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Running Coupling Constant and Transition From Low to High Energies in Quantum Chromodynamics

Alexander Babansky

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RUNNING COUPLING CONSTANT AND TRANSITION
FROM LOW TO HIGH ENERGIES IN QUANTUM
CHROMODYNAMICS

by

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M.S. December 2005. Old Dominion University

A Dissertation Submitted to the Faculty of
Old Dominion University in Partial Fulfillment of the
Requirement for the Degree of

DOCTOR OF PHILOSOPHY

PHYSICS

OLD DOMINION UNIVERSITY
May 2006

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ABSTRACT

RUNNING COUPLING CONSTANT AND TRANSITION FROM LOW TO HIGH ENERGIES IN QUANTUM CHROMODYNAMICS

Alexander Babansky
Old Dominion University. 2006
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The remarkable feature of QCD is that the coupling constant (the strength of the interaction) depends on the energy of the interacting particles. At high energies, the coupling constant vanishes and QCD becomes asymptotically free. However, at lower energy, the running coupling constant increases, leading to the confining strong-coupling theory at energies of the order of hadron masses. We study the essential effects of the running coupling constant and analyze the transition from high to low energies in different processes.
ACKNOWLEDGMENTS

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CHAPTER I
INTRODUCTION

The early attempts in the 1950's to construct field theories of the strong interactions were total failures. In the late 1950's, Landau's theoretical group studied the high energy behavior of quantum electrodynamics. They found the relation between the physical electric charge \( e_R \) and the bare electric charge \( e_0 \) that controls the physics at the ultraviolet cutoff energies \( \Lambda \) [1]:

\[
\frac{1}{e_R^2} - \frac{1}{e_0^2} = \frac{N_f}{6\pi^2} \ln \frac{\Lambda}{m_R},
\]

where \( m_R \) is the renormalized electron mass and \( N_f \) is the number of flavors.

This equation has two obvious implications:
1) Zero charge problem: the physical charge vanishes \( (e_R \to 0) \), for a fixed value of the bare charge, as we let the ultraviolet cutoff become infinitely large \( (\Lambda \to \infty) \).
2) Landau poles problem: the bare charge becomes singular \( (e_0 \to \infty) \), for a fixed value of the physical charge, at \( \Lambda_{\text{Landau}} = m_R \exp[6\pi^2/(N_f e_R)] \).

Landau, in his last paper [2], even argued that quantum field theory had been nullified by the discovery of the zero charge problem. Nowadays, we understand that this problem occurs in any non-asymptotically free theory, such as quantum electrodynamics (QED), which is inconsistent at high energies. For any non-asymptotically free field theory of strong interaction Landau's poles lead to an immediate catastrophe.

The biggest advance in theoretical physics of the early 1960's was the discovery of an approximate symmetry of hadrons, namely \( SU(3) \), by Murray Gell-Mann and Yuval Neeman. In 1963 Gell-Mann and George Zweig argued that structure of the groups could be explained by the existence of three flavors of quarks [3]. Between 1970 and 1973, through theories and experiments, physicists created the Standard Model of particle physics. It is a fundamental theory that describes the strong, weak and electromagnetic interactions.

Today, Quantum chromodynamics (QCD), a part of the Standard Model of particle physics, is believed to be the correct theory of strong interactions. The main

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This dissertation follows the style of The Physical Review Letters.
reasons for this belief are the phenomena of asymptotic freedom [4] and the fact
that the global symmetries of the observed hadronic world are the same as of QCD.
Asymptotic freedom implies that at short distances quarks behave as almost free
particles and, hence, we can apply perturbation theory for hard processes involving
large momentum transfers. Indeed, asymptotic freedom allows us to compare
experiments with the predictions of QCD in the deep inelastic regime. The funda-
mental degrees of freedom of QCD (quarks and gluons) are already well established
even though they cannot be observed as free particles, only in color neutral bound
states. Quark confinement is the other side of the asymptotic freedom. Since the
force between color charges does not decrease with distance, the quarks and gluons
can never be liberated from hadrons. This aspect of the theory was verified within
numerous lattice QCD computations, but has not yet been mathematically proven.

Nielsen has given a simple and satisfying explanation of why non-Abelian gauge
theories are asymptotically free [5]. To understand the dynamics of a field theory,
we have to understand how the coupling constant behaves as a function of distance.
This behavior, in turn, is determined by the response of the vacuum (the ground
state of the theory) to the presence of external charges. The uncertainty principle
allows particle-antiparticle pairs to be present in the vacuum for a period of time
inversely proportional to their energy. In most theories, the charge renormalization
is nothing more than vacuum polarization. In QED, the vacuum contains virtual
electron-positron pairs, and an external electric charge $e_0$ will polarize it. Hence, a
medium with virtual electric dipoles will screen the charge and the actual observable
(effective) charge $e_R$ will differ from $e_0$ as $\frac{1}{\epsilon} e_0$, where $\epsilon$ is the dielectric constant.
Obviously, the effective charge $e_R$ decreases with increasing distance as there is more
medium that screens the charge. Thus, the electromagnetic coupling constant in-
creases toward shorter distances. Using the methods of condensed physics matter
one can calculate the dielectric constant $\epsilon$ in terms of the magnetic permeability
$\mu = \epsilon^{-1}$. According to Lenz's law, in classical physics all media are diamagnetic
($\mu < 1$), which corresponds to electric screening ($\epsilon > 1$). However, in quantum
systems paramagnetism is possible. In QCD gluons carry color charge. Since they
have a larger spin (spin 1) than quarks, the paramagnetic effect dominates. Gluons
behave as permanent color magnetic dipoles that align themselves parallel to an ap-
plied external field increasing its magnitude and producing $\mu > 1$. We can, therefore,
regard the anti-screening of the Yang-Mills vacuum as paramagnetism.
FIG. 1. Quark-gluon vertex (a) is corrected by a loop (b) that results in a correction to the vertex only (c) or to the rest of the diagram (d).

QCD is asymptotically free because the anti-screening of the gluons overcomes the screening due to the quarks.

In terms of the Feynman renormalization procedure, in QED one can find the dependence of the electric charge on distance by resumming (logarithmically divergent and regularized at the distance $r_0$) electron-positron loops dressing the virtual photon propagator. However, to understand the QCD paramagnetism it is very helpful to go back to the classical Lagrangian density for a non-Abelian gauge theory:

$$\mathcal{L} = -\frac{1}{4}G^a_{\alpha\beta}G^{a\alpha\beta} + \bar{\psi}(i\gamma^\nu D_\nu - m)\psi + \phi^4(-D_\nu D^\nu - M^2)\phi + \text{other terms}$$ (2)

where $G^a_{\alpha\beta} \equiv \partial_\alpha A^a_\beta - \partial_\beta A^a_\alpha - g_s f^{abc} A^b_\alpha A^c_\beta$ is the field strength with the structure constants $f^{abc}$. The covariant derivative $D_\nu = \partial_\nu + i g_s A^a_\nu \cdot t^a$, where the $t^a$ are the appropriate representation matrices (i.e. the Pauli matrices $\frac{1}{2}$ for the fundamental representation of $SU(2)$ or the Gell-Mann matrices $\frac{1}{2}$ for the fundamental representation of $SU(3)$). The other terms include scalar field self-interactions, which do not depend on the gauge field. As always, the strength of the interaction part with respect to the kinetic part or between sectors of the interaction part we parametrize by the strong coupling constant $g_s$. By redefining $g_s A \rightarrow A$, one can isolate the coupling constant $g_s$ so it appears only in the first term:

$$\mathcal{L} = -\frac{1}{4g_s^2}G^a_{\alpha\beta}G^{a\alpha\beta} + \bar{\psi}(i\gamma^\nu D_\nu - m)\psi + \phi^4(-D_\nu D^\nu - \mu^2)\phi + \text{other terms}$$ (3)

where now $G^a_{\alpha\beta} \equiv \partial_\alpha A^a_\beta - \partial_\beta A^a_\alpha - f^{abc} A^b_\alpha A^c_\beta$ and $D_\nu = \partial_\nu + i A^a_\nu \cdot t^a$.

If we set the masses of the light quarks to zero, and take the masses of the heavy quarks to be much greater than the scale of QCD, we get

$$\mathcal{L} = \frac{1}{2} (\partial_\mu A^a_\mu)^2 - \frac{1}{4} F^{a\alpha\beta} F_{a\alpha\beta}$$

where $F^{a\alpha\beta} = \partial_\alpha A^a_\beta - \partial_\beta A^a_\alpha - f^{abc} A^b_\alpha A^c_\beta$.
quarks to be infinitely large, then the only important parameter in the Lagrangian is the coupling constant $g_s$.

In a perturbation theory, any physical observable can be calculated as a function of the Lagrangian parameters. However, it is not obvious that these parameters themselves are physically observable quantities, and, hence, we have to rely on the renormalization theory in which one physical observable can be written as a function of another. For instance, in the lowest order of perturbation theory we can redefine $g_s$ to be the strength of the quark-gluon coupling, as shown in Fig. 1(a). However, in higher orders we must include loop corrections, like the one shown in Fig. 1(b) that corrects the vertex. To avoid double-counting we must uniquely decide whether these corrections are part of the vertex only, as in Fig. 1(c), or the rest of the diagram, as in Fig. 1(d). Thus, we have to introduce a renormalization scale $\mu$ and assume that physics at high scales (short distances) above $\mu$ is part of the vertex and physics at lower scales (longer distances) below $\mu$ is part of the rest of the diagram. If we take into account only one-loop corrections and neglect the higher orders, we can solve the renormalization group equation exactly and derive $\alpha_s = \frac{g_s^2}{4\pi}$ at some scale $Q$ as a function of its value at the renormalization scale $\mu$ [4]:

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \frac{1}{4\pi} \alpha_s(\mu^2) b \ln \frac{Q^2}{\mu^2}}, \quad b \equiv \beta_0 = \frac{11}{3} N_c - \frac{2}{3} N_f.$$

(4)

where $N_f$ is the number of quark flavors and the number of quark colors $N_c$ is the eigenvalue of the quadratic Casimir operator in $SU(N)$ (for the fundamental $N_c = N_f$). The famous $\Lambda_{QCD} \approx 200 \text{ MeV} \approx 1 \text{ fm}^{-1}$ defines the scale at which the QCD perturbative solution diverges.

From (4) we can instantly see, that in $SU(3)$ the gluonic contribution to the vacuum polarization reverses the sign of the QCD $\beta$-function $\beta[\alpha_s(\mu)] = \mu^2 \frac{\partial}{\partial \mu^2} \alpha_s(\mu)$ for 16 or fewer flavors of quarks:

$$4\pi \beta(\alpha_s) = -\alpha_s^2 \beta_0 + \alpha_s^3 \beta_1 + \alpha_s^4 \beta_2 + \ldots < 0: \quad N_f < \frac{33}{2} \quad (5)$$

Thus, at high momentum transfer corresponding to short distances, the coupling constant $\alpha_s$ becomes small, and one can apply perturbation theory. QCD correctly predicts $\alpha_s(Q^2)$ for a variety of processes that involve high momentum transfers $Q^2$, such as deep inelastic scattering, $e^+e^-$-annihilation into hadrons and production of heavy quarks (see Fig. 2).
The basic challenge in understanding QCD can be seen very clearly in a spacetime description: it is how to link the phenomena at large distances with one at small distances. While asymptotic freedom implies that the theory becomes simple at short distances, it also tells us that at large distances the coupling becomes very strong. In this regime we have no reasons to believe that perturbation theory is valid. In fact, in the large distance limit (momentum scales smaller than 500-1000 MeV), the QCD dynamics are best described by an effective chiral Lagrangian containing the $\sigma$, $\pi$, and constituent-quark degrees of freedom rather than the partonic quark-gluon degrees of freedom that form the basis of the perturbative QCD (pQCD) approach.

Before the discovery of asymptotic freedom, it was expected that any quantum field theory would fail at high energies because of the flaws in renormalization procedures. In an asymptotically free theory this is no longer the case. At short distances there are no infinities at all: the bare coupling is finite and vanishes, and the perturbative expansion gets better and better when we approach the asymptotically free region. However, it is well-known that Feynman diagrams with loops in QCD

---

**FIG. 2.** The running coupling constant $\alpha_s(Q^2)$ as a function of momentum transfer $Q^2$ determined from a variety of processes.
diverge badly, and we still have to deal with regularization and renormalization problems. A good choice of regularization is crucial for any non-trivial computation: the regularization should preserve as much of the symmetry of the original theory as possible and manipulating regularized Feynman integrals should be simple and straightforward. Cut-off regularization is difficult to define unambiguously, and in gauge theories it breaks gauge invariance. Pauli-Villars regularization also breaks gauge invariance. Lattice regularization preserves it, but breaks Lorentz invariance. The method used in practically all multiloop QCD calculations is dimensional regularization that preserves both Lorentz and gauge invariance with simple rules for manipulating integrals (no boundary terms). As discussed above, the renormalization is another very important concept in QCD. Physical results should be expressed via measurable physical parameters and not via parameters in the bare Lagrangian. The most widely used renormalization procedure in QCD is the minimal subtraction renormalization scheme (MS scheme) that depends on a single renormalization scale parameter $\mu$.

The evaluation of perturbative corrections requires the computation of higher-order Feynman diagrams that involve real and virtual partons. In this computation one has to deal with different singularities. In Chapter II we review the formal and phenomenological aspects of the QCD divergency pattern of perturbative expansions. Small-$x$ physics is a topic at the border of hard and soft hadronic interactions. In Chapter III we summarize the formalism of Wilson lines and outline recent theoretical developments in the small-$x$ dynamics. In Chapter IV we study the scattering of color dipoles in the first two orders of perturbation theory. Finally, we draw conclusions and outline future research plans in Chapter V.
CHAPTER II
HIGH ORDERS OF PERTURBATIVE QCD

We understand Quantum field theories only when the interactions between elementary degrees of freedom are weak. Namely, applying perturbation techniques we can express any observables $O$ in terms of the renormalized interaction strength $\alpha_s$ as a series:

$$O = \sum_n O_n \alpha_s^n$$

It is well known that these series are asymptotic at best. Classical books on diagrammatic techniques describe the construction of Feynman’s diagram series as if their convergence is well defined. However, Dyson [9] first noted that the perturbative series diverge badly for any value of the coupling constant $g$ since they have zero radii of convergence. For instance, we can consider a perturbation series in terms of the coupling constant $g$ for a Fermi gas with delta-function interaction $g\delta(r - r')$. In the case of a repulsive interaction ($g > 0$), the ground state of the system is a Fermi liquid. However, Cooper’s instability leads to a system with superconductivity when we go into the attractive interaction ($g < 0$) region. Obviously, the ground state of the system changes at $g = 0$, where we have a singularity. The radius of convergence of the system is determined by the distance from the origin to the nearest singular point in the complex plain. Since the nearest singular point of the system is located at the origin itself, the convergence radius of the series is zero.

We have to note that Dyson’s argument is unquestionable, and does not depend on the interaction or definition of the coupling constant. Hence, many of the early studies of the high order expansions in perturbation theory have focused on the question of whether a quantum field theory can be constructed non-perturbatively from the perturbative expansions. Beginning in the 1970’s QCD has been growing from a qualitative to a quantitative theory of strong interactions. The third order perturbative calculations became available for $e^+e^-$ annihilation [10, 11] and deep inelastic scattering [12]. The precision of the latest experiments now can be matched by theoretical accuracy. The coupling constant expansion has become an important theoretical instrument that tells us what happens at asymptotically large external momenta. Therefore, it is natural to ask how much we could improve the theoretical
precision and what we could learn about the $O_n$ parameters in the expansion formula (6) and whether asymptotic estimates could be extrapolated to $n \sim 2-3$. Hence, one of the main problems in pQCD nowadays is to find techniques to deal with the infinities in summing of the perturbation series.

It is well known that the coefficients $O_n$ of a typical perturbative expansion (6) grow like $n!$, where $n$ is the order of the perturbation series [8]. According to Lipatov [13], the high orders of perturbation theory are determined by the saddle-point configurations (instantons) of the corresponding functional integrals. When the Lipatov asymptotics are known, and several lowest orders of perturbation theory are found by direct calculation of the Feynman diagrams, one can gain insight into the behavior of the remaining terms of the series. However, 't Hooft has questioned Lipatov's method with discovery of the factorially large contributions from individual diagrams (renormalons) [14]. In 't Hooft's opinion, the instanton contributions do not contain the renormalons. Thus, we have two sources of the $n!$ behavior of a typical perturbative series: all Feynman diagrams are $\sim 1$ but their number is large ($\sim n!$) and there is one abnormally big diagram $\sim n!$.

This Chapter extends the results published in [15] and is organized as follows. In Section II.1 we review the Lipatov's saddle-point approximation approach used in calculations of the high-order terms in perturbation theory. In Section II.2 we review formal and phenomenological aspects of the renormalon divergence. Since infrared renormalons probe large distances, they are closely connected with non-perturbative power corrections in asymptotically free theories. In Section II.3 we demonstrate that from the functional-integral viewpoint renormalons do not correspond to a particular configuration, but manifest themselves as dilatation modes in the functional space.

II.1 LIPATOV'S METHOD

Lipatov’s method [13] was proposed in 1977 as a tool for calculating high-order terms in perturbation theory and making estimates for its divergences.

If a function $F(g)$ can be expanded into a series in $g$:

$$ F(g) = \sum_{N=0}^{\infty} F_N g^N. $$ (7)
then the $N$th expansion coefficient $F_N$ can be determined as:

$$F_N = \int_C \frac{dg}{2\pi i} \frac{F(g)}{g^{N+1}} \equiv \int_C \frac{dg}{2\pi i} F(g) e^{-(N+1)\ln g},$$

(8)

where the contour $C$ encloses the point $g = 0$ in the complex plane.

It is well known that many problems in theoretical physics can be calculated in terms of functional integrals

$$Z(g) = \int D\varphi \exp(-S_0\{\varphi\} - gS_{\text{int}}\{\varphi\})$$

(9)

whose expansion in terms of the coupling constant $g$ gives an ordinary perturbation theory. Taking the integral (9) as $F(g)$ we obtain the expansion coefficients:

$$Z_N = \int_C \frac{dg}{2\pi i} \int D\varphi \exp(-S_0\{\varphi\} - gS_{\text{int}}\{\varphi\} - N\ln g)$$

(10)

Lipatov's idea is to seek the saddle point configuration in (10) with respect to $g$ and $\varphi$ simultaneously, rather than just to $g$. The desired configuration is realized on a localized function $\varphi$ called instanton. The conditions for applicability of the saddle-point approximation are satisfied when $N$ is large.

The Lipatov method was originally applied to scalar theories, such as $\varphi^4$, and then generalized to vector fields [16], scalar electrodynamics [17] and Yang-Mills fields [18, 19].

The typical Lipatov's asymptotics of the expansion coefficients for any $F(g)$ has a form

$$F_N = c a^N \Gamma(N+b) \approx e^{a^N N^{b-1} N!}$$

(11)

where $\Gamma(x)$ is the gamma function and $a$, $b$ and $c$ are theory-dependent parameters. In the framework of any particular theory, $a$ is a universal constant, parameter $b$ depends on $F(g)$ expansion, and $c$ depends on external coordinates.

When a few lowest orders of a perturbation series are found by direct calculation of Feynman's diagrams, and the form of Lipatov's asymptotic is known, we can estimate the contribution of the higher orders of the perturbative expansion, and even sum them in a certain approximation. The most important consequence of this is the possibility of finding the asymptotic behavior of the effective coupling constant.
Expanding the theory into the strong coupling region is required in many fields of theoretical physics. The most important problems in QED and QCD are related to the dependence of the effective coupling constant $g$ on the spatial scale $\Lambda$. Such dependence is determined by the Gell-Mann-Low function $\beta(g)$:

$$
-\frac{dg}{d\ln \Lambda^2} = \beta(g) = \beta_0 g^2 + ...
$$

(12)

In relativistic theories, the first term in the expansion of $\beta(g)$ is quadratic. For a small $g$ the expansion (12) yields to the well-known result:

$$
g(\Lambda) = \frac{g_0}{1 - \beta_0 g_0 \ln (\Lambda^2/\Lambda_0^2)}, \quad g_0 = g(\Lambda_0)
$$

(13)

In QED the constant $\beta_0$ is positive and $g(\Lambda)$ increases at small $\Lambda$. In QCD the sign of $\beta_0$ is negative. Accordingly, the interaction between quarks and gluons is weak at small $\Lambda$ that corresponds to the asymptotic freedom. Moreover, $g(\Lambda)$ increases with $\Lambda$ and, hence, demonstrates a tendency toward confinement.

The main problem is to study the behavior of $g(\Lambda)$ in the case of intermediate and strong coupling. In QCD, the first four terms of the expansion of the beta function are known in the minimal subtraction renormalization scheme (MS scheme):

$$
\beta(g) = -\sum_{n=0}^{\infty} \beta_n g^{n+2} = -\beta_0 g^2 - \beta_1 g^3 - \beta_2 g^4 - ....
$$

(14)

where $g = g_s^2/16\pi^2 = \alpha_s/4\pi$ and $g_s$ is the strong coupling constant in the standard QCD Lagrangian (2) and in case of SU(3), where $N_c = 3$:

$$
\beta_0 = 11 - \frac{2}{3}N_f
$$

(15)

$$
\beta_1 = 102 - \frac{38}{3}N_f
$$

(16)

$$
\beta_2 = \frac{2857}{2} - \frac{5033}{18}N_f - \frac{325}{54}N_f^2
$$

(17)

The asymptotic form of the coefficients in the series (14) coincides with (11):

$$
\beta_N = c \Gamma \left[ N - 2 + 4N_c + \frac{11}{6}(N_c - N_f) \right]
$$

(18)

As was mentioned already, the knowledge of the first few coefficients and their asymptotics allows us to reconstruct the Gell-Mann-Low function and study the problem of confinement.
FIG. 3. Typical renormalon bubble chain diagrams giving the $N!$ contribution.

However, Lautrup in his paper [20] first noted that the Lipatov asymptotics is based on a formal calculation of the functional integral (9) and does not rely on a statistical analysis of diagrams. The asymptotics (11) implies the contribution of a factorially large number of diagrams of the same order $(ag)^N$. However, in QED and QCD we have individual $N$th-order diagrams whose contributions are proportional to $N!$. The latter are diagrams (Fig. 3) that contain long chains of bubbles. Such factorial contributions of individual diagrams were termed renormalons. In 1977 't Hooft raised a question whether the renormalon contributions are taken into account in the instanton asymptotics (11).

II.2 'T HOOFT RENORMALONS

In 1977 't Hooft claimed that the renormalons provide an independent mechanism of the perturbation series divergence. His arguments are based on the analytic properties of Borel transformations, widely used in the theory of divergent series:

$$F(g) = \sum_{N=0}^{\infty} \frac{F_N}{N!} g^N = \sum_{N=0}^{\infty} \frac{F_N}{N!} \int_0^\infty dx \ e^{-x} x^N, \quad F_N = \int_0^\infty dx \ e^{-x} \sum_{N=0}^{\infty} \frac{F_N}{N!} (gx)^N. \quad (19)$$

As one can see from the formula (19), the Borel transformations factorially improve perturbation series convergence. It is convenient to introduce the Borel transform $B(z)$ of the function $F(g)$:

$$F(g) = \int_0^\infty dx \ e^{-z} B(gx), \quad B(z) = \sum_{N=0}^{\infty} \frac{F_N}{N!} z^N. \quad (20)$$
For a function with the expansion coefficients (11) the Borel transform has a singularity at the point \( z = 1/a \):

\[
B_z(z) = \sum_{N} c N^{a-1} z^N \sim (1 - az)^{-b} \quad za \to 1
\]  \hfill (21)

Thus, the parameter \( a \) determines locations of the singular points in the Borel plane.

