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Steady incompressible magnetohydrodynamic flow near a point of reattachment

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The oblique stagnation-point flow of an electrically conducting fluid in the presence of a magnetic field is a highly nonlinear problem whose solution is of interest even in the simplest of geometries. The problem models the flow of a viscous conducting fluid near a point where a separation vortex reattaches itself to a rigid boundary. A similarity solution exists which reduces the problem to a coupled system of four ordinary differential equations which can be integrated numerically. The problem has two independent parameters, the conductivity of the fluid and the strength of the magnetic field. Solutions are tabulated for a variety of cases involving the two parameters. The geometry of the flow as well as that of the induced magnetic field is determined near the point of reattachment. © *1998 American Institute of Physics.* [S1070-6631(98)00406-1]

I. INTRODUCTION

One of the oldest similarity solutions to the full Navier–Stokes equations is stagnation-point flow in which a viscous incompressible fluid flows steadily towards a two-dimensional rigid wall. The flow has a centerline of symmetry and the incoming flow along this centerline is at right angles to the wall. A trio of authors (Stuart,[1] Tamada,[2] and Dorrepaal[3]) has shown more recently that by combining traditional stagnation-point flow with a shear flow directed parallel to the wall, one can come up with a similarity solution describing oblique flow towards the wall. The new problem involves a coupled system of nonlinear ordinary differential equations which yield easily to numerical integration. One interesting feature of the equations is the fact that the angle of incidence of the impinging stream can be scaled out of the problem. As a result the equations can be solved independently of this angle and the existence of constants valid for all angles of incidence can be proven.

Magnetohydrodynamic stagnation-point flow was a popular area of investigation approximately 35 years ago. In this class of problems, the fluid is electrically conducting and its motion towards the wall occurs in the presence of a magnetic field. A variety of cases have been treated depending upon how the applied magnetic field is oriented relative to the wall. The early attempts (Meyer,[4] Poots and Sowerby,[5] and Axford[6]) prescribed a constant magnetic field perpendicular to the wall and then later, Gribben[7] considered both the two-dimensional and axisymmetric problems where the magnetic field was parallel to the wall. In all of the cases considered, the fluid flow was directed orthogonally towards the wall.

In the present paper, two magnetohydrodynamic stagnation-point flows are examined. In the first the flow is directed orthogonally towards the wall and because the wall is impenetrable, the streamlines of the undisturbed flow are rectangular hyperbolas just as in the classical problem. Unlike previous magnetohydrodynamic studies, however, an applied magnetic field is prescribed which is perfectly aligned with the undisturbed flow. It is expected, of course, that the no-slip conditions at the wall and the induced magnetic field will disrupt the alignment of the two fields near the wall. But far from the wall, the alignment of the two fields is preserved.

In the second problem, we consider an oblique stagnation-point flow. Once again the applied magnetic field is aligned with the fluid streamlines far from the wall. As in the electrically inert case, the angle which the incident flow makes with the wall can be scaled from the problem. The solution of magnetohydrodynamic stagnation-point flow for arbitrary angles of incidence serves as a model for the flow of an electrically conducting fluid in a magnetic field near a point on a rigid boundary where a region of separated flow reattaches itself to the boundary.

The asymptotic alignment of the velocity and magnetic fields far from the wall is similar to another problem treated some 35 years ago, namely magnetohydrodynamic Blasius flow. In this problem a viscous electrically conducting fluid moves past a semi-infinite flat plate. Far from the plate both the velocity and magnetic fields are uniform and parallel to the plate. A linear version of the problem was first considered by Greenspan and Carrier,[8] and subsequent authors (Glauert,[9] Reuter and Stewartson,[10] Wilson,[11] and Stewartson and Wilson[12]) treated various aspects of the full nonlinear problem. Some of the techniques developed in these papers are useful in analyzing the problems considered here.

II. ORTHOGONAL MAGNETOHYDRODYNAMIC STAGNATION-POINT FLOW

The equations governing the steady magnetohydrodynamic flow of a viscous electrically conducting incompressible fluid are

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} + \frac{\mu}{\rho}(\mathbf{J}\times\mathbf{H}), \tag{1}$$

$$\mathbf{J} = \sigma[\mathbf{E} + \mu(\mathbf{v}\times\mathbf{H})], \tag{2}$$

$$\mathbf{J} = \nabla\times\mathbf{H}, \tag{3}$$

$$\nabla\cdot\mathbf{v} = \nabla\cdot\mathbf{H} = 0, \tag{4}$$

$$\nabla\times\mathbf{E} = \mathbf{0}. \tag{5}$$

The position-dependent quantities are defined as follows: $\mathbf{v}$, fluid velocity; p, fluid pressure; $\mathbf{H}$, magnetic field intensity; $\mathbf{E}$, electric field intensity; and $\mathbf{J}$, current density. The constants are ρ, fluid density; ν, kinematic viscosity; μ, magnetic permeability; and σ, electrical conductivity.

