Exact Solutions for Orthogonal and Non-Orthogonal Magnetohydrodynamic Stagnation-Point Flow

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Exact Solutions for Orthogonal and Non-Orthogonal
Magnetohydrodynamic Stagnation-Point Flow

by

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ABSTRACT

EXACT SOLUTIONS FOR ORTHOGONAL AND NON-ORTHOGONAL MAGNETOHYDRODYNAMIC STAGNATION-POINT FLOW

Shahrooz Moosavizadeh

Old Dominion University, 1996

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The viscous plane flow of an electrically conducting fluid towards an infinite wall is solved in the presence of a magnetic field which is aligned with the flow far from the wall. The problem has two dimensionless parameters - \( \epsilon \), the magnetic Prandtl number, and \( \beta \), the square of the ratio of Alfven velocity to fluid velocity far from the wall. The problem has a similarity solution which reduces the governing equations to a system of coupled ordinary differential equations which can be solved numerically. For extreme values of \( \epsilon \), both large and small, singular perturbation techniques are used to derive asymptotic expansions for the physically relevant quantities in the flow - the skin friction at the wall and the tangential component of magnetic intensity at the wall. Extensive comparisons are made between the asymptotic predictions and the numerical results with remarkably good agreement being obtained when \( \epsilon \geq 100 \) and \( \epsilon \leq 0.001 \).

The flow is next combined with a shear flow to yield a flow impinging on the wall at some angle of incidence. The problem has a similarity solution and the resulting
system of coupled differential equations is solved numerically. A series solution for the shear component of tangential stress at the wall for small and large values of ε is derived using singular perturbation techniques. The asymptotic expansions obtained are shown to be in excellent agreement with the numerical results when ε ≤ 0.001 and ε ≥ 100. The behavior of the flow near the wall is analyzed and the slope-ratio constant is evaluated for a variety of (ε, β) cases.
Dedicated

to

the memory of my father, Harnid Moosavizadeh

(1940-1989)
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8.1 The dividing streamline \( \psi = 0 \) far from the wall is a straight line, \( l \), with slope \(-\frac{1}{k}\). The line \( l \), whose slope is denoted by \( m_s \), is tangent to the dividing streamline at \( X = 0 \).

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Chapter 1

Introduction

Exact solutions of the Navier-Stokes equation are exceptionally rare in fluid mechanics because of the analytic difficulties associated with nonlinear boundary-value problems. One of the primary difficulties rests in the fact that nonlinear problems do not admit a superposition principle, thereby ruling out the building up of complicated solutions from simple ones. This severely restricts our scope to problems having particularly simple geometries for which a similarity solution exists.

The Navier-Stokes equation describes the motion of a viscous fluid of density $\rho$ and viscosity $\nu$. In the present work these will be taken as constants. In addition all flows considered will be steady. The two-dimensional steady-state Navier-Stokes equation is given by

$$ (\vec{V} \cdot \nabla)\vec{V} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{V}, \quad (1.1) $$
where \( \vec{V}(x, y) \) and \( p(x, y) \) are respectively fluid velocity and pressure. We will assume the fluid is incompressible which adds the constraint

\[
\vec{V} \cdot \vec{V} = 0. \tag{1.2}
\]

Equation (1.2) is satisfied identically by the introduction of a stream function \( \psi(x, y) \) defined by

\[
\vec{V} = \vec{\nabla} \times \{\psi(x, y)\vec{k}\} = \frac{\partial \psi}{\partial y} \vec{i} - \frac{\partial \psi}{\partial x} \vec{j}. \tag{1.3}
\]

This shows that \( \vec{V} \) is perpendicular to \( \vec{\nabla} \psi(x, y) \), and it is a general property that \( \vec{V} \psi(x, y) \) is perpendicular to the curves

\[
\psi(x, y) = \text{constant}.
\]

Hence the velocity \( \vec{V} \) is parallel to the curves \( \psi(x, y) = \text{constant} \) at every point, and so these curves are the streamlines. Figure 1.1 shows a streamline and the directions of \( \vec{V} \) and \( \vec{\nabla} \psi(x, y) \).
After substituting (1.3) into (1.1) and then taking the curls to eliminate the pressure term, we obtain the vorticity-transport equation

\[ \nu \nabla^4 \psi = \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2}. \] (1.4)

At a rigid boundary the normal components of the fluid velocity and the boundary velocity must be identical if we assume that the fluid cannot penetrate into the solid or leave a gap. Also the tangential velocity component in fluid and solid is the same. Assuming that the rigid boundary is at \( y = 0 \), this implies that

\[ \bar{V}(x,0) = 0, \] (1.5)

and it is referred to as 'no-slip condition'. The no-slip condition translates to

\[ \psi(x,0) = \frac{\partial \psi}{\partial y}(x,0) = 0. \] (1.6)

In this dissertation we examine a variety of flows in the presence of a bi-infinite rigid wall occupying the entire \( x \)-axis. There are three known exact solutions of the Navier-Stokes equations involving this geometry: i) Shear flow parallel to the wall, ii) Stagnation-point flow and iii) Non-orthogonal stagnation-point flow.

(i) **Shear Flow Parallel to Wall**

A stream parallel to the \( x \)-axis, whose speed is proportional to distance from the \( x \)-axis has

\[
\begin{align*}
  u &= \frac{\partial \psi}{\partial y} = \alpha y, \\
  v &= -\frac{\partial \psi}{\partial x} = 0,
\end{align*}
\] (1.7)
Figure 1.2: The velocity profile $u = \alpha y$. The slope of the velocity profile (which is a measure of the strength of the shear) is $\frac{1}{\alpha}$.

where $\alpha$ is a constant. Using the definition of stream function (1.3), we obtain

$$\psi(x, y) = \frac{1}{2} \alpha y^2.$$  \hspace{1cm} (1.8)

A 'uniform shear flow' like this is a simple model for some real flows, such as the wind at low levels in the atmosphere which increases in strength as one goes up from ground level. The variation of velocity across the stream is called the 'velocity profile', and is shown in figure 1.2.

It is clear that the stream function given by (1.8) satisfies equation (1.4). In particular the vorticity (whose magnitude is given by $| \text{curl} \vec{V} |$) is $\nabla^2 \psi = \alpha$ and is everywhere constant. In addition the no-slip conditions are satisfied trivially at the wall $y = 0$. Shear flow therefore is a simple parallel flow which satisfies the full Navier-Stokes equations and no-slip boundary conditions exactly.

(ii) Stagnation-Point Flow

Stagnation-point flow is another well-known exact solution which acts as a sort
of orthogonal complement to shear flow. In stagnation-point flow a rigid wall occupies the entire x-axis and the fluid domain is $y > 0$. The boundary conditions on $\tilde{V}(x,y)$ are

\[
\begin{align*}
\tilde{V}(x,0) &= 0, \\
\tilde{V}(x,y) &\sim \gamma x^2 - \gamma y^2, \quad \text{as} \quad y \to \infty,
\end{align*}
\]

where $\gamma$ has units of inverse time. If the variables $(x,y)$ and $\psi(x,y)$ are non-dimensionalized using $(\nu/\gamma)^{\frac{1}{2}}$ and $\nu$ respectively, the scaled vorticity-transport equation is identical with (1.4) except that the viscosity coefficient is missing from the left side of the equation. The non-dimensional boundary conditions are given by

\[
\begin{align*}
\psi(x,0) &= \frac{\partial \psi}{\partial y}(x,0) = 0, \\
\psi(x,y) &\sim xy, \quad \text{as} \quad y \to \infty.
\end{align*}
\]

A similarity solution exists for this problem. The stream function is of the form

\[
\psi(x,y) = xF(y),
\]

where $F(y)$ is known as the Hiemenz function named after the German mathematician who performed exhaustive investigations on the numerical properties of the function (see [1]). The boundary-value problem for $F(y)$ obtained by substituting (1.11) into the governing equation is, after one integration,

\[
\begin{align*}
F'''(y) + F(y)F''(y) - F''(y)^2 &= -1, \\
F(0) &= F'(0) = 0, \\
F'(\infty) &= 1.
\end{align*}
\]
Figure 1.3: Streamlines for the orthogonal stagnation-point flow.

From a numerical solution recorded by Goldstein [2], we observe that

$$F(y) = \frac{1}{2}Cy^2 - \frac{1}{6}y^3 + O(y^5) \quad \text{for small } y,$$  \hspace{1cm} (1.13)

where $C = 1.23259$.

The asymptotic behavior for large $y$ is given by

$$F(y) \sim y - A + O((y - A)^{-4}\exp[-\frac{1}{2}(y - A)^2]),$$ \hspace{1cm} (1.14)

where $A = 0.64790$. The expression in (1.13) shows that the stagnation-point stream function near the wall behaves like $\psi(x, y) = \frac{1}{2}Cx^2$. This has the form of shear flows moving off in opposite directions from a central “point of re-attachment”.

The shear stress along the wall is given by $\frac{\partial^2 \psi}{\partial y^2}(x, 0) = Cx$ and changes sign at the point of re-attachment ($x = 0$).

The expression in (1.14) reveals that far from the wall, the stream function has the form of a simple potential flow, $\psi(x, y) \sim x(y - A)$. The hyperbolic streamlines are clearly evident from this perspective and the presence of a boundary layer at
Figure 1.4: Streamlines for the flow to which non-orthogonal stagnation-point flow asymptotes. The dividing streamline $\psi = 0$ is a straight line with slope $m = -\frac{1}{k}$.

$y = A$ is observed. The constant $A$ is the thickness of the boundary layer.

(iii) Non-Orthogonal Stagnation-Point Flow

What we have seen, then, in these first two exact solutions of the Navier-Stokes equations is that stagnation-point flow describes the motion of a fluid towards a flat wall, while shear flow describes the motion of the fluid along the wall. In his paper, Dorrepaal [3] asks an interesting question: is it possible to combine these two flows in a way which yields a flow impinging on the wall at some angle of incidence? His non-orthogonal stagnation-point flow solution is sketched here for future reference.

Consider the flow in figure 1.4 described by

$$\psi(x, y) = \gamma [xy + \tilde{k}y^2],$$

(1.15)

where $\gamma$ is a dimensional constant of proportionality having units $(time)^{-1}$ and $\tilde{k}$ depends on the angle of incidence. It is trivial to show that this stream function satisfies the vorticity-transport equation (1.4) exactly. We will take the stream
function given in (1.15) as the large-y asymptotic version of a viscous incompressible flow which comes into the boundary \( y = 0 \) obliquely. Equations (1.4) and (1.15) can be non-dimensionalized by introducing the following transformations:

\[
\psi = \nu \tilde{\psi}, \quad x = \left(\frac{\nu}{\gamma}\right)^{\frac{1}{2}} \tilde{x}, \quad y = \left(\frac{\nu}{\gamma}\right)^{\frac{1}{2}} \tilde{y},
\]

(1.16)

where the non-dimensional variables are denoted by bars. After dropping the bars, the vorticity-transport equation and the large-\( y \) behavior of the stream function become

\[
\nabla^4 \psi = \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y},
\]

(1.17)

\[
\psi(x, y) \sim xy + k y^2, \quad y \to \infty.
\]

(1.18)

By regarding the stream function in (1.18) as the far-field behavior of a flow impinging obliquely on a flat wall, we are led to the following similarity solution for such a flow:

\[
\psi(x, y) = x f(y) + m(y).
\]

(1.19)

We shall refer to \( f(y) \) as the normal component of the flow and \( m(y) \) as the tangential component. Both components must satisfy no-slip conditions at the wall and conditions consistent with (1.18) at infinity.

The substitution of (1.19) into equation (1.17) yields fourth-order differential equations for the two functions \( f(y) \) and \( m(y) \). The equation for \( f(y) \) is identical
with that derived by Blasius and one integration of it yields

\[
\begin{align*}
  f'''(y) + f(y)f''(y) - f'(y)^2 + 1 &= 0, \\
  f(0) = f'(0) &= 0, \\
  f'(\infty) &= 1.
\end{align*}
\]

The solution of this equation is the Hiemenz function $F(y)$ discussed in the previous section on stagnation-point flow.

The corresponding boundary-value problem for the tangential component $m(y)$ is linear and is given by

\[
\begin{align*}
  m'''(y) + F(y)m''(y) - F''(y)m'(y) &= 0, \\
  m(0) = m'(0) &= 0, \\
  m''(\infty) &= 2\kappa,
\end{align*}
\]

As in the previous case this equation can be integrated once. The asymptotic behaviors of $F(y)$ and $m(y)$ for large $y$ are used to determine the constant of integration. The resulting third-order equation is

\[
\begin{align*}
  m'''(y) + F(y)m''(y) - F'(y)m'(y) &= -2\kappa A,
\end{align*}
\]

where $A = 0.64790$. The solution of (1.22) is found using the substitution

\[
m'(y) = 2\kappa H(y).
\]

The problem for $H(y)$ is then given by

\[
\begin{align*}
  H''(y) + F(y)H'(y) - F'(y)H(y) &= -A, \\
  H(0) &= 0, \\
  H'(\infty) &= 1.
\end{align*}
\]
Two observations can be made which enable us to solve this equation exactly. First, the function \( AF'(y) \) is a particular solution of the equation and, secondly, \( F''(y) \) is a solution of the corresponding homogeneous problem. Reduction of order is used to generate the second fundamental solution of the homogeneous equation and in this way the general solution of (1.24) is pieced together. Equation (1.24) is integrated numerically using a shooting method to satisfy the far-field condition. This yields the result

\[
H'(0) = 1.406544 = D. \tag{1.25}
\]

We can now use (1.25) in place of the condition at infinity and solve (1.24) as an initial-value problem. The closed-form solution for \( H(y) \) so obtained is

\[
H(y) = AF''(y) + C(D - AC)F''(y) \int_0^y w(t)dt, \tag{1.26}
\]

where

\[
w(t) = F''(t)^{-2} \exp \left\{- \int_0^t F(s)ds \right\}. \tag{1.27}
\]

For large \( y \), the behavior of \( H(y) \) is given by

\[
H(y) \sim y + O\{(y - A)^{-2} \ln (y - A) \exp \left[-\frac{1}{2}(y - A)^2 \right]. \tag{1.28}
\]

It follows from (1.23) that the solution of (1.22) is given by

\[
m(y) = 2k \int_0^y H(\eta)d\eta. \tag{1.29}
\]

When \( y \) is small, we have

\[
m(y) = kDy^2 - \frac{1}{3}kAy^3 + O(y^5). \tag{1.30}
\]
The asymptotic behavior of \( m(y) \) for large \( y \) is given by

\[
m(y) \sim \tilde{k}y^2 + 2Bk + \text{exponentially small terms},
\]

where \( B = 0.215395 \).