By replacing \( \varphi \to \varphi/\sqrt{g} \) and \( x \to x/g \) in Eq. (9), and rewriting the functional integral (9) in the form:

\[
Z(g) = \int D\varphi \exp(-S(\varphi)/g) = \int_0^\infty dz \, e^{-z/g} \, B(z),
\]  \hfill (22)

we can obtain the Borel transform:

\[
B(z) = \int D\varphi \, \delta(z - S(\varphi)) = \int_{z = S(\varphi)} \frac{d\sigma}{|\nabla S(\varphi)|},
\]  \hfill (23)

The factors \( g^n \) are omitted because they cancel out when Green functions are calculated as ratios of two functional integrals. The integration \( d\sigma \) is carried out over the hypersurface \( z = S(\varphi) \).

If an instanton \( \varphi_c(x) \) solution exists for the integral (22) with a finite action \( S(\varphi) \), then \( \delta S\{\varphi_c\} = 0 \) (the partial derivatives \( \partial S/\partial \varphi_i \) with respect to all the variables \( \varphi_i \), comprising \( D\varphi \), vanish and \( \nabla S\{\varphi_c\} = 0 \)). Therefore, the Borel transform function \( B(z) \) has a singularity at:

\[
z = S\{\varphi_c\}
\]  \hfill (24)

The main question that remains is whether the Lipatov’s method takes into account renormalons. To answer this question ’t Hooft has studied the applicability of the saddle-point approximation and found a gap in the mathematical arguments of the Lipatov technique. Let’s consider a function \( f(x) \) with a sharp maximum and a slow tail (see Fig. 4) so that the integral contributions \( \int dx f(x) \) from the vicinity of the maximum \( x_0 \) and from the tail region are comparable. Obviously, the integral calculation in the saddle-point approximation is incorrect since the contribution of the tail will be lost. Hence, the lack of nonsaddle-point methods for calculating functional integrals makes it impossible to investigate the contribution of possible tails.
FIG. 4. Example of a function with incorrect saddle-point approximation.

\[ f(x) \]

\[ x_0 \]

FIG. 5. Bubble chain contributions to the effective coupling.

In QED or QCD renormalon singularities can be easily found if we consider bubble diagrams (Fig. 3). The virtual photon/gluon lines with momentum \( q \) correspond to integration over the range of large momenta:

\[
\int_{q > q_0} d^4 q \, q^{-2n} \ldots \quad n = 3, 4, 5, \ldots
\]

Insertion of the virtual lines corresponds to the screening of the effective coupling (Fig. 5). Summation of the bubble chains corresponds to using a one-loop approximation \( \beta(g) = 3_0g^2 \) of the Gell-Mann-Low beta function (14), and leads to the formula (13) for a coupling constant.

Performing the integration over large momenta \( q > \Lambda_0 \) we obtain

\[
\int_{q > q_0} d^4 q \, q^{-2n} g(q) = g_0 \sum_n \int d^4 q \, q^{-2n} \left( \beta_0 g_0 \ln \frac{q^2}{\Lambda^2} \right)^n \sim g_0 \sum_n N! \left( \frac{\beta_0}{n-2} \right)^N g_0^N
\]

(25)
One can see that Borel summation of a sequence of bubble diagrams gives ultraviolet (UV) singularities at the points

\[ z = \frac{m}{\beta_0}, \quad m = 1, 2, 3, \ldots \]  \hspace{1cm} (26)

The constant \( \beta_0 \) is positive in a non-asymptotically free theory such as QED or \( \varphi^4 \) (12) or negative for an asymptotically free theory such as QCD (14)-(15).

Analogous infrared renormalon (IR) singularities arise in the integral over a small momentum region:

\[ z = -\frac{m}{\beta_0}, \quad m = 2, 3, 4, \ldots \]  \hspace{1cm} (27)

Hence, according to 't Hooft, in QCD we have two types of singularities: renormalons (ultraviolet or infrared) and instanton-induced singularities (Fig. 6).

II.3 RENORMALONS AS DILATATION MODES IN THE FUNCTIONAL SPACE

Consider the Adler's \( D \)-function in massless QCD:

\[ D(q^2) = 4\pi^2 q^2 \frac{d}{dq^2} \frac{1}{3q^2} \Pi(q^2). \]  \hspace{1cm} (28)

where \( \Pi(q^2) \) is the vacuum polarization operator in the Euclidean region \( (q^2 < 0) \) of the current \( j^\mu(x) \):

\[ \Pi(q^2) = \int dx e^{iqx} \int \tilde{D} \tilde{C} \tilde{D} \tilde{A} \tilde{j}_\mu(x)j_\mu(0) \ e^{-S_{QCD}} \]  \hspace{1cm} (29)

Suppose we write down \( D(q^2) \) as a Borel integral

\[ D(\alpha_s(q)) = \int_0^\infty dt \ D(t) \ e^{-\frac{\alpha_s}{\beta_0} t} \]  \hspace{1cm} (30)

The divergent behavior of the original series \( D(\alpha_s(q)) \) is encoded in the singularities of its Borel transform (Fig. 6).

In QCD, we have two types of singularities in the Borel plane: UV/IR renormalons and instanton-induced singularities. From (26), the UV renormalons are located at \( t = -m/\beta_0 \), where \( m = 1, 2, \ldots \) and \( \beta_0 = 11 - \frac{2}{3}N_f \) (see (15)). In terms of Feynman
diagrams they come from the regions of hard momenta in renormalon bubble chain. From (27), the IR renormalons are located at $t = (m + 1)/\beta_0$ and come from the region of soft momenta in the bubble chain diagrams.

The instanton singularities are located at $t = 1, 2, 3, ...$ and they correspond to the large number of graphs summed up to the contribution of instanton-antiinstanton $\Pi \bar{\Pi}$ configurations in the functional space. The instanton itself is not related to the divergence of the perturbative series since it belongs to a different topological sector. The first topologically trivial classical configuration, which contributes to the divergence of perturbation theory, is a weakly coupled instanton-antiinstanton pair [24].

We show that unlike the instanton-type singularities, the renormalons do not correspond to a particular configuration but manifest themselves as dilatation modes in the functional space.

II.3.1 Valleys and Borel plane in quantum mechanics

The interpretation of renormalons as dilatation modes is based upon the similarity of the functional integral in the vicinity of a valley in the functional space [25] to the Borel representation (30). At first, we will consider the quantum mechanical example without renormalons, and then demonstrate that in a field theory the same integral along the valley leads to renormalon singularities. With QCD in view, we consider the double-well anharmonic oscillator described by the functional integral

$$Z = \int D\phi \ e^{-\frac{S_{\phi}}{\hbar^2}} \tag{31}$$
where

\[ S(\phi) = \int dt \left[ \frac{\dot{\phi}^2}{2} + \frac{\phi^2(1 - \phi)^2}{2} \right] \]  

(32)

The large-order behavior in this model is determined by the instanton-antiinstanton (II) configuration. The II valley for the double-well system may be chosen as

\[ f_\alpha(t - \tau) = \frac{1}{2} \tanh \frac{t - \tau + \alpha}{2} - \frac{1}{2} \tanh \frac{t - \tau - \alpha}{2} \]  

(33)

It satisfies the valley equation [27, 28]:

\[ \frac{12}{\xi^2} w_\alpha(t) f'_\alpha(t) = L_\alpha(t) \left\{ \begin{array}{c} \xi \equiv \epsilon^\alpha \\ f'_\alpha \equiv \frac{\partial}{\partial \phi} f_\alpha \\ L_\alpha(t) = \frac{\delta S}{\delta \phi} |_{\phi=f_\alpha(t)} \end{array} \right. \]  

(34)

where \( w_\alpha(t) = \frac{1}{2} \sinh \alpha [\cosh \alpha \cosh t + 1]^{-1} \) is the measure in the functional space, so that \( (f, g) \equiv \int dt w_\alpha(t) f(t) g(t) \). The II valley (33) connects two classical solutions: perturbative vacuum at \( \alpha = 0 \) and infinitely separated II pair at \( \alpha \to \infty \). The II separation \( \alpha \) and position of the II pair \( \tau \) are two collective coordinates of the valley.

In order to integrate over small fluctuations near the configuration (33), we introduce two \( \delta \)-functions, namely: \( \delta(\phi(t) - f_\alpha(t - \tau), \dot{f}_\alpha(t - \tau)) \) and \( \delta(\phi(t) - f_\alpha(t - \tau), \dot{f}_\alpha(t - \tau)) \) restricting the integration along the two collective coordinates. Following the standard procedure (Faddeev-Popov trick), we insert

\[ 1 = \int d\alpha \left[ (f'_\alpha \cdot \dot{f}'_\alpha) + (\phi - f_\alpha \cdot \dot{f}'_\alpha + \frac{w'}{w} f_\alpha) \right] \delta(\phi - f_\alpha \cdot \dot{f}'_\alpha) \]  

(35)

and

\[ 1 = \int d\tau \left[ (\dot{f}'_\alpha \cdot \dot{f}_\alpha) + (\phi - f_\alpha \cdot \dot{f}_\alpha + \frac{w'}{w} f_\alpha) \right] \delta(\phi - f_\alpha \cdot \dot{f}_\alpha) \]  

(36)

into the functional integral (31). After that, we make a shift \( \phi(t) \to \phi(t) + f_\alpha(t - \tau) \), an expansion in the quantum deviations \( \phi(t) \) and the gaussian integration in the first nontrivial order in perturbation theory. After the shift, the linear term in the exponent \( \int dt \phi(t) I_\alpha(t) \) vanishes due to the valley equation (34) (recall \( \delta(\phi, f'_\alpha) \) in the integrand coming from the equation (35)) so that the functional integral (31) for the vacuum energy in the leading order in \( g^2 \) reduces to the following

\[ Z \sim T \int d\alpha \int D\phi(f'_\alpha \cdot \dot{f}'_\alpha)(\dot{f}_\alpha \cdot f'_\alpha) \delta(\phi, \dot{f}'_\alpha) \delta(\phi, \dot{f}_\alpha) e^{-\frac{1}{2} \left[ S_\alpha + \frac{1}{2} \int dt \phi(t) I_\alpha(t) \right]} + O(g^2). \]  

(37)
Here $T$ is the total volume in one space-time dimension, the result of a trivial integration over $\tau$. The operator of the second derivative of the action $\Box_{\alpha} = -\partial^2 + 1 - 6f_{\alpha}(1 - f_{\alpha})$ and

$$S_{\alpha} \equiv S(\xi) = \frac{6\xi^4 - 14\xi^2}{(\xi^2 - 1)^2} - \frac{17}{3} + \frac{12\xi^2 + 4}{(\xi^2 - 1)^3} \ln \xi$$  \hspace{1cm} (38)

is the action of $I\bar{I}$ valley. Performing the Gaussian integrations we obtain

$$Z = T \frac{1}{g^2} \int_0^\infty d\alpha \ e^{-\frac{1}{4}S_{\alpha}} F(\alpha).$$  \hspace{1cm} (39)

At $\alpha \rightarrow \infty$ we have widely separated $I$ and $\bar{I}$. The determinant of the $I\bar{I}$ configuration factorizes into a product of the two one-instanton determinants (with zero modes excluded) so that $F(\alpha) \rightarrow \text{const}$ at $\alpha \rightarrow \infty$. The divergent part of the integral (39) corresponds to the second iteration of the one-instanton contribution to the vacuum energy, therefore it must be subtracted from the $I\bar{I}$ contribution to $F_{\text{vac}}$:

$$g^2F_{\text{vac}}(g^2) = \int d\alpha \left[ e^{-\frac{1}{g^2}S_{\alpha}} F(\alpha) - e^{-\frac{1}{g^2}2S_{\alpha}} F(\infty) \right].$$  \hspace{1cm} (41)

where $S_I$ is the one-instanton action. Since $S_{\alpha}$ is a monotonous function of $\alpha$, we can invert Eq. (38) and obtain the desired form of the Borel integral (29) for the vacuum energy:

$$g^2 F_{\text{vac}}(g^2) = \int 2S_{\bar{I}} dS \ e^{-\frac{1}{g^2}S} F(S) \left[ \frac{1}{S - 2S_{\bar{I}}} \right]_+$$  \hspace{1cm} (42)

Thus, the leading singularity for $F_{\text{vac}}(S)$ is $F(2S_I)/(S - 2S_I)$. We have to note, that even though our semiclassical calculation determines the leading singularity in the Borel plane, it does not give us the complete answer for the Borel transform of the vacuum energy since $F(S) \neq F_{\text{vac}}(S)$ in general and we threw away the higher quantum fluctuations around the $I\bar{I}$ pair in Eq. (37).

II.3.2 $I\bar{I}$ valley and the Borel integral in QCD

The situation with the instanton-induced asymptotics of perturbative series in a field theory such as QCD is pretty much similar to the case of quantum mechanics, with one notable exception: in QCD there is an additional dimensional parameter.
ρ. Namely, the overall size of the $II$ configuration. The classical $II$ action does not depend on this parameter, but the quantum determinant does, leading to the following replacement:

$$e^{-\frac{1}{2}S} \rightarrow e^{-\frac{1}{2}\rho S}$$

(43)

Hence, the Borel integrand have the following generic form:

$$F(S) \sim \int d\rho \ e^{-\frac{1}{2}\rho S} \mathcal{F}(\rho)$$

(44)

The divergence of this integral at either large or small $\rho$ determines the position of the $F(s)$ singularities. Focusing only on the dimensional analysis, we will demonstrate that $\mathcal{F}(\rho) \sim \rho^{-5}$ at $\rho \rightarrow \infty$ and $\mathcal{F}(\rho) \sim \rho$ at $\rho \rightarrow 0$ leading to the IR renormalon at $S = \frac{32\pi^2}{b}$ and the UV renormalon at $S = -\frac{16\pi^2}{b}$ respectively.

The $II$ valley in QCD can be chosen as a conformal transformation of the spherical configuration [27]:

$$A_\mu(x) = -i \frac{\sigma_{\mu} x - x_\mu}{x^2} f_\mu(t), \quad t = \ln \frac{x^2}{d^2}.$$  

(45)

where $d$ is an arbitrary scale.

To derive the $II$ configuration with a separation $R$ and arbitrary sizes $p_1$ and $p_2$, one can perform a shift $x \rightarrow x - a$, an inversion $x \rightarrow \frac{d^2}{x^2} x$ and a second shift $x \rightarrow x - x_0$ (see Fig. 7). The resulting valley is the sum of the $I$ and $\bar{I}$ in the singular gauge in a maximum attractive orientation plus an additional term, which is small at large $II$ separations.

$$A_\mu^v(x) = A_\mu^I(x - x_0) + A_\mu^I(x - x_0) + B_\mu(x - x_0)$$

(46)

where

$$A_\mu^I(x) = -i \rho_1^2 \frac{\sigma_{\mu} \hat{x} - x_\mu}{x^2(x^2 + \rho_1^2)}$$

(47)

$$A_\mu^\bar{I}(x) = -i \rho_2^2 \frac{R \sigma_\mu (x - R) \hat{R} - (x - R)_\mu}{R^2(x - R)^2((x - R)^2 + \rho_2^2)}.$$  

The explicit expression for $B_\mu$ can be found in [28]. The action of the $II$ valley (46) is equal to the action of the spherical configuration (45) given by (38):

$$S^v(z) = 48\pi^2 S(\xi), \quad \xi = z + \sqrt{z^2 - 1}.$$  

(48)

1In its present form, our analysis of renormalons is applicable to the off-shell processes which can be related to the Euclidean correlation functions of two (or more) currents. An example of a different type is the IR renormalon in the pole mass of a heavy quark located at $S = \frac{16\pi^2}{b}$ [26].

2We use the standart notation $x = x_\mu \sigma_\mu$: $\hat{x} = x_\mu \bar{\sigma}_\mu$, where $\sigma_\mu = (1, -i \bar{\sigma})$; $\bar{\sigma}_\mu = (1, i \bar{\sigma})$. 

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where the conformal parameter \( z \) is given by

\[
z = \frac{\rho_1^2 + \rho_2^2 + R^2}{2\rho_1 \rho_2}.
\] (49)

Let us now calculate the polarization operator (29) in the valley background. The valley of a general color orientation has the form \( O_{ab} A^b \), where \( O \) is an arbitrary \( SU(3) \) matrix. Hence, the collective coordinates are the instanton sizes \( \rho_i \) with separation \( R \), overall position \( x_0 \), and the orientation in the color space. The structure of Gaussian result for the polarization operator in the valley background has the following form:

\[
\int_0^\infty \frac{d\rho_1}{\rho_1^3} \frac{d\rho_2}{\rho_2^3} d^4 R d^4 x_0 dO \ \Pi'(q) \frac{1}{g_1 7} \frac{- \nabla_{<x>}}{\xi_2^2} \Delta(\rho_1, R).
\] (50)

where 17 is a total number of the collective coordinates. \(^3\) Here \( \Pi'(q) \) is a Fourier transform of the form

\[
\Pi'(x) = \left( \sum_i c_q^i \right) \text{Tr} [\gamma_\mu G(x, 0) \gamma_\mu G(0, x)].
\]

where \( G(x, y) \) is the Green function in the valley background. The factor \( \Delta(\rho_1, R) \) in Eq. (50) is the quantum determinant, the result of Gaussian integrations near the valley (46) with the additional factors due to the restricted integrations along the collective coordinates (cf. Eq. (40)). For our purposes, it is convenient to introduce

\(^3\)The mode corresponding to the relative orientation in the color space for weakly interacting \( I \) and \( \bar{I} \) pair becomes non-gaussian at \( z \to \infty \). Hence, we should introduce an additional collective coordinate corresponding to this quasi-zero mode. However, we are interested only in \( z \sim 1 \), where this mode is still Gaussian and is taken into account in the quantum determinant \( \Delta \).
the conformal parameter \( z \) and the average size \( \rho = \sqrt{\rho_1 \rho_2} \) as collective coordinates in place of \( \rho_1 \) and \( \rho_2 \). We have

\[
\int dz \frac{d\rho}{\rho^9} d^4 R d^4 x_0 \Pi^v(q) \frac{1}{g^{17}\rho} F(z, R^2/\rho^2) e^{-\frac{c^v(z)}{2g^{17}}}.
\]

(51)

where \( F(z, R^2/\rho^2) \) contains the \( \theta(z - 1 - \frac{R^2}{2\rho^2}) \) function (see Eq. (49)). We have included in \( F \) the trivial integral over color orientation, which gives the volume of \( SU(3) \) group.

The main contribution of the quantum determinant \( \Delta \) is the replacement of the bare coupling constant \( g^2 \) in Eq. (50) by the effective coupling constant \( g^2(\rho) \). The remaining function \( F \) is the dimensionless function of the ratio \( R^2/\rho^2 \) and the conformal parameter. This is obvious from the renormalizability of the theory since the only dimensional parameters are \( \rho \) and \( R \). Formally, one can prove that due to the conformal anomaly rescaling of the configuration by a factor \( \lambda (\rho \rightarrow \lambda \rho \text{ and } R \rightarrow \lambda R) \) leads, at the one-loop level, to multiplication of the determinant by a factor \( \lambda^{b_{\Delta}/2\pi^2} \) (see Appendix B).

II.3.3 IR renormalon from the dilatation mode of the \( \Pi \) valley

Consider the singularities of the integral (51). The function \( F(z, R^2/\rho^2) \) is non-singular since the singularity in \( \Phi (\equiv \text{singularity in } \Delta) \) would mean a non-existing zero mode in the quantum determinant. Moreover, the integration over \( R \) is finite due to the function \( \theta(z - 1 - \frac{R^2}{2\rho^2}) \), indicating that the only source of singularity at finite \( z \) is the divergence of the \( \rho \) integral at either large or small \( \rho \). (At \( z \rightarrow \infty \), similar to the derivation of Eq. (42), we obtain the first instanton-type singularity located at \( t = 1 \) [31]).

Let us demonstrate that the singularity of the integral (51) at large \( \rho \gg \frac{1}{q} \) corresponds to the IR renormalon. The polarization operator \( \Pi(q) \) in the background of the large-scale vacuum fluctuation reduces to [32] (see Fig. 8)

\[
\Pi(q) = \int d^3 x \ e^{iqx} \langle j_\mu(x)j_\mu(0) \rangle_A \rightarrow -\sum \frac{c^2}{16\pi^2} \left[ \frac{G^2(0)}{q^2} + \cot \frac{G^3(0)}{q^4} + \ldots \right]
\]

(52)

where \( G^2 \equiv 2 \text{Tr} \ G_{\xi\eta} G_{\xi\eta} \), \( G^3 \equiv 2 \text{Tr} \ G_{\xi\eta} G_{\eta\sigma} G_{\sigma\xi} \), and \( c \) is an (unknown) constant. The coefficient in front of \( G^3 \) vanishes at the tree level [34]. Let's consider the leading
FIG. 8. Expansion of the polarization operator in large-scale external fields.

term in this expansion. Since the field strength for the valley configuration (46) depends only on $x - x_0$:

$$
\int d^4x_0 \text{Tr} \ G_{\xi_0}^r(0)G_{\xi_0}^r(0) = 4S^r(z).
$$

(53)

the integral (51) takes the form

$$
\frac{1}{q^2} \int_1^\infty dz \int_0^\infty \frac{dp}{\rho^2} d^4R \frac{1}{g^{17}(\rho)} F(z, R^2/\rho^2) \ e^{-\frac{S^r(z)}{4\pi\rho^2}}.
$$

(54)

where we have included the factor $\frac{1}{4\pi^2} \sum e_i^2 S^r(z)$ in $F$. The (finite) integration over $R$ can be performed resulting in an additional dimensional factor $\rho^4$:

$$
\int d^4R \ F(\rho, z, R) = \rho^4 \Phi(z).
$$

The function $\Phi$ is dimensionless so it can depend only on $z$. Finally, we get

$$
\frac{1}{q^2} \int_1^\infty dz \int_0^\infty \frac{dp}{\rho^2} d^4R \frac{1}{g^{17}(\rho)} \Phi(z) \ e^{-\frac{S^r(z)}{4\pi\rho^2}}.
$$

(55)

Inverting Eq. (38), we can write the corresponding contribution to Adler's function as an integral over the valley action ($t = \frac{S}{16\pi^2}$):

$$
D(q^2) \ \simeq \ \frac{1}{3q^4} \int_0^1 dt \int_0^\infty \frac{dp}{\rho^2} \frac{1}{g^{17}(\rho)} \Phi(t) \ e^{-\frac{4\pi}{\alpha_s(\rho)} t}.
$$

(56)

The extra $\frac{1}{g^2(\rho)}$ in the numerator can be eliminated using the integration by parts:

$$
\frac{16\pi^2}{g^2(\rho)} \int_0^1 dt \ e^{-\frac{4\pi}{\alpha_s(\rho)} t} \Phi(t) = \int_0^1 dt \ \Phi(t) e^{-\frac{4\pi}{\alpha_s(\rho)}} - \Phi(1)e^{-\frac{4\pi}{\alpha_s(\rho)}} + \Phi(0).
$$

(57)

The last two terms are irrelevant for the renormalon singularity at $t = \frac{2}{b}$ since the term $\sim e^{-\frac{4\pi}{\alpha_s(\rho)}}$ corresponds to the $I\bar{I}$ singularity and the term $\sim \Phi(0)$ does not
depend on the coupling constant. Likewise, the extra $g(\rho)$ can be absorbed by the Laplace transformation
\[ g(\rho) \int_0^1 dt \Phi(t) e^{-\frac{4\pi}{\alpha_s(\rho)}t} = 4\sqrt{\pi} \int_0^1 dt \frac{e^{-\frac{4\pi}{\alpha_s(\rho)}t}}{\rho^3} \int_0^t dt' (t-t')^{-1/2} \Phi(t'). \] (58)

After nine integrations by parts and the Laplace transformation we have:
\[ D(q^2) = \frac{1}{q^4} \int_0^1 dt \int_0^\infty \frac{dp}{p^3} e^{-\frac{4\pi}{\alpha_s(\rho)}t} \Psi(t) \] (59)
\[ \Psi(t) = \frac{1}{3\sqrt{\pi}(4\pi)^{1/2}} \int_0^t dt' (t-t')^{-1/2} \Phi^{(0)}(t') \]

Using the two-loop formula for $\alpha_s$ we have
\[ D(t) \approx \Psi(t) \int_0^1 \frac{dp}{q^4 p^3} \left[ \frac{\alpha_s(\rho)}{\alpha_s(q)} \right]^{\frac{2\lambda'}{b'}} \left[ \alpha_s(q) \right]^{\frac{2\lambda'}{b'}} \left[ \alpha_s(\rho) \right] \times 
\]
\[ b' = 51 - \frac{19}{3} N_f \] (60)

It is clear now that the integral over $\rho$ diverges in the IR region starting from $t = \frac{2}{b}$ leading to the singularity in $D(t)$ at $t = \frac{2}{b}$. This singularity is the first IR renormalon.