The standard two-dimensional stagnation-point geometry will be assumed; viz., the fluid velocity $\mathbf{v}$ and the magnetic field $\mathbf{H}$ are each perpendicular to the z direction. A rigid wall lies in the plane $y=0$ and the fluid occupies the half space $y>0$. Far from the wall, the flow is directed towards the wall and follows hyperbolic streamlines. The magnetic field lines are aligned with the flow when $y \gg 1$.

One of the consequences of the two-dimensionality of the flow is that the current density $\mathbf{J}$ and the cross product $\mathbf{v}\times\mathbf{H}$ in Eq. (2) are both directed perpendicular to the plane of the flow. It follows that $\mathbf{E}$ is in the z direction. This coupled with equation (5) implies that $\mathbf{E} = -E_0\hat{k}$ where E_0 is constant. Following Greenspan and Carrier,[8] we take $E_0 = 0$. The flow which we obtain corresponds to a stagnation-point flow between two plates at $z = \pm z_0$, ($z_0 \gg 1$) which are perfectly conducting and joined together by a wire of zero resistance. There is no potential difference between the plates and therefore no electric field affecting the flow.

The solenoidal nature of both the velocity and magnetic fields permits the definition of scalar potential functions for each field as follows:

$$\mathbf{v} = \nabla\times\{\psi(x,y)\hat{k}\}, \quad \mathbf{H} = \nabla\times\{\phi(x,y)\hat{k}\}. \tag{6}$$

When the current density is eliminated between Eqs. (2) and (3), we obtain the magnetic diffusion equation

$$\eta(\nabla\times\mathbf{H}) = \mathbf{v}\times\mathbf{H}, \tag{7}$$

where $\eta = 1/\sigma\mu$ is the magnetic diffusivity (or magnetic viscosity) of the fluid. The scalar version of (7) is found by substituting (6) into (7) to obtain

$$\eta\nabla^2\phi + \frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial x} = 0. \tag{8}$$

A second equation relating the two scalar potentials is found by eliminating $\mathbf{J}$ between Eqs. (1) and (3) and then taking the curl to eliminate the pressure term. After considerable simplification, the momentum equation appears as follows:

$$\nu\nabla^4\psi + \frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x} - \frac{\mu}{\rho}\left[\frac{\partial\phi}{\partial x}\frac{\partial\nabla^2\phi}{\partial y} - \frac{\partial\phi}{\partial y}\frac{\partial\nabla^2\phi}{\partial x}\right] = 0. \tag{9}$$

The no-slip conditions at the wall translate into boundary conditions on the stream function; viz., $\psi(x,0) = \psi_y(x,0) = 0$. In addition, the normal component of $\mu\mathbf{H}$ must be continuous across the boundary $y=0$. We assume the magnetic field in the fluid and the induced field in the solid region $y<0$ are both parallel to the wall when $|y| \ll 1$. This is consistent with the fact that the applied magnetic field in the fluid has hyperbolic field lines which asymptote to the wall as $|x| \to \infty$. As a result, the normal component of the magnetic field must vanish at the wall which means $\phi(x,0) = 0$. Far from the wall, the velocity and magnetic fields have the form

$$\mathbf{v} \sim \gamma(x\hat{i} - y\hat{j}), \quad \mathbf{H} \sim H_\infty(x\hat{i} - y\hat{j}), \tag{10}$$

where γ and H_∞ are dimensional constants of proportionality having units $(\text{time})^{-1}$ and $(\text{charge})\cdot(\text{length})^{-1}\cdot(\text{time})^{-1}$, respectively.

Equations (8) and (9) can be nondimensionalized using $(\nu/\gamma)^{1/2}$ as the length scale. If nondimensional variables are denoted by bars, then

$$x = \left(\frac{\nu}{\gamma}\right)^{1/2}\bar{x}, \quad y = \left(\frac{\nu}{\gamma}\right)^{1/2}\bar{y}, \quad \psi = \nu\bar{\psi}, \quad \phi = \frac{H_\infty\nu}{\gamma}\bar{\phi}. \tag{11}$$

After dropping the bars, the two equations of motion for $\psi(x,y)$ and $\phi(x,y)$ are

$$\nabla^4\psi + \frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x} - \beta\left\{\frac{\partial\phi}{\partial x}\frac{\partial\nabla^2\phi}{\partial y} - \frac{\partial\phi}{\partial y}\frac{\partial\nabla^2\phi}{\partial x}\right\} = 0, \tag{12}$$

$$\nabla^2\phi + \epsilon\left\{\frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial x}\right\} = 0, \tag{13}$$

where $\beta = \mu H_\infty^2/\rho\gamma^2$ is the square of the ratio of the Alfvén velocity to the fluid velocity far from the wall, and $\epsilon = \nu/\eta = \nu\sigma\mu$ is the magnetic Prandtl number. Reuter and Stewartson[10] have shown that no solution exists in magnetohydrodynamic Blasius flow when $\beta > 1$. A similar result holds for the present problem.