By substituting the small-\( y \) expansions for \( f(y) \) and \( m(y) \) into (1.19), Dorrepaal [3] analyzes the flow near the wall. If a new horizontal coordinate \( X \) is defined in the following way

\[
X = x + \frac{2kD}{C},
\]

then the resulting expansion for the stream function is given by

\[
\psi(X, y) = \frac{\tilde{k}}{3C(D - AC)}(D(y + \frac{3C^2X}{2k(D - AC)} - \frac{CXy}{2k(D - AC)}) + O(y^3)).
\]

From (1.15) we see that far from the wall the dividing streamline \( \psi = 0 \) is a straight line with slope

\[
m = -\frac{1}{\tilde{k}}
\]

which if extended would intersect the wall at \( x = 0 \). Equation (1.33) shows that in fact the streamline \( \psi = 0 \) meets the wall at \( X = 0 \) as shown in figure 1.5. The distance, \( d \), between these two locations is

\[
|d| = |x - X| = \frac{2kD}{C}.
\]

An interesting result follows from (1.35) and (1.34):

\[
h_0 = |d \cdot m| = \frac{2D}{C} = 2.282262,
\]
Figure 1.5: The dividing streamline $\psi = 0$ meets the wall at $X = 0$ and has slope $m_s$ at that point. The ratio $\frac{m_s}{m}$ and the height $h_0$ are the same for all angles of incidence.

which is independent of $\tilde{k}$. The slope $m_s$ of the dividing streamline at the wall, $X = 0$, is given by

$$m_s = \frac{-3C^2}{2k(D - AC)}.$$  \hfill (1.37)

Thus the ratio $\frac{m_s}{m}$,

$$\frac{m_s}{m} = \frac{3C^2}{D - AC} = 3.748513,$$  \hfill (1.38)

is also found to be independent of $\tilde{k}$. The slope of the dividing streamline $\psi = 0$ at the wall divided by its slope at infinity, $m$, is the same for all non-orthogonal stagnation-point flows; furthermore, the same is true for the distance, $h_0$, as shown in figure (1.5). Non-orthogonal stagnation-point flow is an exact solution of the Navier-Stokes equation which contains two constants which are totally independent of the angle of incidence of the impinging flow.

In the following chapters, we intend to study the stagnation-point and non-
orthogonal stagnation-point flows in the presence of a magnetic field. Our primary goals are: i) to develop series solutions for the skin friction and the tangential component of magnetic intensity at the wall in the limiting cases when the fluid is a poor conductor of electricity and when the fluid is an excellent conductor, and ii) to analyze the non-orthogonal stagnation-point flow in the vicinity of the wall when a conducting fluid is experiencing a magnetic field. In particular we investigate the role that the electromagnetic parameters play in the values of $\frac{m^2}{m}$ and $h_0$. 

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Chapter 2

Incompressible Magnetohydrodynamic Stagnation-Point Flow

2.1 Introduction

The equations of magnetohydrodynamics describe the flow of a viscous electrically conducting fluid in the presence of a magnetic field. The subject possesses an intrinsic interest because it exhibits an interplay between fluid mechanical effects and electrodynamic effects which can involve significant modifications to both the fluid velocity field and the applied magnetic field. The literature contains numerous examples documenting the changes predicted by the equations in simple viscous flows. In reference [4], Roberts discusses a few of these, including plane and circular
Couette flow, as well as Hartmann flow which is the magnetohydrodynamic equivalent of plane Poiseuille flow. Flows past spheres and cylinders have been treated using various linear approximations by Ludford and Murray [5], Barthel and Lykoudis [6], Riley [7], Tamada and Sone [8] and Ranger [9]. Ranger [10] uses a Stokes approximation to examine creeping magnetohydrodynamic Jeffery - Hamel flow in a wedge - shaped channel.

A nonlinear problem which has commanded some interest is magnetohydrodynamic Blasius flow. In their paper Greenspan and Carrier [11] considered the flow of a viscous incompressible electrically conducting fluid of constant properties onto and along a semi-infinite rigid plate.

One successful technique used by Greenspan & Carrier [11] to treat the problem was a direct extension of the asymptotic method which led to the classical Blasius result; another was the modified Oseen technique (Lewis & Carrier [12]). The authors found that when $\beta \geq 1$ no steady flow which is uniform at large distances from the plate exists.

Reuter and Stewartson [13] later proved the nonexistence of solution to the governing boundary value problem equations when $\beta > 1$ and $f''(0) \geq 0$. The nonexistence theorem implies that the problem is incorrectly posed when $\beta > 1$. Physically this corresponds to the fact that when $\beta > 1$, disturbances can travel an appreciable distance upstream of the plate, so that the notion of boundary layer originating at the leading edge is no longer appropriate.

In his paper, Wilson [14] showed that when $\beta$ is slightly less than unity, two
solutions to the boundary value problem exist for $\epsilon$ less than unity. The more
detailed work on the existence of dual solutions was later that year presented by
Stewartson and Wilson [15]. They found that when $\epsilon < 1$, there are two critical
values $\beta_0$, $\beta_1$ such that $0 < \beta_0 < \beta_1 < 1$. For $\beta < \beta_0$ there is a unique solution while
if $\beta > \beta_1$ no solutions can be found. The physical explanation of the nonexistence
if $\beta_1 < \beta < 1$ is not clear because there is still no upstream propagation of small
disturbances possible. It is noted that $\beta_1 > \approx 0.95$ for all $\epsilon < 1$ so that this zone
is quite narrow. Finally, if $\beta_0 < \beta < \beta_1$, two solutions can be found of which one
has an extremely small skin friction and the boundary layer is virtually detached
from the wall. Presumably the solution with the larger skin friction and which is
the continuation of the solution with $\beta < \beta_0$ occurs in practice.

In our problem we intend to examine a second magnetohydrodynamic boundary
layer flow; viz. magnetohydrodynamic stagnation-point flow. The electrically inert
version of this flow is a well-known exact solution of the full Navier-Stokes equation
as was discussed earlier in chapter 1. The magnetohydrodynamic stagnation-point
flow possesses a similarity solution involving two single-variable functions which
satisfy a coupled system of ordinary differential equations. The problem yields
to both numerical and asymptotic analyses and provides unique insights into the
governing equations of magnetohydrodynamics. The solution is one of the very few
exact solutions to the full nonlinear system of equations which describes the flow
of a conducting fluid in the presence of a magnetic field.
It might be expected that some of the more interesting magnetohydrodynamic phenomena occur when the fluid is a good conductor and finds itself in the presence of a very strong magnetic field. The results of this paper support this expectation. It is also unfortunately true, however, that observation of some of these phenomena in earth-bound laboratories is next to impossible because of the great difficulty in producing some of the large parameter values necessary to trigger the effects. On the other hand, strong magnetic fields are known to exist in astrophysical settings. But then imagining a scenario in outer space which could be described by simple two-dimensional geometry seems a bit ludicrous as well. The correct way to view our problem is to see it as a reasonably complete theoretical and numerical treatment of a fully nonlinear problem in magnetohydrodynamics whose governing equations admit very few exact analytic solutions. The predictions of the theory are traceable directly from the governing equations and are not tempered or restricted by dropping terms or making other kinds of approximations.

2.2 Equations of Motion

The well known Maxwell's equations governing the electrodynamic field are

\[ \nabla \cdot \vec{B} = 0, \]  \hspace{1cm} (2.1)

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \]  \hspace{1cm} (2.2)

\[ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}, \]  \hspace{1cm} (2.3)
where $\vec{E}$ is the electric field; $\vec{B}$ is the magnetic field; $\vec{J}$ and $\frac{\partial \vec{D}}{\partial t}$ are respectively the current density and the displacement current. We assume isotropic constitutive equations for the magnetic induction $\vec{H}$ and the electric induction $\vec{D}$:

$$\vec{H} = \frac{\vec{B}}{\mu}, \quad \vec{D} = \varepsilon \vec{E}. \quad (2.4)$$

The constants $\varepsilon$ and $\mu$ represent respectively the permittivity (dielectric constant) and the magnetic permeability of the fluid. Here we are interested in materials which are sufficiently conducting that the charge relaxation time is much shorter than the transit time of electromagnetic waves (molten metals belong to this category); the displacement current $\frac{\partial \vec{D}}{\partial t}$ is then negligible with respect to $\vec{J}$ and $\vec{\nabla} \times \vec{H}$. Consequently, the Maxwell's equations, having substituted $\mu \vec{H}$ for $\vec{B}$, are reduced to $\nabla \cdot \vec{H} = 0$, $\nabla \times \vec{H} = \vec{J}$ and

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}. \quad (2.5)$$

For all the materials in which we are interested, Ohm's law represents the current density in terms of the fluid velocity, electric field and magnetic field and is given by

$$\vec{J} = \sigma [\vec{E} + (\vec{V} \times \vec{B})]$$

where $\vec{V}$ is the fluid velocity and $\sigma$ is the electrical conductivity. Ohm's law can be rewritten using (2.4) as

$$\vec{J} = \sigma [\vec{E} + \mu (\vec{V} \times \vec{H})].$$

In fluids of our interest such as simple liquids, two variables such as pressure and temperature are enough to define the thermodynamic state. Among the equations
of state of a liquid, the simplest is

\[ \rho = \text{Constant}, \quad (2.6) \]

where \( \rho \) is the density. It means that the fluid is both incompressible (\( \rho \) independent of the pressure) and undilatable (\( \rho \) independent of the temperature).

The principle of conservation of mass applied to a given material domain \( D \) requires that

\[ \frac{d}{dt} \int_D \rho d\nu = \int_D [\partial \frac{\rho}{\partial t} + \nabla \cdot (\rho \vec{V})] d\nu = 0. \quad (2.7) \]

At each point, the velocity field \( \vec{V} \) and the density \( \rho \) are then related by the equation

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0. \quad (2.8) \]

The equation of state (2.6) reduces this equation to \( \nabla \cdot \vec{V} = 0 \).

The Navier-Stokes equation modified for a conducting fluid in the presence of a magnetic field has the form:

\[ \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \vec{V} + \frac{\mu}{\rho} (\vec{J} \times \vec{H}). \quad (2.9) \]

The equations governing the steady magnetohydrodynamic flow of a viscous electrically-conducting incompressible fluid are given by

\[ (\vec{V} \cdot \nabla) \vec{V} = -\nabla P + \nu \nabla^2 \vec{V} + \frac{\mu}{\rho} (\vec{J} \times \vec{H}), \quad (2.10) \]

\[ \vec{J} = \sigma [\vec{E} + \mu (\vec{V} \times \vec{H})], \quad (2.11) \]

\[ \vec{V} \times \vec{H} = \vec{J}, \quad (2.12) \]
\[ \nabla \cdot \vec{V} = \nabla \cdot \vec{H} = 0, \quad (2.13) \]
\[ \nabla \times \vec{E} = 0. \quad (2.14) \]

The standard two-dimensional stagnation-point geometry will be assumed; viz., the fluid velocity \( \vec{V} \) and the magnetic field \( \vec{H} \) are each perpendicular to the \( z \)-direction. A rigid wall lies in the plane \( y = 0 \) and the fluid occupies the half-space \( y > 0 \). Far from the wall, the flow is directed towards the wall and follows hyperbolic streamlines. The magnetic field lines are aligned with the flow when \( y \gg 1 \).

The solenoidal nature of both the velocity and magnetic fields permits the definition of scalar potential functions for each field as follows:

\[ \vec{V} = \nabla \times \{ \psi(x, y)\hat{k} \}, \quad \vec{H} = \nabla \times \{ \phi(x, y)\hat{k} \}. \quad (2.15) \]

When the current density is eliminated between equations (2.11) and (2.12), we obtain the magnetic diffusion equation

\[ \eta (\nabla \times \vec{H}) = \frac{1}{\mu} \vec{E} + (\vec{V} \times \vec{H}), \quad (2.16) \]

where \( \eta = \frac{1}{\sigma \mu} \) is the magnetic diffusivity (or magnetic viscosity) of the fluid. Equations (2.15) and (2.12) imply that

\[ \vec{V} \times \vec{H} = -\left( \frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial x} \frac{\partial \phi}{\partial y} \right)\hat{k}, \quad (2.17) \]
\[ \vec{J} = -\left( \nabla^2 \phi \right)\hat{k}. \]

So that \( \vec{E} \) must be of the form

\[ \vec{E} = -E_0(x, y)\hat{k}, \quad (2.18) \]
in order to satisfy equation (2.11). It can be shown from (2.14) that

\[
\frac{\partial E_0}{\partial y}(x, y) = \frac{\partial E_0}{\partial x}(x, y) = 0, \tag{2.19}
\]

implying that

\[
E_0(x, y) = \text{constant}. \tag{2.20}
\]

The applied electric field, \( \vec{E} \), is hence of the form

\[
\vec{E} = -E_0 \hat{k}. \tag{2.21}
\]

The scalar version of (2.16) is found by substituting (2.15) and (2.17) into (2.16) to obtain

\[
\eta \nabla^2 \phi + \frac{\partial \psi}{\partial x} \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \phi}{\partial x} = \frac{1}{\mu} E_0. \tag{2.22}
\]

A second equation relating the two scalar potentials is found by eliminating \( \vec{J} \) between equations (2.10) and (2.12) and then taking the curl to eliminate the pressure term. After considerable simplification, the linear momentum equation appears as follows:

\[
\nu \nabla^4 \psi + \frac{\partial \psi}{\partial x} \frac{\partial \nabla^2 \psi}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \nabla^2 \psi}{\partial x} - \frac{\mu}{\rho} \left[ \frac{\partial \phi}{\partial x} \frac{\partial \nabla^2 \phi}{\partial y} - \frac{\partial \phi}{\partial y} \frac{\partial \nabla^2 \phi}{\partial x} \right] = 0. \tag{2.23}
\]

Far from the wall, we require that the stream function and magnetic potential function, \( \psi(x, y) \) and \( \phi(x, y) \) respectively, have the form

\[
\psi(x, y) \sim \gamma xy, \quad \phi(x, y) \sim H_\infty xy, \tag{2.24}
\]

where \( \gamma \) and \( H_\infty \) are dimensional constants of proportionality having units \((\text{time})^{-1}\) and \((\text{charge})(\text{length})^{-1}(\text{time})^{-1}\) respectively. It follows, therefore, that the velocity
and magnetic fields are of the form

\begin{align*}
\vec{V} & \sim \gamma (x \hat{i} - y \hat{j}), \\
\vec{H} & \sim H_\infty (x \hat{i} - y \hat{j}),
\end{align*}

as \( y \to \infty \). (2.25)

It can be shown, from equation (2.24), that as \( y \to \infty \)

\begin{align*}
\nabla^2 \phi & \sim 0, \\
\nabla^2 \psi & \sim 0,
\end{align*}

(2.26)

Consequently, we require that \( E_0 \) be zero in order to satisfy equation (2.22).