At the one-loop level we can drop the ratio $(\alpha_s(\rho)/\alpha_s(q))^{\frac{2\lambda'}{b'}}$ so that the renormalon singularity is a simple pole $\frac{1}{t-2/b}$. To understand this singularity at the two-loop level, we have to recall that an extra $\alpha(p)$ factor does not affect the singularity, while an extra $\frac{1}{\alpha(q)}$ factor changes it from $(t - \frac{2}{b})^{\lambda}$ to $(t - \frac{2}{b})^{\lambda-1}$. One can easily see that from the integration by parts 5:
\[ \int_0^1 \frac{16\pi^2}{g^2(q)} \int_0^1 dt e^{-\frac{4\pi}{\alpha_s(\rho)}t} \left( t - \frac{2}{b} \right)^{\lambda} = \int_0^1 dt \lambda \left( t - \frac{2}{b} \right)^{\lambda-1} e^{-\frac{4\pi}{\alpha_s(\rho)}t} + ... \] (61)

Therefore, the extra factor $(\alpha_s(\rho)/\alpha_s(q))^{\frac{2\lambda'}{b'}}$ converts the simple pole $\frac{1}{t-2/b}$ into a branching point singularity $(t - \frac{2}{b})^{-1-1} = (t - \frac{2}{b})^{-1} \frac{6}{b^3}$ [33].

It is easy to see that the second term in the expansion (52) gives the second renormalon singularity located at $t = \frac{3}{b}$. Higher terms of the expansion of the polarization operator (52) will give the subsequent IR renormalons located at $t = \frac{4}{b}, \frac{5}{b}, \frac{6}{b}$, etc.

\[ ^{4\text{Strictly speaking, at finite } q \text{ we cannot get to the singularity since our semiclassical approach is valid up to } \rho < \Lambda_{\text{QCD}}, \text{ which translates into } \frac{2}{b} - t \geq \frac{1}{in_{\text{QCD}}}. \text{ Therefore, we must take the limit } q^2 \to \infty \text{ as well.}} \]
\[ ^{5\text{In proving that extra } \frac{1}{\alpha(\rho)} \text{ does not change the singularity, we used the fact that before the } \rho \text{ integration the function } \Phi(t), \text{ which is constructed from valley determinants, can be singular only at } t \to 1, \text{ where these determinants acquire zero modes.}} \]
II.3.4 UV renormalon as a dilatation mode

Let's demonstrate that the divergence of the integral (51) at small $\rho$ leads to the UV renormalon. In order to find the polarization operator in the background of a very small vacuum fluctuation (46) we have to recall that this fluctuation is an inversion

$$ I(x - x_0) = \frac{\rho^2}{(x - x_0)^2} \left( \delta_{\mu\alpha} - 2 \frac{(x - x_0)\mu(x - x_0)\alpha}{y^2} \right) A_\alpha^s \left( \frac{\rho^2}{(x - x_0)^2} (x - x_0) \right) $$

of the spherical configuration (45) (gauge rotated by $x/\sqrt{x^2}$)

$$ A_\alpha^s(x) = -i[\sigma_{\alpha}(x - \tilde{a}) - (x - a)\alpha] \left( \frac{1}{(x - a)^2 + \rho^2/\xi} + \frac{\rho^2/\xi}{(x - a)^2((x - a)^2 + \rho^2/\xi)} \right). $$

where $a = R(\xi - 1/\xi)$ (we chose $d = \rho$ for the inversion).

The corresponding transformation of the polarization operator has the form

$$ \Pi^s(x) = \left( \delta_{\mu\alpha} - 2 \frac{y_{\mu}y_{\alpha}}{y^2} \right) \langle j_{\alpha}(\frac{\rho^2}{y^2}y) j_{\beta}(- \frac{\rho^2}{x_0^2}x_0) \rangle_A \left( \delta_{\mu\alpha} - 2 \frac{x_0\mu x_0\alpha}{x_0^2} \right). $$

where we use the notation $y \equiv x - x_0$. In the limit $\rho \to 0$ we need the polarization operator in the background of $A^s$ at small distances (see Fig. 9), so we can use the expansion (52) in coordinate space

$$ \langle j_{\mu}(x)j_{\nu}(0) \rangle_A \to - \sum \frac{\xi^2}{384\pi^4} \left[ \frac{2x_\mu x_\nu + x^2 \delta_{\mu\nu}}{x^4} G^2(0) + 2 \xi \xi_0 G^3(0)(2 \frac{x_\mu x_\nu}{x^2} - 3 \delta_{\mu\nu} \ln x^2) + ... \right] $$

To derive the Eq. (46) literally, this inversion must be accompanied by the gauge rotation with the matrix $x(x - R)R/\sqrt{x^2(x - R)^2R^2}$, see Ref. [28].
and obtain

$$\Pi^i(x) = \sum_{n=0}^{\infty} \frac{e^2}{12\pi^4} \left\{ \frac{\rho^2 C^2_n(0)}{y^4 x^2} \left[ 1 - \frac{4(x_0 y)^2}{x_0 y^2} \right] + \rho^2 C^2_n(0) \right\}$$

The field strength $G^\mu_\nu(x)$ of the spherical configuration (63) is calculated at the origin. After the integration over $x$ and $x_0$ the first term $\sim C^2_2$ vanishes and the second one gives us

$$\int dx e^{i\rho x} \int dx_0 \frac{1}{y^6 x_0^2} \left[ \left( 3 \ln \frac{x^2}{y^2 x_0^2} \right) - 1 \right] = \frac{\pi^4}{32} (6 \ln \pi + 3C - 2) q^4 \ln^2 q^2$$

leading to

$$\int dx dx_0 e^{i\rho x} \Pi^i(x) = c' \alpha_s(q) \sum_{n=0}^{\infty} \frac{e^2}{64} \rho^2 G(z, R^2/\rho^2) q^4 \ln^2 q^2$$

Here $G$ is a dimensionless non-singular function of $z$ and $R^2/\rho^2$. and this explicit expression can be easily found from Eq. (63)

$$G(z, R^2/\rho^2) = \rho^6 C^2_3(0)$$

The argument of $\alpha_s$ in Eq. (66) is determined by the characteristic momenta in the loop with one extra gluon in the valley background. After inversion, the characteristic distances in the loop diagram determining the coefficient in front of $G^3$ are $\sim \rho^2 \sqrt{z^2/\rho^2}$, which means that the characteristic momenta in the loop were $\sim q$ before the inversion ($x^2 \sim x_0^2 \sim y^2 \sim q^{-2}$ in the integral).

Performing the integration over $R$ we derive the analog of Eq. (59) for the UV renormalon

$$D(q^2) \approx q^2 \alpha_s(q) \int_0^1 dt \int_0^\infty d\rho \rho (\ln q^2 \rho^2)^2 e^{-\frac{4\pi}{\alpha_s(q)}} \Psi(t)$$

leading to

$$D(t) \approx \Psi(t) q^2 \int_0^\infty d\rho (q^2 \rho^2)^h (\ln q^2 \rho^2) \left[ \frac{\alpha_s(\rho)}{\alpha_s(q)} \right]^{\frac{2}{\rho}}$$

Note that the extra $\alpha_s(q)$ in Eq. (70) is compensated by one power of $\ln q^2 \rho^2$. The integral over $\rho$ diverges at $t = -\frac{1}{b}$ which is the position of the first UV renormalon. At the one-loop level this renormalon is a double pole $(1 + t/b)^{-2}$ in agreement with
the perturbative analysis [23]. Subsequent terms in the expansion (52) correspond
to the UV renormalons located at $t = -\frac{2}{b}, -\frac{3}{b}, -\frac{4}{b}, \ldots$. It should be mentioned that
Eq. (71) cannot reproduce the strength of the first UV renormalon at the two-loop
level [35]. The reason is that in Eqs.(52),(66) we have neglected the anomalous
dimensions of the operators $\sim [\alpha_s(q)/\alpha_s(\rho)]^{\frac{3}{2}}$. Such factors can change the strength
of the singularity. For the IR renormalon this does not matter since the operator $G^2$
is renorm-invariant ($\gamma = 0$) and for the subsequent renormalons we can easily correct
our results by corresponding $\gamma$'s. Unfortunately, for the UV renormalons we do not
know how to use the conformal invariance with the anomalous dimensions included.

II.4 CONCLUSIONS

We have demonstrated that integration along the dilatation modes in the functional
space near the $\Pi$ configuration leads to the renormalon singularities in the Borel plane. It is very important to note that we have never used the explicit form
of the valley configuration. Instead, we have applied only three facts:

- the conformal anomaly (43): rescaling of the vacuum fluctuation with an action $S$ by a factor $\lambda$ multiplies the Diract determinant by $\lambda^{bs/8\pi^2}$;

- the expansion of the polarization operator in slowly varying fields (52);

- the conformal invariance of QCD at the tree level (for the UV renormalon we
  wrote down the small-size configuration as an inversion of a large-scale vacuum
  fluctuation).

These properties hold true for any vacuum fluctuations so that we can derive the
same Eq. (60) and Eq. (71) for an arbitrary valley.  

\footnote{We don't see the singularity at $t = 1$ proposed in [36]. However, this singularity is due to
the small-size monopoles that are beyond the scope of our interest. Our results are based on the
gaussian integration near the finite-action vacuum fluctuations, while monopoles have an infinite
action.}

\footnote{The only difference is that the upper limit in the integral over $t$ will be $\infty$ rather than 1 since
an arbitrary valley starts at the perturbative vacuum and goes to infinity with constantly increasing
action.}
The form of the polarization operator in a valley background suggests the following parametrization of the Adler’s function as a double integral in $S$ and $\rho$

$$D(\alpha_s(q)) = \int_0^\infty dt \int \frac{d\rho^2}{\rho^2} d(q^2 \rho^2, t) e^{-\frac{4\alpha_s t}{\alpha_s(\rho^2)}}, \quad (72)$$

where

$$d(q^2 \rho^2, t) \sim \frac{1}{(q^2 \rho^2)^2} \text{ at } \rho \to \infty. \quad d(q^2 \rho^2, t) \sim (q^2 \rho^2) \text{ at } \rho \to 0 \quad (73)$$

The function $d(q^2 \rho^2, t)$ contains only instanton-induced singularities in $t$, whereas the IR and UV renormalons emerge from the divergence of the integral (72) at large or small $\rho$, respectively. If we neglect the running coupling constant (i.e. consider the conformal theory without $\beta$-functions), the representation of the Adler’s function takes the following form:

$$D^{\text{conf}}(\alpha_s) = \int_0^\infty dt e^{-\frac{4\alpha_s t}{\alpha_s}} D^{\text{conf}}(t) \quad (74)$$

$$D^{\text{conf}}(t) = \int \frac{d\rho^2}{\rho^2} d(q^2 \rho^2, t)$$

The integral over $\rho$ converges due to Eq. (73). In real QCD, the function $D^{\text{conf}}(\alpha_s)$ defines the conformal expansion of the Adler’s function with coefficients coming from the skeleton diagrams in terms of the usual perturbation theory [37].

As mentioned above, our renormalon analysis is applicable only to the off-shell processes, which can be related to the Euclidean correlation functions of two (or more) currents. One of the possible future problems is to generalize the analysis to include renormalons in the on-shell processes, as intensively discussed in the current literature [23].

Our calculations are in good agreement with a conventional approach for the position and strength of the renormalon singularity, but not for the coefficient in front of it. The coefficient in front of the first IR renormalon is extremely important since it determines the numerical value of the asymptotics of perturbative series for $R_{e+e^- \to \text{hadrons}}$. In principle, for a given valley it is possible to calculate this coefficient in the first order in coupling constant $g^2(\rho)$ since it is given by a product of the determinants in the valley background, which can be computed numerically. However, an extra $g^2(\rho)$ does not change the position and/or character of the singularity, which means that all terms in the perturbative series in $g^2(\rho)$ produce equal contributions.
This is closely related to the fact that the choice of the valley is not unique. Namely, when we change the valley, the new leading-order coefficient, coming from the determinants in the background of a new configuration, is given by an infinite perturbative series in $g^2(\rho)$ (coming from quantum corrections) in terms of the original valley. As a result, deriving the coefficient in front of the leading IR singularity is extremely difficult since it requires integration over all possible valleys.
CHAPTER III
SMALL-X EVOLUTION OF COLOR DIPOLES

The theoretical analysis of high-energy cross sections in pQCD has a long history. The standard tools in the QCD description of deeply inelastic scattering (DIS) for moderate values of the Bjorken variable $x_B$ are the factorization theorems and Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations [38]. These equations describe the change of the DIS observables with the change of a resolution scale (virtuality $Q^2$ of the probe) and have two remarkable properties: they are linear equations and the evolution at high $Q^2$ is not affected by the non-perturbative physics. The study of the $Q^2$ behavior of the DIS structure functions serves as a tool allowing us to penetrate deeply inside the hadron substructure and observe its constituents with transverse size $\delta x_\perp \sim Q^{-2}$ and longitudinal extent $\sim 1/R$.

The DGLAP $Q^2$-dynamics is based on the following separation of the DIS amplitude into two parts: a hard part coming from the transverse momenta $k_\perp^2 > \mu^2$ and a soft part coming from low $k_\perp^2 < \mu^2$: where $\mu^2$ is the factorization scale that divides short and long distance physics. The contributions from hard momenta give the coefficient functions in front of the light-cone operators formed by the soft contributions. The factorization scale $\mu^2$ serves as a normalization point for these operators. Taking $\mu^2 = Q^2$, we obtain the usual result that the $Q^2$ dynamics is governed by the renormalization group equations for the light-cone operators. A very important property of this factorization that was mentioned earlier is that the coefficient functions are purely perturbative which leads to the summation of the contributions of the type $(\alpha_s \ln Q^2)^n$, etc., in the momentum transfer. Indeed, the effective coupling constant is determined by characteristic transverse momenta, so that the contributions coming from large $k_\perp^2 > \mu^2$ are perturbative as long as $\mu^2$ is sufficiently large. The non-perturbative physics becomes important only when we lower the normalization point $\mu^2$ down to the typical hadronic scale of the order $\sim 1$ GeV. The higher order terms of the perturbative expansion (for both the coefficient functions and the anomalous dimensions of the light-cone operators) lie in the same framework of linear evolution, and lead to the corrections $\sim \alpha_s, \alpha_s^2$, etc. Therefore, to compare experimental measurements of structure functions $F(x_B, Q^2)$ at different $Q^2$, we rely only on the
pQCD, while the linear character of the DGLAP equations makes this comparison especially simple.

The situation for the small-$x$ DIS is more complicated. The DGLAP evolution leads to a strong rise of the DIS structure function $F(x_B, Q^2)$ at small values of the Bjorken variable $x_B$:

$$x_B F(x_B, Q^2) \sim \exp \left( \ln \frac{1}{x_B} \ln \ln Q^2 \right)^{1/2}.$$  \hspace{1cm} (75)

If one relies on using the DGLAP evolution for smaller and smaller $x_B$, higher loop contributions become enhanced by additional factors $\ln \frac{1}{x_B}$, and the perturbative expansion of the coefficient functions and anomalous dimensions breaks down, calling for the small-$x_B$ resummation. Recall that the DGLAP equation sums logarithms of the hard scale to all orders, i.e. terms such as $(\alpha_s \ln Q^2)^n$ and $(\alpha_s \ln Q^2 \ln \frac{1}{x_B})^n$.

It takes only a single logarithm or none of the energy for a power of the coupling constant. Thus, it fails for very low $x_B$ when $\alpha_s \ln \frac{1}{x_B} \gg 1$ and these contributions have to be summed over. In pQCD the small-$x_B$ asymptotic behavior is described in the leading logarithmic approximation (LLA) by the Balitsky-Fadin-Kuraev-Lipatov (BFKL) pomeron [39], which sums up the leading energy logarithms $(\alpha_s \ln \frac{1}{x_B})^n$.

It is possible to reformulate the BFKL equation as an evolution equation where the relevant operators are Wilson lines - infinite gauge links [41]. Indeed, at high energies the particles move so fast that their trajectories can be approximated by straight lines collinear to their velocities. ¹ The proper degrees of freedom for fast particles moving along the straight lines are the infinite gauge factors ordered along straight line [42, 43]. The two-Wilson-line operator corresponding to the fast-moving quark-antiquark pair is called color dipole [44, 45]. The evolution of this dipole with respect to the slope of Wilson lines reproduces the BFKL equation.

Unfortunately, the theoretical status of the BFKL evolution is not as clear as that of the DGLAP (see [46] for a review). The biggest problem is a lack of unitarity. Namely, the power behavior of the BFKL cross section due to BFKL dynamics

$$x_B F(x_B, Q^2) \sim x_B^{\alpha_{BFKL}}, \hspace{1cm} \alpha_{BFKL} = 1 + 4 \chi_c \frac{\alpha_s}{\pi} \ln 2.$$  \hspace{1cm} (76)

violates the so-called Froissart theorem stating that a cross section may grow at most as $\ln^2 \frac{1}{\varepsilon_B}$ at $x_B \rightarrow 0$. Therefore, in order to get the true asymptotics at small

¹Strictly speaking, if we produce in a high-energy collision two clusters of particles with different rapidities, they perceive each other as moving at a great speed along the straight lines.
In the DGLAP case, the sub-leading logarithms follow the same general pattern of the linear DGLAP equation, and we solve a purely technical problem of calculating loop corrections to the kernels. In the case of small-$x$ evolution there are also $\alpha_s$ corrections to the BFKL kernel [47], and, in addition, there are the unitarity corrections which lie outside of the framework of the BFKL equation. At small $\alpha_s$ and $x$, these corrections seem to dominate over the NLO BFKL effects [48].

Another problem with the BFKL evolution is its infrared instability. We can safely apply pQCD to the small-$x$ DIS only if the characteristic transverse momenta of the gluons $k_\perp$ in the gluon ladder are large. For the first few diagrams, one can check by explicit calculation that the characteristic $k_\perp^2$ are of the order $\sim Q^2$. However, as $x$ decreases, it turns out that the characteristic transverse momenta in the middle of the gluon ladder drift towards $\Lambda_{\text{QCD}}$, making the application of pQCD questionable. This is related to the fact that the operator expansion for the high-energy scattering in terms of Wilson line operators, which represent quarks moving with almost the velocity of light, is based on the factorization in the rapidity $\eta \equiv \ln(\frac{1}{x})$ space [49], rather than the transverse momentum. Unlike the usual light-cone expansion, the high-energy expansion in Wilson operators does not admit an additional meaning of the perturbative vs non-perturbative separation. Both the coefficient functions and the matrix elements have perturbative and non-perturbative parts. This happens because the coupling constant in a scattering process is determined by the scale of the transverse momenta. When we use the factorization in hard ($k_\perp > \mu$) and soft ($k_\perp < \mu$) momenta, we calculate the coefficient functions perturbatively since $\alpha_s(k_\perp > \mu)$ is small, whereas the matrix elements are non-perturbative.

Conversely, when we factorize the amplitude in rapidity, both fast and slow parts have contributions coming from the regions of large and small $k_\perp$. In this sense, the small-$x$ evolution in QCD is not protected from the IR side in the same way as the DGLAP evolution is. In order to compare two structure functions measured at different small values of $x$, the pQCD may be insufficient and, in order to explain the small-$x$ behavior of structure functions, it may be necessary to take into account the interplay between the hard and soft pomerons.

Recently, an idea has emerged that these two difficulties may cancel each other out. Consider DIS from a heavy nuclei where the large density sets the saturation...
scale $Q_s$ [50] which effectively cuts the integration over $k_\perp$ even at relatively low energy. The small-$x$ evolution in this case is nonlinear, which leads to the growth of the saturation scale with energy [51, 52]. It is natural to assume that even for the DIS from the nucleon, where there is no saturation at low energies, the saturation scale at sufficiently small $x$ may be generated by the nonlinear evolution itself. Indeed, the linear BFKL evolution, which describes the parton splitting, leads to the gluon density increasing as $x^{-12\ln 2\alpha_s/\pi}$ at small $x$. This growth, however, cannot last forever as it would violate the unitary bound. Thus, at some point the parton recombination, described by the nonlinear evolution, must balance the effects of parton splitting so the partons will reach the state of saturation [50, 51, 53, 54]. In this high-density regime the coupling constant is small but the characteristic fields are large, making a perfect case for the application of the semiclassical QCD methods [55]. The high-density regime of QCD can serve as a bridge between the domain of pQCD and the real non-perturbative QCD regime governed by the physics of confinement.

**III.1 HIGH-ENERGY ASYMPTOTICS AS A SCATTERING FROM THE SHOCK-WAVE FIELD**

The amplitude of $\gamma p$ scattering is given by the matrix element

$$i \int d^4z \, d^4x \, e^{-ip_A \cdot x + ir \cdot z} \langle p_B' | T \{ j_A(x + z) j_A^*(z) \} | p_B \rangle = (2\pi)^4 \delta(p_B - p_B' - r) T^{AA'}(s,t)$$

(77)

where $r = p_B - p_B'$. $p_B^2 = m_A^2$. The DIS structure functions are given by the imaginary parts of the forward scattering amplitude

$$W^{AA'} = -\frac{1}{\pi} \Im T^{AA'}(s,0).$$

(78)

where $x_B = \frac{p_A^2}{s}$, $p_A^2 = -Q^2$. It is convenient to start with the upper part of the Feynman diagram (see Fig. 10), which allows us to study how fast quarks move in an external gluon field. At small $x_B$, gluon exchanges are mandatory and, therefore, functional integration over the slow gluon fields will reproduce the typical ladder diagrams.