The well-known similarity solution for viscous two-dimensional stagnation-point flow can be modified to give a similarity solution to the system of equations given in (12) and (13). We let

$$\psi(x,y) = xf(y), \quad \phi(x,y) = xg(y) \tag{14}$$

and substitute. The resulting system of differential equations is as follows:

$$f''' + ff'' - (f')^2 + 1 - \beta\{gg'' - (g')^2 + 1\} = 0, \tag{15}$$

$$g'' + \epsilon\{fg' - f'g\} = 0, \tag{16}$$

where $f(0) = f'(0) = g(0) = 0$, and $f'(\infty) = g'(\infty) = 1$.

The challenge is to find values for $f''(0)$ and $g'(0)$ which, when combined with the boundary conditions at $y=0$, give a solution to the system of equations (15) and (16) which satisfies the conditions at infinity. The difficulties posed by a simultaneous search for $f''(0)$ and $g'(0)$ in a coupled system can be averted, however, by using the same

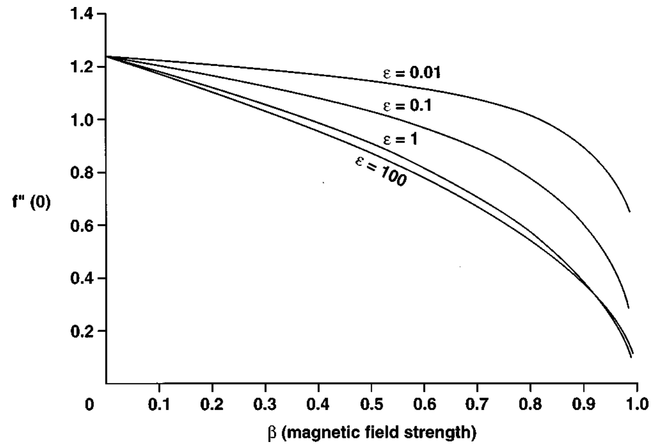

FIG. 1. Plot of $f''(0)$ vs β for various values of ϵ. When $\beta=0$, $f''(0)=1.232\,588$.

transformation as that employed by Wilson[11] in the Blasius problem. We define $F(\eta)$ and $G(\eta)$ as follows:

$$F(\eta)=A^{1/2}f(A^{1/2}\eta), \quad G(\eta)=A^{-1/2}Bg(A^{1/2}\eta), \tag{17}$$

where A and B are constants. The conditions at infinity on f and g indicate that $A=\lim_{\eta\to\infty} F'(\eta)$ and $B=\lim_{\eta\to\infty} G'(\eta)$, which suggests that A and B are proportional to fluid velocity and Alfvén velocity, respectively, at infinity. It follows that

$$\beta=\frac{B^2}{A^2}. \tag{18}$$

When these relations are substituted into (15) and (16), the resulting system of equations for $F(\eta)$ and $G(\eta)$ is given by

$$F'''+FF''-(F')^2-[GG''-(G')^2]+K=0, \tag{19}$$

$$G''+\epsilon[FG'-F'G]=0, \tag{20}$$

where $K=A^2-B^2$. The boundary conditions at the wall ($\eta=0$) are given by

$$F(0)=F'(0)=G(0)=0, \quad G'(0)=1. \tag{21}$$

The value of $G'(0)$, taken here to be 1, can be chosen arbitrarily because of the linearity of Eq. (20) with respect to $G(\eta)$. Fixing the parameters K and ϵ, a shooting method is used to find the value of $F''(0)=C$ which makes $F'(\eta)$ constant as $\eta\to\infty$. This constant, of course, is A. For large η, Eq. (19) simplifies to

$$GG''-(G')^2+B^2=0, \tag{22}$$

whose solution is $G'(\eta)=B$. Therefore the numerical integration of Eqs. (19) and (20) using (21) and $F''(0)=C$ as initial conditions yields values for the constants A, B, and β [from (18)]. The corresponding solution to Eqs. (15) and (16) is found from (17). In particular, we have

$$f''(0)=C/A^{3/2}, \quad g'(0)=1/B. \tag{23}$$

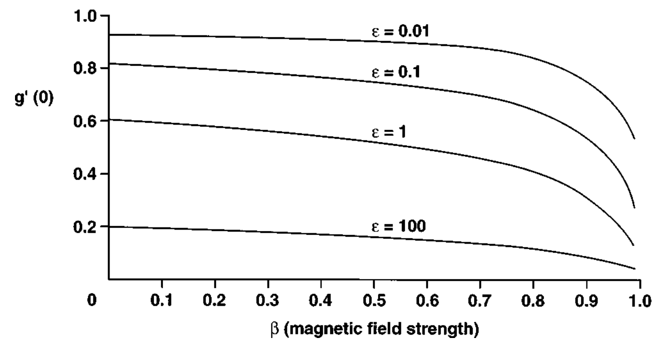

FIG. 2. Plot of $g'(0)$ vs β for various values of ϵ.