The no-slip conditions at the wall translate into the boundary conditions

\( \psi(x, 0) = \psi_r(x, 0) = 0 \). In addition, the normal component of the magnetic field
must vanish at the boundary which means \( \phi(x, 0) = 0 \).

Equations (2.22)-(2.23) can be non-dimensionalized using \((\nu/\gamma)^{\frac{1}{2}}\) as the length
scale. If non-dimensional variables are denoted by bars, then

\[ x = \left(\frac{\nu}{\gamma}\right)^{\frac{1}{2}} \hat{x}, \quad y = \left(\frac{\nu}{\gamma}\right)^{\frac{1}{2}} \hat{y}, \quad \psi = \nu \bar{\psi}, \quad \phi = \frac{H_\infty \nu}{\gamma} \bar{\phi}. \]

(2.27)

After dropping the bars, the two equations of motion for \( \psi(x, y) \) and \( \phi(x, y) \) are

\begin{align*}
\nabla^4 \psi + \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x^2} - \beta \left\{ \frac{\partial \phi}{\partial x} \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial \phi}{\partial y} \frac{\partial^2 \phi}{\partial x^2} \right\} &= 0, \\
\nabla^2 \phi + \epsilon \left\{ \frac{\partial \psi}{\partial x} \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \phi}{\partial x} \right\} &= 0,
\end{align*}

(2.28) (2.29)

where \( \beta = \frac{\epsilon H_\infty^2}{\nu^2} \) is the square of the ratio of the Alfven velocity to the fluid velocity
far from the wall, and \( \epsilon = \frac{\nu}{\eta} = \nu \sigma \mu \) is the magnetic Prandtl number. Our numerical
treatment of the problem supports the claim that no solution to the boundary-value
problem exists when \( \beta > 1 \).
The well-known similarity solution (1.11) for viscous two-dimensional stagnation-point flow can be modified to give a similarity solution to the system of equations given in (2.28)-(2.29). We let

\[ \psi(x,y) = xf(y), \quad \phi(x,y) = xg(y), \quad (2.30) \]

and substitute. The resulting system of ordinary differential equations is as follows:

\[ f''' + f'' + f'^2 + 1 - \beta \{ gg'' - g'^2 + 1 \} = 0, \quad (2.31) \]
\[ g'' + \epsilon \{ fg' - f'g \} = 0, \quad (2.32) \]

with boundary conditions \( f(0) = f'(0) = g(0) = 0 \) and \( f'(\infty) = g'(\infty) = 1 \).

The challenge is to find values for \( f''(0) \) and \( g'(0) \) which, when combined with the boundary conditions at \( y = 0 \), give a solution to the system of equations (2.31-2.32) which satisfies the conditions at infinity. The difficulties posed by a simultaneous search for \( f''(0) \) and \( g'(0) \) in a coupled system can be averted, however, by using the same transformation as that employed by Wilson [14] in the magnetohydrodynamic Blasius problem. We define \( F(\eta) \) and \( G(\eta) \) as follows:

\[ F(\eta) = A^{\frac{1}{2}} f(A^{\frac{1}{2}} \eta), \quad G(\eta) = A^{-\frac{1}{2}} B g(A^{\frac{1}{2}} \eta), \quad (2.33) \]

where \( A \) and \( B \) are constants. The conditions at infinity on \( f \) and \( g \) indicate that

\[ A = \lim_{\eta \to \infty} F'(\eta) \quad \text{and} \quad B = \lim_{\eta \to \infty} G'(\eta), \quad (2.34) \]

which suggests that \( A \) and \( B \) are proportional to fluid velocity and Alfven velocity respectively at infinity. It follows that

\[ \beta = \frac{B^2}{A^2}. \quad (2.35) \]

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When these relations are substituted into (2.31-2.32), the resulting system of
equations for $\mathcal{F}(\eta)$ and $\mathcal{G}(\eta)$ is given by:

$$
\mathcal{F}''' + \mathcal{F}' - (\mathcal{F}')^2 - [\mathcal{G}'' - (\mathcal{G}')^2] + K = 0, \quad (2.36)
$$

$$
\mathcal{G}'' + \epsilon[\mathcal{F}\mathcal{G}' - \mathcal{F}'\mathcal{G}] = 0, \quad (2.37)
$$

where $K = A^2 - B^2$. The numerical procedure now requires that we fix $K$ and $\epsilon$, and solve the system of equations in (2.36-2.37) subject to the boundary conditions

$$
\begin{aligned}
\mathcal{F}(0) &= \mathcal{F}'(0) = \mathcal{G}(0) = 0, \\
\mathcal{G}'(0) &= 1, \\
\mathcal{F}'(\infty) &= A,
\end{aligned} \quad (2.38)
$$

where $A > 0$ is an eigenvalue. Even though both boundary conditions on $\mathcal{G}(\eta)$ are invoked at $\eta = 0$, it is easily shown that for large $\eta$, $\mathcal{G}'(\eta)$ approaches a constant which we call $B$. Therefore the solution of equations (2.36-2.37) involves only a single shooting procedure to determine the value of $\mathcal{F}''(0)$ which corresponds to $\mathcal{F}''(\infty) = 0$. If we call this value $C(K, \epsilon)$, then we find that equations (2.36-2.37) subject to the conditions in (2.38), with the condition at infinity replaced by

$$
\mathcal{F}''(0) = C(K, \epsilon), \quad (2.39)
$$

results in a solution which determines $A(K, \epsilon)$, $B(K, \epsilon)$ and $\beta(K, \epsilon)$. The corresponding solution to equations (2.31-2.32) and the boundary layer thickness, $B_\mathcal{C}$, are found from (2.33). In particular, we find that

$$
f''(0) = \frac{C}{A^2}, \quad g'(0) = \frac{1}{B}, \quad B_\mathcal{C} = \frac{1}{A^2} \lim_{\eta \to \infty} [A\eta - \mathcal{F}(\eta)]. \quad (2.40)
$$
The results of our numerical investigations are presented in Tables 2.1, 2.2 and 2.3. Tables 2.1 and 2.2 contain values of $f''(0)$ and $g'(0)$ respectively for various combinations of $(\epsilon, \beta)$ while Table 2.3 exhibits the corresponding values for $B_C$. Even though the numerical procedure described above gives $\beta$ only after $K$ and $\epsilon$ are fixed, we were able to calculate data for specific values of $\beta$ by modifying the ordinary differential equation solver package which iteratively adjusted $K$ until the desired value of $\beta$ was obtained. The ordinary differential equation solver package which uses the Hamming’s Predictor-Corrector Method with fourth order Runge-Kutta initialization is used throughout the dissertation to carry out all numerical calculations. A shooting method is employed to satisfy the far-field condition. The results for $\beta = 0$ corresponds to cases where the Alfvén velocity far from the wall is negligible as compared with the fluid velocity. In these cases, $f(y)$ reduces to the Hiemenz function.
Table 2.1: $f''(0)$ for various values of $(\epsilon, \beta)$

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<th>0.6</th>
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Table 2.2: $g'(0)$ for various values of $(\epsilon, \beta)$

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Table 2.3: Boundary layer thickness, $B_e$, for various values of $(e, \beta)$

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Chapter 3

Asymptotic Analysis for Small Conductivity

3.1 Equations for Small Conductivity

In the work which follows, we will be using the Hiemenz function defined in (1.12) extensively. At the same time, we wish to continue the practice of using the function names $f$, $F$, $\mathcal{F}$ to represent velocity-related functions and $g$, $G$, $\mathcal{G}$ to represent magnetic-related functions. For future reference therefore, we give here the full definition of the Hiemenz function which heretofore will be represented by $H(y)$:

\[
\begin{cases}
    H'''(y) + H(y)H''(y) - H'(y)^2 = -1, \\
    H(0) = H'(0) = 0, \\
    H''(0) = 1.23259, \quad H'(\infty) = 1.
\end{cases}
\] (3.1)

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The asymptotic behavior for large \( y \) is given by

\[
H(y) \sim y - 0.64790 + \text{exponentially small terms.} \quad (3.2)
\]

For fluids which are non-conducting, the parameter \( \epsilon \) vanishes and the equations (2.31-2.32) uncouple. The solutions are easily found to be

\[
f(y) = H(y), \quad g(y) = y, \quad (3.3)
\]

where \( H(y) \) is the Hiemenz function. If \( \epsilon \) is small but non-zero, the magnetic diffusion equation (2.32) is not satisfied by (3.3) and the perturbation problem is singular. The asymptotics are worth pursuing, however, because this case corresponds very nicely to values of \( \epsilon \) which are easily duplicated in a laboratory. Mercury at 20°C, for example, has a magnetic Prandtl number of \( 1.51 \times 10^{-7} \) and liquid sodium at 100°C exhibits a slightly larger value of \( 9.56 \times 10^{-6} \) (see [4]).

The inner solution is generated by equations (2.31-2.32) together with the boundary conditions \( f(0) = f'(0) = g(0) = 0 \). The outer solution must involve a scaling which renders all three terms of the magnetic diffusion equation of comparable magnitude for arbitrary small \( \epsilon \). Such a scaling is provided by the following definition of the outer functions \( F(\xi) \) and \( G(\xi) \):

\[
f(y) = \epsilon^{-\frac{1}{2}} F(\epsilon^{\frac{1}{2}} y), \quad g(y) = \epsilon^{-\frac{1}{2}} G(\epsilon^{\frac{1}{2}} y). \quad (3.4)
\]

On substituting these into (2.31-2.32), we obtain the generating equations for the outer solution; viz.

\[
\epsilon F''' + FF'' - (F')^2 + 1 - \beta \{GG'' - (G')^2 + 1\} = 0, \quad (3.5)
\]
\[ G'' + FG' - F'G = 0, \quad (3.6) \]

where \( F'(\infty) = G'(\infty) = 1. \)

### 3.2 Solutions for Small Conductivity

The first approximations, \( f_0(y) \) and \( g_0(y) \), to the inner solution are found by taking \( \varepsilon \) to be zero in (2.31-2.32) and are given in (3.3). The matching is carried out by comparing the inner functions when \( y \) is large with the outer functions when \( \xi = \varepsilon^{\frac{1}{2}}y \) is small. It follows that the first approximations to the outer solution are

\[ F_0(\xi) = G_0(\xi) = \xi. \quad (3.7) \]

It turns out that both inner and outer solutions proceed in powers of \( \varepsilon^{\frac{1}{2}} \) and appear as follows:

**Outer Solution**

\[
\begin{align*}
F_0(\xi) &= \xi + \varepsilon^{\frac{1}{2}}f_1(\xi) + \varepsilon f_2(\xi) + O(\varepsilon^{\frac{1}{2}}), \\
G_0(\xi) &= \xi + \varepsilon^{\frac{1}{2}}G_1(\xi) + \varepsilon G_2(\xi) + O(\varepsilon^{\frac{1}{2}}),
\end{align*}
\]

**Inner Solution**

\[
\begin{align*}
f(y) &= H(y) + \varepsilon^{\frac{1}{2}}f_3(y) + \varepsilon f_4(y) + O(\varepsilon^{\frac{1}{2}}), \\
g(y) &= y + \varepsilon^{\frac{1}{2}}g_1(y) + \varepsilon g_2(y) + O(\varepsilon^{\frac{1}{2}}).
\end{align*}
\]

On substituting (3.8) into (3.5-3.6) and dropping terms of \( O(\varepsilon) \), we obtain the following system of equations:

\[
\begin{align*}
F_0F_1'' + F_1F_0'' - 2F_0F_1' - \beta(G_0G_1'' + G_1G_0'' - 2G_0'G_1') &= 0, \quad (3.10) \\
G_1'' + F_0G_1' + F_1G_0' - G_0F_1' - G_1F_0' &= 0, \quad (3.11)
\end{align*}
\]
\[ f'' + f_0 f'' + f_1 f_0'' - 2f_0 f_1' - \beta (g_0 g_1'' + g_1 g_0'' - 2g_0 g_1) = 0, \quad (3.12) \]

\[ g_1'' = 0. \quad (3.13) \]

Using (3.7), equations (3.10) and (3.11) become

\[ \xi F'' - 2F' - \beta \{ \xi G'' - 2G' \} = 0, \quad (3.14) \]

\[ G'' + \xi G' + F_1 - \xi F_1 - G = 0. \quad (3.15) \]

The boundary conditions at infinity are \( F'_{1}(\infty) = G'_{1}(\infty) = 0 \). The boundary conditions at \( \xi = 0 \) are obtained by recalling from (3.4) that

\[ F(\xi) = e^{\xi} f(y) = e^{\xi} f_0(y) + O(\epsilon) \]

\[ \sim e^{\xi} (y - c) + O(\epsilon) \quad \text{for large } y \]

\[ = \xi - \epsilon e^{\xi} + O(\epsilon) \quad \text{for small } \xi, \quad (3.17) \]

where \( c=0.64790 \). On comparing with (3.8), we conclude that \( F_{1}(0) = -c \) and in a similar way that \( G_{1}(0) = 0 \).