The Regge limit $s \to \infty$ with $p_A^2$ fixed corresponds to the following rescaling of
the virtual photon momentum:

$$p_A = \lambda p_1^{(0)} + \frac{P^2_A}{2\lambda p_1^{(0)} \cdot p_2} \cdot p_2.$$  \hspace{1cm} (79)

with $p_B$ fixed. This is equivalent to

$$p_1 = \lambda p_1^{(0)}, \quad p_2 = p_2^{(0)}.$$  \hspace{1cm} (80)

where $p_1^{(0)}$ and $p_2^{(0)}$ are fixed light-like vectors so that $\lambda$ is a large parameter associated with the center-of-mass energy $s = 2\lambda p_1^{(0)} \cdot p_2^{(0)}$.

We study the asymptotics of high-energy $\gamma^* \gamma^*$ scattering from the fixed external field created by the nucleon

$$\int d^4 x d^4 z \ e^{-i p_A x + i r z} \langle T \{j_\mu(x + z) j_\nu(z)\}\rangle_A.$$  \hspace{1cm} (81)

Instead of rescaling the incoming photon’s momentum (79), it is convenient to boost the external field:

$$\int d^4 x d^4 z \ e^{-i p_A x + i r z} \langle T \{j_\mu(x + z) j_\nu(z)\}\rangle_A = \int d^4 x d^4 z \ e^{-i p_A x + i r z} \langle T \{j_\mu(x + z) j_\nu(z)\}\rangle_B.$$  \hspace{1cm} (82)

where $p_A^{(0)} = p_1^{(0)} + \frac{p_A^2}{m^0 p_2}$. The boosted field $B_\mu$ has the form

$$B_\mu(x_0, x_\star, x_\parallel) = \lambda A_0 \left( \frac{x_0}{\lambda} \cdot x_\star \cdot x_\parallel \right), \quad B_\star(x_0, x_\star, x_\parallel) = \frac{1}{\lambda} A_\star \left( \frac{x_\star}{\lambda} \cdot x_\star \cdot x_\parallel \right).$$  \hspace{1cm} (83)

$^2$The transverse components do not change: $B_\perp(x_0, x_\star, x_\parallel) = A_\perp \left( \frac{x_\perp}{\lambda} \cdot x_\star \cdot x_\parallel \right)$. 

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Here we use notation $x_0 \equiv x^\mu p_\mu^{(0)}, x_\perp \equiv x^\mu p_\perp$. The original external field in the coordinates independent of $\lambda$ has the form

$$A_\mu(x_0, x_\perp) = A_\mu \left( \frac{2}{s_0} x_0 p_1^{(0)} + \frac{2}{s_0} x_\perp p_2 + x_\perp \right)$$

(84)

Therefore, we may assume that the scales of $x_0, x_\perp$ in the function (84) are $\sim O(1)$. First, it is easy to see that at large $\lambda$ the boosted field $B_\mu(x)$ does not depend on $x_0$. Moreover, in the large $\lambda$ limit the field $B_\mu$ has the form of a shock wave $\sim \delta(x_\perp)$. One can see this if we write down the field strength tensor $G_{\mu\nu}$ for the boosted field. If we assume that the field strength $F_{\mu\nu}$ for the field $A_\mu$ vanishes at infinity, we obtain

$$G_{\mu\nu}(x_0, x_\perp) = \lambda F_{\mu\nu} \left( \frac{2}{s_0} x_0 x_\perp + x_\perp \right) - \delta(x_\perp) G_{\mu\nu}(x_\perp)$$

(85)

while all the other components of the field strength tensor $G_{\mu\nu}$ vanish. The only component surviving the infinite boost exists only within the thin wall near $x_\perp = 0$. In the rest of the space the field $B_\mu$ is a pure gauge. Let us denote by $\Omega$ the corresponding gauge matrix and by $B_\perp^{\Omega}$ the rotated gauge field which vanishes everywhere except the thin wall:

$$B_\perp^\Omega = \lim_{\lambda \to \infty} \frac{\partial}{\partial x_\perp} G_{\perp i}^{\Omega}(0, \lambda x_\perp) \rightarrow \delta(x_\perp) \frac{\partial}{\partial x_\perp} G_{\perp i}(x_\perp). \quad B_\perp = 0.$$  

(86)

Let's consider the propagator of the scalar particle (for example, the Faddeev-Popov ghost) in the shock-wave background. In Schwinger’s notations we write down formally the propagator in the external gluon field $A_\mu(x)$ as

$$G(x, y) = \left( \left| x \right| \left( \left| p^2 + m^2 \right| \right) \right) = \left( \left| x \right| \left( \left| (p + gA)^2 + m^2 \right| \right) \right).$$

(87)

where $\left( \left| x \right| \left| y \right| \right) = \delta(x - y)$ and

$$\left( \left| x \right|^\mu \left| y \right| \right) = -i \frac{\partial}{\partial y^\mu} \delta(x - y). \quad \left( \left| x \right| A_\mu \left| y \right| \right) = A_\mu(x) \delta(x - y).$$

(88)

Here $|x\rangle$ are the eigenstates of the normalized coordinate operator $\mathcal{X} |x\rangle = x |x\rangle$ (For details, see Appendix A in [56]). From Eq. (88), it is also easy to see that the eigenstates of the free momentum operator $p$ are the plane waves $|p\rangle = \int d^4x e^{-ipx} |x\rangle$. The path-integral representation of a scalar particle Green function in the external
field has the form:

\[
\left( \left| \frac{1}{p^2} \right| g \right) = -i \int_0^\infty dl \left( \left| e^{ilp^2} \right| g \right)
\]

\[
= -i \int_0^\infty dlN^{-1} \int_{\Gamma(0)=y} dx(t)e^{-il\int_0^t dt' \frac{x(t')}{l(t')}} \text{exp}\left\{ig \int_0^t dt' (B^\Omega_{\mu}(x(t)))x^\mu(t)\right\}.
\]

where \( l \) is Schwinger's proper time. It is clear that the interaction with the external field \( B^\Omega_{\mu} \) occurs at the intersection of the particle path with the shock wave (see Fig. 11). Therefore, it is convenient to rewrite at first the bare propagator marking the point of the intersection of the integration path with the plane \( z_\gamma = 0 \). After some algebra (see the review [57]) we arrive at the following representation of the bare propagator (in the case of \( x_\gamma > 0, y_\gamma < 0 \):

\[
\left( \left| \frac{1}{p^2 + i\nu} \right| g \right) = \frac{2}{s} \int dz^\gamma dz_\perp \frac{1}{4\pi^2(x-z)^2 \pi^2(z-y)^4}
\]

where \( z = \frac{2}{s} z^\gamma p^\gamma + z_\perp \) is the mentioned above point of the intersection of the particle path with the shock wave. Let's recall that our particle moves in the shock-wave external field and, therefore, each path in the functional integral (89) is weighted with an additional gauge factor \( Pe^{ig\int H_\mu dz^\mu} \). Since the external field exists only within the infinitely thin wall at \( x_\gamma = 0 \), we can replace the gauge factor along the actual path \( x^\mu(t) \) by that along the straight-line path shown in Fig. 11. It crosses the plane \( z_\gamma = 0 \) at the same point \((z_\gamma, z_\perp)\) at which the original path does. Since the shock-wave field outside the wall vanishes, we may formally extend the limits of this segment to infinity.
and write the corresponding gauge factor as $U^{\Omega}(z) = [\infty p_1 + z_1, -\infty p_1 + z_1]^3$. The error brought by replacement of the original path inside the wall with a segment of straight line parallel to $p_1$ is $\sqrt{\frac{m^2}{\pi}}$. Indeed, the time of the particle transition through the wall is proportional to the wall thickness $\sim \frac{m^2}{\pi}$, and the particle can deviate in the perpendicular directions inside the wall only within the distances $\sqrt{\frac{m^2}{\pi}}$. Thus, if the particle intersects this wall at some point $(z_1, z_\perp)$ the gauge factor $P_{\nu} = \int_0^1 d\nu A_\nu(x + (1 - \nu)y)$ reduces to $U^{\Omega}(z_\perp)$.

In the region $x_\perp > 0, y_\perp < 0$:

$$
\left(\left| \frac{1}{p^2} \right| y \right) = \int dz \delta(z_\perp) \frac{1}{4\pi^2(x - z)^2} U^{\Omega}(z_\perp) \frac{y_\perp}{\pi^2(z - y)^2} 
$$

It is easy to see that the propagator in the region $x_\perp < 0, y_\perp > 0$ differs from Eq. (92) only by the replacement $U^{\Omega} \rightarrow U^{\Omega}$. Also, the propagator outside the shock-wave wall at $x_\perp, y_\perp < 0$ or $x_\perp, y_\perp > 0$ coincides with the bare propagator. The final answer for the Green function of a scalar particle in the $B^{\Omega}$ background can be written down as:

$$
\left(\left| \frac{1}{p^2} \right| y \right) = \frac{i}{4\pi^2(x - y)^2} \theta(x_\perp y_\perp) + \int dz \delta(z_\perp) \frac{1}{4\pi^2(x - z)^2} \times \left\{ U^{\Omega}(z_\perp) \theta(-y_\perp) - U^{\Omega}(z_\perp) \theta(y_\perp) \right\} \frac{y_\perp}{\pi^2(z - y)^2}.
$$

We see that the propagator in a shock-wave background is a convolution of the free propagation up to the plane $z_\perp = 0$, instantaneous interaction with the shock wave described by the Wilson-line operator $U^{\Omega} (U^{\Omega})$ and another free propagation from $z$ to the final point (see Fig. 11)

In order to find the propagator in the original field $B_\mu$, we must perform the gauge rotation with the $\Omega$ matrix. It is convenient to represent the result in the following form:

$$
\left(\left| \frac{1}{p^2} \right| y \right) = \frac{i}{4\pi^2(x - y)^2} [x, y] \theta(x_\perp y_\perp) + \int dz \delta(z_\perp) \frac{1}{4\pi^2(x - z)^2} \times \left\{ U^{\perp}(z_\perp x, y) \theta(-y_\perp) - U^{\perp}(z_\perp x, y) \theta(y_\perp) \theta(-x_\perp) \right\} \frac{y_\perp}{\pi^2(z - y)^2}.
$$

\footnote{We use the following notation for a straight-line gauge link between points $x$ and $y$:

$$
[x, y] = \exp \left\{ i \theta \int_0^1 dv (x - y)^\mu A_\mu (x + (1 - v)y) \right\}
$$

(91)}
where

$$U(z_\perp; x, y) = [x, z_\perp][y, z_\perp][x, y]$$

$$z_x \equiv \left( \frac{\delta}{\delta_0} z_\perp + \frac{2}{\delta_0} x \cdot p_2 + \frac{q}{\delta_0} \right),$$

$$z_y \equiv z_x (x \cdot y),$$

is a gauge factor for the contour made from the straight line segments as shown in Fig. 11. Since the field $B_\mu$ outside the shock-wave wall is a pure gauge, the precise form of the contour does not matter as long as it starts at the point $x$, then intersects the wall at the point $z$ in the direction collinear to $p_2$, and ends at the point $y$. We have chosen this contour in such a way that the gauge factor (95) is the same for the field $B_\mu$ and for the original field $A_\mu$ (see Eq. (83)).

The quark propagator in a shock-wave background can be calculated in a similar way [58]

$$\begin{align*}
(x | \frac{1}{p^2} | y) &= - \frac{\delta - \gamma}{2\pi^2(x - y)^4} [x, y] \theta(x, y) + i \int dz \delta(z) \frac{\delta - \gamma}{2\pi^2(x - z)^4} \times \\
&\times \left\{ U(z_\perp; x, y) \theta(x) \theta(-y) - U^\dagger(z_\perp; x, y) \theta(y) \theta(-x) \right\} \frac{\delta - \gamma}{2\pi^2(z - y)^4}. \\
&= \frac{1}{4\pi^4(x - y)^8} \gamma_{\mu} \gamma_{\nu} \theta(x, y) \theta(-x, y) \int dz dz' \delta(z) \delta(z') \times \\
&\times \frac{\delta - \gamma}{2\pi^2(x - z)^4} \frac{\delta - \gamma}{2\pi^2(y - z')^4} \frac{\delta - \gamma}{2\pi^2(z' - x)^4} U(z_\perp; z_\perp'),
\end{align*}$$

For the quark-antiquark amplitude in the shock-wave field (see Fig. 11) we have

$$\begin{align*}
\text{Tr} \gamma_{\mu} \left( \left| \frac{1}{p^2} \right| y \right) \gamma_{\nu} \left( \left| \frac{1}{p^2} \right| x \right) &= \\
= \frac{1}{4\pi^4(x - y)^8} \gamma_{\mu} \gamma_{\nu} \theta(x, y) \theta(-x, y) \int dz dz' \delta(z) \delta(z') \times \\
&\times \frac{\delta - \gamma}{2\pi^2(x - z)^4} \frac{\delta - \gamma}{2\pi^2(y - z')^4} \frac{\delta - \gamma}{2\pi^2(z' - x)^4} U(z_\perp; z_\perp'),
\end{align*}$$

where the gauge factor $U(z_\perp; z_\perp') = U(z_\perp; x, y) U^\dagger(z_\perp'; y, x)$ is a product of two infinite Wilson-line operators connected by the gauge segments at $\pm \infty$.

$$U(z_\perp; z_\perp') = U[z_\perp; z_\perp'] U[z_\perp; z_\perp'].$$

We use the following space-saving notations$^3$

$$[x_\perp, y_\perp]_+ \equiv [\infty p_1 + x_\perp, \infty p_1 + y_\perp], \quad [x_\perp, y_\perp]_- \equiv [-\infty p_1 + x_\perp, -\infty p_1 + y_\perp].$$

As mentioned above, the precise form of the connecting contour at infinity does not matter as long as it is outside the shock wave. We have chosen this contour in

$^3$By definition, $[un, vn]_+ \equiv [un + x_\perp, vn + x_\perp]$. 

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such a way that the gauge factor (98) is the same for the field $B_\mu$ and for the original field $A_\mu$ (see Eq. (83)). Using our result for the quark-antiquark propagation (97) in the right-hand side of Eq. (81), we obtain
\[
\int d^4x \int d^4z \ e^{-ip_{A\perp}x + r_z} \langle T \{j_A(x + z)j_A^\dagger(z)\} \rangle_A = \int d^4x \int d^4z \ e^{-ip_{A\perp}x + r_z} \langle T \{j_A(x + z)j_A^\dagger(z)\} \rangle_A = 2\pi\delta(\alpha_r) \sum \epsilon_i^2 \int \frac{d^2k_{\perp}}{4\pi^2} I^A(k_{\perp}, r_{\perp}) \ Tr \{U(k_{\perp})U^\dagger(r_{\perp} - k_{\perp})\},
\]
where the impact factor $I^A$ is an explicit function of $Q^2$ and $k_{\perp}, r_{\perp}$ [57]. For simplicity, we have omitted the end gauge factors (99).

Formula (100) describes fast quark and antiquark moving through an external gluon field. After integrating over gluon fields in the functional integral we can find the amplitude (77) in the factorized form:
\[
(2\pi)^2\delta(r_{\perp} - p'_{B\perp}) 2\pi\delta(\beta_r) \ T(s, t) = \int d^2k_{\perp} \ I^A(k_{\perp}, r_{\perp}) \langle p_B' \ Tr\{U(k_{\perp})U^\dagger(r_{\perp} - k_{\perp})\} | p_B \rangle.
\]
where $U(k_{\perp})$ denotes the Fourier transform $\int dx_{\perp} U(x_{\perp}) e^{-i(k_{\perp}x_{\perp})}$. The gluon fields in $U$ and $U^\dagger$ have been promoted to operators, and we replaced $U$ by $\hat{U}$, etc. It is easy to see that the $\delta$-function of the transverse momenta is also present in the r.h.s. of Eq. (101) so that we can cancel it at both sides of the equation.

\[
2\pi\delta(\beta_r) \ T(s, t) = \int d^2k_{\perp} \ I^A(k_{\perp}, r_{\perp}) \int dx_{\perp} e^{-i(k_{\perp}x_{\perp})} \langle p_B' \ Tr\{\hat{U}(x_{\perp})\hat{U}^\dagger(0)\} | p_B \rangle.
\]
This matrix element describes the propagation of the color dipole through a nucleon.

The remaining $\delta$-function reflects the fact that the matrix element of the operator $U(x_{\perp})U^\dagger(0)$ contains the unrestricted integration along $p_1$. It is convenient to define the reduced matrix element:
\[
\langle p_B' \ Tr\{\hat{U}(x_{\perp})\hat{U}^\dagger(0)\} | p_B \rangle = 2\pi \delta \left(\frac{s}{2} \beta_r\right) \langle \langle Tr\{\hat{U}(x_{\perp})\hat{U}^\dagger(0)\} \rangle \rangle
\]
In the case of DIS (when $p_B' = p_B$), this matrix element is related to the non-integrated gluon distribution.
III.2 REGULARIZED WILSON-LINE OPERATORS

In the Regge limit $s \to \infty$ (79), we have formally obtained the operators $\hat{U}$ ordered along the light-like lines. Matrix elements of such operators contain divergent longitudinal integrations which reflect the fact that light-like gauge factor corresponds to a quark moving with the speed of light (i.e., with infinite energy). This divergence can be already seen at the one-loop level if one calculates the contribution to the matrix element of the two-Wilson-line operator $\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)$ between the virtual photon states. As mentioned above, the reason for this divergence is that we have replaced the fast-quark propagators in the external field (represented by two gluons coming from the bottom part of the diagram in Fig. 12a) with the light-like Wilson lines (see Fig. 12b).

The integration over rapidities of the gluon $q_p$ in the matrix element of the light-like Wilson-line operator $\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)$ is formally unbounded. Consequently, we need some regularization of the Wilson-line operator to cut off the fast gluons. It can be done by changing the slope of the supporting line [41].

FIG. 12. A typical Feynman diagram for the $\gamma^*\gamma^*$ scattering amplitude (a) and the corresponding two-Wilson-line operator (b).
If we wish to stop the longitudinal integration at \( \eta = \eta_0 \), we should order our gauge factors \( U \) along a line parallel to \( p_c = p_1 + \zeta p_2 \), where \( \eta_0 = \ln \zeta \). We define

\[
\hat{U}^c(t_\perp) \equiv [\infty p^c + x_\perp, -\infty p^c + x_\perp]. \quad \hat{U}^\dagger c(t_\perp) \equiv [-\infty p^c + x_\perp, \infty p^c + x_\perp].
\]

The matrix elements of these operators coincide with matrix elements of the operators \( \hat{U} \) and \( \hat{U}^\dagger \) calculated with the restriction \( \alpha < \sigma = \sqrt{p^2_1 s \zeta} \) imposed inside internal loops and external tails. Let us demonstrate this using a simple example of the matrix element of the operator \( \hat{U}^c(k_\perp)\hat{U}^\dagger c(r_\perp - k_\perp) \) sandwiched between the virtual photon states. The contribution from the diagram shown in Fig. 12 has the form

\[
\frac{-i g^6}{2 \pi} \int \frac{d\alpha_p}{16 \pi^2} \int \frac{d\alpha_{k'}}{16 \pi^2} \frac{[(\alpha_p - 2\alpha_k')\delta_k^c s - (\vec{k} + \vec{k}')^2]\Phi^B(k')}{[(\alpha_p - \alpha')\delta_{k'(k - k')}]\delta^c(k - \vec{k'})^2 + ic^]{}}\times
\]

where the numerator comes from the product of two three-gluon vertices \( \Gamma^c_{\mu\nu} \):

\[
4 s^2 \epsilon^a (k, -k') \Gamma^c_{\mu\nu}(k, -k') = (\alpha_k - 2\alpha_k')\delta_k^c s - (\vec{k} + \vec{k}')^2 (106)
\]

where the three-gluon vertex \( \Gamma^c_{\mu\nu}(k, k') = (k + k')^c \delta_{\mu\nu} + (k - 2k')_c \delta_{\mu}^c + (k - 2k')_c \delta_{\nu}^c \).

As we shall see below, the logarithmic contribution comes from the region \( m^2 \gg \alpha_k \gg \alpha_k' \sim \frac{m^2}{c s} \) (cf. Eq. (18)). The lower limit of this logarithmical integration is provided by the matrix element itself \( (3k - 1) \) in the lower quark bulb), while the upper limit at \( \alpha_k' \sim m^2/\zeta s \) is enforced by the non-zero \( \zeta \). The result has the form

\[
\langle \text{Tr}\hat{U}^c(k_\perp)\hat{U}^\dagger c(-k_\perp) \rangle_{\text{Fig. 12}} = \frac{g^6}{8\pi} \ln \left( \frac{\kappa}{m^2 \zeta} \right) \int \frac{d^2k'_{\perp}}{4\pi^2} \frac{\vec{k}^2_{\perp} + p^2_{\perp}}{\vec{k}^2_{\perp} + p^2_{\perp}} \Phi^B(k').
\]

(108)
Similarly to the case of the usual light-cone expansion, we expand the amplitude in a set of the regularized Wilson-line operators $\hat{U}^\zeta$ (see Fig. 13: the double Wilson line corresponds to a fast-moving gluon):

$$A(p_A, p_B \rightarrow p_A', p_B') = \sum \int d^2x_1...d^2x_n \zeta(x_1,...,x_n; \zeta)$$

$$\times \langle p_B | \text{Tr} \{ \hat{U}^\zeta(x_1)\hat{U}^\zeta(x_2)\ldots\hat{U}^\zeta(x_{n-1})\hat{U}^\zeta(x_n)| p_B' \rangle. $$

The coefficient functions in front of the Wilson-line operators (impact factors) contain logarithms $\sim g^2 \ln 1/\sigma$ and matrix elements $\sim g^2 \ln \frac{s}{m^2}$. Similar to the DIS, for amplitude calculation we add terms $\sim g^2 \ln \frac{s}{m^2}$ (coming from the coefficient functions, Fig. 13b) to the terms $\sim g^2 \ln \frac{s}{m^2}$ (coming from the matrix elements, Fig. 13a) so that the dependence on the rapidity divide $\sigma$ cancels resulting in the usual high-energy factors $g^2 \ln \frac{s}{m^2}$ responsible for the BFKL pomeron.

In the LLA, the light-like operators $\hat{U}$ and $\hat{U}^\dagger$ in Eq. (101) should be replaced by the Wilson-line operators $\hat{U}^\zeta$ and $\hat{U}^\zeta$ ordered along $n \parallel p_A$. Indeed, let us compare the matrix element (108) shown in Fig. 12b to the corresponding physical amplitude shown in Fig. 12a. The Feynman integral for this amplitude is similar to the one for the matrix element of the operator (108), except that there is now a factor of the upper quark bulb and the integral over $p_\perp$ (see Ref. [58] for details):

$$\sim i \frac{g^6}{4\pi} \ln \left( \frac{s}{m^2} \right) \int \frac{d^2k_1}{4\pi^2} \frac{d^2k'_1}{4\pi^2} \frac{d^2k_2}{4\pi^2} \frac{d^2k'_2}{4\pi^2} \frac{d^2p_\perp}{4\pi^2} \frac{d^2p'_\perp}{4\pi^2}$$

$$\times t_A(k_{1\perp}) t_B(k'_{1\perp})$$

The result (110) agrees with the with estimate Eq. (108) if we set $\zeta = \frac{p_A^\perp}{s}$. This corresponds to making the line in the path-ordered exponential collinear to the momentum of the photon.