Even though the procedure gives β only after K and ϵ are fixed, it can be modified to provide desired values of β by iterating on K.

Results are presented in Figs. 1 and 2. Figure 1 is a plot of $f''(0)$ vs β for various values of ϵ. Figure 2 gives similar plots for $g'(0)$ vs β.

The conditions on f and g at infinity indicate that

$$f,g\sim y-b \quad \text{as } y\to\infty, \tag{24}$$

where b is displacement thickness. From (17), we have

$$b=A^{-1/2}\lim_{\eta\to\infty}[A\eta-F(\eta)]. \tag{25}$$

Figure 3 is a plot of b vs β for various values of ϵ.

III. DISCUSSION OF THE ORTHOGONAL FLOW

The orthogonal problem was solved for a full range of ϵ values and for $0\leq\beta<1$. The parameter ϵ is proportional to fluid conductivity and β is a measure of the strength of the applied magnetic field. The nonexistence of a solution for $\beta>1$ is traceable to the fact that when $\beta>1$, disturbances are no longer contained within a boundary layer along the wall, but can travel an infinite distance away from the wall. This means that boundary conditions can no longer be prescribed at infinity and the problem therefore becomes ill-

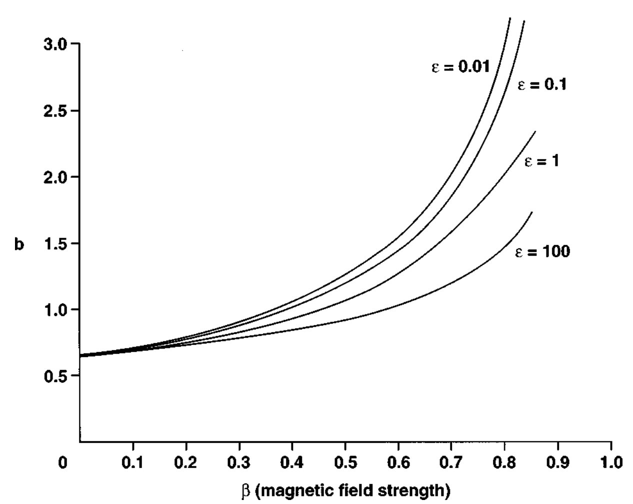

FIG. 3. Plot of displacement thickness b vs β for various values of ϵ. When $\beta=0$, $b=0.647\,900$.

posed. When $\beta<1$, the disturbances are contained within a boundary layer and the problem is well defined. What is interesting are the predictions of the equations as $\beta \to 1^-$.

The simplest way to study this limit is to consider the special case when the fluid is a perfect conductor $(\epsilon=+\infty)$. When ϵ is infinite, Eq. (16) together with the boundary conditions indicates that $f(y)\equiv g(y)$. Equation (15) then becomes

$$f''' + (1-\beta)\{ff'' - (f')^2 + 1\} = 0, \tag{26}$$

whose solution is $f(y)=(1-\beta)^{-1/2}H[(1-\beta)^{1/2}y]$ where $H(\eta)$ is the classical Hiemenz function defined by

$$H''' + HH'' - (H')^2 + 1 = 0 \tag{27}$$

with boundary conditions $H(0)=H'(0)=0$, $H'(\infty)=1$.

Rosenhead[13] has shown that for small η,

$$H(\eta) = \tfrac{1}{2}C\eta^2 - \tfrac{1}{6}\eta^3 + O(\eta^5), \tag{28}$$

where $C=1.232\,588$, and that for large η,

$$H(\eta) \sim \eta - c + O\{(\eta-c)^{-4}\exp[-\tfrac{1}{2}(\eta-c)^2]\}, \tag{29}$$

where $c=0.647\,900$.

As $\beta\to 1^-$, the functions $f(y)$ and $g(y)$ take on the form $f(y)=g(y)\sim\frac{1}{2}C(1-\beta)^{1/2}y^2$ and both vanish in the limit. As the applied magnetic field is strengthened therefore, the flow is totally brought to rest. A similar result is obtained in Blasius flow. Greenspan and Carrier[8] explain that by increasing β, the induced current in the z direction produces a strengthening countermagnetic field which ultimately plugs the flow and annuls the applied magnetic field.

Even when the fluid is not a perfect conductor, the results in Figs. 1 and 2 indicate that a similar phenomenon occurs as $\beta\to 1^-$. The decrease in the values of $f''(0)$ and $g'(0)$ is moderate when β is small, but steepens dramatically when β is very close to 1.

Figure 3 indicates that the strengthening of the magnetic field as $\beta\to 1^-$ is also accompanied by a significant increase in displacement thickness which of course is proportional to boundary layer thickness. In fact, when ϵ is infinite, we have from (29) that

$$f,g \sim y - c(1-\beta)^{-1/2} \quad \text{as} \quad y\to\infty. \tag{30}$$

In a perfectly conducting fluid, therefore, displacement thickness becomes infinite as $\beta\to 1^-$.