The system of equations (3.14-3.15) can be simplified by defining the function \( u(\xi) = F_{1}(\xi) - \beta G_{1}(\xi) \). Then the equation (3.14) and the boundary conditions indicate that \( u(\xi) \equiv -c \) and this enables \( F_{1}(\xi) \) to be expressed in terms of \( G_{1}(\xi) \). Equation (3.15) then becomes

\[ G''_{1} + (1 - \beta) \xi G'_{1} - (1 - \beta) G_{1} = c, \quad (3.19) \]

whose solution is expressible in terms of the complementary error function defined as follows:

\[ \text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_{z}^{\infty} \exp(-t^2)dt. \quad (3.20) \]
The final results are

\[ F_1(\xi) = -c + \beta G_1(\xi), \]  
(3.21)

\[ G_1(\xi) = \frac{-c}{1 - \beta} \left[ 1 + \sqrt{\frac{\pi}{2}} (1 - \beta)^{\frac{3}{4}} \text{erfc}\left( \frac{1}{\sqrt{2}} (1 - \beta)^{\frac{1}{2}} \xi \right) \right] - \exp\left( -\frac{1}{2} (1 - \beta) \xi^2 \right). \]  
(3.22)

After simplification the differential equations for the inner functions \( f_1(y) \) and \( g_1(y) \) are

\[ f''_1 + Hf''_1 - 2H'f'_1 + H''f_1 + 2\beta G'_1 = 0, \]  
(3.23)

\[ g''_1 = 0, \]  
(3.24)

where \( H(y) \) is the Hiemenz function defined in (3.1). The boundary conditions are

\[ f_1(0) = f'_1(0) = g_1(0) = 0 \]  
(3.25)

\[ f'_1(\infty) = F'_1(0) = -\sqrt{\frac{\pi}{2}} \frac{c\beta}{(1 - \beta)^{\frac{3}{4}}}, \]  
(3.26)

\[ g'_1(\infty) = G'_1(0) = -\sqrt{\frac{\pi}{2}} \frac{c}{(1 - \beta)^{\frac{3}{4}}}. \]  
(3.26)

Despite the complexity of (3.23), the equations can be solved exactly with the results being given by

\[ f_1(y) = -\frac{\sqrt{2\pi}}{4} \frac{c\beta}{(1 - \beta)^{\frac{3}{4}}} \{ H(y) + y H'(y) \}, \]  
(3.27)

\[ g_1(y) = -\sqrt{\frac{\pi}{2}} \frac{c}{(1 - \beta)^{\frac{3}{4}}} y. \]  
(3.28)

The analysis can be extended to \( O(\epsilon) \) but the differential equations now require numerical solution. The procedures followed in obtaining the \( O(\epsilon^{\frac{1}{2}}) \) functions are basically repeated to get the \( O(\epsilon) \) results. The differential equations are obtained in the usual manner and the boundary-value problems for the outer
functions \( F_2(\xi) \) and \( G_2(\xi) \) are solved first. The conditions at infinity are simply \( F'_2(\infty) = G'_2(\infty) = 0 \), and the conditions at \( \xi = 0 \) are determined from that part of the inner solution which has already been found. In particular, \( f(y) = H(y) + \varepsilon^{1/2} f_1(y) + O(\varepsilon) \) is expanded for large \( y \) and the resulting expansion is substituted into \( F(\xi) = F(\varepsilon^{1/2} y) = \varepsilon^{1/2} f(y) \). When the change of variables \( y = \varepsilon^{-1/2} \xi \) is made, the resulting small-\( \xi \) expansion for \( F(\xi) \) is

\[
F(\xi) \sim \xi - \frac{1}{\varepsilon^{1/2}} \left\{ \frac{c^2 \beta}{2 (1 - \beta)^{1/2}} \xi + \frac{\sqrt{2\pi}}{4} \frac{c^2 \beta}{(1 - \beta)^{1/2}} \right\} + O(\varepsilon^{1/2}). \tag{3.29}
\]

By comparing (3.29) with (3.8), we see that

\[
F_2(0) = \frac{\sqrt{2\pi}}{4} \frac{c^2 \beta}{(1 - \beta)^{1/2}}, \tag{3.30}
\]

and in a similar way we find that \( G_2(0) = 0 \). The appropriate differential equations for this stage of our problem are given by

\[
F_0'''' + F_0'F_2'' + F_0''F_2' - 2F_0'F_2' + F_1F_1'' - (F_1')^2
- \beta\{G_0G_2'' + G_0''G_2 - 2G_0'G_2' + G_1'G_1'' - (G_1')^2\} = 0, \tag{3.31}
\]

\[
G_2'' + F_0G_2' + F_2G_0' - G_0F_2' - G_2F_0' + F_1G_1' - G_1F_1' = 0, \tag{3.32}
\]

\[
f_2'''' + f_0f_2'' + f_2f_0'' - 2f_0'f_2' + f_1f_1'' - (f_1')^2
- \beta\{g_0g_2'' + g_0''g_2 - 2g_0'g_2' + g_1g_1'' - (g_1')^2\} = 0, \tag{3.33}
\]

\[
g_2'' + g_0'f_0 - f_0'g_0 = 0. \tag{3.34}
\]

The equations (3.31) and (3.32) uncouple if we define

\[
\beta U(\xi) = F_2(\xi) - \beta G_2(\xi). \tag{3.35}
\]
The boundary-value problems then become

\[ \xi U'' - 2U' = cG_1'' + (1 - \beta)\{G_1G_1'' - (G_1')^2\}, \quad (3.36) \]

\[ G_2'' + (1 - \beta)\xi G_2' - (1 - \beta)G_2 = cG_1' + \beta\{\xi U'' - U\}, \quad (3.37) \]

where

\[
\begin{aligned}
U(0) &= \frac{\sqrt{2\pi}}{4} \cdot \frac{c^2}{(1-\beta)^2}, & U'(\infty) &= 0, \\
G_2(0) &= 0, & G_2'(\infty) &= 0. 
\end{aligned}
\]

(3.38)

The small-\(\xi\) expansion for \(U'(\xi)\) is

\[
U'(\xi) \sim \frac{1}{2} \left( \frac{\pi}{2} \right)^{-1} - 1)c^2 - \sqrt{\frac{\pi}{2}(1 - \beta)^{\frac{3}{4}}} c^2 \xi - (1 - \beta)c^2 \xi^2 \ln \xi
\]

\[
+ d_1(\beta)\xi^2 + \frac{\sqrt{2\pi}}{12}(1 - \beta)^{\frac{3}{4}}c^2 \xi^3 + O(\xi^4), \quad (3.39)\]

where \(d_1(\beta)\) is obtained numerically. This implies that \(U'(0) = \frac{1}{2}(\frac{\pi}{2} - 1)c^2\). The general solution of (3.37) is given by

\[
G_2(\xi) = d_2(\beta)\xi + d_3(\beta)[\sqrt{\frac{\pi}{2}(1 - \beta)^{\frac{3}{4}}} \text{erf} \left( \frac{1}{\sqrt{2}}(1 - \beta)^{\frac{3}{4}} \xi \right)
\]

\[
- \exp\left[ -\frac{1}{2}(1 - \beta)\xi^2 \right] + G_{2p}(\xi), \quad (3.40)\]

where \(d_2(\beta)\) and \(d_3(\beta)\) are coefficients of the homogeneous solution and \(G_{2p}(\xi)\) is any particular solution of \(G_2(\xi)\). The small-\(\xi\) expansion for \(G_{2p}(\xi)\) in terms of \(d_1(\beta)\) under the conditions

\[
G_{2p}(0) = G_{2p}'(0) = 0, \quad (3.41)\]

is derived to be

\[
G_{2p}'(\xi) \sim c^2\left\{ -\sqrt{\frac{\pi}{2}}(1 - \beta)^{-\frac{1}{4}} \left[ 1 + \frac{\beta}{2}\xi + \frac{1}{2}\xi^2 + \frac{1}{6}\sqrt{\frac{\pi}{2}}(1 - \beta)^{\frac{3}{2}} \left[ 1 - \frac{\beta}{2} \right] \xi^3 
\right.
\]

\[
- \frac{1}{6}\beta(1 - \beta)\xi^4 \ln \xi + \left( -\frac{1}{8} + \frac{d_1(\beta)}{6} + \frac{1}{72}\beta \right)(1 - \beta)\xi^4 + O(\xi^5) \right\}. \quad (3.42)\]
It is clear that $d_3(\beta)$ must identically be zero while $G'_2(0) = d_2(\beta) = -G'_{2p}(\infty)$ under the conditions (3.38) and (3.41). For selected values of $\beta$, the corresponding values of $d_1(\beta)$ are found by integrating (3.36) numerically using a shooting method to satisfy the far-field condition. Due to an apparent problem at $\xi = 0$, the integration procedure, with assistance from (3.39), is carried out from $\xi = 0.01$. The results are then used to compute $G_{2p}(0.01)$ and $G'_{2p}(0.01)$ for specific values of $\beta$. It now remains to integrate (3.37) numerically to obtain different values of $d_2(\beta)$. Seeking a relation to represent $d_2(\beta)$ analytically for all values of $\beta$, we analyze the collected data and find

$$G'_2(0) = d_2(\beta) = 0.23961 + \frac{0.59665\beta}{1 - \beta}, \quad (3.43)$$

which implies that

$$F'_2(0) = \beta (U''(0) + G'_2(0)) = 0.35941\beta + \frac{0.59665\beta^2}{1 - \beta}. \quad (3.44)$$

In our future study, where we need to derive a similar series expansion for the boundary layer thickness, $B_{\ell}$, the value of $F_2(\xi)$ far from the wall plays an important role. It is then worth investigating the $\beta$-dependency of the outer function $F_2(\xi)$ at infinity at this stage. We find

$$F_2(\infty) = 0.29314 \frac{\beta}{(1 - \beta)^{\frac{3}{2}}}. \quad (3.45)$$

The $O(\epsilon)$ terms of the inner solution are now within reach. The equations for the inner functions given by (3.33) and (3.34) are

$$f''_2 + H f''_2 - 2H' f''_2 + H'' f_2 = (f'_1)^2 - f_1 f''_1 + \beta \{y g''_2 - 2g'_2 - (g'_1)^2\}, \quad (3.46)$$
\[ g''_2 = y H' - H. \] \hspace{1cm} (3.47)

Focusing on \( g_2(y) \) first, the matching analysis indicates that far from the wall,

\[ g'_2(y) \sim cy + G'_2(0). \] \hspace{1cm} (3.48)

One integration of equation (3.47) then yields

\[ g'_2(y) = y H(y) - 2 \int_0^y H(\eta) d\eta + g'_2(0). \] \hspace{1cm} (3.49)

The far-field condition (3.48) suggests that \( g_2(y) \) is of the form

\[ g_2(y) = g_{2,1}(y) + g_{2,2}(y) \frac{\beta}{1 - \beta}, \] \hspace{1cm} (3.50)

where the functions \( g_{2,1}(y) \) and \( g_{2,2}(y) \) are defined as follows:

\[ \begin{align*}
g'_{2,1}(y) &= y H(y) - 2 \int_0^y H(\eta) d\eta + g'_{2,1}(0), \\
g_{2,1}(0) &= 0, \quad g'_{2,1}(y) \sim cy + 0.23961 \quad \text{as} \quad y \to \infty,
\end{align*} \] \hspace{1cm} (3.51)

and

\[ \begin{align*}
g'_{2,2}(y) &= 0.59665, \\
g_{2,2}(0) &= 0, \quad g'_{2,2}(y) \sim 0.59665 \quad \text{as} \quad y \to \infty.
\end{align*} \] \hspace{1cm} (3.52)

The constant \( g'_{2,2}(0) \) is found analytically from (3.52) while (3.51) is integrated numerically to obtain the value of \( g'_{2,1}(0) \). The results are

\[ \begin{align*}
g'_{2,1}(0) &= 0.95865, \\
g'_{2,2}(0) &= 0.59665.
\end{align*} \] \hspace{1cm} (3.53)

Equation (3.46) can now be examined because the right-side functions are all known. The boundary conditions at the wall are \( f_2(0) = f'_2(0) = 0 \). The matching
analysis indicates that when \( y \) is large,
\[
f'_2(y) \sim \beta cy + F'_2(0). \tag{3.54}
\]
A study of (3.46) suggests that \( f''_2(0) \) is of the form
\[
f''_2(0) = \frac{\beta r}{1 - \beta} + \frac{\beta^2 t}{1 - \beta}, \tag{3.55}
\]
where \( r \) and \( t \) are constants. The following differential equations
\[
f'''_{2,1} + H f'''_{2,1} - 2H' f'_{2,1} + H'' f_{2,1} = y^2 H' - 3y H \\
+ 4 \int^y_0 H(\eta) d\eta - 2g'_{2,1}(0) - \frac{\pi c^2}{2}, \tag{3.56}
\]

and
\[
f'''_{2,2} + H f'''_{2,2} - 2H' f'_{2,2} + H'' f_{2,2} = \frac{\pi c^2}{8} \left\{ (4(H')^2 + y H' H'' + y^2 (H'')^2 \\
- 3H H'' - y H H''' - y^2 H' H''') - y^2 H' + 3y H \\
- 4 \int^y_0 H(\eta) d\eta + 2[g'_{2,1}(0) - g'_{2,2}(0)] \right\}, \tag{3.57}
\]
together with the initial conditions
\[
f_{2,1}(0) = f'_{2,1}(0) = 0, \quad f_{2,2}(0) = f'_{2,2}(0) = 0, \tag{3.58}
\]
and the far-field conditions
\[
f'_{2,1}(y) \sim cy + 0.35941, \quad f'_{2,2}(y) \sim 0.23724 - cy, \tag{3.59}
\]
are integrated to determine \( r = f''_{2,1}(0) \) and \( t = f''_{2,2}(0) \). We obtain
\[
\begin{align*}
\begin{cases}
  r &= 2.51983, \\
  t &= -1.11192.
\end{cases}
\end{align*} \tag{3.60}
\]
The inner solution is now completely determined to $O(\epsilon)$. The quantities of physical interest are the skin friction at the wall which is proportional to $f''(0)$, and the tangential component of magnetic intensity at the wall which is given by $g'(0)$. From (3.9), (3.27) and (3.55), we have

$$f''(0) = 1.23259 - 1.50133\beta\left(\frac{\epsilon}{1-\beta}\right)^{\frac{1}{2}} + [2.51983\beta - 1.1192\beta^{2}](\frac{\epsilon}{1-\beta}) + O\left((\frac{\epsilon}{1-\beta})^{\frac{3}{2}}\right). \quad (3.61)$$

The expansion for $g'(0)$ is obtained from (3.9), (3.28), (3.50) and (3.53):

$$g'(0) = 1 - 0.81202\left(\frac{\epsilon}{1-\beta}\right)^{\frac{1}{2}} + [0.95865\epsilon - 0.36201\beta](\frac{\epsilon}{1-\beta}) + O\left((\frac{\epsilon}{1-\beta})^{\frac{3}{2}}\right). \quad (3.62)$$

Tables 3.1 and 3.2 compare the asymptotic predictions with the numerical results provided earlier in tables 2.1 and 2.2 respectively.