![Diagram](image-url)
III.3 ONE-LOOP EVOLUTION OF WILSON-LINE OPERATORS

As we demonstrated in the previous Section, with the LLA accuracy, the improved version of the factorization formula Eq. (100) has the operators $\hat{U}$ and $\hat{U}^\dagger$ regularized at $\zeta \sim \frac{p_A}{s}$:

$$\int d^4x \int d^4z \, e^{-ip_A x + mr_z} \, \mathcal{T}\{j(x)j(z)\} = $n_A(x+z)j_A(z)\} = 2\pi \delta(\alpha_s) \sum_i e_i^2 \int \frac{d^2k}{4\pi^2} \, I^A(k_\perp, r_\perp) \, \text{Tr}\{\hat{U} j^{x=\frac{m_1}{s}}(k) \hat{U} j^{z=\frac{m_2}{s}}(r-k)\}.$$  \hfill (111)

In the next-to-leading order (NLO) in $\alpha_s$ we have corrections $\sim \alpha_s \text{Tr}\hat{U}(x)\hat{U}^\dagger(y)\text{Tr}\hat{U}(y)\hat{U}^\dagger(z)$ (see Fig. 13).

Our next step is to derive the evolution equation of these operators in the LLA with respect to the slope $\zeta$. In order to find dependence of the light-cone operator on the normalization point $\mu$ in case of the ordinary Wilson OPE, we integrate over the momenta with virtualities $\mu_1^2 > p^2 > \mu_2^2$. Likewise, in order to find the behavior of the matrix elements of the operators $\hat{U} j^{x}(x)\hat{U}^\dagger j^{z}(y)$ with respect to the slope $\zeta$, we must take the matrix element of this normalized at $\zeta_1$ operator and integrate over the momenta with $\sigma_1 = \sqrt{m_1^2} = \alpha > \sigma_2 = \sqrt{m_2^2}$. The result will be the operators $\hat{U}$ and $\hat{U}^\dagger$ normalized at the slope $\zeta_2$ times the coefficient functions determining the evolution equation kernel. The calculation of the kernel is essentially identical to the calculation of the impact factor with the only difference of having initial gluons instead of quarks.

In the first order in $\alpha_s$ there are two one-loop diagrams for the matrix element of the operator $\hat{U}(x)\hat{U}^\dagger(y)$ in an external field (see Fig. 14). Let us start with the diagram shown in Fig. 14a. Our goal is to calculate the one-loop evolution of...
the operator $\hat{U}(x_\perp) \times \hat{U}^\dagger(y_\perp) \equiv \{\hat{U}(x_\perp)\}^i_j \{\hat{U}^\dagger(y_\perp)\}_f^k$ with the non-convoluted color indices.

Following Ref. [41], we use the gauge $A_\perp = 0$ for our calculations. In this gauge, the gluon propagator in the external field has the following form $^5$:

$$iG^{ab}_{\mu\nu}(x, y) = \left( \delta^{\xi}_{\mu} - \mathcal{P}^\xi_{\mu} \frac{\xi}{\mathcal{P}^\xi_{\nu}} \right) \left[ \delta^{\eta}_{\xi} - 2i \frac{1}{\mathcal{P}^2} F_{\xi\eta} \frac{1}{\mathcal{P}^2} + \mathcal{O}(\eta) \right] \left( \delta^{\eta}_{\nu} - \frac{\eta}{\mathcal{P}^\xi_{\nu}} \mathcal{P}^\xi_{\nu} \right) + \ldots \quad (112)$$

where the operator $\mathcal{O}$ stands for

$$\mathcal{O}_{\mu\nu} = \frac{4}{\mathcal{P}^2} F_\xi^\xi_{\mu} \frac{1}{\mathcal{P}^2} F_{\xi\nu}^{\xi} \frac{1}{\mathcal{P}^2} \mathcal{P}_\xi^\xi_{\nu} - \frac{1}{\mathcal{P}^2} (D^\alpha F_{\alpha\mu} \frac{p_{2\nu}}{p \cdot p_2} + \frac{p_{2\mu}}{p \cdot p_2} D^\alpha F_{\alpha\nu} - \frac{p_{2\mu}}{2p \cdot p_2} D^\alpha F_{\alpha\nu} \frac{p_{2\nu}}{2p \cdot p_2}) \frac{1}{\mathcal{P}^2}. \quad (113)$$

In the LLA, the slope $\hat{p}_\perp$ of the operators $\hat{U}$ can be replaced by $p_1$ so that we need the $(\bullet \bullet)$ component of this propagator. It is easy to see that we may drop the terms proportional to $\mathcal{P}_\perp$ in the parenthesis as they lead to the terms proportional to the integrals of total derivatives $[57]$:

$$\langle \hat{U}(x_\perp) \times \hat{U}^\dagger(y_\perp) \rangle_A = -ig^2 \int du \left[ \propto p_1, \nu p_1 \right] L^a \left[ \nu p_1, -\propto p_1 \right] \propto \int dv \left[ -\propto p_1, \nu p_1 \right]_y t^{b} \left[ \nu p_1, \propto p_1 \right]_y \left( \left[ u p_1 + x_\perp \right] \mathcal{O}_{\bullet \bullet} \left[ \nu p_1 + y_\perp \right] \right)_{ab} \quad (114)$$

Similar to the calculation of the quark propagator, it is convenient to go to the rest frame of the fast gluons, where the slow gluons form a thin pancake shown in Fig. 15. Let us consider the case $x_\perp > 0, y_\perp < 0$. We can rewrite Eq. (114) as follows:

$$\langle \hat{U}_x \times \hat{U}_y^\dagger \rangle_A = -ig^2 \int dy \left[ \propto p_1, \nu p_1 \right] \int_0^\infty dl \int_{-\infty}^\infty dv \left( \left( \nu p_1 + x_\perp \right) \mathcal{O}_{\bullet \bullet} \left[ \nu p_1 + y_\perp \right] \right)_{ab} \quad (115)$$

Using the thin-wall approximation we obtain

$$\left( \left[ x \left| \mathcal{O}_{\bullet \bullet} \right| y \right] \right) = \frac{s^2}{2} \int dz \delta(z_\perp) \frac{\ln(x - z)^2}{16\pi^2 x_\perp} \frac{1}{4\pi^2 (z - y)^2}. \quad (116)$$

where

$$[DF](x_\perp) \equiv \int du \left[ \propto p_1, \nu p_1 \right] D^\alpha F_{\bullet \bullet}(\nu p_1 + x_\perp) \left[ \nu p_1, -\propto p_1 \right]_x,$$

$$[FF](x_\perp) \equiv \int du \int dv \Theta(u - v) \left[ \propto p_1, \nu p_1 \right] F^{\xi}_\bullet(\nu p_1 + x_\perp) \times \left[ \nu p_1, \nu p_1 \right]_x F^{\xi}_\bullet(\nu p_1 + x_\perp) \left[ \nu p_1, -\propto p_1 \right]_x. \quad (117)$$

$^5$It can be demonstrated that further terms in expansion in powers of gluon propagator beyond those given in Eq. (112) do not contribute in the LLA.
FIG. 15. Wilson-line operators in the shock-wave field background.

It is easy to see that the operators in braces are, in fact, the total derivatives of $U$ and $U^\dagger$ with respect to translations in the perpendicular directions ($\partial^2_{\perp} U = -\partial^2_{\perp} U^\dagger$):

$$\partial^2_{\perp} U_r \equiv \frac{\partial^2}{\partial x_r \partial x_i} U_r = -i[DF](x_{\perp}) + 2[FF](x_{\perp}).$$

$$\partial^2_{\perp} U^\dagger_r \equiv \frac{\partial^2}{\partial x_r \partial x_i} U^\dagger_r = i[DF](x_{\perp}) + 2[FF](x_{\perp}).$$

(118)

Technically, it is easier to find the derivative of Eq. (115) with respect to $x_{\perp}$:

$$-ig^2 \int_0^\infty du \int_{-\infty}^0 dv \left( \left| pp^{(0)}_A + p_i \mathcal{O}^{\bullet} | pp^{(0)}_A + y_{\perp} \right\rangle \right)_{ab} = \frac{g^2}{16\pi^4} \int dz_{\perp} \times$$

$$\int_0^\infty \frac{du}{u} \int_{-\infty}^0 dv \int dz_{\perp} \left[ (u\zeta_s - 2z_{\perp})^2 - (\bar{\tau} - \bar{z})^2_{\perp} - i\tau \right][v(\epsilon\zeta_s + 2z_{\perp}) - (\bar{\tau} - \bar{z})^2_{\perp} - i\tau].$$

The integration over $z_{\perp}$ can be performed by taking the residue:

$$-ig^2 \int_0^\infty \frac{du}{u} \int_{-\infty}^0 dv \left( \frac{z_{\perp} - z_{\perp}}{[(\bar{\tau} - \bar{z})^2_{\perp} v + (\bar{\tau} - \bar{z})^2_{\perp} v - uv(u + v)\zeta_s + i\tau]} \right).$$

(120)

This integral diverges logarithmically when the emission of quantum gluon occurs in the vicinity of a shock wave ($u \to 0$). It is convenient to replace $u$ and $v$ by $w = u + v$ and $\alpha = \frac{w}{u+v}$ ($\bar{\alpha} = 1 - \alpha$). The lower limit of the logarithmic integral over $w$ is the size of the shock wave $z_{\perp} \sim m^{-1} \zeta_s$, where $1/m$ is the characteristic transverse size. Hence, in the LLA we have

$$-i \frac{g^2}{16\pi^3} \ln \frac{\sigma_1}{\sigma_2} \int dz_{\perp} \int_0^1 d\alpha \frac{(x_{\perp} - z_{\perp})}{\alpha [\bar{\tau} - \bar{z}]^2_{\perp} \alpha + (\bar{\tau} - \bar{z})^2_{\perp} \alpha]} =$$

$$= -i \frac{g^2}{16\pi^3} \ln \frac{\sigma_1}{\sigma_2} \left( \left| \frac{p_i}{p^2_{\perp}} \left( \frac{\partial^2_{\perp}}{p^2_{\perp}} \right) \left| y_{\perp} \right\rangle \right)_{ab}. $$

(121)
Thus, the contribution of the diagrams shown in Fig. 15 in the LLA takes the form
\[
\langle \hat{U}_x \otimes \hat{U}_y \rangle_A = - \left( \frac{g^2}{2\pi} \ln \frac{\sigma_1}{\sigma_2} \right) \left\{ t^a U_x \otimes t^b U_y \left( x_\perp \left| \frac{1}{p_1^\perp} (\partial_\perp^2 U) \frac{1}{p_2^\perp} y_\perp \right) \right\}_{ab} + U_x t^b \otimes U_y t^a \left( y_\perp \left| \frac{1}{p^\perp_1} (\partial_\perp^2 U) \frac{1}{p_2^\perp} x_\perp \right) \right\}_{ab},
\]  
(122)
where we have included the term coming from \( x_+ < 0, y_+ > 0 \) (see Fig. 15b).

Likewise, for the diagrams shown in Fig. 16 the result can be obtained by comparing the space-time picture Fig. 16a with Fig. 15a:
\[
\langle \hat{U}_x \otimes \hat{U}_y \rangle_A = \left( \frac{g^2}{2\pi} \ln \frac{\sigma_1}{\sigma_2} \right) \left\{ t^a U_x \otimes t^b U_y \left( x_\perp \left| \frac{1}{p_1^\perp} (\partial_\perp^2 U) \frac{1}{p_2^\perp} y_\perp \right) \right\}_{ab} + U_x t^b \otimes U_y t^a \left( y_\perp \left| \frac{1}{p_1^\perp} (\partial_\perp^2 U) \frac{1}{p_2^\perp} x_\perp \right) \right\}_{ab}.
\]  
(123)

The total result for the one-loop evolution of two-Wilson-line operator is the sum of Eqs. (122) and (123) [41]:
\[
\zeta \frac{\partial}{\partial \zeta} U(x_\perp, y_\perp) = - \frac{\alpha_s N_c}{4\pi^2} \int d\zeta \left\{ U(x_\perp, z_\perp) + U(z_\perp, y_\perp) - U(x_\perp, y_\perp) - U(x, z) U(z, y) \right\} \frac{\bar{f}_1 (\bar{y})^2}{(\bar{x}_\perp - \bar{z}_\perp)^2 (\bar{z}_\perp - \bar{y}_\perp)^2},
\]  
(124)
where
\[
U(x_\perp, y_\perp) \equiv - \frac{1}{N_c} \left\{ \text{Tr} \left\{ U(x_\perp) | x_\perp, y_\perp \right\} U^\dagger(y_\perp) | y_\perp, x_\perp \right\} \right\} - N_c
\]  
(125)

This equation describes the multiplication of pomerons due to so-called fan diagrams. The r.h.s. of this equation is IR and UV finite \(^6\). It is clear that the result of

---

\(^6\)The IR finiteness is due to the fact that Tr \{UU^\dagger\} corresponds to the colorless state in t-channel.
the two-line operator $\text{Tr}\{U^U\}$ evolution is the same operator (times the kernel) plus
the four-line operator $\text{Tr}\{U^U\}\text{Tr}\{U^U\}$. Accordingly, the result of the evolution of
the four-line operator is the same operator (times some kernel) plus the six-line op­
erator of the type $\text{Tr}\{U^U\}\text{Tr}\{U^U\}\text{Tr}\{U^U\} + \text{Tr}\{U^U\}\text{Tr}\{U^U\}\text{Tr}\{U^U\}$ and so on.

III.4 QUARK CONTRIBUTION TO THE SMALL-X EVOLUTION OF
COLOR DIPOLE

The non-linear equation for the small-$x$ evolution has the form

$$\frac{d}{dt}\text{Tr}\{U_x \otimes U^*_x\} = \frac{\alpha_s}{4\pi} \int d^2z \frac{(x-y)^2}{(x-z)(z-y)^2} \text{Tr}\{U_x U^*_z\}\text{Tr}\{U_z U^*_x\} - \lambda_c \text{Tr}\{U^*_z U^*_x\}$$

(126)

The argument of the coupling constant in Eq. (126), that was left undetermined in
our calculation, is very important from the both theoretical and experimental points
of view. From the theoretical viewpoint, it is essential whether the coupling constant
is determined by the size of the original dipole $|x - y|$ or by the size of the produced
dipoles ($|x - z|$ and/or $|z - y|$) since they lead to a very different behavior of the
equation solution. On the experimental side, the cross section is proportional to some
power of the coupling constant so that the argument determines how big/small the
cross section is. The typical argument of $\alpha_s$ is the characteristic transverse momenta

As a consequence, the IR divergent parts coming from the diagrams in Figs. 15 and 16 cancel out. However, the result is IR divergent in case of the exchange by color state in t-channel [57].
of the process. For high enough energies, they are believed to be of the order of the saturation scale $Q_s \sim 2 - 3 \text{ GeV for the LHC collider}$. Thus, one can see that even the difference between $\alpha(Q_s)$ and $\alpha(2Q_s)$ can make a huge impact on the cross section.

The argument of the coupling constant cannot be determined in the LLA so that the NLO calculation is in order. In the next-to-leading order the non-linear equation (126) looks as follows

$$
\frac{d}{dy} \mathcal{R} \{ U_x \otimes U_y^\dagger \} = \frac{1}{4\pi} \int d^2z \left\{ \alpha_s \frac{(x-y)^2}{(x-z)^2(z-y)^2} + \alpha_s^2 K_{\text{NLO}}(x,y,z) \right\} \times \\
\times \mathcal{R} \{ U_x U_y^\dagger \} \mathcal{R} \{ U_x U_y^\dagger \} - N_c \mathcal{R} \{ U_x U_y^\dagger \} + \\
+ \alpha_s^2 \int d^2z \int d^2z' \left( K_4(x,y,z,z') \{ U_x U_y^\dagger, U_x U_y^\dagger \} + \\
+ K_6(x,y,z,z') \{ U_x U_y^\dagger, U_x U_y^\dagger, U_x U_y^\dagger \} \right).$$

where $K_{\text{NLO}}$ is the next-to-leading order correction to the dipole kernel and $K_4$ and $K_6$ are the coefficients in front of the (tree) four- and six-Wilson line operators with arbitrary white arrangements of color indices. Note that $K_{\text{NLO}}$ must describe the known LO BFKL contribution. For now, we have completed the calculation of the quark part of the kernel and calculation of the gluon part is still in progress.

It should be mentioned, that knowing the NLO result does not lead automatically to the argument of the coupling constant. In order to find this argument, we can use the renormalon-based ones. As was discussed in Chapter II, in the leading logarithmic approximation the quark part of the running coupling constant in front of the leading term in Eq. (127) comes from the bubble chain of quark loops. This bubble chain will be summed up in Section III.4.2, but, at first, we need to present the results of the next-to-leading order calculation.

### III.4.1 Quark contribution to the NLO kernel

There are two types of quark contribution in the NLO: with quarks in the loop interacting with the shock wave (Fig. 17) or without (Fig. 20). The contribution of the diagram shown in Fig. 17 has the following form

$$
U_x \otimes U_y^\dagger \in 4\alpha_s^2 N_f \Delta \eta \left[ t^a U_x \otimes t^a U_y^\dagger + U_x t^b \otimes U_y^\dagger t^a \right] \int d^2k_1 d^2k_2 d^2k'_1 d^2k'_2 \times
$$
where $N_f$ is a number of light quarks ($N_f = 3$ for the momenta $Q, 1 \div 2$ GeV) and $\Delta \eta$ is the rapidity interval. To calculate this diagram we use the dimensional regularization and change the dimension of the transverse space to $d = 2 + \epsilon$. The calculation yields to

\[
4\alpha_s^2 N_f \Delta \eta \left[ t^a U_x \otimes t^b U_y + U_x t^b \otimes U_y t^a \right] \frac{\mu^{2-d}}{4\pi} \int dp \, dq \, dq' \, e^{i(p \cdot x) - i(p-q \cdot y)} \times
\]

\[
\times \int_0^1 du \frac{(k_1 \cdot k_2)(k'_1 \cdot k'_2) + (k_1 \cdot k'_2)(k'_1 \cdot k_2) - (k_1 \cdot k'_2)(k'_1 \cdot k_2)}{(k_1 + k'_2)^2(k_2 + k'_1)^2(k_1^2 u + k'_1^2 u + k_2^2 \bar{u})}.
\]

(128)

where $P = p - (q + q') u$, $Q^2 \equiv q^2 \bar{v} + q'^2 v$.

The contribution of two diagrams in Fig. 18 is

\[
2\alpha_s^2 N_f \Delta \eta \left[ t^a U_x \otimes t^b U_y + U_x t^b \otimes U_y t^a \right] \frac{\mu^{2-d}}{12\pi} \int dp \, dl \, e^{i(p \cdot x) - i(p-l \cdot y)} \times
\]

\[
\times \frac{1}{p^2} \left( \frac{\Gamma(1 - \frac{d}{2})}{(p_\perp^2)^{1-\frac{d}{2}}} + \frac{\Gamma(1 - \frac{d}{2})}{((p-l)^2)^{1-\frac{d}{2}}} \right) \frac{1}{p^2} i l^{ab}(l_\perp).
\]

(129)

The sum of the contributions (128) and (129) takes the form

\[
2\alpha_s^2 N_f \Delta \eta \left[ t^a U_x \otimes t^b U_y + U_x t^b \otimes U_y t^a \right] \frac{\mu^{2-d}}{4\pi} \left( \int dp \, dl \, e^{i(p \cdot x) - i(p-l \cdot y)} \frac{1}{p^2} \right)
\]
FIG. 19. Quark loop inside the shock wave.

\[ \times \left[ \frac{\Gamma(1 - \frac{d}{2})}{(p^2)^{1-\frac{d}{2}}} + \frac{\Gamma(1 - \frac{d}{2})}{(p - l)^{2}} \right] - \int_0^1 du \frac{\Gamma(1 - \frac{d}{2})}{(p^2 u + (p - l)^2 u)^{1-\frac{d}{2}}} \frac{1}{(p - l)^{2}} \partial^2 U^{ab}(l) + \]

\[ + \int d q \frac{2\Gamma(2 - \frac{d}{2})}{(P^2 + Q^2 u)^{2-\frac{d}{2}}} \left\{ 2P^2[\bar{u}v(q, l - q) - \bar{u}uQ^2 + 2\bar{u}u\bar{v}(q^2 + (l - q)^2)] - \\
+ 2\bar{u}(1 - 2u)[\bar{u}v(q, l - q)(P, l) + \bar{v}q(P, l - q) + v(l - q)^2(P, q)] - \\
- 4\bar{u}\bar{u}v(P, q)(P, l - q) + 2\bar{u}^2Q^2l^2 \right\} \text{Tr}^a l^a U(q)l^b l^c(q') \].

(130)

Subtracting the pole at \( d = 2 \), as required by the dimensional regularization, we derive

\[ \alpha_s \Delta \eta \left[ t^a U_x \otimes t^b U_y + U_x t^b \otimes U_y t^a \right] \int d p d l \ e^{i(p,x)l - (p - l,y)} \left( \frac{\alpha_s}{6\pi} N_f \right) \frac{1}{p^2} \times \\
\left[ \int_0^1 du \ln \left( \frac{p^2 u + (p - l)^2 u}{\mu^2} \right) - \ln \left( \frac{p^2}{\mu^2} \right) \right] \frac{1}{(p - l)^{2}} \partial^2 U^{ab}(l) + \]

\[ + \frac{\alpha_s}{\pi} N_f \int d q \frac{1}{p^2 + Q^2} \left\{ 2P^2[\bar{u}v(q, l - q) - \bar{u}uQ^2 + 2\bar{u}u\bar{v}(q^2 + (l - q)^2)] + \\
+ 2\bar{u}(1 - 2u)[\bar{u}v(q, l - q)(P, l) + \bar{v}q(P, l - q) + v(l - q)^2(P, q)] - \\
- 4\bar{u}\bar{u}v(P, q)(P, l - q) + 2\bar{u}^2Q^2l^2 \right\} \text{Tr}^a l^a U(q)l^b l^c(q') \].

(131)

We see that the first (UV) term proportional to \( \ln(...)^2 \mu^2 \) has the same structure as the zero-order contribution (126). In the next section we will use it to determine the argument of the running coupling constant in Eq. (126).