IV. OBLIQUE MAGNETOHYDRODYNAMIC STAGNATION-POINT FLOW

The oblique flow problem involves Eqs. (1)–(5) with the velocity and magnetic fields again represented by (6). The stream function $\psi(x,y)$ and the magnetic potential $\phi(x,y)$ satisfy the same homogeneous conditions at $y=0$ as before. The difference between the orthogonal and oblique problems lies in the prescribed boundary conditions at infinity. In the orthogonal case we had

$$\left.\begin{aligned}\psi(x,y) &\sim \gamma xy\\ \phi(x,y) &\sim H_\infty xy\end{aligned}\right\} \text{as} \quad y\to\infty, \tag{31}$$

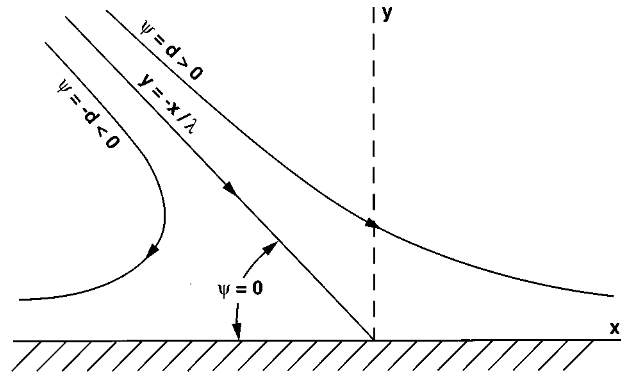

FIG. 4. The undisturbed oblique flow. The dividing streamline has equation $y=-x/\lambda$.

where γ and H_∞ are the dimensional proportionality constants referred to in Eq. (10). In the oblique problem, the far-field boundary conditions are given by

$$\left.\begin{aligned}\psi(x,y) &\sim \gamma(xy+\lambda y^2)\\ \phi(x,y) &\sim H_\infty(xy+\lambda y^2)\end{aligned}\right\} \text{as} \quad y\to\infty, \tag{32}$$

where λ is a dimensionless constant related to the angle of incidence of the impinging stream. Figure 4 gives a picture of the undisturbed oblique flow as well as the undisturbed magnetic field lines, the assumption being that the two are aligned. From Eq. (32) it is clear that the undisturbed streamline $\psi(x,y)=0$ consists of two straight lines; the wall $y=0$ and the dividing streamline $x+\lambda y=0$. The slope of the dividing streamline far from the wall, therefore, is $m_\infty = -1/\lambda$. The same is true of the undisturbed magnetic field line $\phi(x,y)=0$.

From Eq. (6) the magnetic field intensity vector $\mathbf{H}(x,y)$ is related to the magnetic potential function $\phi(x,y)$ via

$$\mathbf{H} = \nabla\times\{\phi(x,y)\hat{k}\}. \tag{33}$$

It follows from Eq. (3) that the current density is given by

$$\mathbf{J} = \nabla\times\mathbf{H} = -\hat{k}\nabla^2\phi, \tag{34}$$

which in the case of the applied magnetic field [Eq. (32)] reduces to

$$\mathbf{J} = -2\lambda H_\infty\hat{k}. \tag{35}$$

Thus the current density far from the wall in oblique stagnation-point flow is nonzero.

This is in marked contrast to orthogonal flow where the current density far from the wall was zero. As a result we were able to take $\mathbf{E}=\mathbf{0}$ in Eq. (2). In oblique flow the presence of nonzero current density requires the existence of an electric field in the z direction to support the applied magnetic field in the xy plane. The electric field is constant and has the form $\mathbf{E}=-E_0\hat{k}$. Its strength is, from (35),

$$E_0 = \frac{2\lambda H_\infty}{\sigma}, \tag{36}$$

where σ is electrical conductivity and λ is the angle of incidence constant. Conditions for such a flow can be established by maintaining an appropriate potential difference across plates situated at $z=\pm z_0$ where $z_0\gg 1$.

The existence of an applied electric field perpendicular to the plane of the flow has a modifying effect on the magnetic diffusion equation. When the current density is eliminated between Eqs. (2) and (3), and when $\mathbf{E}$ is replaced by $-E_0\hat{k}$, we obtain

$$\nabla\times\mathbf{H}=-2\lambda H_\infty\hat{k}+\frac{1}{\eta}(\mathbf{v}\times\mathbf{H}), \tag{37}$$

where $\eta=1/\sigma\mu$ is the magnetic diffusivity of the fluid. After substituting (6) into (37) and simplifying, we have

$$\nabla^2\phi+\frac{1}{\eta}\left[\frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial x}\right]=2\lambda H_\infty. \tag{38}$$

The momentum equation, on the other hand, has exactly the same form as that given in (9).

