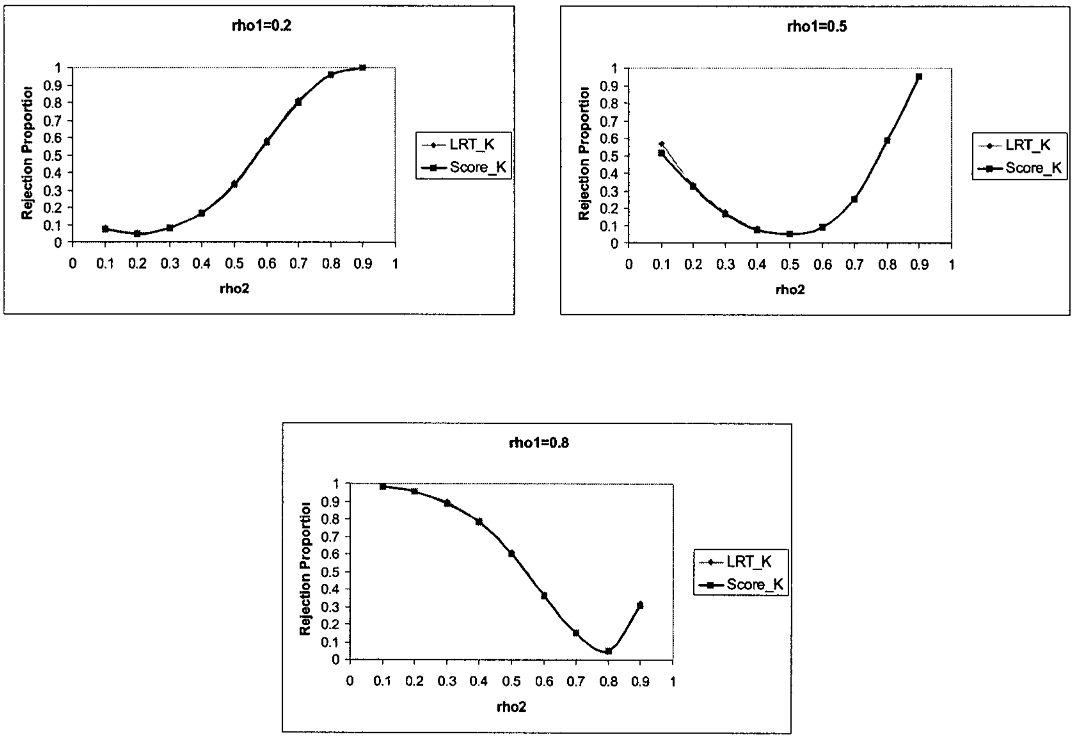


Figure 28. Power estimation using $\alpha = 0.05$ for testing $H_0 : \rho_{ss1} = \rho_{ss2} = \rho_{ss}$ when simulation is from Kotz type distribution.