In an attempt to write the boundary layer thickness, $B_{\xi}$, in a series expansion, to $O(\epsilon)$, we must evaluate the $O(\epsilon^{\frac{3}{2}})$ term of the outer function, $F(\xi)$, far from the wall. Such expansion of $B_{\xi}$ is required for our study in chapter 6. We denote the $O(\epsilon^{\frac{3}{2}})$ terms of the outer solutions, $F(\xi)$ and $G(\xi)$, by $F_{3}(\xi)$ and $G_{3}(\xi)$ respectively. The differential equations which determine these functions are:

$$F''_{1} + F_{0}F''_{3} + F_{3}F''_{0} + F_{1}F''_{2} + F_{2}F''_{1} - 2F_{0}'F_{3}' - 2F_{1}'F_{2}' - \beta\{G_{0}G''_{3} + G_{3}G''_{0} + G_{1}G''_{2} + G_{2}G''_{1} - 2G_{0}'G_{3}' - 2G_{1}'G_{2}'\} = 0, \quad (3.63)$$

$$G''_{3} + F_{0}G''_{3} + F_{3}G''_{0} + F_{1}G''_{2} + F_{2}G''_{1} - F_{3}'G_{0}' - F_{1}'G_{2}' - F_{2}'G_{1}' = 0. \quad (3.64)$$
Table 3.1: Comparison of $f''(0)$ actual (numerical) with $f''(0)$ asymptotic.

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$\beta$</th>
<th>Actual (Num.)</th>
<th>Asymptotic</th>
<th>Error</th>
<th>%-Error</th>
</tr>
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<td>1.23259</td>
<td>0</td>
<td>0</td>
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<td>1.22255</td>
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<td>$1.10 \times 10^{-2}$</td>
</tr>
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</tr>
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<td>1.23259</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>$1.55 \times 10^{-1}$</td>
</tr>
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</tr>
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<td>1.23259</td>
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<td>0</td>
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Table 3.2: Comparison of $g'(0)$ actual (numerical) with $g'(0)$ asymptotic.

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$\beta$</th>
<th>Actual(Num.)</th>
<th>Asymptotic</th>
<th>Error</th>
<th>%-Error</th>
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<td>0.97240</td>
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</tr>
<tr>
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<td>0.4</td>
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<td>0.96821</td>
<td>$4.85 \times 10^{-5}$</td>
<td>$5.01 \times 10^{-3}$</td>
</tr>
<tr>
<td>0.001</td>
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<td>0.96125</td>
<td>$6.83 \times 10^{-5}$</td>
<td>$7.11 \times 10^{-3}$</td>
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<tr>
<td>0.001</td>
<td>0.8</td>
<td>0.94577</td>
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</tr>
<tr>
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<td>$1.12 \times 10^{-3}$</td>
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</tr>
<tr>
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<td>$1.22 \times 10^{-3}$</td>
<td>$1.33 \times 10^{-1}$</td>
</tr>
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<td>0.90873</td>
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<td>0.89014</td>
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</tr>
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<td>$3.15 \times 10^{-2}$</td>
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<tr>
<td>0.1</td>
<td>0.8</td>
<td>0.64691</td>
<td>0.76034</td>
<td>$1.13 \times 10^{-1}$</td>
<td>17.53</td>
</tr>
</tbody>
</table>
Equations (3.63) and (3.64) can be rewritten with help from (3.7) and (3.21) as

\[ m_1 \frac{\partial}{\partial \xi} \left( \frac{1}{\beta} \right) - 2(2 \xi - \xi') + \beta G''_1(F_2 - G_2) + \beta G_1(F''_3 - G''_2) - 2\beta G'_1(F'_3 - G'_2) + \beta G''_1 - cF''_2 = 0, \]  

(3.65)

\[ G''_3 - \xi(F'_3 - G'_3) + (F_3 - G_3) - G_1(F'_2 - \beta G'_2) + G'_1(F_2 - \beta G_2) - cG'_2 = 0. \]  

(3.66)

Equations (3.65) and (3.66) can be uncoupled by defining

\[ \beta \phi(\xi) = F_3(\xi) - \beta G_3(\xi). \]  

(3.67)

The boundary value problems then become

\[ \xi \phi'' - 2\phi' = (1 - \beta)\{G_2 G''_1 + G_1 G''_2 - 2G'_1 G'_2 + \xi G''_1\} \]

\[ -\beta \{U G''_1 + U''G_1 - 2U'G'_1\} + c(U'' + G''_2), \]  

(3.68)

\[ G''_3 + \xi(1 - \beta)G'_3 - (1 - \beta)G_3 = \beta \{\xi \phi' - \phi + G_1 U' - G'_1 U\} + cG'_2, \]  

(3.69)

where the function \( U(\xi) \) is defined in (3.35). The conditions at infinity are \( F_3'(\infty) = G_3'(\infty) = 0 \), and the conditions at \( \xi = 0 \) are once again determined from the known inner solutions. The expansions of \( g'_2(y) \) and \( f'_2(y) \) for large \( y \), already found in (3.48) and (3.54) respectively, imply that as \( y \to \infty \),

\[ f_2(y) \sim \frac{1}{2} \beta cy^2 + yF'_2(0) + a_3(\beta), \]  

(3.70)

\[ g_2(y) \sim \frac{1}{2} cy^2 + yG'_2(0) + a_4(\beta), \]  

(3.71)

where \( a_3(\beta) \) and \( a_4(\beta) \) are unknown functions of \( \beta \) and \( G'_2(0) \) and \( F'_2(0) \) are given by (3.43) and (3.44) respectively.
Following the procedure used to arrive at (3.30), the inner solution

\[ f(y) = H(y) + \epsilon^{\frac{1}{2}} f_1(y) + \epsilon f_2(y) + O(\epsilon^{\frac{3}{2}}), \]

is first expanded for large \( y \). The resulting expansion is then substituted into

\[ F(\xi) = e^{\frac{1}{2} f(e^{-\frac{1}{2} \xi})} \]

to obtain the small-\( \xi \) expansion for \( F(\xi) \):

\[
F(\xi) \sim -\epsilon^\frac{1}{2}\left\{ c + \sqrt{\frac{\pi}{2}} \frac{c\beta}{\sqrt{1-\beta}} \xi - \frac{1}{2} c\beta \xi^2 \right\} \\
+ \epsilon \left\{ \frac{\sqrt{2\pi}}{4} \frac{\beta c^2}{\sqrt{1-\beta}} + F'_2(0) \xi \right\} + \epsilon^{\frac{3}{2}}\{ a_3(\beta) \} + O(\epsilon^2). \tag{3.72}
\]

Similarly, we find the small-\( \xi \) expansion for \( G(\xi) \) to be

\[
G(\xi) \sim \xi + \epsilon^\frac{1}{2}\left\{ -\sqrt{\frac{\pi}{2}} \frac{c}{\sqrt{1-\beta}} \xi + \frac{1}{2} c\xi^2 \right\} + \epsilon\{ G'_2(0) \xi \} + \epsilon^{\frac{3}{2}}\{ a_4(\beta) \} + O(\epsilon^2). \tag{3.73}
\]

A simple comparison of equations (3.72) and (3.73) with (3.8) yields the followings:

\[
F_3(0) = a_3(\beta), \quad G_3(0) = a_4(\beta). \tag{3.74}
\]

Equation (3.70) suggests that \( a_3(\beta) \) is of the form

\[
a_3(\beta) = \alpha_1 \beta + \alpha_2 \frac{\beta^2}{1-\beta}, \tag{3.75}
\]

where \( \alpha_1 \) and \( \alpha_2 \) are constants independent of \( \beta \) - that is,

\[
a_3(\beta) = \lim_{y \to \infty} \left\{ f_2(y) - \frac{1}{2} \beta cy^2 - yF'_2(0) \right\} \\
= \lim_{y \to \infty} \left\{ [f_{2,1}(y) - \frac{1}{2} cy^2 - 0.35941] \beta \\
+ [f_{2,2}(y) + f_{2,1}(y) - 0.59665] \frac{\beta^2}{1-\beta} \right\}, \tag{3.76}
\]

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where \( f_{2,1}(y) \) and \( f_{2,2}(y) \) are defined in (3.56-3.60). The constants \( \alpha_1 \) and \( \alpha_2 \) are then determined numerically using equations (3.76) and (3.56-3.60):

\[
\begin{align*}
\alpha_1 & \approx 0.56428, \\
\alpha_2 & \approx -0.13988,
\end{align*}
\]  

(3.77)

and hence, \( a_3(\beta) \) is completely determined to be

\[
a_3(\beta) = 0.56428\beta - 0.13988\frac{\beta^2}{1 - \beta}.
\]  

(3.78)

In a similar way we proceed to determine the value of \( a_4(\beta) \). From equations (3.71), (3.43) and (3.50-3.53), we find

\[
a_4(\beta) = \lim_{y \to \infty} \{g_2(y) - \frac{1}{2}cy^2 - yG_2'(0)\}
\]

\[= \lim_{y \to \infty} \{g_{2,1}(y) - \frac{1}{2}cy^2 - 0.95865y\} = 0.53147. \]  

(3.79)

It follows from (3.67) that

\[
\phi(0) = \frac{1}{\beta} [F_3(0) - \beta G_3(0)] = 0.032802 - 0.13988\frac{\beta}{1 - \beta}. \]  

(3.80)

In order to integrate equations (3.68) and (3.69) numerically, it is imperative that we derive the small-\( \xi \) expansions of the functions \( \phi(\xi) \) and \( G_3(\xi) \). The right hand side of (3.68) is completely known; therefore, its small-\( \xi \) expansion, which we denote by \( I_4(\xi) \), can easily be found. The expansion of \( \phi(\xi) \), when \( \xi \) is small, is then obtained from the relation

\[
\phi(\xi) \sim \phi(0) + \int_0^\xi \eta^2 \int_0^n \frac{1}{\delta^2} I_4(\delta)d\delta d\eta + \frac{1}{3}\xi^3 e(\beta) \]  

(3.81)
where $e(\beta)$ is obtained numerically. A similar expansion for $G_3(\xi)$ can now be derived from (3.69) for small-$\xi$ that involves, in addition to $e(\beta)$, yet another unknown quantity, $b(\beta)$. We integrate equations (3.68) and (3.69) numerically to obtain the values of $e(\beta)$ and $b(\beta)$ for a fixed $\beta$. Making use of the small-$\xi$ expansions found for $\phi(\xi)$ and $G_3(\xi)$, the integrations are carried out from $\xi = 0.01$. Using a shooting method to satisfy the far-field condition—namely, that $\phi'(\infty) = 0$, the equation (3.68) is integrated to obtain the value of $e(\beta)$ for a specific $\beta$. The procedure is then repeated for selected values of $\beta$ in order to construct a table of results for $e(\beta)$. These results along with the small-$\xi$ expansion of $G_3(\xi)$ are then used to determine $G_3(0.01)$ in terms of $b(\beta)$. The corresponding values of $b(\beta)$ are then obtained by integrating (3.69) using a shooting method to satisfy the condition $G_3'(\infty) = 0$. To insure accuracy and reliability, the integration procedures are repeated for various small values of $\xi$ and the results are compared. The result, which interests us, is

$$F_3(\infty) = \{-0.15441 + 0.0076621/3\} e^{\beta/2}. \quad (3.82)$$

The resulting series expansion for the boundary layer thickness, $B_L$, to $O(\epsilon)$ is given by

$$B_L \sim e^{-1/3}\{\xi - F(\xi)\} \quad \text{as} \quad \xi \to \infty$$

$$\sim \{-F_1(\xi) + \epsilon^{1/3}F_2(\xi) + \epsilon F_3(\xi) + O(\epsilon^{1/3})\} \quad \text{as} \quad \xi \to \infty$$

$$\sim \frac{1}{1 - \beta}\{0.64790 - 0.29314\sqrt{1 - \beta}\epsilon^{1/3}$$

$$+ \{0.15441 - 0.0076621\beta\}\frac{\beta}{1 - \beta}\epsilon\} + O(\epsilon^{3/2}). \quad (3.83)$$

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Table 3.3 compares the numerical results provided in Table 2.3 with the asymptotic predictions.

Table 3.3: Comparison of $B_C$ actual (numerical) with $B_C$ asymptotic.

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$\beta$</th>
<th>Actual (Num.)</th>
<th>Asymptotic</th>
<th>Error</th>
<th>%-Error</th>
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<td>0.64790</td>
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<td>0</td>
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<td>1.07202</td>
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<td>3.15955</td>
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<td>3.00696</td>
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<td>0.64790</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>0.78874</td>
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Chapter 4

Asymptotic Analysis for Large Conductivity

4.1 Equations for Large Conductivity

One of the results emerging from the previous section is that the small parameter which arises naturally in the asymptotic expansions for \( f''(0) \) and \( g'(0) \) is \( \frac{\epsilon}{1-\beta} \). It turns out that in the opposite extreme when the fluid possesses large conductivity, the same parameter is prominent but now with large values. It is convenient therefore to re-scale the governing equations so that this naturally-occurring large parameter is present. Accordingly we define

\[
\lambda = \frac{\epsilon}{1-\beta},
\]

\[
f(y) = (1 - \beta)^{-\frac{1}{\lambda}} P[(1 - \beta)^{\frac{1}{\lambda}} y]\quad \text{and} \quad g(y) = (1 - \beta)^{-\frac{1}{\lambda}} Q[(1 - \beta)^{\frac{1}{\lambda}} y]. \quad (4.1)
\]
On substituting these transformations into equations (2.31-2.32), we obtain the system of equations which determines the outer solution of the singular perturbation problem which arises when fluid conductivity is large. The equations are as follows:

\[(1 - \beta)P''' + PP'' - (P')^2 + 1 - \beta\{QQ'' - (Q')^2 + 1\} = 0, \tag{4.2}\]

\[Q'' + \lambda\{PQ' - P'Q\} = 0. \tag{4.3}\]

The boundary conditions carry over unchanged:

\[
\begin{align*}
P(0) &= P'(0) = Q(0) = 0, \\
P'(\infty) &= Q'(\infty) = 1. 
\end{align*}
\tag{4.4}\]

For a perfectly conducting fluid where \(\lambda\) is infinite, equation (4.3) reduces to

\[PQ' - P'Q = 0. \tag{4.5}\]

The only solution of (4.5) satisfying the boundary conditions is

\[P(\theta) = Q(\theta), \tag{4.6}\]

where \(\theta = (1 - \beta)^{\frac{1}{2}}y\). From (4.2), this implies that

\[P''' + PP'' - (P')^2 + 1 = 0,
\]

which has the Hiemenz function as defined in (3.1) as its solution; hence,

\[P(\theta) = Q(\theta) = H(\theta). \tag{4.7}\]

But when \(\lambda\) is finite, this solution is incorrect in the vicinity of the wall because (4.7) predicts \(Q''(0) = 1.23259\) which contradicts the fact that \(Q''(0) = 0\) which is
obtained when the boundary conditions are applied to equation (4.3). According to (4.7), when $\theta$ is small $P(\theta) = \frac{1}{2} a_0 \theta^2 - \frac{1}{6} \theta^3 + O(\theta^5)$ where $a_0 = 1.23259$ which implies that $P'(\theta) = O(\theta)$ and $P''(\theta) = O(1)$ and similarly for $Q(\theta)$, and so the terms of (4.3) become of comparable magnitudes when $\theta = O(\lambda ^{-\frac{1}{3}})$. A re-scaling of equations (4.2-4.3) is therefore necessary which will render all terms of the magnetic diffusion equation of comparable magnitude for large $\lambda$. The inner functions $p(\xi)$ and $q(\xi)$ are defined as follows:

$$P(\theta) = \lambda^{-\frac{2}{3}} p(\lambda^{\frac{1}{3}} \theta) \quad \text{and} \quad Q(\theta) = \lambda^{-\frac{2}{3}} q(\lambda^{\frac{1}{3}} \theta). \quad (4.8)$$

The inner solution is generated by the following equations:

$$(1 - \beta) \lambda p'' + pp'' - (p')^2 + \lambda^{\frac{5}{3}} - \beta \{qq'' - (q')^2 + \lambda^{\frac{5}{3}}\} = 0, \quad (4.9)$$

$$q'' + pq' - p'q = 0, \quad (4.10)$$

where $p(0) = p'(0) = q(0) = 0$.