It can be demonstrated that there are no additional terms coming from inside
III.4.2 Bubble chain and the argument of the coupling constant

To find the argument of the coupling constant, we can trace the quark part of the J-function (proportional to $N_f$). Since the quark part of the J-function in the LLA comes from the bubble chain of quark loops, we can either have no intersection of quark loops with the shock wave (Fig. 20), or we may have one of the loops in the shock-wave background (Fig. 21). It is easy to see that the sum of these diagrams yields to

$$\alpha_s(\mu) \Delta \eta \left[ t^a U_x \otimes t^b U_y + U_x U_y \otimes U_y U^a \right] \int d^2 p \, d^2 l \, e^{i(p, r) - i(p - l, y)} \times$$

$$\times \sum_{m=0}^{\infty} \left( \frac{\alpha_s(\mu) N_f}{6 \pi} \ln \frac{\mu^2}{\mu^2} \right)^m 1 \left[ 1 + \frac{\alpha_s(\mu) N_f}{6 \pi} \int_0^1 du \, \ln \frac{p_+^2 \bar{u} + (p - l)^2 u}{\mu^2} \right] \times$$

$$\times \sum_{n=0}^{\infty} \left( \frac{\alpha_s(\mu) N_f}{6 \pi} \ln \frac{(p - l)^2}{\mu^2} \right)^n \frac{1}{(p - l)^2} \delta_{l}^{ab} (l_\perp), \quad (132)$$

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where we left only the UV part of the contribution (131) proportional to the leading term (126) multiplied by the logarithms of \((\text{momenta})^2/\mu^2\). Replacing the quark part of the \(\beta\)-function \(\frac{\alpha_s}{\pi} N_f \ln \frac{p^2}{\mu^2}\) by the total contribution \(-\frac{\alpha_s}{\pi} \beta_0 \ln \frac{p^2}{\mu^2}\) we derive

\[
\Delta \eta \left[ \left( t^a U_x \otimes t^b U_y \right) + U_x t^b \otimes t^a U_y \right] \int d^2 p \ d^2 l \ e^{i(p_x l_x - i(p_y l_y)_+} \times \right.
\]

\[
\times \frac{\alpha_s(p^2)}{p^2} \int_0^1 \frac{du}{\alpha_s(p_x^2 u + (p - l)^2 u)} \frac{\alpha_s((p - l)^2)}{(p - l)_+^2} \delta^2_l U^{ab}(l_+) \quad (133)
\]

Therefore, our final result for the argument of the coupling constant in the nonlinear equation (126) is

\[
\frac{\partial}{\partial \eta} U_x \otimes U_y = \left[ \left( t^a U_x \otimes t^b U_y \right) + U_x t^b \otimes t^a U_y \right] \int d^2 p \ d^2 l \ e^{i(p_x l_x - i(p_y l_y)_+} \times \right.
\]

\[
\times \frac{\alpha_s(p^2)}{p^2} \left( \int_0^1 \frac{du}{\alpha_s(p_x^2 u + (p - l)^2 u)} \frac{\alpha_s((p - l)^2)}{(p - l)_+^2} \right) \delta^2_l U^{ab}(l_+) \quad (134)
\]

### III.5 CONCLUSIONS AND OUTLOOK

Gribov, Levin and Ryskin, in their pioneering papers [40, 53] first discussed the QCD evolution process and have suggested the first nonlinear evolution equation (the GLR equation). Later, a number of studies have led to the most general nonlinear evolution equation in the large-\(N_c\) limit

\[
\frac{d}{d \eta} N(x, y) = \int d^2 z \ K_1(x, z, y) \left\{ N(x, z) + N(z, y) \right\} +
\]

\[
+ \int d^2 z \ K_2(x, z, y) \ N(x, z) N(z, y) +
\]

\[
+ \int d^2 z d^2 z' K_3(x, z, z', y) \ N(x, z) N(z, z') N(z', y) + \ldots \quad (135)
\]

where the evolution kernels \(K_1\) and \(K_2\) are known in the leading order of perturbation theory. Obviously, the LO \(K_1\) kernel coincides with the BFKL evolution kernel. The LO \(K_2\) kernel for the first nonlinearity has also been found in a number of studies. We have derived the result for the argument of the coupling constant in the nonlinear evolution of dipoles using the quark part of the \(\beta\)-function. This argument is essential since it has a huge impact on the cross section and leads to a different
behavior of the equation solution. However, it would be extremely interesting to confirm this result by calculating the diagrams with gluon loops. That study is in progress.
CHAPTER IV

SCATTERING OF COLOR DIPOLES

Let us consider a classical example of the scattering of virtual photons in QCD. The photons decompose into quark-antiquark pairs that interact by exchanging gluons. As was mentioned in Chapter III, at high energies quarks move very fast so that their propagation in the background fields of the exchange gluons is reduced to the gauge factor of $U^n(x_\perp)$:

$$U^n(x_\perp) = P e^{ig \int_{x_\perp}^{x_\perp} d\nu \, n_\mu A^\mu(x_\perp + \nu)}.$$  \hspace{1cm} (136)

ordered along a classical trajectory of the particle, i.e. a straight line collinear to the velocity $n$. Here $x_\perp$ is the transverse position (an impact parameter) of the fast quark, that does not change in the collision. The propagation of a quark-antiquark pair is described by the color dipole [59] formed from the two Wilson lines

$$W^n(x_\perp, y_\perp) = \text{Tr} \{ U^n(x_\perp) U^n(y_\perp) \}.$$  \hspace{1cm} (137)

where $x_\perp$ and $y_\perp$ are the transverse positions (impact parameters) of the quark and antiquark. The high-energy $\gamma^*\gamma^*$ scattering reduces then to the dipole-dipole amplitude integrated over dipole sizes and separations (see Fig. 22, where the wavy lines are gluons and the dashed ones are the Wilson gauge factors):

$$A(s) = \frac{s}{2} \int d^2a_\perp d^2b_\perp \, I^A(a_\perp) I^B(b_\perp) \, T(a_\perp, b_\perp; s)$$  \hspace{1cm} (138)

where $I^A(a_\perp)$ is the so-called impact factor (see Appendix C) and $T(a_\perp, b_\perp; s)$ is the dipole-dipole scattering amplitude:

$$T(a_\perp, b_\perp; s) = -i \int d^2z_\perp \langle W^n(a_\perp + z_\perp, 0_\perp) \, W^r(b_\perp, 0_\perp) \rangle.$$  \hspace{1cm} (139)

Here $e$ is the unit vector in the direction of the motion of the second virtual photon. The collision energy is related to the relative rapidity $\eta$ (\equiv angle between the dipoles):

$$\frac{p_A \cdot p_B}{\sqrt{p_A^2 p_B^2}} = n \cdot e = \cosh \eta.$$  \hspace{1cm} (140)

The energy dependence of the $\gamma^*\gamma^*$ amplitude is governed by evolution of the dipoles with respect to the slope of the Wilson lines determined by $\eta$ [41].
We see that the dipole-dipole amplitude is an elementary high-energy scattering process. In theories without asymptotic particle states (for instance, N=4 SYM) the dipole-dipole scattering is the only way to access the high-energy behavior of amplitudes. Moreover, there have been numerous efforts to estimate the high-energy behavior of hadron-hadron amplitudes using non-perturbative models for the dipole-dipole scattering. Unfortunately, the hadron impact factors cannot be calculated in pQCD and can only be related to the hadron wave functions.

The usual phenomenological approach to the dipole-dipole scattering in pQCD is to take the light-like dipoles. However, the matrix elements of the light-like Wilson operators contain longitudinal divergencies. The usual LLA technique to take the light-like dipoles and to cut all arising divergences [44, 60] seems to be avoiding ambiguities in the next-to-leading order too [61]. However, to go beyond the perturbation theory we have to consider the scattering of the off-light-cone dipoles. The dipole-dipole amplitude is then a function of the dipole separations and the angle $\theta$ ($\sim$ relative rapidity $\eta$) between the dipoles. If this amplitude is an analytic function of $\theta$, one can calculate the dipole-dipole scattering (a correlation function of the two infinite rectangular Wilson loops) in the Euclidean region and continue to the Minkowski space. In a recent series of papers [62, 63] the dipole-dipole scattering was calculated in the Euclidean space using the AdS-CFT correspondence and continued analytically as a function of the angle $\theta$ to the Minkowski space, where $\theta \rightarrow i\eta$. Another example of such analytic continuation is calculations of the instanton-induced
To be able to obtain the high-energy behavior from the Euclidean amplitudes it is crucial to verify that the dipole-dipole amplitude is an analytic function of $\theta$. In Ref. [65] this statement was proved for the correlation function of the two Wilson rectangles with a finite longitudinal length $T$ (see also Ref. [66]). At finite $T$, the amplitude is an analytic function of the angle $\theta$ between the rectangles. In the limit $T \to \infty$, one recovers the dipole-dipole scattering amplitude. However, whether the $T \to \infty$ limit commutes with the analytic continuation from the Euclidean to the Minkowski space is still an open question. We will demonstrate with explicit calculations in the first two orders of perturbation theory that the dipole-dipole amplitude is, indeed, an analytic function of the angle between the dipoles and dipole sizes. Moreover, the analyticity survives the $T \to \infty$ limit in the first two orders of perturbation theory and, presumably, it happens in all the other higher orders as well, so that the dipole-dipole amplitude appears to be an analytic function of $\theta$.

The second important result concerns the rate at which the dipole-dipole amplitude approaches the BFKL asymptotics. At sufficiently high energies, the cross section for the unpolarized dipole-dipole scattering is given by the BFKL formula

$$\sigma(a, b, \eta) \sim \alpha_s^2 \int d\nu \left(\frac{a}{b}\right)^{2\nu} \eta^{\gamma_2 N_c \chi(\nu)}.$$

(141)

where $\chi(\nu) = -\text{Re}\left(\frac{1}{2} + i\nu\right) - C$ is the position of the hard pomeron [39] (for a review, see Ref. [67]). The asymptotics of the $\gamma^*\gamma^*$ scattering is then given by Eq. (138). The formula (141) is obtained in the LLA approximation, when $\alpha_s \ll 1; \alpha_s \ln s \sim \alpha_s \eta \sim 1$. Unfortunately, the power behavior $s^{\gamma_2 N_c \ln 2}$ of the LLA result (141) violates the Froissart bound $\ln^2 s$ at asymptotically large $s$. Thus, the BFKL pomeron gives us only the pre-asymptotic behavior at what we can call moderately high energies. It is a common belief that the true asymptotics at $s \to \infty$ (where $\alpha \ln s \gg 1$) comes from the unitarization of the BFKL pomeron. Despite numerous efforts, the high-energy asymptotics satisfying the Froissart bound is still an unsolved problem in QCD (for a recent discussion, see Refs. [68, 69, 72]). In this Chapter we address a different problem that is important for mid-energy accelerators. Namely, at what energies (1, 10, or 100 GeV?) does the LLA asymptotics start making sense? To answer this question one should calculate the amplitude exactly and compare it to the BFKL asymptotics (141). Since it seems to be an impossible task at present,
we will calculate the dipole-dipole scattering in the first two orders of perturbation theory exactly and compare it to the asymptotic form. Our main result is that the asymptotics starts rather late, at $\eta \sim 5$, which translates into $\sqrt{s} \sim 10$ GeV for the scattering of dipoles with the $\rho$-meson size $a \sim 0.3$ fm.

This Chapter is based on results published in [73] and is organized as follows. In Section IV.1 we calculate the dipole-dipole amplitude and the cross section in the leading order of perturbation theory. In the second order, we have calculated independently both the dipole-dipole amplitude and its imaginary part proportional to the cross section. Since the structure of the diagrams and the result is much more transparent for the cross section, in Section IV.2 we present only results for the amplitude in the second order and discuss all details in Section IV.4, where we give the detailed calculation of the dipole-dipole cross section. We discuss the effects due to the running coupling constant in Section IV.3. In Section IV.5 we provide the numerical estimates of the dipole-dipole cross section. The explicit form of the second-order amplitudes is given in Appendix D.

IV.1 DIPOLE-DIPOLE SCATTERING IN THE BORN APPROXIMATION

Let's consider the dipole-dipole scattering amplitude defined as follows

$$T(x_{1\perp}, x_{2\perp}; x_{3\perp}, x_{4\perp}; \eta) = -i\langle W^n(x_{1\perp}, x_{2\perp}) W^c(x_{3\perp}, x_{4\perp}) \rangle. \quad (142)$$

Strictly speaking, the Wilson lines forming dipoles are connected with gauge links at infinity, so that

$$T(x_{1\perp}, x_{2\perp}; x_{3\perp}, x_{4\perp}; \eta) = -i \lim_{T \to \infty} \langle W^n_T(x_{1\perp}, x_{2\perp}) W^c_T(x_{3\perp}, x_{4\perp}) \rangle. \quad (143)$$

where $W^n_T$ is a rectangular Wilson loop of longitudinal length $T$.

$$W^n_T(x_{1\perp}, x_{2\perp}) = \text{Tr} \left\{ U^n_T(x_{1\perp}) [x_{1\perp} - \frac{T}{2} n, x_{2\perp} - \frac{T}{2} n] U^n_T(x_{2\perp}) [x_{1\perp} + \frac{T}{2} n, x_{2\perp} + \frac{T}{2} n] \right\} \quad (144)$$

Since the gluons reduce to the pure gauge fields at infinity, the precise form of a contour connecting the points $x_{1\perp} \pm \frac{T}{2} n$ and $x_{2\perp} \pm \frac{T}{2} n$ does not matter. Moreover, we

---

1By definition, $U^n_T(x_{1\perp}) \equiv \frac{T}{2} n + x_{1\perp} - \frac{T}{2} n + x_{1\perp}$
use the Feynman gauge for the gluon propagator. In this gauge the links at infinity do not contribute to the amplitude and we can omit them to simplify the notation.

IV.1.1 Cross section of the dipole-dipole scattering and the optical theorem for dipoles

The total cross section of the scattering of a dipole of size $a$ on a dipole of size $b$ may be defined as

$$\sigma(a_\perp, b_\perp; \eta) = \int dz_\perp \varsigma(z_\perp; a_\perp, b_\perp; \eta)$$  \hspace{1cm} (145)

where

$$\varsigma(z_\perp; \eta) = \sum_{N \neq 0} \langle 0| W^{n*}_{ij}(z_{1\perp}, z_{2\perp}) W^{n*}_{kl}(z_{3\perp}, z_{4\perp}) |X\rangle \times$$

$$\times \langle X| W_{ij}^{n*}(x_{1\perp}, x_{2\perp}) W_{jk}^{n*}(x_{3\perp}, x_{4\perp}) |0\rangle$$  \hspace{1cm} (146)

is the total amplitude of the dipole-dipole transition into hadrons. Here the summation goes over all the intermediate states except for the vacuum and

$$W^{n}_{ij}(z_{1\perp}, z_{2\perp}) = \lim_{T \to \infty} \left( U_{ij}^{n}(x_{1\perp}) [U_{ij}^{n}(x_{2\perp} - \frac{T}{2} n, x_{2\perp} - \frac{T}{2} n)] U_{ij}^{n*}(x_{2\perp}) \right)_{ij}.$$  \hspace{1cm} (147)

If we include the intermediate vacuum state in the r.h.s. of Eq. (146), we obtain $N_{\gamma}^2$ due to the completeness relation. Separating the non-interacting contribution in a usual way

$$\langle 0| W^{n*}(x_{1\perp}, x_{2\perp}) W^{n}(x_{3\perp}, x_{4\perp}) |0\rangle = N_{\gamma}^2 + iT(x, \eta)$$  \hspace{1cm} (148)

we have the optical theorem for the dipole-dipole scattering:

$$\Im T(x_{1\perp}, x_{2\perp}; x_{3\perp}, x_{4\perp}; \eta) = \frac{1}{2} \varsigma(x_{1\perp}, x_{2\perp}; x_{3\perp}, x_{4\perp}; \eta).$$  \hspace{1cm} (149)

It is worth noting that the high-energy $\gamma^*\gamma^*$ cross section reduces to the dipole cross section (146) integrated with impact factors, similar to Eq. (138) for the amplitude:

$$A(s) = \frac{1}{4} \int d^2 a_\perp d^2 b_\perp I^A(a_\perp) I^B(b_\perp) \sigma(a_\perp, b_\perp; \eta)$$  \hspace{1cm} (150)

We will calculate the dipole-dipole cross section in two ways: directly as the r.h.s. of Eq. (146) or via the optical theorem as the imaginary part of Eq. (142).
IV.1.2 Lowest order dipole-dipole amplitude

In the leading order of perturbation theory, the correlation function of the two Wilson-line operators is proportional to the massless two-dimensional propagator \(^2\):

\[
\langle U^n(x_\perp) U^r(y_\perp) \rangle = - \langle U^n(x_\perp) U^{rt}(y_\perp) \rangle = ig^2 \coth \eta \int \frac{d^2 k_\perp}{k_\perp^2} e^{i(k, x-y)_\perp} . \quad (151)
\]

The contribution of the relevant Feynman diagrams shown in Fig. 23. has the following form:

\[
T(x_{1\perp}, x_{2\perp}; x_{3\perp}, x_{4\perp}; \eta) = ig^4 \coth^2 \eta \frac{N_c^2 - 1}{8} \int \frac{d^2 k_{1\perp}}{16\pi^2} \frac{d^2 k_{2\perp}}{k_{1\perp}^2 k_{2\perp}^2} \times \\
\times \left( e^{-i(k_1, x_1)_\perp} - e^{-i(k_1, x_2)_\perp} \right) \left( e^{i(k_1, x_3)_\perp} - e^{i(k_1, x_4)_\perp} \right) \times \\
\times \left( e^{-i(k_2, x_1)_\perp} - e^{i(k_2, x_2)_\perp} \right) \left( e^{-i(k_2, x_3)_\perp} - e^{-i(k_2, x_4)_\perp} \right) . \quad (152)
\]

Integrating over the transverse momenta can be performed explicitly, resulting in

\[
T(x_{1\perp}, x_{2\perp}; x_{3\perp}, x_{4\perp}; \eta) = i \frac{N_c^2 - 1}{8} \alpha_s^2 \left( \ln \frac{x_{1\perp}^2 x_{2\perp}^2}{x_{3\perp} x_{4\perp}} \right)^2 \coth^2 \eta . \quad (153)
\]

where \(x_{12} \equiv x_{1\perp} - x_{2\perp}\) (cf. Ref.[72]).

IV.1.3 Dipole-dipole cross section in the Born approximation

As mentioned above, the dipole-dipole cross section can be found in two ways: via the optical theorem as the imaginary part of Eq. (153) or directly as the r.h.s. of Eq. (148). Since the amplitude in the lowest order in \(\alpha_s\) is purely imaginary, the \textit{unintegrated cross section} (146) is

\[
\zeta(x_{1\perp}; \eta) = g^4 \frac{N_c^2 - 1}{4} \coth^2 \eta \left( \ln \frac{x_{1\perp}^2 x_{2\perp}^2}{x_{3\perp} x_{4\perp}} \right)^2 . \quad (154)
\]

For future use, it is instructive to calculate \(\zeta(x_{1\perp}; \eta)\) directly as the r.h.s. of Eq. (146). In this case, the Feynman rules for calculation of the cross sections can be reproduced by a functional integral over the double set of fields [70]: (+) to the right of the cut

---

\(^2\)We use the \(h\)-inspired notations \(d^n k \equiv \frac{d^n k}{(2\pi)^n}\) and \(\delta^{(n)}(k) \equiv (2\pi)^n \delta^{(n)}(k)\).

As usual, \((a, b)_\perp\) denotes the (positive) scalar product of two-dimensional vectors \(\vec{a}_\perp\) and \(\vec{b}_\perp\).
FIG. 23. Dipole-dipole scattering in the lowest order of perturbation theory.
FIG. 24. Cross section of the dipole-dipole scattering in the lowest order of perturbation theory.
and (-) to the left with the propagators $A_{\pm}(x)$ corresponding to the Cutkovsky rules for a cross section:

$$A_{\pm}^{ab}(x)A_{\pm}^{cd}(y) = \int \frac{dk}{16\pi^2} e^{-ik(x-y)} \frac{ig^{\mu\nu} \delta^{ab}}{\mp k^2 - i\epsilon}$$

$$A_{\mp}^{ab}(x)A_{\mp}^{cd}(y) = -\int \frac{dk}{16\pi^2} e^{-ik(x-y)} g^{\mu\nu} \delta^{ab} 2\pi \delta(k^2) \theta(k_0)$$

In this notation, the dipole-dipole cross section reads (cf. [71])

$$\langle x_1, \eta \rangle = \langle \text{Tr} \{ W_{(+)}^n(x_1, x_2, y_1)W_{(+)}^n(x_1, x_2, y_1) \} \rangle \times$$

$$\times \langle \text{Tr} \{ W_{(-)}^r(x_3, x_4, y_1)W_{(-)}^r(x_3, x_4, y_1) \} \rangle$$

The relevant diagrams are shown in Fig. (24). To emphasize that here we have only the gauge links at $T = -\infty$, we draw them explicitly.

The leading-order correlation function of Wilson lines in the (+) sector is given by Eq. (151) and in the (-) sector by the same Eq. (151) with a different sign.

$$\langle U_{(+)}^r(x_1, y_1)U_{(+)}^r(y_1) \rangle = -\langle U_{(-)}^r(x_1, y_1)U_{(-)}^r(y_1) \rangle$$

$$= \langle U_{(-)}^r(x_1, y_1)U_{(-)}^r(y_1) \rangle$$

$$= \langle U_{(-)}^r(x_1, y_1)U_{(-)}^r(y_1) \rangle$$

$$= ig^2 \coth \eta \int \frac{d^2k_\perp}{4\pi^2 k_\perp^2} e^{i(k, x-y) \perp}$$

Using Eq. (157) it is easy to see that

$$\frac{4}{N_c^2 - 1} \langle x_1, x_2, x_3, x_4, \eta \rangle = g^4 \coth^2 \eta \int \frac{d^2k_1 \perp d^2k_2 \perp}{16\pi^2 k_1 \perp^2 k_2 \perp^2} (e^{-i(k_1, x_1) \perp} - e^{-i(k_1, x_2) \perp}) \times$$

$$\times (e^{i(k_1, x_3) \perp} - e^{i(k_1, x_4) \perp})(e^{i(k_2, x_1) \perp} - e^{i(k_2, x_2) \perp}) \times$$

$$\times (e^{-i(k_2, x_3) \perp} - e^{-i(k_2, x_4) \perp})$$

where the $2^4 = 16$ terms correspond to 16 diagrams shown in Fig. 24. Integrating over $k_\perp$, we reproduce the optical theorem result (154).

### IV.2 SECOND-ORDER AMPLITUDE

In the next order of perturbation theory there are too many diagrams to present them all. Roughly speaking, we must take each diagram in Fig. 23 and add an extra
gluon line in all possible ways. For instance, if we insert a gluon line in all possible ways to connect the left and the right parts of the diagram in Fig. 23(a), we'll have the diagrams shown in Fig. 25. Also, we can insert a gluon line in the left or right part only and get the disconnected diagrams shown in Fig. 26.