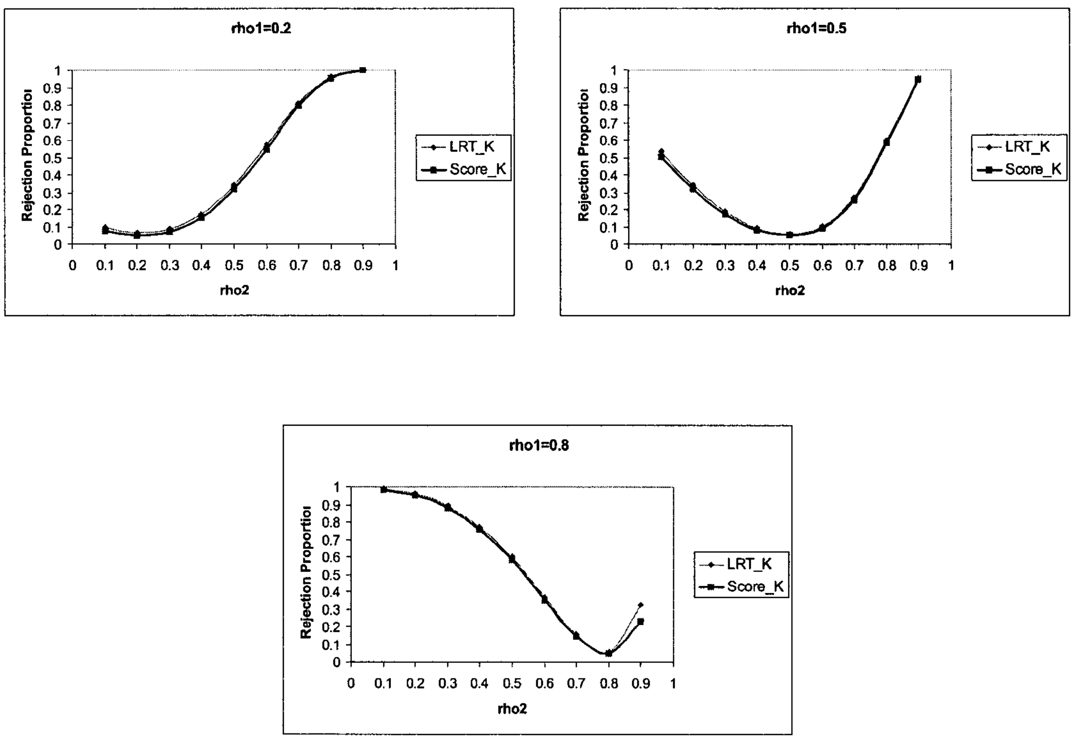


Figure 29. Power estimation using $\alpha = 0.05$ for testing $H_0 : \rho_{ss1} = \rho_{ss2} = \rho_{ss}$ when simulation is from T distribution, with df=5.

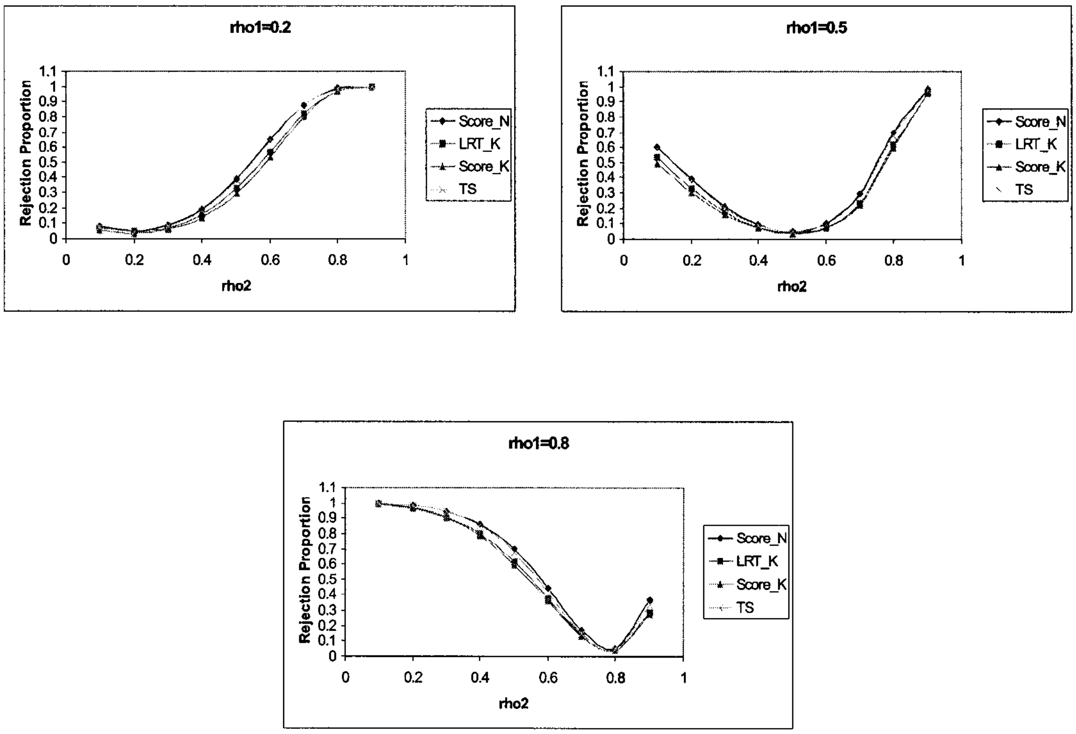


Figure 30. Power estimated with nominal level $\alpha = 0.05$ for testing $H_0 : \rho_{ss1} = \rho_{ss2} = \rho_{ss}$ when simulation is from normal distribution.