We nondimensionalize the two governing equations as well as the boundary conditions exactly as before [see Eq. (11)]. The defining equations for oblique magnetohydrodynamic stagnation-point flow are therefore given by

$$\nabla^4\psi+\frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x}$$
$$-\beta\left\{\frac{\partial\phi}{\partial x}\frac{\partial\nabla^2\phi}{\partial y}-\frac{\partial\phi}{\partial y}\frac{\partial\nabla^2\phi}{\partial x}\right\}=0, \tag{39}$$

$$\nabla^2\phi+\epsilon\left\{\frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial x}\right\}=2\lambda, \tag{40}$$

where β and ϵ are defined in Eqs. (12) and (13). The nondimensionalized far-field boundary conditions take the form

$$\left.\begin{matrix}\psi(x,y)\\ \phi(x,y)\end{matrix}\right\}\sim xy+\lambda y^2 \quad \text{as } y\to\infty. \tag{41}$$

V. A SIMILARITY SOLUTION FOR OBLIQUE FLOW

Equation (41) suggests a similarity solution for oblique flow having the form

$$\begin{aligned}\psi(x,y)&=xf(y)+2\lambda r(y),\\ \phi(x,y)&=xg(y)+2\lambda s(y).\end{aligned} \tag{42}$$

Upon substituting these into the governing equations (39) and (40), we obtain exactly the same two equations (15) and (16) for $f(y)$ and $g(y)$ as we had in the orthogonal problem. The functions $f(y)$ and $g(y)$, therefore, act as the orthogonal components of the full solution. The functions $r(y)$ and $s(y)$ constitute the components of the solution parallel to the wall. In fact, the function $r(y)$ behaves like a shear flow with a shear rate close to the wall which differs from its shear rate at infinity. In the same way, $s(y)$ corresponds to a magnetic field having a linear shear profile. The governing equations for $r(y)$ and $s(y)$ are given by

$$r'''+fr''-f'r'-\beta\{gs''-g's'\}+(1-\beta)b=0, \tag{43}$$

$$s''+\epsilon\{fs'-gr'\}=1, \tag{44}$$

where $r(0)=r'(0)=s(0)=s'(0)=0$ and $r''(\infty)=1$.

The constant term in Eq. (43) contains the displacement thickness b discussed in Secs. II and III. The substitution of (42) into Eq. (39) results in a fourth-order differential equation for $r(y)$ which is integrable. The integration reduces the equation to third order and the evaluation of the constant of integration follows from the far-field behaviors of the various functions. We use

$$\left.\begin{matrix}f,g\sim y-b+\text{exp. small terms}\\ r,s\sim\frac{1}{2}y^2+a+\text{exp. small terms}\end{matrix}\right\}\text{as } y\to\infty, \tag{45}$$

where a is constant. The value of a varies with the parameters (ϵ,β) but is only of numerical interest and does not affect the asymptotics of Eqs. (43) and (44).

In order to facilitate the numerical solution of (43) and (44), we introduce a transformation similar to that employed in Sec. II. We define

$$R(\eta)=A^{-1}r(A^{1/2}\eta),\quad S(\eta)=A^{-2}Bs(A^{1/2}\eta), \tag{46}$$

where A and B are the same constants used in Eq. (17). When these relations are substituted into Eqs. (43) and (44), we obtain the following system of equations for $R(\eta)$ and $S(\eta)$:

$$R'''+FR''-F'R'-\{GS''-G'S'\}+bA^{-3/2}K=0, \tag{47}$$

$$S''+\epsilon\{FS'-GR'\}=B/A, \tag{48}$$

where $K=A^2-B^2$ as before. The boundary conditions transform to

$$\begin{aligned}&R(0)=R'(0)=S(0)=S'(0)=0,\\ &R''(\infty)=1.\end{aligned} \tag{49}$$

The numerical solution of the oblique problem proceeds as follows. The parameters K and ϵ are fixed and Eqs. (19) and (20) are solved subject to boundary conditions (21) to obtain the orthogonal components $F(\eta)$ and $G(\eta)$ of the full solution. From these the values of the constants A, B, b and $\beta=B^2/A^2$ are determined. If a particular value of β is desired, a feedback mechanism is set up to repeatedly adjust K until the desired value of β is obtained to within suitable tolerances. Once the orthogonal flow has been fully determined, the parameters ϵ, K, A, B, b are all substituted into Eqs. (47) and (48). The coefficient functions $F(\eta)$ and $G(\eta)$ are known and the system becomes a nonhomogeneous system of linear equations for $R'(\eta)$ and $S'(\eta)$. Using shooting methods we find the value of $R''(0)$ which corresponds to $R''(\infty)=1$. The solution of Eqs. (43) and (44) is obtained from transformation (46) and we find, in particular, that $r''(0)=R''(0)$. Figure 5 is a plot of $r''(0)$ vs β for various values of ϵ.