Not all the diagrams for the correlator of the two Wilson loops (142) contribute to the dipole-dipole scattering amplitude. There are two exceptional classes of diagrams proportional to the total time of the evolution $T$. First, there are those shown in Fig. 26(q)-(t) that describe mass renormalization of Wilson lines. Their contribution has the usual form of $\delta mL$, where $\delta m$ is the self-energy correction and $L \approx 2T$ is the perimeter of the Wilson loop corresponding to a dipole. Similarly, the diagrams in Fig. 25(m)-(p) contribute to the binding energy $\epsilon$ of the dipole and lead to $\epsilon T$ terms. These two contributions exponentiate so that one obtains:

$$-i\langle \text{Tr}\{U_T^n(x_1)U_T^{n\dagger}(x_2)\}\text{Tr}\{U_T^n(x_3)U_T^{n\dagger}(x_4)\} \rangle = e^{-4\delta m T - i(\epsilon_{12} + \epsilon_{34}) T} T(x_{1\perp}, x_{2\perp}, x_{3\parallel}, x_{4\parallel}, \eta, T).$$

(159)

where $T(x_{1\perp}, \eta, T)$ describes the dipole-dipole scattering at the time $T$. The resulting amplitude

$$T(x_{1\perp}, \eta) = \lim_{T \to \infty} T(x_{1\perp}, \eta, T)$$

(160)

$$= \lim_{T \to \infty} -i\langle \text{Tr}\{U_T^n(x_1)U_T^{n\dagger}(x_2)\}\text{Tr}\{U_T^n(x_3)U_T^{n\dagger}(x_4)\} \rangle e^{4\delta m T - i(\epsilon_{12} + \epsilon_{34}) T}$$

is finite as $T \to \infty$. In the second order of perturbation theory, the multiplication in the r.h.s. of Eq. (160) reduces to the subtraction of the corresponding terms ($4\delta m T$ and $2\epsilon T$). The result is that the diagrams in Fig. 25(m)-(p) and Fig. 26(q)-(t) are left out and in the diagrams in Fig. 25(i)-(l) and Fig. 26(m)-(p) the color factor $-\frac{1}{2N_c}$ is replaced by $-\frac{1}{2N_c} - e_F = -\frac{N_c}{2}$. Thus, we are left only with the diagrams shown in Fig. 25(a)-(j) and Fig. 26(a)-(r).

There are $\sim 20$ times more diagrams obtained from the extra gluon addition to the graphs in Fig. 23(b),(c),...(s). The calculation of these diagrams is standard but rather lengthy and some details are shown in Section IV.4. Here we present only the final result for the amplitude and discuss the contribution of different diagrams.

We have performed a calculation of the dipole-dipole amplitude in both the Euclidean and Minkowski spaces. In the Euclidean space, this amplitude is a correlation
FIG. 25. Typical connected diagrams for the dipole-dipole amplitude.
FIG. 26. *Disconnected* diagrams for the dipole-dipole amplitude.
function of the two Wilson rectangles at angle $\theta$ (such as $\cos \theta = n \cdot c$). In both cases, the $g^6$ amplitude has the following form (in the Euclidean space $\eta = i\theta$):

$$T(x_{1\perp}, \eta) = \frac{ig^6 N_c (N_c^2 - 1)}{4\pi} \coth^2 \eta \times$$

$$\times \left\{ \int d^2 k_1 d^2 k_2 d^2 k_1' d^2 k_2' \left( e^{-i(k_1 \cdot x_1)} - e^{-i(k_1' \cdot x_1')} \right) (e^{-i(k_2 \cdot x_2)} - e^{-i(k_2' \cdot x_2')}) \times \right.$$  

$$\times (e^{i(k_2 \cdot x_2)} - e^{i(k_2' \cdot x_2')} (e^{i(k_1' \cdot x_3)} - e^{i(k_1 \cdot x_3')}) \left\{ \delta^2 (k_1 + k_1' - k_2 - k_2') \times \right.$$  

$$\times \left[ \frac{A(k_1; \eta)}{(k_1 + k_1')^6} + i\pi \frac{(N_c^2 - 4) \coth \eta}{N_c^2 (k_1 + k_1')^2} \left( \frac{1}{k_1^2 k_1'^2} + \frac{1}{k_2^2 k_1'^2} \right) \right] +$$  

$$+ \delta^2 (k_1 + k_1') \delta^2 (k_2 + k_2') \left\{ - \left( \eta - \frac{i\pi}{2} + \frac{i\pi N_c^2 - 4}{2 N_c^2} \right) \times \right.$$  

$$\times \coth \eta \int d^2 p \left\{ \frac{k_1^2}{p^2 (k_1 - p)^2} + \frac{k_2^2}{p^2 (k_2 - p)^2} \right\} +$$  

$$+ \frac{1}{2\pi} \ln \frac{\mu^2}{|k_1||k_2|} \left( \frac{5}{3} - \frac{N_f}{2N_c} \right) + \frac{1}{\pi} \ln \mu^2 |x_{1\perp}| |x_{3\perp}| \right\} \right\}$$

Here $\mu$ is the normalization point of the $\overline{\text{MS}}$ scheme. The explicit form of the function $A(k_1, k_1', k_2, k_2'; \eta)$ is given in Appendix D. Note that the $A(k_i; \eta)$ are analytical functions of $\eta$ and $k_i^2$ with singularities at $\eta \to 0$ corresponding to the bound state of two parallel dipoles.

The r.h.s. of Eq. (161) consists of two terms. The first term\(^3\) comes from the connected diagrams of the type shown in Fig. 25. The first expression of the second term comes from the gluon reggeization diagrams shown in Fig. 26(a)-(d), while the second expression $\sim \left( \frac{5}{3} - \frac{2N_f}{3N_c} \right)$ results from the gluon and quark self-energy diagrams shown in Fig. 26(e)-(h). The diagrams in Fig. 26(i)-(l) vanish (cf. Ref. [74]). while the terms coming from Fig. 26(m)-(p) are the pure divergency $\sim \int \frac{d^2 k}{k^2}$ and vanish in the framework of the dimensional regularization. It is easy to see that the apparent divergency at $k + k' \to 0$ in the gluon reggeization terms cancels with the corresponding contribution to $A(k_i; \eta)$ (see Eq. (195)).

The main conclusion from the Eq. (161) is that the dipole-dipole amplitude is an analytic function of the angle $\eta$. It is easy to check that the functions $A(k_i; \eta)$ become real as $\eta \to i\theta$ (see the Appendix D) and so does the amplitude $iT(x_{1\perp}, i\theta)$.

It is worth noting that in the LLA (as $\eta \to \infty$) we reproduce the first iteration

\(^3\)The $N_c^2 - 4$ odderon contribution comes from the structure $d^{abc}d^{abc}$. 

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of the BFKL kernel

$$T(x_{1+}, \eta) \rightarrow \frac{i g^6 \eta N_c (N_c^2 - 1)}{4 \pi} \int d^2 k_1 d^2 k_2 d^2 k_1' d^2 k_2' \times$$

$$\times (e^{-i(k_1 x_1)} - e^{-i(k_1' x_1)}) (e^{-i(k_2 x_2)} - e^{-i(k_2' x_2)}) \times$$

$$\times (e^{i(k_2 x_2)} - e^{i(k_2' x_2)}) (e^{i(k_2' x_2)} - e^{i(k_2' x_2)}) \times$$

$$\times \delta^2(k_1 + k_1' - k_2 - k_2') \left[ - \frac{(k_1 - k_2)^2}{k_1 k_1 k_2 k_2 k_2 k_2} + \frac{k_1^2 k_2^2 k_2^2}{k_1^2 k_1^2 k_1^2} + \frac{k_2^2 k_2^2}{k_2^2 k_1^2 k_1^2} \right] - \delta^2(k_1 + k_1') \int d^2 p \left\{ \frac{k_1^2}{p^2(k_1 - p)^2} + \frac{k_2^2}{p^2(k_2 - p)^2} \right\}.$$  \hspace{1cm} (162)

IV.3 RUNNING COUPLING CONSTANT

As we shall see below, the diagrams contributing to the renormalization of the coupling constant are somewhat unusual: as in the case of a heavy-quark potential $\frac{\alpha_s(r)}{r}$, the coefficient \( \left( \frac{11}{3} N_c - \frac{2 N_f}{3} \right) \) in front of the $\ln \mu$ comes from both the UV and IR-divergent diagrams (cf. \cite{74}). To show that, it is instructive to start with a scattering of the light-like dipoles, where the diagrams for the coupling constant renormalization are the usual UV-divergent vertices and self-energies shown in Fig. 27.

As was mentioned above, the scattering amplitude for the light-like dipoles is not a well-defined quantity because of the longitudinal divergencies corresponding to $\eta = \infty$ coming from the diagrams in Fig. 25(a)-(h) and Fig. 26(a)-(d). By cutting these divergencies, we find the asymptotical contribution, i.e. Eq. (162). Apart from this contribution, the only non-zero diagrams are those shown in Fig. 26(e)-(h). They lead to the renormalization of the coupling constant in Eq. (153). Standard computation of these diagrams yields to the contribution

$$\left[ \left( \frac{5}{3} N_c - \frac{2 N_f}{3} \right) + 2 N_c \right] \frac{g^2}{16 \pi^2} \ln \frac{\mu^2}{k^2_{1+}}.$$  \hspace{1cm} (163)

where the $\left( \frac{5}{3} N_c - \frac{2 N_f}{3} \right)$ term comes from the diagram in Fig. 27(a) and the similar diagram with the quark loop in Fig. 26(o), while $2 N_c$ term comes from the diagram in Fig. 27(b). It turns out that the diagram in Fig. 27(c) vanishes, while the diagram

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in Fig. 27(d) is irrelevant since it contributes only to the mass renormalization \( \delta m \) of the Wilson line (see Eq. (159)).

Surprisingly, the structure of the diagrams contributing to the running coupling constant is quite different for even slightly off-the-light-cone dipoles: the diagram in Fig. 27(a) is the same as for the light-like dipoles, the diagram in Fig. 27(b) vanishes, and the diagram in Fig. 27(c) is a pure divergency of the type \( \int \frac{dp^+}{p^+} \), that is set to zero in the dimensional regularization approach. Following Ref. [74], it is convenient to write down this divergent term as

\[
\frac{g^2 N_c}{8\pi^2} \left( \frac{2}{4-d} - \frac{2}{d-4} \right)
\]

The UV pole \( \frac{2}{4-d} \) together with the pole coming from the diagram in Fig. 27(a) forms the term \( \frac{2}{4-d} \frac{g^2}{16\pi^2} \left( \frac{11}{3} N_c - \frac{2}{3} N_f \right) \), while the IR pole \( \frac{2}{d-4} \) cancels with similar poles in the diagrams in Fig. 28.

Due to the cancellation between the UV and IR contributions to the diagram shown in Fig. 27(c), in order to reproduce the running-coupling coefficient

\[
\text{FIG. 27. Running coupling constant for the light-like dipoles.}
\]
FIG. 28. IR divergent diagrams contributing to the renormalization of the coupling constant.

\((\frac{11}{3}N_c - \frac{2}{3}N_f)\) in front of \(\ln \mu^2\) in the dimensional regularization, one should take into account not only the usual UV divergent diagrams in Fig. 27 but also the IR-divergent diagrams in Fig. 28 (cf. [75]). As a result, the coefficient \(\left(\frac{5}{3}N_c - \frac{2N_f}{3}\right)\) in front of the \(\frac{q^2}{16\pi^2}\) \(\ln \mu^2\) term comes from the diagram in Fig. 27(a)\(^4\), while the \(\frac{q^2N_f}{8\pi^2}\) \(\ln \mu^2\) term comes from the diagrams in Fig. 28.

This mixture of the IR and UV divergencies could have been a source of different problems. The cornerstone of the BFKL approach is the assumption that all the \((\ln s)\)-terms come from the longitudinal integrations, while the integrations over the transverse momenta are convergent at \(k_{\perp}^2 \sim m^2\), where \(m\) is of the order of masses (or virtualities) of the colliding particles\(^5\). This assumption is not true for the diagrams leading to the coupling constant renormalization since they are UV divergent. However, the structure of these diagrams is very simple. Namely, they are 1-loop vertex and self-energy corrections. After subtraction of the \(\frac{1}{4-d}\) poles, the integrals over \(k_{\perp}^2\) are bound with \(\mu^2\). In our case, the structure of the integrals over the transverse momenta is more complicated due to the mixture of the IR and UV divergencies. There are individual diagrams, where the upper cutoff in the integrals over \(k_{\perp}\) is the energy \(\sqrt{s}\) rather than \(\frac{1}{a}\) or the normalization scale \(\mu\). Such contributions give the additional non-BFKL \((\ln s)\) terms coming from the logarithmical integrals over the transverse momenta \(\int \frac{dk}{k^A}\). Fortunately, such terms cancel out in the final sum of all diagrams and the remaining transverse integrals are cut off by the dipoles’ sizes (or

\(^4\)All the other diagrams shown in Fig. 27 vanish.

\(^5\)After summation of the BFKL ladder we have diffusiveness in the transverse momenta leading to \(\ln k_{\perp}^2 \sim \sqrt{\ln \frac{\mu^2}{m^2}}\), but such terms do not spoil the power counting in \((\ln s)^n\).
by the normalization point \( \mu \)).

**IV.4 CROSS SECTION**

In case of the cross section (146), the structure of the diagrams is more transparent. There are two types of Feynman diagrams: with and without a gluon emission. Typical diagrams of the first type are shown in Fig. 29. The sum of these diagrams is proportional to the square of the finite-energy Lipatov vertex \( L^\mu(k_1, k'_1) L_\mu(k_2, k'_2) \), where the Lipatov vertex

\[
L(k_1, k'_1) = (k_1 - k'_1)(\text{nc}) + \left[ 2(k'_1 n) + \frac{(\text{nc})}{(k'_1 n)} k'_1 \right] c - \left[ 2(k_1 c) + \frac{(\text{nc})}{(k_1 n)} k_1 \right] n \quad (165)
\]

is a sum of all possible emissions of the real gluon with momentum \((k_1 + k'_1)\) from the left part of the diagrams in Fig. 29. This vertex is gauge-invariant:

\[
(k_1 + k'_1)^\mu L(k_1, k'_1) = 0 \quad (166)
\]

The expression (165) at infinite energies \((\text{nc}) \rightarrow \infty\) reduces to the usual asymptotic form of the Lipatov vertex \([67]\). The sum of the diagrams in Fig. 29 has the form

\[
\frac{4}{N_c^2 - 1} \delta(x_1, x_2, x_3, x_4) \delta(\eta) \int d^4k_1 d^4k'_1 d^4k_2 d^4k'_2 \times (167)
\]

\[
\times \delta(\gamma(k_1 + k'_1 - k_2 - k'_2) \delta(k_1 + k'_1)^2 \theta(k_1 + k'_1) \delta(k_1 \cdot n) \delta(k_2 \cdot n) \times
\]

\[
\times \delta(k'_1 \cdot c) \delta(k'_2 \cdot c) (e^{-i(k_1 \cdot x_1)} - e^{-i(k_1 \cdot x_2)}) \times
\]

\[
\times (e^{i(k_2 \cdot x_1)} - e^{i(k_2 \cdot x_2)}) (e^{i(k_2 \cdot x_1)} - e^{i(k_2 \cdot x_2)}) \times
\]

\[
\times (e^{-i(k_2 \cdot x_1)} - e^{-i(k_2 \cdot x_2)}) \times (167)
\]

The explicit form of the product of two Lipatov vertices is

\[
-L(k_1, k'_1) L(k_2, k'_2) = \frac{(\text{nc})^2}{k_{1e} k_{1n}} (k^2_1 k^2_2 + k^2_2 k^2_1) + 2(k_1 - k_2)^2 (\text{nc})^2 - 4(k^2_1 - k^2_2) - 2(\text{nc}) \left[ k_{1n}^2 (k^2_1 + k^2_2) + k_{1e}^2 (k^2_1 + k^2_2) \right] - (\text{nc})^2 \left[ \frac{k^2_1 k^2_2}{k_{1e}^2} + \frac{k^2_1 k^2_2}{k_{1n}^2} \right] \quad (168)
\]

where \(k_{1e} \equiv (k_1 \cdot c)\) and \(k_{1n} \equiv (k'_1 \cdot n)\). Apart from the diagrams with emission of a real gluon, there are diagrams with an extra virtual gluon. For example, if we take the diagram in Fig. 24(h) and insert an extra gluon line in the left \((-\) sector in all possible ways, we will derive the diagrams shown in Fig. 30. We do not display the
FIG. 29. Typical dipole-dipole cross section diagrams with a gluon emission.
diagrams corresponding to $\delta m$ or $\delta \epsilon$. The contribution of these diagrams vanishes since the phase factors $e^{-i(2m+1)T}$ and $e^{i(2m+1)T}$ coming from the left and right parts of the cut cancel out in the cross section. Once again, this leads to the replacement of the color factor $-\frac{1}{2N_c}$ by $-\frac{1}{2N_c} - c_F = -\frac{N_c}{2}$ in the diagrams shown in Fig. 29(m) and Fig. 30(f)-(i). The calculation yields to:

$$
\frac{4}{N_c^2 - 1} \left( x_{1,2,3} \right)_{\text{Fig. 30}} = \frac{g^4 N_c \coth^2 \eta}{4\pi} \int \frac{d^2 k_1}{16\pi^3 k_1^2 k_2^2} \times \left[ \right.
$$

where the first gluon regularization term comes from the diagrams in Fig. 30(a),(b) and the second term comes from the diagrams in Fig. 30(n),(o). As in the case of the amplitude, the diagrams in Fig. 30(j)-(m) vanish and the diagrams in Fig. 30(h),(i) produce a pure divergence $\sim \int \frac{d k^2}{16\pi^3}$ that does not contribute to the final result in the framework of the dimensional regularization. As was discussed in Section IV.3, such divergence leads to the mixing of the IR and UV singularities. Finally, the diagrams in Fig. 30(c)-(e) cancel out with the corresponding diagrams with an extra gluon to the right of the diagram cut.

The diagrams in Fig. 30(f),(g) should be regularized by finite $T$ before calculation. After the summation of all the relevant diagrams shown in Fig. 30(f),(h) and Fig. 29(g),(m),(n), the limit $T \to \infty$ becomes regular. Performing the remaining integrations over the longitudinal momenta in Eq. (167) and over $p$ in Eq. (169), one obtains

$$
\left( x_{1,2,3} \right)_{\text{Fig. 30}} = \frac{g^6 N_c (N_c^2 - 1)}{16\pi \tanh^2 \eta} \int d^2 k_1 d^2 k_2 d^2 k_1' d^2 k_2' \times \left[ \right.
$$

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FIG. 30. Typical dipole-dipole cross section diagrams with an extra virtual gluon.
\[
\times \left( e^{i(k_2 \cdot x_1)} - e^{i(k_2 \cdot x_2)} \right) \left( e^{-i(k_2 \cdot x_3)} - e^{-i(k_2 \cdot x_4)} \right) \times \\
\times \left[ -\eta \int d^2p \left\{ \frac{k_1^2}{p^2(k_1 - p)^2} + \frac{k_2^2}{p^2(k_2 - p)^2} \right\} + \\
+ \frac{1}{2\pi} \ln \frac{\mu^2}{|k_1||k_2|} \left( \frac{5}{3} - \frac{N_f}{2N_c} \right) + \frac{1}{\pi} \ln \mu^2 |x_{12}| |x_{34}| \right].
\]

Here
\[a(k_1; \eta) = \sum_{j=1}^{5} a_j(k_1; \eta) \] (171)

and \(a_1, a_2, \ldots, a_5\) correspond to the first, second, \ldots, fifth terms in Eq. (168) respectively. It should be mentioned that the fifth term in the r.h.s. of Eq. (168) is IR divergent. After the regularization, in addition to \(a_5(k_1; \eta)\), it produces the \(\ln \mu^2 x_{12} x_{34}\) term, which forms \(\left( \frac{11}{3} N_c - \frac{2}{3} N_f \right)\) together with a contribution from the self-energy diagrams in Fig. 30(n),(o). The explicit form of the functions involved is:

\[a_i(k_1, k_2, k_1', k_2'; \eta) = \Re A_i(k_1, k_2, k_1', k_2'; \eta). \] (172)

where \(A_i\) can be found in Appendix D. Hence, the un-integrated cross section (170) is given by the imaginary part of the amplitude (161) (see the optical theorem for dipoles (149)).

The cross section of dipole-dipole scattering is given by the integral (145)

\[\sigma(\hat{a}, \hat{b}; \eta) = \int d^2z \varsigma(\hat{a} + \hat{z}, \hat{z}, \hat{b}; \eta)\] (173)

IV.4.1 Asymptotics of the cross section

As \(\eta \to \infty\), the cross section (173) reduces to

\[\sigma_{\text{asy}}(\hat{a}, \hat{b}; s) = g^4(N_c^2 - 1) \int \frac{d^2k}{k'^2} \frac{d^2k'}{k'^2} \frac{4}{\sin^2 \frac{(k, a)}{2}} \frac{\sin^2 \frac{(k', b)}{2}}{2} \times \] (174)

\[\times \left[ 1 + \frac{g^2N_c}{2\pi} \left\{ -\eta \int d^2p \frac{k^2}{p^2(k - p)^2} + \\
+ \frac{1}{4\pi} \ln \frac{\mu^2}{k^2} \left( \frac{5}{3} - \frac{2N_f}{3N_c} \right) + \frac{1}{2\pi} \ln \mu^2 ab \right\} \delta(k + k') + \\
+ \frac{g^2N_c}{2\pi} \left\{ \frac{2\eta}{(k + k')^2} + \frac{\ln(k + k')^2/k^2}{(k + k')^2 - k^2 \left( \frac{2k^2}{k^2} - 1 \right)} + \\
+ \frac{\ln(k + k')^2/k^2}{(k + k')^2 - k^2 \left( \frac{2k^2}{k^2} - 1 \right)} - 2 \frac{k^2 + k'^2}{k^2 k'^2} \right\} \right] + O(e^{-\eta})
\]

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It is easy to see that the integrals over the transverse momenta converge at \( k_\perp^2 \sim a^{-2} \). The integrals over \( k_\perp^2 \) diverge in the individual diagrams and cut off by \( s \) in the corresponding exact expressions (173) and (170). However, as was discussed in the Section IV.3, in the sum of all diagrams such terms cancel out.

The structure of the asymptotic result (174) has the form \( \eta \sigma(a, b) + \sigma'(a, b) \). The term \( \sim \eta \) is the first iteration of the BFKL kernel. After the integration over \( k \) and \( k' \), it gives the first term in the expansion of \( \eta^{\alpha_s(x)} \) in powers of \( \alpha_s \). The second term may be called the dipole impact factor. Indeed, the standard representation of the BFKL asymptotics of the cross section has the form

\[
\sigma_{\text{asy}} = \int \frac{d^2k}{k^2} \frac{d^2k'}{k'^2} I(a, k_\perp) \eta^K_{\text{BFKL}} I(b, k'_\perp). \tag{175}
\]

where \( K_{\text{BFKL}} \) is the BFKL kernel (the LLA kernel + NLO BFKL kernel +...). The kernel terms come from the central region of the rapidity, while the impact factors come from the fringes of the longitudinal integral, where the momentum of the emitted gluon is almost collinear to \( n \) or \( c \). However, we do not access the NLO BFKL terms [76] that would correspond to the amplitude contributions \( \sim \alpha_s^4 \ln s \). All the non-LLA terms in Eq. (174) should be included into the dipole impact factors \( I(b, k_\perp) \) and \( I(a, k'_\perp) \), where

\[
I(b, k_\perp) = \sin^2 \frac{(k, b)}{2} + 2\alpha_s k^2 \int \frac{d^2k'}{k'^2} \left[ \frac{\ln(k + k')}{(k + k')^2} - \frac{2k^2}{k^2} \right] \sin^2 \frac{(k', b)}{2} \tag{176}
\]

The impact factor (176) comes from the region of the longitudinal momenta in the emitted gluon close to \( n \). The integral in the r.h.s of Eq. (176) is convergent and behaves like \( \ln k'^2\beta^2 \) for \( k \to \infty \). Numerically, the \( \alpha_s \) correction to the impact factor (176) is quite significant. It is responsible for the difference between the dotted and dash-dot lines in Fig. 31. Recently calculated, the NLO corrections to the virtual photon impact factor also appear to be big (see Refs. [77, 78]).