Table IV.30

Size of testing $H_0 : \rho_{ps1} = \rho_{ps2} = \rho_{ps}$ when simulation is from normal distribution, with $\alpha = 0.05$

ρ_{ps}	LRT_N	$Wald_N$	$Score_N$	TP	TP^*
.1	.061±.005	.07±.004	.057±.004	.056±.004	.07±.005
.3	.059±.001	.065±.001	.055±.001	.054±.002	.064±.002
.5	.06±.001	.061±.005	.054±.001	.05±.003	.059±.001

Table IV.31

Size of testing $H_0 : \rho_{ps1} = \rho_{ps2} = \rho_{ps}$ when simulation is from Kotz type distribution, with $\alpha = 0.05$

ρ_{ps}	LRT_N	$Wald_N$	$Score_N$	TP	TP^*
.1	.113±.002	.12±.002	.11±.007	.102±.003	.119±.003
.3	.11±.004	.114±.005	.11±.002	.1±.005	.113±.004
.5	.11±.001	.11±.001	.11±.001	.096±.001	.104±.008

Table IV.32

Size of testing $H_0 : \rho_{ps1} = \rho_{ps2} = \rho_{ps}$ when simulation is from Kotz type distribution, with $\alpha = 0.05$

ρ_{ps}	LRT_K	$Wald_K$	$Score_K$
.1	.059±.001	.069±.001	.055±.001
.3	.06±.002	.067±.003	.056±.001
.5	.058±.002	.06±.002	.054±.002

Table IV.33

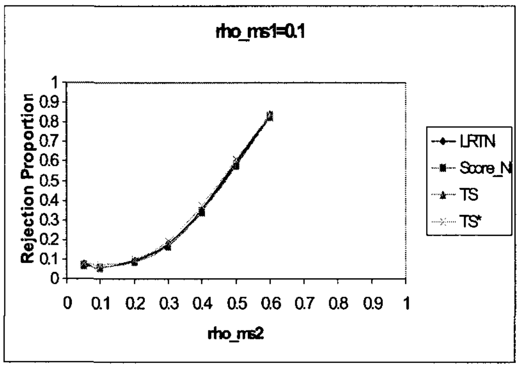
Size of testing $H_0 : \rho_{ps1} = \rho_{ps2} = \rho_{ps}$ when simulation is from T, with df=5 and $\alpha = 0.05$

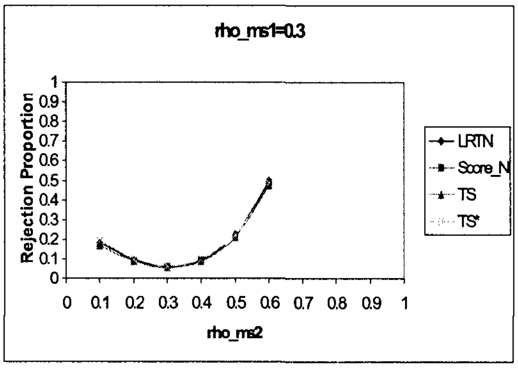
ρ_{ps}	LRT_N	$Wald_N$	$Score_N$	LRT_K	$Wald_K$	$Score_K$	TP	TP^*
.1	.1535	.1656	.1481	.067±.003	.078±.003	.061±.003	.1483	.1638
.3	.1568	.1666	.1513	.065±.003	.073±.003	.059±.002	.1526	.1666
.5	.1549	.1561	.147	.064±.001	.066±.001	.058±.001	.1417	.1507