When $\beta=0$, there is no magnetic field and the solution reverts back to the electrically inert case treated by Dorrepaal[3] where $r''(0)=1.406\,544$. On the other hand as $\epsilon\to 0$, the fluid ceases to conduct electricity and therefore does not respond to a magnetic field. Once again the system reverts to the electrically inert case and this is evidenced by the approach of $r''(0)$ to the limiting value 1.406 544. Figure 5 reveals, however, that for finite conductivities, the dependence of $r''(0)$ on β can exhibit different behaviors depending upon the value of ϵ. When ϵ is small, increasing β causes $r''(0)$ to steadily decrease. When $\epsilon=1$, $r''(0)$ increases to

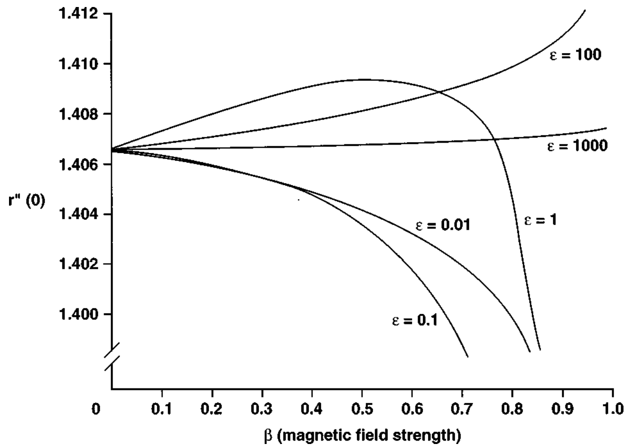

FIG. 5. Plot of $r''(0)$ vs β for various values of ϵ. When $\beta=0$, $r''(0)=1.406\,544$.

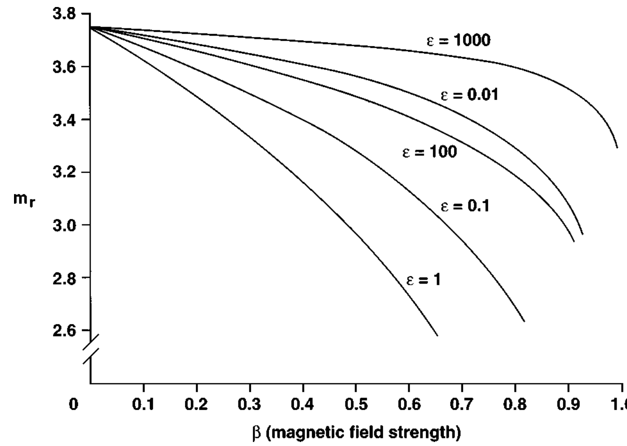

FIG. 6. Plot of slope ratio m_r vs β for various values of ϵ. When $\beta=0$, $m_r=3.748\,513$.

some maximum value and then decreases as the magnetic field intensifies. When ϵ is large, $r''(0)$ increases with β over its entire range.

In the limiting case when the fluid becomes a perfect conductor ($\epsilon \to +\infty$), it is known from Sec. III that

$$f(y)=g(y)=(1-\beta)^{-1/2}H[(1-\beta)^{1/2}y], \tag{50}$$

where $H(\eta)$ is the Hiemenz function. It follows from Eq. (44) that $s'(y)=r'(y)$. Equation (43) then simplifies as follows:

$$r'''+(1-\beta)\{fr''-f'r'+b\}=0, \tag{51}$$

where $b=0.647\,900(1-\beta)^{-1/2}$.

The coefficient $(1-\beta)$ can be scaled out of Eq. (51) using the transformation

$$r(y)=(1-\beta)^{-1}D[(1-\beta)^{1/2}y]. \tag{52}$$

This coupled with (50) reduces (51) to the equation

$$D'''+HD''-H'D'+0.647\,900=0, \tag{53}$$

which is treated in Ref. 3. There Dorrepaal shows that $D''(0)=r''(0)=1.406\,544$ irrespective of the value of β. The data in Fig. 5 bears this out with the values of $r''(0)$ on the curve $\epsilon=1000$ being uniformly close to 1.406 544.

VI. BEHAVIOR OF THE FLOW NEAR THE WALL

One of the major advantages of having the exact solution to a magnetohydrodynamic problem of this complexity is that the fluid streamlines and the magnetic field lines near the wall can be readily obtained and analyzed. We simply need the Maclaurin series for the four basic functions in Eq. (42). These are easily obtained from the respective differential equations. We find that

$$f(y)=\tfrac{1}{2}f''(0)y^2-\tfrac{1}{6}\{(1-\beta)+\beta g'(0)^2\}y^3+O(y^5), \tag{54}$$

$$r(y)=\tfrac{1}{2}r''(0)y^2-\tfrac{1}{6}(1-\beta)by^3+O(y^5), \tag{55}$$

$$g(y)=g'(0)y+\tfrac{1}{24}\epsilon f''(0)g'(0)y^4+O(y^5), \tag{56}$$

$$s(y)=\tfrac{1}{2}y^2+\tfrac{1}{12}\epsilon r''(0)g'(0)y^4+O(y^5). \tag{57}$$

When expansions (54) and (55) are substituted into the expression for the stream function in (42), we have

$$\psi(x,y)=\lambda r''(0)y^2+\tfrac{1}{2}f''(0)xy^2-\tfrac{1}{3}\lambda(1-\beta)by^3 -\tfrac{1}{6}\{(1-\beta)+\beta g'(0)^2\}xy^3+O(y^5). \tag{58}$$