### IV.4.2 Unpolarized dipole-dipole scattering

The unpolarized dipole-dipole scattering corresponds to the cross section (173) averaged over the orientation of the dipoles :

\[
\sigma(a, b; s) = \int \frac{d\theta_a}{2\pi} \frac{d\theta_b}{2\pi} \sigma(\vec{a}, \vec{b}; s) \tag{177}
\]
Note that in the unpolarized $\gamma^* \gamma^*$ scattering the impact factor (193) depends only on $|\vec{a}|$ and $|\vec{b}|$ so that the cross section of the unpolarized $\gamma^* \gamma^*$ scattering (138) is given by the unpolarized dipole-dipole cross section in Eq. (177) integrated over the dipole sizes with weights being impact factors

$$\sigma(a, b; \eta) = g^4(N_c^2 - 1) \coth^2 \eta \left\{ \frac{g^2 N_c}{2\pi} \int d^2 k \, d^2 k' \left[ 1 - J_0(ka) \right] \left[ 1 - J_0(k'b) \right] \times \right.$$

$$\times \frac{a_f(k, k'; \eta)}{(k + k')^6} + \int \frac{d^2 k}{k^2} \left[ 1 - J_0(ka) \right] \left[ 1 - J_0(k'b) \right] \times$$

$$\times \left[ 1 + \frac{g^2 N_c}{2\pi} \left[ -\eta \coth \eta \int d^2 p \, \frac{k^2}{p^2(k - p)^2} + \frac{1}{4\pi} \left( \frac{5}{3} - \frac{2N_f}{3N_c} \right) \ln \frac{\mu^2}{k^2} + \frac{1}{2\pi} \ln \mu^2 ab \right] \right\} + \frac{1}{2\pi} \ln \frac{\mu^2}{k^2} \right\}.$$  (178)

The explicit form of the function $a_f(k, k'; \eta)$ is presented in Appendix D.

**IV.5 NUMERICAL ESTIMATES OF THE DIPOLE-DIPOLE CROSS SECTION**

Let us consider the unpolarized scattering of equal-size dipoles. In this case we have only one-scale problem and it is natural to take $\mu = a^{-1}$, where $a$ is a dipole size. The amplitude (178) reduces to

$$\sigma(a, \eta) = 16\pi a^2 \alpha_s^2(a) \coth^2 \eta \left[ 1 + 6\alpha_s(a) \Phi(\eta) \right]$$  (179)

where

$$\Phi(\eta) = 8\pi \int d^2 k d^2 k' \left[ 1 - J_0(k) \right] \left[ 1 - J_0(k') \right] \frac{a_f(k, k'; \eta)}{(k + k')^6} - \frac{1}{2\pi} \ln \frac{\mu^2}{k^2}.$$  (180)

where we have rescaled $k$ and $k'$ by $a$ and took $N_f = N_c = 3$. This expression should be compared to the asymptotical form (174), which gives

$$\sigma_{asy}(a, \eta) = 16\pi \alpha_s^2(a) a^2 \left[ 1 + 6\alpha_s(a) \Phi_{LLA+IF}(\eta) \right].$$  (181)

where

$$\Phi_{LLA+IF} = 8\pi \int \frac{d^2 k \, d^2 k'}{k^2 k'^2} \left[ 1 - J_0(k) \right] \left[ 1 - J_0(k') \right].$$  (182)
\[ \times \left\{ \frac{2\eta}{(k + k')^2} + \frac{\ln(k + k')^2/k^2}{(k + k')^2 - k^2} \left( \frac{2k^2}{k^2 - 1} \right) + \ln\left( \frac{(k + k')^2/k^2}{(k + k')^2 - k^2} \right) \left( \frac{2k^2}{k^2 - 1} \right) - \frac{2k^2 + k'^2}{k^2 k'^2} \right\} - \left\{ \eta \int \frac{d^2k}{k^4} \right\} \] 

is a sum of the LLA term \( \sim \eta \) and the impact factor term \( \sim \text{const.} \) Numerically, \( \Phi_{\text{LLA+IF}}(\eta) \approx 1.65(\eta - 3.25) \) so that

\[
\sigma_{\text{asy}}(a, a; \eta) = 16\pi a^2 \alpha_s^2(a) \left[ 1 + 9.88 \alpha_s(a)(\eta - 3.25) \right].
\]  

(183)

where \( \eta \) corresponds to the LLA approximation and the coefficient \(-3.25\) comes from the dipole impact factor (176). We have not calculated the next \( \frac{1}{s} \) term in the asymptotic expansion (183), but the fit to Fig. 31 suggests a coefficient of the order of 100 in front of it.

On Fig. 31, we have depicted the comparison between the function \( \Phi(\eta) \) (solid line), describing the exact second-order cross section, its asymptotics \( \Phi_{\text{LLA+IF}}(\eta) \) Eq. (182) (dotted line), and the pure BFKL asymptotics \( \Phi_{\text{LLA}}(\eta) = 1.64\eta \) (dashed line).

We have used the VEGAS algorithm of the Monte-Carlo technique with 15% accuracy to calculate the function \( \Phi(\eta) \). From Fig. 31 we see that the BFKL asymptotics starts rather late, at \( \eta \sim 5 \), which translates into \( \sqrt{s} \sim 10 \text{GeV} \) for the scattering of dipoles with the \( \rho \)-meson size \( a \sim 0.25 \text{fm} \). This is, perhaps, not surprising since even within the NLO BFKL approximation itself the asymptotics starts relatively late, at \( \eta \sim 6 \div 8 \) for the realistic dipoles [79]. It should be emphasized, however, that these are two different theoretical questions relevant to the BFKL description of the experimental cross sections: how good is the BFKL approximation to the true high-energy asymptotics and when this true asymptotics makes sense. For the second-order dipole-dipole scattering, the first question is when the impact factor correction \( \sim \text{const.} \) can be neglected in comparison to the LLA \( \ln s \) term, whereas the second question is when the \( \frac{1}{s} \) corrections disappear. We can mainly address the second question and the conclusion is that the true asymptotics, represented by \( \Phi_{\text{LLA+IF}} \), starts rather late.
IV.6 CONCLUSIONS

Our main conclusion is that the scattering of dipoles is as suitable process to study the energy behavior of QCD amplitudes as, for instance, the scattering of virtual photons. This result could have been different because the infinite length of the Wilson lines forming dipoles leads to the mixture of the IR and UV divergencies. Moreover, there are individual diagrams, where the upper cutoff in the integrals over the transverse momenta is $s$ rather than $\frac{1}{a}$ or the normalization scale $\mu$. Such contributions lead to additional non-BFKL $\ln s$ terms. Fortunately, in the final sum of all diagrams such terms cancel out, so that the transverse integrals are cut off by the dipole sizes or normalization point $\mu$. The resulting dipole-dipole amplitude is an analytic function of the angle between dipoles and, in principle, one can derive the BFKL kernel from the Euclidean calculation.

The next conclusion, based on numerical calculations, is that the high energy dipole-dipole scattering asymptotics starts rather late, at $\eta \sim 5$. It would be interesting to check this statement for a scattering of virtual photons rather than dipoles.
However, the corresponding photon-photon scattering diagram contains four loops so that one should not expect this calculation to be performed in the near future.
CHAPTER V
CONCLUSIONS AND OUTLOOK

In this dissertation we have studied three major aspects of Quantum Chromodynamics. Namely, in Chapter II we have discussed crucial challenges of the renormalon divergence. Renormalons have been a very active area of research over the past years since they expand our understanding of the large-order behavior and power corrections to particular processes in QCD. Moreover, the true asymptotics of perturbation theory in theories with a running coupling constant is determined by the renormalons. We have demonstrated that from the functional-integral viewpoint the renormalons do not correspond to a particular configuration, but manifest themselves as dilatation modes in the functional space.

In Chapter III we have addressed the second major aspect of QCD related to the small-$x$ dynamics. In pQCD the small-$x$ asymptotic behavior is described in the leading logarithmic approximation by the BFKL pomeron that sums up the leading energy logarithms. However, it is possible to reformulate the BFKL equation as an evolution equation of Wilson-line operators. We have derived the result for the argument of the coupling constant in the non-linear evolution of dipoles using the quark part of the $J$-function, following the bubble chain argument discussed in Chapter II. We plan to confirm this result by calculating the diagrams with gluon loops. The calculations are in progress.

In Chapter IV we have studied the scattering of color dipoles. We calculated the dipole-dipole scattering amplitude exactly in the first two orders of perturbation theory. Our main conclusion is that the scattering of dipoles is, indeed, a suitable process to study the energy behavior of QCD amplitudes. The second conclusion is that for the dipole-dipole scattering of equal-size dipoles, the high-energy asymptotics starts rather late, at $\eta \sim 5$. In many cases, such as deep inelastic scattering, one of the dipoles is small. The conventional assumption is that the propagation amplitude for the small-size dipole through a hadron is proportional to the area of the dipole $a^2$ multiplied by the gluon parton density normalized at $\mu^2 = 1/a^2$. This formula is valid for the asymptotically small dipoles, but it is unknown how big/small the dipoles are, and at what energies the asymptotics makes sense. This is an important
question since in the analysis of DIS from a nucleus based on the non-linear equation for the dipole evolution we assume that at the energy of a few GeV and size of the order of the saturation scale (2–3 GeV for LHC) the dipole matrix element between the nucleon states satisfies this approximation $a_2^2 G(a^{-2})$. Since at present we cannot perform model-independent calculations of the dipole-hadron amplitudes, we can consider the target to be a second dipole with a size of a typical hadron. For this DIS from a color dipole we can answer the question of at what size of the spectator dipole the $a_2^2 G$ approximation makes sense. These studies are in progress.
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APPENDIX A

GLOSSARY

The following abbreviations are often used in the thesis:

DD ....................... double distribution
DIS ....................... deeply inelastic lepton scattering
DVCS ....................... deeply virtual Compton scattering
GPD ....................... generalized parton distribution
II ....................... instanton – antiinstanton configuration
LLA ....................... leading logarithmic approximation
MS ....................... minimal subtraction renormalization
\bar{MS} ....................... modified MS
NLO ....................... next – to – leading order
OPE ....................... operator product expansion
PDF ....................... parton distribution function
QCD ....................... quantum chromodynamics
QED ....................... quantum electrodynamics
APPENDIX B  

DIRAC DETERMINANT: ONE-LOOP RESCALING

We want to prove that rescaling of an arbitrary configuration by a factor $\lambda (\rho \to \lambda \rho)$ at the one-loop level leads to the multiplication of the Dirac determinant by a factor of

$$\lambda^{bS/8\pi^2} \quad (184)$$

where $S$ is the action of this configuration. For a determinant of the Dirac operator in the background of a field $A_\mu(x; \rho)$ characterized by the size $\rho$ we will prove that

$$\sqrt{\det \hat{\mathcal{P}}^2} = \sqrt{\det \mathcal{P}^2} \lambda^{\frac{S}{12\pi^2}} \quad (185)$$

where $\mathcal{P}_\mu = i\partial_\mu + A_\mu(x)$ is the operator of the covariant momentum for our configuration and $\hat{\mathcal{P}}_\mu = i\partial_\mu + \hat{A}_\mu(x)$ is the covariant momentum in the background of the stretched configuration $\hat{A}_\mu(x) = A_\mu(x; \lambda \rho) = \lambda^{-1} A_\mu(\lambda^{-1} x; \rho)$. Using Schwinger's notations we can write down

$$\ln \det \hat{\mathcal{P}}^2 = - \int_0^\infty \frac{ds}{s} \int d^4x \ Tr(x|e^{-s\hat{\mathcal{P}}^2}|x) \quad (186)$$

where $|x\rangle$ are the eigenstates of the coordinate operator normalized according to $(x|y) = \delta^{(4)}(x - y)$. The integral in Eq. (186) diverges for the Dirac operator, so that we assume a cutoff $s > \epsilon$ and take $\epsilon \to 0$ in the final results. After the change of variables $x \to \lambda^{-1} x$, the integral in the r.h.s. of Eq. (186) reduces to

$$\ln \det \hat{\mathcal{P}}^2 = - \int_0^\infty \frac{ds}{s} \int d^4x \ Tr(x|e^{-s\mathcal{P}^2}|x) \quad (187)$$

so that

$$\ln \det \hat{\mathcal{P}}^2 - \ln \det \mathcal{P}^2 = \int_0^\infty \frac{ds}{s} \int d^4x \ Tr(x|e^{-s\mathcal{P}^2} - e^{-s\mathcal{P}^2}|x)$$

$$= - \lim_{\epsilon \to 0} \int_{\lambda^{-2}\epsilon}^{\lambda^{-1}} \frac{ds}{s} \int d^4x \ Tr(x|e^{-s\mathcal{P}^2}|x) \quad (188)$$

Using the well-known DeWitt-Seeley coefficients for the Dirac operator \[30\]

$$\text{Tr}(x|e^{-s\mathcal{P}^2}|x) = \frac{3}{4\pi^2 s^2} + \frac{1}{48\pi^2} G^a_{\mu\nu}(x) G^a_{\mu\nu}(x). \quad (189)$$

For simplicity, we assume that the Dirac operator has no zero modes, as in the case of the $\Sigma$ valley. For a treatment of the Dirac operator with zero modes see e.g. \[29\].
we obtain

\[ \ln \det \hat{P}^2 = \ln \det P^2 - \frac{\ln \lambda}{24\pi^2} \int d^4 x \, C^a_{\mu \nu} C^a_{\mu \nu} \]  

(190)

which corresponds to Eq. (185). In the case of \( N_f \) quark flavors, the coefficient \( \frac{1}{12\pi^2} \) in the r.h.s. of Eq. (185) will be multiplied by \( N_f \) leading to the factor \( \lambda^{-\frac{1}{2}} N_f S/8\pi^2 \). This factor is easily recognized as a quark part of the one-loop coefficient \( b \) for the Gell-Mann-Low \( \beta \)-function in Eq. (184). Similarly, the rescaling of the gluon (or ghost) determinants reproduces the gluon (or ghost) part of the one-loop coefficient \( b \) in Eq. (184). Thus, we have shown that \( e^{-S} \rightarrow e^{-S_{(\rho)}} \) with one-loop accuracy. Using the two-loop formulas for the Seeley coefficients, one could prove that \( e^{-S_{(\rho)}} \) reproduces at the two-loop level and demonstrate that \( g^2 \rightarrow g^2(\rho) \) in the pre-exponential factor \( g^{-17} \) as well (as it follows from the renormalizability of the theory).
APPENDIX C
VIRTUAL PHOTON IMPACT FACTOR

In the momentum representation, the $\gamma^*$ impact factor for the transverse photons with the initial polarization $e_A$ and final $e_A'$ is given by

$$I^A(k_{\perp}) = \frac{1}{2} \int d\alpha d\beta \frac{1}{k_{\perp}^2 \alpha^2 - p_{A,\perp}^2 \beta^2} \times \left\{ (1 - 2\alpha^2)(1 - 2\beta^2) (e_A e_A') k_{\perp}^2 + 4\alpha^2 \beta^2 \left[ 2(e_A, k)(e_A', k) - (e_A, e_A') k_{\perp}^2 \right] \right\}.$$  \hfill (191)

where $\tilde{x} \equiv 1 - x$. If we average over the transverse polarizations, the last term drops so that

$$I^A(k_{\perp}) = \int d\alpha d\beta \frac{(1 - 2\alpha^2)(1 - 2\beta^2) k_{\perp}^2}{2(k_{\perp}^2 \alpha^2 - p_{A,\perp}^2 \beta^2)}.$$  \hfill (192)

In the coordinate representation, the impact factor (192) takes the form

$$I^A(x_{\perp}) = \int \frac{d^2k}{4\pi^2} \epsilon^{(k, x_{\perp})} I^A(k_{\perp})$$

$$= \int d\alpha d\beta \frac{(1 - 2\alpha^2)(1 - 2\beta^2)}{4\pi \alpha^2} \sqrt{-p_{A,\perp}^2 x_{\perp}^2 \frac{\alpha^2}{\beta^2}} \times \left[ 2K_1 \left( \sqrt{-p_{A,\perp}^2 x_{\perp}^2 \frac{\alpha^2}{\beta^2}} \right) + \sqrt{-p_{A,\perp}^2 x_{\perp}^2 \frac{\alpha^2}{\beta^2}} K_2 \left( \sqrt{-p_{A,\perp}^2 x_{\perp}^2 \frac{\alpha^2}{\beta^2}} \right) \right] \hfill (193)$$

where $K_1$ are the McDonald functions. Note that the spin-averaged impact factor (193) depends only on $x_{\perp}^2$. 

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APPENDIX D

SECOND-ORDER AMPLITUDES

The scattering amplitude is a sum of five functions coming from the five terms in the product of two Lipatov vertices

$$A(k_1;\eta) = \sum_{i=1}^{5} A(k_i;\eta)$$  \hspace{1cm} (194)

The explicit form of the first function is

$$A_1(k_{i\perp}) = A_1(r_1, r_2') + A_1(r_1', r_2).$$  \hspace{1cm} (195)

$$A_1(r_1, r_2) = \frac{2\eta \coth \eta}{r_1 r_2} + \frac{\cosh^2 \eta}{4r_1 r_2 \cosh^2 \eta - (1 - r_1 - r_2)^2} \times \left[ \left( l_1 - \frac{2\eta \tanh \eta}{r_1} \right) \frac{4r_1 \sinh^2 \eta + 3 + r_1 - r_2}{(1 - r_1) \sinh^2 \eta} + \left( l_2 - \frac{2\eta \tanh \eta}{r_2} \right) \frac{4r_2 \sinh^2 \eta + 3 + r_2 - r_1}{(1 - r_2) \sinh^2 \eta} + \frac{4\eta(1 + r_1 + r_2)}{r_1 r_2 \sinh 2\eta} \right] -$$

$$\frac{i\pi \coth \eta}{r_1 r_2} \left( \frac{r_1 r_2}{(1 - r_1)(1 - r_2)} - \frac{1/2}{\sqrt{r_1 r_2} - 1 + r_1 + r_2} \right)$$

where $$r_i \equiv \frac{k_i^2}{(k_1 + k_i)^2}$$ and

$$l_i = \frac{\ln \left[ -1 - 2 \sinh^2 \eta \left( r_i + \sqrt{r_i^2 + \frac{r_i}{\sinh^2 \eta} + i\epsilon} \right) \right]}{\sqrt{r_i^2 + \frac{r_i}{\sinh^2 \eta} + i\epsilon}}.$$  \hspace{1cm} (196)

The second function is

$$A_2 = -2r_i \frac{M(r_1, r_1') - M(r_2, r_2') - M(r_1, r_2') + M(r_2, r_2')}{(r_1 - r_2)(r_1' - r_2')}.$$  \hspace{1cm} (197)

where

$$M(r_1, r_2) = \frac{1}{4 \cosh^2 \eta r_1 r_2 - (1 - r_1 - r_2)^2} \times$$

$$\times \left[ (2r_i \cosh^2 \eta + 1 - r_i - r_j) l_i + (2r_j \cosh^2 \eta + 1 - r_i - r_j) l_j - \right.$$

$$\left. - 2\eta \sinh 2\eta + i\pi \left( 2 \cosh \eta + \frac{1 - r_i - r_j}{\sqrt{r_i r_j}} \right) \sinh \eta \right]$$

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and $r_t \equiv \frac{-(k_1 - k_2)^2}{(k_1 + k_1')^2}$. Note that the second function does not contribute to the forward scattering and, therefore, to the total cross section.

The third and the fourth functions are given by

$$A_3 = \frac{4 \tanh^2 \eta}{(r_1 - r_2)(r'_1 - r'_2)} \left[ (r_1 + r_1')M(r_1, r_1') - (r_1 + r_2')M(r_1, r_2') - (r_2 + r_1')M(r_2, r_1') - (r_2 + r_2')M(r_2, r_2') \right] \quad (198)$$

$$A_4 = -\frac{2 \tanh \eta}{r_1 - r_2} \left[ N(r_1, r_1) + N(r_2, r_1) - (r_1 \leftrightarrow r_2) \right] - \frac{2 \tanh \eta}{r'_1 - r'_2} \left[ N(r_1, r'_1) + N(r_2, r'_1) - (r'_1 \leftrightarrow r'_2) \right] \quad (199)$$

where

$$N(r_1, r_j) = \frac{1}{4r_i r_j \cosh^2 \eta - (1 - r_i - r_j)^2} \times$$

$$\times \left[ \frac{4r_i \cosh^2 \eta - r_j}{r_i - 1} (r_i \coth \eta - 2\eta + i\pi) + 2\eta - i\pi + (1 + r_i - r_j) l_j \coth \eta + 3i\pi \sqrt{\frac{r_i \cosh^2 \eta}{r_i \sinh^2 \eta + 1} - 2i\pi \sqrt{\frac{r_i}{r_j} \cosh \eta}} \right].$$

Finally,

$$A_5 = -\frac{1}{r_1 r_2} \left( \frac{1}{r'_1} + \frac{1}{r'_2} \right) - \frac{1}{r'_1 r'_2} \left( \frac{1}{r_1} + \frac{1}{r_2} \right) +$$

$$+ \frac{l'_1 r'_2 - l'_2 r'_1}{r_1 r_2 (r_1 - r_2) \sinh^2 \eta} + \frac{l'_2 r'_2 - l'_2 r'_1}{r'_1 r'_2 (r'_1 - r'_2) \sinh^2 \eta}. \quad (200)$$

All the functions $A_i$ are analytical functions of the energy ($\equiv \eta$). In the Euclidean region $\eta$ should be replaced by $i\theta$, where $\theta$ is the angle between the dipoles (Wilson rectangles). It is easy to see that the functions $A_i$ become real after such a replacement.

In case of the dipole-dipole cross section, we need only the real parts of the functions (195) - (200) at $k_1^2 = k_2^2$ and $k_1'^2 = k_2'^2$. After some algebra we find

$$a_f(r, r'; \eta) = f_1(r, r'; \eta) + f_2(r, r'; \eta) + f_4(r, r'; \eta) + f_5(r, r'; \eta). \quad (201)$$

where

$$f_1(r, r'; \eta) = \Re A(r, r'; \eta)$$
\[ f_3(r, r'; \eta) = 2 \tanh^2 \eta \frac{\partial^2}{\partial r \partial r'} (r + r') m(r, r'; \eta) \]

\[ f_4(r, r'; \eta) = -2 \tanh \eta \left[ \frac{\partial}{\partial r} n(r', r; \eta) + \frac{\partial}{\partial r'} n(r, r'; \eta) \right] \]

\[ f_5(r, r'; \eta) = \Re \frac{l - r \frac{\partial}{\partial r}}{2 r^2 \sinh^2 \eta} + \Re \frac{l - r' \frac{\partial}{\partial r'}}{2 r'^2 \sinh^2 \eta} - \frac{r + r'}{r^2 r'^2}. \]

Here \( m(r, r'; \eta) \equiv \Re M(r, r'; \eta) \) and \( n(r, r'; \eta) \equiv \Re N(r, r'; \eta) \).
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