Table IV.34

Size of testing $H_0 : \rho_{ps1} = \rho_{ps2} = \rho_{ps}$ when simulation is from normal distribution, with $\alpha = 0.05$

ρ_{ps}	LRT_N	$Wald_N$	$Score_N$	LRT_K	$Wald_K$	$Score_K$	TP	TP^*
.1	.061	.07	.057	.049±.004	.058±.003	.044±.004	.056	.07
.3	.059	.065	.055	.045±.002	.051±.002	.041±.002	.054	.064
.5	.06	.061	.054	.046±.001	.05±.001	.041±.001	.05	.059





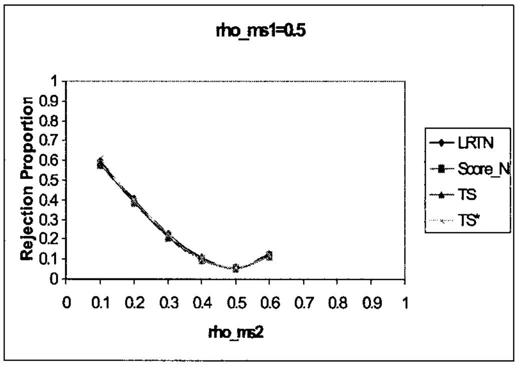


Figure 31. Power estimation of mom-sib using $\alpha = 0.05$ for testing $H_0 : \rho_{ps1} = \rho_{ps2} = \rho_{ps}$ when simulation is from normal distribution.

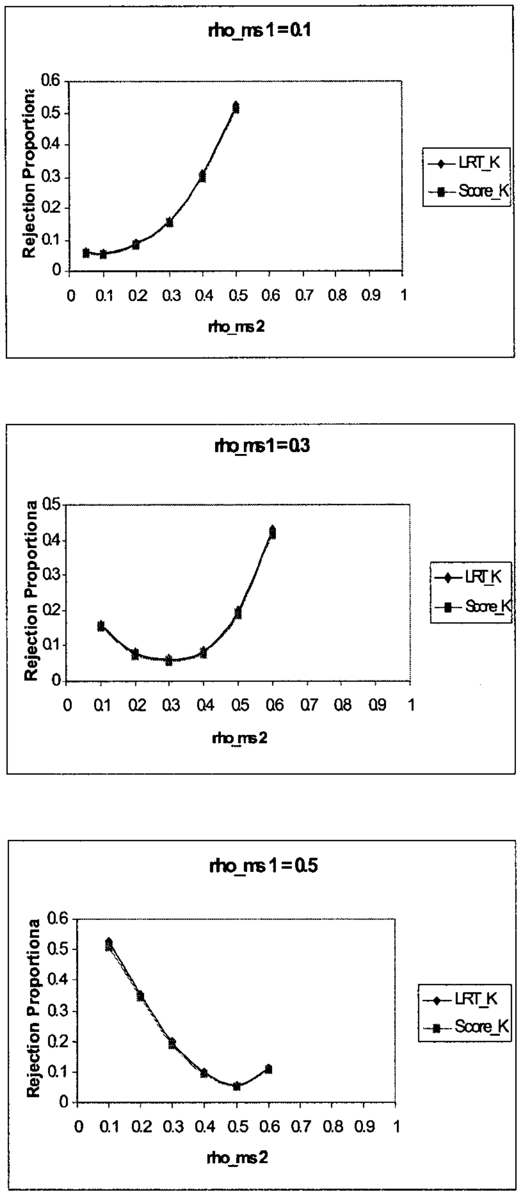


Figure 32. Power estimation of mom-sib using $\alpha = 0.05$ for testing $H_0 : \rho_{ps1} = \rho_{ps2} = \rho_{ps}$ when simulation is from Kotz type distribution.