### 4.2 Solutions for Large Conductivity

Clearly the first terms in the outer expansions, $P_0(\theta)$ and $Q_0(\theta)$, are given by (4.7). If $p_0(\xi)$ and $q_0(\xi)$ are the leading terms in the inner expansion, then we have

$$p_0''' = 0, \quad (4.11)$$

$$q_0'' + p_0 q_0' - p_0' q_0 = 0. \quad (4.12)$$
The solution of (4.11) which satisfies the boundary conditions at the wall is given by \( p_0(\xi) = \frac{1}{2} c_0 \xi^2 \) where \( c_0 \) is a constant. For \( \theta \) small,

\[
P_0(\theta) = Q_0(\theta) = \frac{1}{2} a_0 \theta^2 - \frac{1}{6} \theta^3 + \frac{1}{120} A^2 \theta^5 + O(\theta^6),
\]

(4.13)

and from (4.8), for large \( \xi \),

\[
p(\xi) = \lambda^{\frac{5}{3}} P(\lambda^{-\frac{1}{3}} \xi) \sim \lambda^{\frac{5}{3}} H(\lambda^{-\frac{1}{3}} \xi) \sim \lambda^{\frac{5}{3}} \left\{ \frac{1}{2} a_0 (\lambda^{-\frac{1}{3}} \xi)^2 \right\}
\]

\[- \frac{1}{6} (\lambda^{-1} \xi)^3 + \frac{a_0^2}{120} (\lambda^{-\frac{1}{3}} \xi)^5 + O[(\lambda^{-\frac{1}{3}} \xi)^9] \}
\]

\[= \frac{1}{2} a_0 \xi^2 - \frac{1}{6} \lambda^{-\frac{1}{3}} \xi^3 + \frac{a_0^2}{120} \lambda^{-1} \xi^5 + O(\lambda^{-\frac{1}{3}} \xi^6),
\]

(4.14)

where \( a_0 = 1.23259 \). It follows that the condition \( c_0 = a_0 \) is required to balance the inner and outer solutions and hence,

\[p_0(\xi) = \frac{1}{2} a_0 \xi^2.
\]

(4.15)

Equation (4.12) now becomes a second-order linear ordinary differential equation with variable coefficients,

\[q_0'' + \frac{1}{2} a_0 \xi^2 q_0' - a_0 \xi q_0 = 0.
\]

(4.16)

Seeking a solution of the form \( q_0(\xi) = k_1 \xi \phi(\xi) \), we introduce the following change of variables:

\[z = -\frac{1}{6} a_0 \xi^3, \quad \phi(\xi) = \Omega(z).
\]

(4.17)

On substituting (4.17) into (4.16), we arrive at

\[z \Omega''(z) + \left( \frac{4}{3} - z \right) \Omega'(z) + \frac{1}{3} \Omega(z) = 0.
\]

(4.18)
The general solution is

\[ \Omega(z) = c_1 M\left(-\frac{1}{3}, \frac{4}{3}, z\right) + c_2 U\left(-\frac{1}{3}, \frac{4}{3}, z\right), \quad (4.19) \]

where \( M(a, b, z) \) and \( U(a, b, z) \) are confluent hypergeometric functions discussed by Abramowitz and Stegun [16]. The general solution to (4.16) is then given by

\[ q_0(\xi) = k\xi M\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{6}a_0\xi^3\right) + \hat{k}\xi U\left(-\frac{1}{3}, -\frac{4}{3}, \frac{1}{6}a_0\xi^3\right) \quad (4.20) \]

where \( k \) and \( \hat{k} \) are constants. Abramowitz and Stegun show that \( M\left(-\frac{1}{3}, \frac{4}{3}, 0\right) = 1 \) and that \( U\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{6}a_0\xi^3\right) = 0(\xi^{-1}) \) when \( \xi \) is small. It follows that we must set \( \hat{k} = 0 \) in (4.20) in order to satisfy the boundary condition \( q_0(0) = 0 \).

The constant \( k \) is determined by the matching condition which requires that

\[ q_0''(\infty) = a_0. \quad (4.21) \]

The large-\( \xi \) behavior of the confluent hypergeometric function, \( M(a, b, z) \),

\[ M\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{6}a_0\xi^3\right) = \frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{7}{3}\right)} \frac{1}{6}a_0\xi^3 \left\{ 1 + O\left(\frac{1}{6}a_0\xi^{-3}\right)^{-1} \right\}, \]

assists us in deriving a similar expansion for \( q_0''(\xi) \). We find

\[ q_0''(\xi) = k\left\{ \frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{5}{3}\right)} \left(\frac{1}{6}a_0\right)^{\frac{1}{3}} + O\left(\frac{1}{6}\xi^{-3}\right) \right\} \text{ as } \xi \to \infty, \quad (4.22) \]

which implies that

\[ q_0''(\infty) = k\left\{ \frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{5}{3}\right)} \left(\frac{1}{6}a_0\right)^{\frac{1}{3}} \right\}. \quad (4.23) \]

From (4.21) and (4.23) we obtain the value of \( k \) to be

\[ k = \frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} \left(6a_0^2\right)^{\frac{1}{3}} \simeq 1.05590. \quad (4.24) \]

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This shows that \( q_0(0) = k \).

The asymptotic form of the confluent hypergeometric function for large \( \xi \) is given by

\[
q_0 \sim \frac{1}{2} a_0 \xi^2 + \frac{2}{3} \xi^{-1} + \frac{4}{9} a_0^{-1} \xi^{-4} + O(\xi^{-7}).
\]  

(4.25)

From (4.8), we have

\[
Q(\theta) = \lambda^{-\frac{2}{3}} q(\lambda^{\frac{1}{3}} \theta) \sim \lambda^{-\frac{2}{3}} \left\{ \frac{1}{2} a_0 (\lambda^{\frac{1}{3}} \theta)^2 + \frac{2}{3} (\lambda^{\frac{1}{3}} \theta)^{-1} + \frac{4}{9} a_0^{-1} (\lambda^{\frac{1}{3}} \theta)^{-4} + O((\lambda^{\frac{1}{3}} \theta)^{-7}) \right\}
\]

\[
= \frac{1}{2} a_0 \theta^2 + \frac{2}{3} \lambda^{-1} \theta^{-1} + \frac{4}{9} a_0^{-1} \lambda^{-2} \theta^{-4} + O(\lambda^{-3} \theta^{-7}).
\]  

(4.26)

The expansion in (4.14) suggests that the inner solution proceeds in powers of \( \lambda^{-\frac{1}{3}} \). In order to extend the matching, we consider the following expansions:

\[
\begin{align*}
P(\theta) &= P_0 + \lambda^{-\frac{1}{3}} P_{\frac{2}{3}} + \lambda^{-\frac{2}{3}} P_{\frac{4}{3}} + \lambda^{-1} \ln \lambda P_{L} + \lambda^{-1} P_{1} + O(\lambda^{-\frac{1}{3}} \ln \lambda), \\
p(\xi) &= p_0 + \lambda^{-\frac{1}{3}} p_{\frac{2}{3}} + \lambda^{-\frac{2}{3}} p_{\frac{4}{3}} + \lambda^{-1} \ln \lambda p_{L} + \lambda^{-1} p_{1} + O(\lambda^{-\frac{1}{3}} \ln \lambda),
\end{align*}
\]  

(4.27)

and similarly for \( Q(\theta) \) and \( q(\xi) \). Replacing the functions \( P, Q, p \) and \( q \) in (4.2-4.3) and (4.9-4.10) with their corresponding expansions and equating to zero the coefficients of successive powers of \( \lambda \) result in obtaining the following equations:

\[
Q_0' + P_0 Q_1' + P_1 Q_0' + P_{\frac{2}{3}} Q_{\frac{2}{3}}' + P_{\frac{4}{3}} Q_{\frac{4}{3}}' - P_0' Q_1 - P_1' Q_0 - P_{\frac{2}{3}}' Q_{\frac{2}{3}} - P_{\frac{4}{3}}' Q_{\frac{4}{3}} = 0,
\]  

(4.28)

\[
P_0 Q_1' + P_1 Q_0' - P_0' Q_1 - P_1' Q_0 = 0,
\]  

(4.29)

\[
P_L Q_0' + P_0 Q_L' - P_L' Q_0 - P_0' Q_L = 0,
\]  

(4.30)

\[
P_0 Q_{\frac{2}{3}}' + P_{\frac{2}{3}} Q_0' - P_0' Q_{\frac{2}{3}} - P_{\frac{2}{3}}' Q_0 = 0,
\]  

(4.31)
\[(1 - \beta)p'''_1 + p_0p''_0 - (p'_0)^2 - \beta\{q_0q''_0 - (q'_0)^2\} = 0, \quad (4.32)\]

\[p''_L = 0, \quad (4.33)\]

\[p'''_{\frac{1}{3}} + 1 = 0, \quad (4.34)\]

\[p''_{\frac{2}{3}} = 0, \quad (4.35)\]

\[q''_L + p_0q'_{L} + p_Lq'_0 - p'_0q_L - p'_Lq_0 = 0, \quad (4.36)\]

\[q'''_1 + p_0q'_1 + p_1q'_0 + p_{\frac{1}{3}}q_{\frac{1}{3}} - p_{\frac{2}{3}}q_{\frac{1}{3}} - p_{1}q_{1} - p'_{1}q_{0} - p'_{1}q_{3} - p'_{1}q_{3} = 0, \quad (4.37)\]

\[q''_{\frac{1}{3}} + p_0q''_{\frac{1}{3}} + p_{\frac{1}{3}}q'_{\frac{1}{3}} - p_{\frac{2}{3}}q_{\frac{1}{3}} - p_{\frac{1}{3}}q_{0} = 0, \quad (4.38)\]

\[q''_{\frac{2}{3}} + p_0q''_{\frac{2}{3}} + p_{\frac{2}{3}}q_{\frac{2}{3}} - p_{\frac{1}{3}}q_{0} = 0. \quad (4.39)\]

Equation (4.29) can be uncoupled by introducing the following transformation:

\[U(\theta) = Q_{\frac{1}{3}}(\theta) - P_{\frac{1}{3}}(\theta). \] The resulting equation has a general solution of the form

\[U(\theta) = bP_0(\theta)\] where \(b\) is a constant. Invoking the boundary conditions, \(P_{\frac{1}{3}}(0) = P_{\frac{1}{3}}'(0) = Q_{\frac{1}{3}}(0) = 0\) and \(P_{\frac{1}{3}}'(\infty) = Q_{\frac{1}{3}}'(\infty) = 0\), yields that the only solution of (4.29) satisfying the boundary conditions is

\[P_{\frac{1}{3}}(\theta) = Q_{\frac{1}{3}}(\theta) = 0. \quad (4.40)\]

Similarly we can show that the only solutions to (4.30) and (4.31) satisfying their corresponding boundary conditions are

\[P_L(\theta) = Q_L(\theta) = P_{\frac{2}{3}}(\theta) = Q_{\frac{2}{3}}(\theta) = 0. \quad (4.41)\]
The solution of (4.34) after invoking the boundary conditions is found to be of the form

\[ p_\delta(\xi) = -\frac{1}{6}\xi^3 + \frac{1}{2}c_1\xi^2, \tag{4.42} \]

where \( c_1 \) is a constant. The first term is as required by (4.14) and the second term implies that \( P''(\theta) \) must have a contribution \( c_1 \lambda^{-\frac{1}{2}} \) for small \( \theta \). But \( P_\delta(\theta) \) is found to be identically zero; hence, we require that \( c_1 = 0 \) to balance the inner and outer solutions. Thus, the inner function \( p_\delta(\xi) \) is given by

\[ p_\delta(\xi) = -\frac{1}{6}\xi^3. \tag{4.43} \]

Equation (4.35) and its boundary conditions yield a solution of the form

\[ p_\delta(\xi) = \frac{1}{2}b_1\xi^2 \]

where \( b_1 \) is a constant. This implies that for \( \theta \)-small \( P''(\theta) \) must have a contribution \( b_1 \lambda^{-\frac{3}{2}} \). For a similar reason as in the case of \( p_\delta(\xi) \) we proceed to require the matching condition \( b_1 = 0 \) since \( P_\delta(\theta) \) is shown to be identically zero. The coefficient of \( \lambda^\frac{3}{2} \) in the inner solution expansion is therefore

\[ p_\delta(\xi) = 0. \tag{4.44} \]

Equations (4.38) and (4.39) can now be analyzed numerically. The equations are

\[ q_\frac{1}{3}'' + \frac{1}{2}a_0\xi^2 q_\frac{1}{3}' - a_0\xi q_\frac{1}{3} = \frac{1}{6}\xi^3 q_0' - \frac{1}{2}\xi^2 q_0 \tag{4.45} \]

and

\[ q_\frac{2}{3}'' + \frac{1}{2}a_0\xi^2 q_\frac{2}{3}' - a_0\xi q_\frac{2}{3} = \frac{1}{6}\xi^3 q_\frac{2}{3}' - \frac{1}{2}\xi^2 q_\frac{2}{3}. \tag{4.46} \]
The asymptotic form of \( q_3 \) and \( q_3^i \) for \( \xi \) large shows that

\[
q_3(-\xi) \sim -\frac{1}{6} \xi^3 - \frac{5}{9a_0} + \frac{28}{135a_0^2} \xi^{-3} + O(\xi^{-4})
\]  

(4.47)

and

\[
q_3^i(-\xi) \sim -\frac{5}{9a_0^2} \xi + \frac{14}{135a_0^3} \xi^{-2} + O(\xi^{-3}).
\]  

(4.48)

The homogeneous differential equation corresponding to (4.45), as seen in the solution of (4.16), has only one solution analytic at \( \xi = 0 \) and that is given by

\[
q_{3h}(\xi) = c_2 q_0(\xi),
\]  

(4.49)

where \( c_2 \) is a constant and \( q_0(\xi) \) is defined in (4.20). The general solution of (4.45), therefore, is of the form

\[
q_3(\xi) = c_2 q_0(\xi) + q_{3p}(\xi),
\]  

(4.50)

where \( q_{3p}(\xi) \) is a particular solution of (4.45). We define \( q_{3p}(\xi) \) by requiring that

\[
q_{3p}(0) = q'_{3p}(0) = 0.
\]  

(4.51)

Equation (4.45) is then integrated numerically under the initial conditions (4.51) to obtain the value of \( q''_{3p}(\xi) \) far from the wall. This allows us to obtain the constant \( c_2 \) from the relation

\[
c_2 = \lim_{\xi \to \infty} \frac{1}{a_0} \left[ q''_3(\xi) - q''_{3p}(\xi) \right],
\]  

(4.52)

where the asymptotic expansion (4.47) is used to obtain the large-\( \xi \) behavior of \( q''_3(\xi) \). Similarly for \( q_3^i(\xi) \), we find that the constant \( c_3 \) in \( q_3^i(\xi) = c_3 q_0(\xi) + q_{3p}^i(\xi) \)
can be found numerically from the relation
\[
c_3 \simeq -\frac{1}{a_0} \lim_{\xi \to \infty} \frac{q_2''(\xi)}{q_2',(\xi)}, \tag{4.53}
\]
having required the following conditions: \(q_{3_0}(0) = q_{3_0}'(0) = 0\). The key results are as follows:
\[
q_3(0) = -0.47570k; \quad q_3'(0) = -0.31832k, \tag{4.54}
\]
where \(k\) is defined in (4.24).