In the vicinity of the origin ($x=0$), the flow is a linear shear. Separation occurs where the tangential stress along the wall vanishes [$\psi_{yy}(x,0)=0$]. From Eq. (58), the dividing streamline meets the wall at

$$x=-2\lambda r''(0)/f''(0). \tag{59}$$

If we define a new horizontal coordinate $X=x+2\lambda r''(0)/f''(0)$ to be centered at this separation point, then the stream function in (58) can be written in the form

$$\psi(X,y)=Ly^2\{y+MX+O(Xy)\}, \tag{60}$$

where

$$L=\tfrac{1}{3}P/f''(0),$$

$$M=\tfrac{3}{2}f''(0)^2/P,$$

$$P=\lambda(1-\beta)\{r''(0)-bf''(0)\}+\lambda\beta r''(0)g'(0)^2.$$

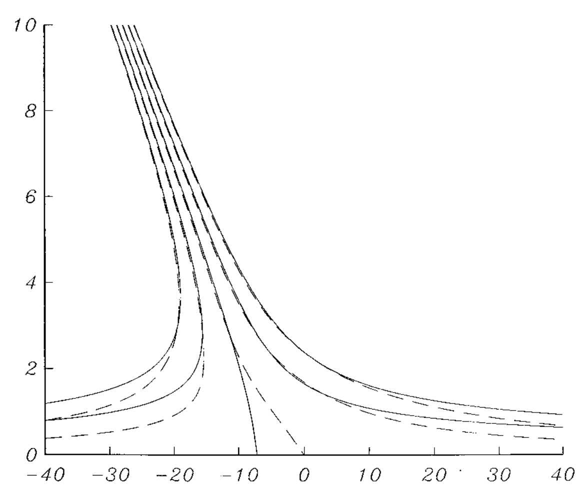

FIG. 7. Graph of streamlines (solid) and magnetic field lines (dashed) near wall for $\epsilon=1$, $\beta=0.4$.

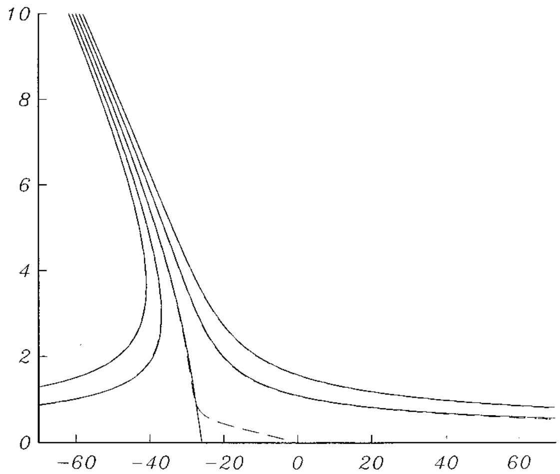

FIG. 8. Graph of streamlines (solid) and magnetic field lines (dashed) near wall for $\epsilon=100$, $\beta=0.8$.

From (60), the slope of the dividing streamline ($\psi=0$) at the wall is $m_w=-M$.

In a previous paper,[3] it has been shown that in the flow of an electrically inert Newtonian fluid, the ratio of the slope of the dividing streamline at the wall to its slope at infinity is independent of λ, the angle of incidence constant. The same is true here. The slope ratio for oblique magnetohydrodynamic stagnation-point flow is given by

$$m_r=m_w/m_\infty=\tfrac{3}{2}f''(0)^2/[(1-\beta)\{r''(0)-bf''(0)\} + \beta r''(0)g'(0)^2]. \tag{61}$$

The variations of m_r with respect to the two parameters (ϵ,β) are depicted in Fig. 6. The slope ratios never exceed the electrically inert value of 3.748 513 and, for any particular β, they appear to reach a minimum at $\epsilon\approx 1$.

When expansions (56) and (57) are substituted into the expression for magnetic potential in (42), we obtain

$$\phi(x,y)=\lambda y\{y+g'(0)x/\lambda+O(y^3)\}. \tag{62}$$

The dividing magnetic field line $\phi=0$ comes into the wall at the origin having a slope of $n_w=-g'(0)/\lambda$. The magnetic slope ratio is therefore

$$n_r=n_w/m_\infty=g'(0), \tag{63}$$

which from Fig. 2 is always less than unity.

Figures 7 and 8 give streamlines and magnetic field lines for the cases $\epsilon=1$, $\beta=0.4$ and $\epsilon=100$, $\beta=0.8$, respectively. The alignment of the two fields for y large is evident. In fact, Fig. 8 reveals that when a strong magnetic field is applied to a highly conducting fluid, the aligning of the two fields occurs very rapidly.

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