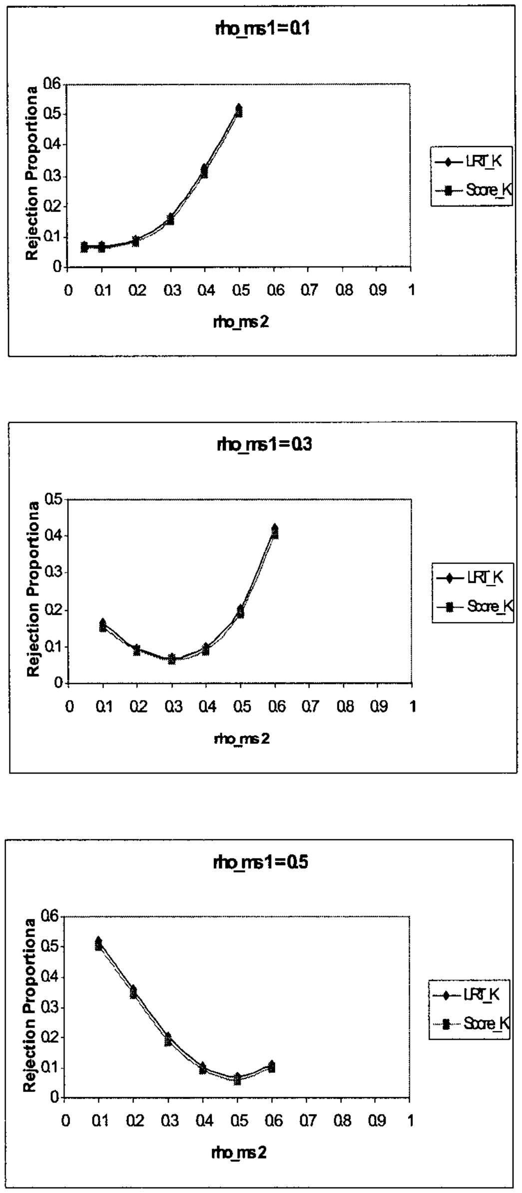
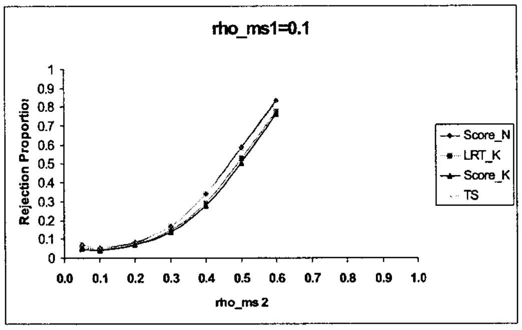
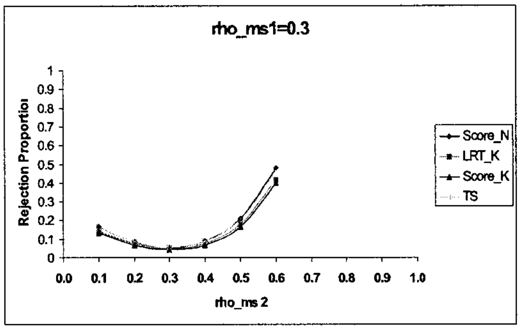


Figure 33. Power estimation of mom-sib using $\alpha = 0.05$ for testing $H_0 : \rho_{ps1} = \rho_{ps2} = \rho_{ps}$ when simulation is from T distribution, with df=5.





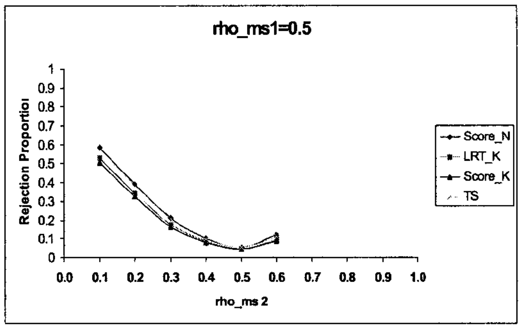


Figure 34. Power estimated with nominal level $\alpha = 0.05$ for testing $H_0 : \rho_{ps1} = \rho_{ps2} = \rho_{ps}$ when simulation is from normal distribution.

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