Turning our attention now to the outer solution, expansions are reduced to have the following form:
\[
\begin{align*}
P(\theta) &= H(\theta) + \lambda^{-1}P_1(\theta) + O(\lambda^{-2}), \\
Q(\theta) &= H(\theta) + \lambda^{-1}Q_1(\theta) + O(\lambda^{-2}).
\end{align*} \tag{4.55}
\]
When these are substituted into equations (4.2-4.3), the resulting equations can be uncoupled by defining:
\[
\begin{align*}
V(\theta) &= P_1(\theta) - Q_1(\theta), \\
\beta W(\theta) &= P_1(\theta) - \beta Q_1(\theta).
\end{align*} \tag{4.56}
\]
After simplification, we obtain
\[
\begin{align*}
H(\theta)V'(\theta) - H'(\theta)V(\theta) &= H''(\theta), \tag{4.57} \\
W'''(\theta) + H(\theta)W''(\theta) - 2H'(\theta)W'(\theta) + H''(\theta)W(\theta) &= V'''(\theta). \tag{4.58}
\end{align*}
\]
The proper boundary condition for (4.57) is \(V(\infty) = 0\). The solution is obtained numerically but the primary focus for later matching requirements is the expansion...
of $V(\theta)$ for small values of $\theta$. The same is true for $W(\theta)$. After finding these expansions, we can use (4.56) to obtain the corresponding small-$\theta$ expansions for $P''(\theta)$ and $Q''(\theta)$:

\begin{equation}
P''(\theta) = \frac{8a_0\beta}{3(1-\beta)} \ln \theta + \frac{a_1\beta}{1-\beta} + O(\theta \ln \theta), \quad (4.59)
\end{equation}

\begin{equation}
Q''(\theta) = \frac{4}{3} \theta^{-3} + \left\{ \frac{8a_0\beta}{3(1-\beta)} - \frac{8a_0}{15} + \frac{20}{27a_0^3} \right\} \ln \theta + \left\{ \frac{a_1\beta}{1-\beta} + a_2 \right\} + O(\theta \ln \theta), \quad (4.60)
\end{equation}

where $a_1$ and $a_2$ are constants. The outer expansions therefore have the form indicated in (4.55) and when these are expanded for small $\theta = \lambda^{-\frac{1}{2}} \xi$, we obtain

\begin{equation}
P''(\lambda^{-\frac{1}{2}} \xi) = H''(\lambda^{-\frac{1}{2}} \xi) + \lambda^{-1}P''(\lambda^{-\frac{1}{2}} \xi) + O(\lambda^{-2})
\end{equation}

\begin{align*}
&\sim a_0 - \lambda^{-\frac{1}{2}} \xi + \frac{1}{6}a_0^2 \lambda^{-1} \xi^3 + O(\lambda^{-\frac{1}{2}}) + \lambda^{-1}\left\{ \frac{8a_0\beta}{9(1-\beta)} \ln \lambda \right. \\
&\quad + \frac{8a_0\beta}{3(1-\beta)} \ln \xi + \frac{a_1\beta}{1-\beta} + O(\lambda^{-\frac{1}{2}} \ln \lambda) \right\} + O(\lambda^{-2}) \\
&= a_0 - \lambda^{-\frac{1}{2}} \xi - \frac{8a_0\beta}{9(1-\beta)} \lambda^{-1} \ln \lambda + \lambda^{-1}\left\{ \frac{1}{6}a_0^2 \xi^3 \\
&\quad + \frac{8a_0\beta}{3(1-\beta)} \ln \xi + \frac{a_1\beta}{1-\beta} \right\} + O(\lambda^{-\frac{1}{2}} \ln \lambda). \quad (4.61)
\end{align*}

We also find a similar expansion for $Q''(\lambda^{-\frac{1}{2}} \xi)$:

\begin{equation}
Q''(\lambda^{-\frac{1}{2}} \xi) \sim a_0 + \frac{4}{3} \xi^{-3} - \lambda^{-\frac{1}{2}} \xi + \left\{ \frac{8a_0\beta}{9(1-\beta)} + \frac{8a_0}{45} - \frac{20}{27a_0^3} \right\} \lambda^{-1} \ln \lambda \\
\quad + \lambda^{-1}\left\{ \frac{1}{6}a_0^2 \xi^3 + \left[ \frac{8a_0\beta}{3(1-\beta)} - \frac{8a_0}{15} + \frac{20}{27a_0^3} \right] \ln \xi \\
+ \left[ \frac{a_1\beta}{1-\beta} + a_2 \right] \right\} + O(\lambda^{-\frac{1}{2}} \ln \lambda). \quad (4.62)
\end{equation}

Equations (4.61) and (4.62) clearly demonstrate the need for $\lambda^{-1} \ln \lambda$ term in the expansions in order to balance the inner and outer solutions. The transformations
given in (4.8) indicate that \( P''(\lambda^{-\frac{1}{2}}\xi) = p''(\xi) \) and \( P''(\lambda^{-\frac{1}{2}}\xi) = q''(\xi) \); hence, equations (4.61) and (4.62) represent the behaviors of \( p''(\xi) \) and \( q''(\xi) \) for \( \xi \) large. We have as \( \xi \to \infty \),

\[
\begin{align*}
p''(\xi) &\sim \frac{-8a_0\beta}{9(1-\beta)}, \\
p''(\xi) &\sim \frac{1}{6}a_0^2\xi^3 + \frac{8a_0\beta}{3(1-\beta)} \ln \xi + \frac{a_1\beta}{1-\beta},
\end{align*}
\tag{4.63}
\tag{4.64}
\]

In a similar way it follows that for \( \xi \) large

\[
\begin{align*}
q''(\xi) &\sim \frac{8a_0}{45} - \frac{20}{81a_0^3} - \frac{8a_0\beta}{9(1-\beta)}, \\
q''(\xi) &\sim \frac{1}{6}a_0^2\xi^3 + \left\{ \frac{8a_0\beta}{3(1-\beta)} - \frac{8a_0}{15} + \frac{20}{27a_0^3} \right\} \ln \xi + \left\{ \frac{a_1\beta}{1-\beta} + a_2 \right\}.
\end{align*}
\tag{4.65}
\tag{4.66}
\]

Having invoked the boundary conditions, we find the solution of (4.33) to be of

the form

\[
p_L(\xi) = \frac{1}{2}d_1\xi^2,
\tag{4.67}
\]

where \( d_1 \) is a constant. Comparing (4.67) and (4.63), we deduce that \( d_1 = \frac{-8a_0\beta}{9(1-\beta)} \)
in order for the inner and outer solutions to match. We have

\[
p''_L(0) = \frac{-8a_0\beta}{9(1-\beta)}.
\tag{4.68}
\]

Equation (4.36) may then be written as

\[
q''_L + \frac{1}{2}a_0\xi^2q'_L - a_0\xi q_L = -\frac{8\beta}{9(1-\beta)}q''_0.
\tag{4.69}
\]

The general solution to (4.69) is given by

\[
q_L(\xi) = d_2q_0(\xi) - \frac{8\beta}{27(1-\beta)}\xi q'_0(\xi),
\tag{4.70}
\]
where \( d_2 \) is constant. A simple comparison of equations (4.70) and (4.65) allows us to determine the value of \( d_2 \). The exact solution of (4.36) is then determined to be
\[
g_L(\xi) = \left\{ \frac{8}{45} - \frac{20}{81a_5^2} - \frac{8\beta}{27(1 - \beta)} \right\} q_0(\xi) - \frac{8\beta}{27(1 - \beta)} \xi q'_0(\xi),
\]
(4.71)
implying that
\[
g'_L(0) = \left\{ \frac{8}{45} - \frac{20}{81a_5^2} - \frac{16\beta}{27(1 - \beta)} \right\} k,
\]
(4.72)
where \( k \) is defined in (4.24).

A study of equations (4.32) and (4.37) reveals that for \( \xi \)-large, \( p'_i(\xi) \) and \( q'_i(\xi) \) have the following behaviors:
\[
p'_i(\xi) \sim \frac{1}{6}a_0^2\xi^3 + \left\{ \frac{8}{3}a_0 \ln \xi + \alpha_2 \right\} \frac{\beta}{1 - \beta}
\]
(4.73)
and
\[
q'_i(\xi) \sim \frac{1}{6}a_0^2\xi^3 + \left\{ \frac{8a_0\beta}{3(1 - \beta)} - \frac{8a_0}{15} + \frac{20}{27a_0^3} \right\} \ln \xi + \alpha_3,
\]
(4.74)
where \( \alpha_2 \) and \( \alpha_3 \) are constants. To balance the inner and outer solutions, we require that \( \alpha_2 = a_1 \) and \( \alpha_3 = \frac{a_1 \beta}{1 - \beta} + a_2 \) where \( a_1 \) and \( a_2 \) are constants appearing in (4.64) and (4.66). To complete our study of the inner functions \( p_i \) and \( q_i \), equations (4.57) and (4.58) are integrated numerically, with great assistance from (4.59) and (4.60), to find
\[
a_1 = 3.61575, \quad a_2 = 0.51228.
\]
(4.75)

As in the case of small conductivity there is a need for the series expansion of the boundary layer thickness, \( B_C \), when we consider the non-orthogonal flow later on in our study. Since \( B_C \) is related to the function \( f(y) \) far from the wall, the
value of the outer solution, $P(\theta)$, when $\theta$ is large, is of interest. It is therefore worth pursuing the value of $P_1(\infty)$ at this stage of our problem. The boundary condition, $V(\infty)$, together with the relations defined in (4.56) imply that

$$P_1(\infty) = \frac{\beta}{1 - \beta} W(\infty), \quad (4.76)$$

where $W(\infty) = 1.64326$ is obtained numerically from equations (4.57) and (4.58).

Equation (4.32) may be written as

$$p_1'' = \frac{1}{2} a^2 \xi^2 \frac{1}{1 - \beta} + \left\{ q_0 q_0'' - (q_0')^2 \right\} \frac{\beta}{1 - \beta}. \quad (4.77)$$

Equations (4.77) and (4.73) suggest a solution of the form

$$p_1(\xi) = \frac{1}{1 - \beta} p_{1,1}(\xi) + \frac{\beta}{1 - \beta} p_{1,2}(\xi), \quad (4.78)$$

where functions $p_{1,1}(\xi)$ and $p_{1,2}(\xi)$ are defined by the following boundary value problems:

$$p_{1,1}''(\xi) = \frac{1}{2} a^2 \xi^2, \quad (4.79)$$

$$p_{1,2}''(\xi) = q_0 q_0'' - (q_0')^2, \quad (4.80)$$

under the conditions

$$p_{1,1}(0) = p_{1,1}'(0) = 0, \quad p_{1,2}(0) = p_{1,2}'(0) = 0, \quad (4.81)$$

and for $\xi$-large,

$$p_{1,1}''(\xi) \sim \frac{1}{6} a_0^2 \xi^3, \quad p_{1,2}''(\xi) \sim -\frac{1}{6} a_0^2 \xi^3 + \frac{8}{3} a_0 \ln \xi + 3.61575. \quad (4.82)$$
The value of \( p''_{1,1}(0) \) is determined analytically to be zero while \( p''_{1,2}(0) \), from numerical integration, is found to be 5.63340. This implies that

\[
p''_{1,2}(0) = 5.63340 \frac{\beta}{1 - \beta}.
\]  
(4.83)

It only remains then to derive a differential equation for \( g_1(\xi) \). Equation (4.37) with the help of equations (4.43) and (4.44) can be rewritten as

\[
q''_1 + \frac{1}{2} a_0 \xi^2 q'_1 - a_0 \xi q_1 = p'_1 q_0 - p_1 q'_0 - \frac{1}{2} a_0 \xi^2 q'_0 - \frac{1}{6} a_0 \xi^3 q'_0 + \frac{1}{6} a_0 \xi^3 q'_1,
\]  
(4.84)

where the behavior of \( q''_1(\xi) \) for \( \xi \)-large is given by (4.74) and the condition \( q_1(0) = 0 \) is also known. Using the fact that \( q_1(\xi) \) can be written in the form

\[
q_1(\xi) = q_{1,1}(\xi) \frac{1}{1 - \beta} + q_{1,2}(\xi) \frac{\beta}{1 - \beta},
\]

we proceed to determine \( q'_1(0) \) by integrating the following differential equations numerically:

\[
q''_{1,1} + \frac{1}{2} a_0 \xi^2 q'_{1,1} - a_0 \xi q_{1,1} = \frac{1}{24} a_0^2 \xi^4 q_0 - \frac{1}{120} a_0^2 \xi^4 q'_0 - \frac{1}{2} a_0 \xi^2 q'_0 + \frac{1}{6} a_0 \xi^3 q'_0,
\]  
(4.85)

\[
q''_{1,2} + \frac{1}{2} a_0 \xi^2 q'_{1,2} - a_0 \xi q_{1,2} = p'_{1,2} q_0 - p_{1,2} q'_0 + \frac{1}{2} a_0 \xi^2 q'_0 - \frac{1}{6} a_0 \xi^3 q'_0,
\]  
(4.86)

under the conditions

\[
q_{1,1}(0) = 0, \quad q''_{1,1} \sim \frac{1}{6} a_0^2 \xi^3 + \left( \frac{20}{27 a_0^3} - \frac{8 a_0}{15} \right) \ln \xi + 0.51228 \quad \text{as} \quad \xi \to \infty,
\]  
(4.87)

\[
q_{1,2}(0) = 0, \quad q''_{1,2} \sim -\frac{1}{6} a_0^2 \xi^3 - \left( \frac{20}{27 a_0^3} - \frac{16 a_0}{5} \right) \ln \xi + 3.10347 \quad \text{as} \quad \xi \to \infty,
\]  
(4.88)

We obtain

\[
q'_1(0) = [0.58746 + 2.91233 \frac{\beta}{1 - \beta}] k.
\]  
(4.89)
The inner solution is now completely determined to $O(\lambda^{-1})$. As in the case of small conductivity, the quantities of physical interest are the skin friction at the wall, $f''(0)$, and the tangential component of magnetic intensity at the wall, $g'(0)$. From (4.1) and (4.8), we have

$$f''(0) = (1 - \beta)^{\frac{1}{2}} p''(0), \quad g'(0) = \lambda^{-\frac{1}{2}} q'(0). \quad (4.90)$$

It follows from (4.27) and (4.90) that the asymptotic expansion for $f''(0)$ is given by

$$f''(0) = (1 - \beta)^{\frac{1}{2}} \{1.23259 - 1.09563 \frac{\beta}{\epsilon} \ln\left(\frac{\epsilon}{1 - \beta}\right) + 5.63340 \frac{\beta}{\epsilon} + O[\epsilon^{-\frac{1}{2}} \ln\left(\frac{\epsilon}{1 - \beta}\right)]\}. \quad (4.91)$$

In a similar way we deduce the expansion for $g'(0)$ to be

$$g'(0) = (1 - \beta)^{\frac{1}{2}} \frac{k}{\epsilon} \{1 - 0.47570 (\frac{1 - \beta}{\epsilon})^{\frac{1}{3}} - 0.31832 (\frac{1 - \beta}{\epsilon})^{\frac{1}{3}}$$

$$+ \left[\frac{0.070305 - 0.66340 \beta}{\epsilon}\right] \ln\left(\frac{\epsilon}{1 - \beta}\right)$$

$$+ \left[\frac{0.58746 + 2.32487 \beta}{\epsilon}\right] + O[\epsilon^{-\frac{1}{2}} \ln\left(\frac{\epsilon}{1 - \beta}\right)]\}, \quad (4.92)$$

where $k$ is defined in (4.24).

The resulting series expansion for $B_\ell$ is deduced from

$$B_\ell \simeq y - f(y) \quad \text{as } y \to \infty$$

$$= (1 - \beta)^{-\frac{1}{2}} \{ \theta - P(\theta) \} \quad \text{as } \theta \to \infty$$

$$= (1 - \beta)^{-\frac{1}{2}} \{ \theta - [H(\theta) + \lambda^{-1} P_1(\theta) + O(\lambda^{-\frac{1}{2}} \ln \lambda)] \} \text{ as } \theta \to \infty, \quad (4.93)$$

and it is given by

62
\[ B_\varepsilon = (1 - \beta)^{-\frac{1}{2}} \{ 0.64790 + \lambda^{-1} (1.64326 \frac{\beta}{1 - \beta}) + O(\lambda^{-\frac{3}{2}} \ln \lambda) \} \]
\[ = (1 - \beta)^{-\frac{1}{2}} \{ 0.64790 + 1.64326 \frac{\beta}{\varepsilon} + O(\varepsilon^{-\frac{3}{2}} \ln (\frac{\varepsilon}{1 - \beta})) \}. \] (4.94)

Tables 4.1-4.3 compare the asymptotic predictions with the numerical results provided earlier in tables 2.1-2.3.
Table 4.1: Comparison of \( f''(0) \) actual (numerical) with \( f''(0) \) asymptotic.

<table>
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<tr>
<th>( \epsilon )</th>
<th>( \beta )</th>
<th>Actual(Num.)</th>
<th>Asymptotic</th>
<th>Error</th>
<th>%-Error</th>
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<td>0</td>
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<td>1.23259</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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</tr>
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<td>0.77844</td>
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<td>( 2.70 \times 10^{-2} )</td>
</tr>
<tr>
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</tr>
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<td>1.23259</td>
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<td>0</td>
</tr>
<tr>
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<td>( 5.74 \times 10^{-1} )</td>
</tr>
<tr>
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<td>( 3.26 \times 10^{-3} )</td>
<td>( 6.00 \times 10^{-1} )</td>
</tr>
<tr>
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<td>1.23259</td>
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<td>0</td>
</tr>
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<td>1.03380</td>
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Table 4.2: Comparison of \( g'(0) \) actual (numerical) with \( g'(0) \) asymptotic.

<table>
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<tr>
<th>( \epsilon )</th>
<th>( \beta )</th>
<th>Actual (Num.)</th>
<th>Asymptotic</th>
<th>Error</th>
<th>%-Error</th>
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</thead>
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<td>0.10034</td>
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<td>( 1.28 \times 10^{-2} )</td>
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</tr>
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<td>0.05987</td>
<td>( 1.49 \times 10^{-5} )</td>
<td>( 2.49 \times 10^{-2} )</td>
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</tbody>
</table>

10 0 0.20270 0.20289 \( 1.91 \times 10^{-4} \) \( 9.41 \times 10^{-2} \)

10 0.2 0.18949 0.18999 \( 4.97 \times 10^{-4} \) \( 2.62 \times 10^{-1} \)

10 0.4 0.17358 0.17427 \( 6.89 \times 10^{-4} \) \( 3.97 \times 10^{-1} \)

10 0.6 0.15316 0.15391 \( 7.44 \times 10^{-4} \) \( 4.85 \times 10^{-1} \)

10 0.8 0.12323 0.12384 \( 6.10 \times 10^{-4} \) \( 4.95 \times 10^{-1} \)

10 0.37785 0.38506 \( 7.21 \times 10^{-3} \) 1.91

10 0.2 0.35565 0.37560 \( 1.99 \times 10^{-2} \) 5.61

10 0.4 0.32818 0.35632 \( 2.81 \times 10^{-2} \) 8.58

10 0.6 0.29179 0.32247 \( 3.07 \times 10^{-2} \) 10.52

10 0.8 0.23629 0.26146 \( 2.52 \times 10^{-2} \) 10.65

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Table 4.3: Comparison of $B_\epsilon$ actual (numerical) with $B_\epsilon$ asymptotic.

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<th>$\beta$</th>
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<th>Asymptotic</th>
<th>Error</th>
<th>%-Error</th>
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Chapter 5

Magnetohydrodynamic
Non-Orthogonal
Stagnation-Point Flow

5.1 Equations of motion

The equations governing the steady magnetohydrodynamic flow of a viscous electrically-conducting incompressible fluid as discussed in chapter 2 are

\[
(\mathbf{V} \cdot \mathbf{V}) \mathbf{V} = -\frac{1}{\rho} \mathbf{V} \rho + \nu \nabla^2 \mathbf{V} + \frac{\mu}{\rho} (\mathbf{J} \times \mathbf{H}),
\]

\( (5.1) \)

\[
\mathbf{J} = \sigma (\mathbf{E} + \mu (\mathbf{V} \times \mathbf{H})),
\]

\( (5.2) \)

\[
\mathbf{V} \times \mathbf{H} = \mathbf{J},
\]

\( (5.3) \)

\[
\mathbf{V} \cdot \mathbf{V} = \mathbf{V} \cdot \mathbf{H} = 0,
\]

\( (5.4) \)
\[ \nabla \times \vec{E} = 0. \quad (5.5) \]

The position-dependent quantities are defined as follows: \( \vec{V} \), fluid velocity; \( p \), fluid pressure; \( \vec{H} \), magnetic field intensity; \( \vec{E} \), electric field intensity; \( \vec{J} \), current density. The constants are \( \rho \), fluid density; \( \nu \), kinematic viscosity; \( \mu \), magnetic permeability and \( \sigma \), electrical conductivity.

The standard two-dimensional stagnation-point geometry will be assumed; viz., the fluid velocity \( \vec{V} \) and the magnetic field \( \vec{H} \) are each perpendicular to the \( z \)-direction. A rigid wall lies in the plane \( y = 0 \) and the fluid occupies the half-space \( y > 0 \). Far from the wall, the flow is directed obliquely towards the wall and follows streamlines as shown in figure 1.4. The magnetic field lines are aligned with the flow when \( y \gg 1 \).

Following the procedure in §2.2, we define the stream function and magnetic potential function as follows:

\[ \vec{V} = \nabla \times \{ \psi(x, y) \hat{k} \}, \quad \vec{H} = \nabla \times \{ \phi(x, y) \hat{k} \}. \quad (5.6) \]

The applied electric field, \( \vec{E} \), is shown in §2.2 to be of the form \( \vec{E} = -\vec{E}_0 \hat{k} \) where \( \vec{E}_0 \) is a constant. Equations (2.22) and (2.23) are then reproduced in the same way and are given by

\[ \eta \nabla^2 \psi + \frac{\partial \psi}{\partial x} \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \phi}{\partial x} = \frac{1}{\mu} \vec{E}_0, \quad (5.7) \]

\[ \nu \nabla^4 \psi + \frac{\partial \psi}{\partial x} \frac{\partial \nabla^2 \psi}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \nabla^2 \psi}{\partial x} - \frac{\mu}{\rho} \left[ \frac{\partial \phi}{\partial x} \frac{\partial \nabla^2 \phi}{\partial y} - \frac{\partial \phi}{\partial y} \frac{\partial \nabla^2 \phi}{\partial x} \right] = 0. \quad (5.8) \]
Far from the wall, we require that the fluid velocity and magnetic potential functions have the form

\[
\begin{align*}
\psi(x, y) &\sim \gamma [xy + \tilde{k}y^2], \\
\phi(x, y) &\sim H_\infty [xy + \tilde{k}y^2],
\end{align*}
\tag{5.9}
\]

as \( y \to \infty \), where \( \gamma \) and \( H_\infty \) are dimensional constants of proportionality having units \((\text{time})^{-1}\) and \((\text{charge})(\text{length})^{-1}(\text{time})^{-1}\) respectively. It then follows from (5.9) that as \( y \) approaches infinity

\[
\begin{align*}
\nabla^2 \phi &\sim 2\tilde{k}H_\infty, \\
\nabla^2 \psi &\sim 2\tilde{k}\gamma,
\end{align*}
\tag{5.10}
\]

A simple study of (5.7) and (5.10) shows that the following relation

\[
2\eta \tilde{k}H_\infty = \frac{1}{\mu} \tilde{E}_0, \tag{5.11}
\]

must hold at infinity in order to satisfy equation (5.7). The relation (5.11) implies that

\[
\tilde{E}_0 = \frac{2\tilde{k}H_\infty}{\sigma}, \tag{5.12}
\]

where \( \sigma = \frac{1}{\eta \mu} \).

Equations (5.7-5.8) can be non-dimensionalized using the following transformations

\[
\begin{align*}
x &= \left(\frac{\nu}{\gamma}\right)^{\frac{1}{2}} \bar{x}, \\
y &= \left(\frac{\nu}{\gamma}\right)^{\frac{1}{2}} \bar{y}, \\
\psi &= \nu \bar{\psi}, \\
\phi &= \frac{H_\infty \nu}{\gamma} \bar{\phi},
\end{align*}
\tag{5.13}
\]

where non-dimensional variables are denoted by bars. After dropping the bars, the two equations of motion for \( \psi(x, y) \) and \( \phi(x, y) \) are

\[
\nabla^4 \psi + \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial y \partial x} - \beta \left\{ \frac{\partial \phi}{\partial x} \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial \phi}{\partial y} \frac{\partial^2 \phi}{\partial x \partial y} \right\} = 0 \tag{5.14}
\]
and
\[ \nabla^2 \phi + \epsilon \left\{ \frac{\partial \psi}{\partial x} \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \phi}{\partial x} \right\} = 2k, \tag{5.15} \]
where \( \beta = \frac{\mu H^2}{\rho c^2} \) and \( \epsilon = \frac{\nu}{\eta} = \nu \sigma \mu \) are, as in the orthogonal problem, the square of the ratio of the Alfven velocity to the fluid velocity far from the wall and the magnetic Prandtl number respectively. The non-dimensional boundary conditions are given by
\[
\begin{align*}
\psi(x,0) &= \frac{\partial \psi}{\partial y}(x,0) = 0, \\
\psi(x,y) &\sim xy + k y^2 \quad \text{as} \quad y \to \infty,
\end{align*}
\tag{5.16}
\]
and
\[
\begin{align*}
\phi(x,0) &= 0, \\
\phi(x,y) &\sim xy + \bar{k} y^2 \quad \text{as} \quad y \to \infty.
\end{align*}
\tag{5.17}
\]
Guided by the electrically inert version of our problem, we introduce the following similarity solution
\[
\begin{align*}
\psi(x,y) &= xf(y) + m(y), \\
\phi(x,y) &= xg(y) + n(y),
\end{align*}
\tag{5.18}
\]
to the system of equations given in (5.14-5.15). The resulting differential equations are
\[
\begin{align*}
f''' + f f'' - f'^2 + 1 - \beta \{gg'' - g'^2 + 1\} &= 0, \\
g'' + \epsilon \{fg' - f'g\} &= 0, \\
m'' + f m''' - f''m' - \beta \{gn'' - g''n'\} &= 0, \\
n'' + \epsilon \{fn' - m'g\} &= 2\bar{k},
\end{align*}
\tag{5.19-5.22}\]
under the conditions

\[
\begin{align*}
  f(0) &= f'(0) = 0, \quad f'(\infty) = 1, \\
  g(0) &= 0, \quad g'(\infty) = 1, \\
  m(0) &= m'(0) = 0, \quad m''(\infty) = 2\bar{k}, \\
  n(0) &= 0, \quad n''(\infty) = 2\bar{k}.
\end{align*}
\]  

(5.23-5.26)

It is readily seen that equations (5.19) and (5.20) together with the conditions (5.23-5.24) form the system of equations which describes the magnetohydrodynamic stagnation-point flow. The functions \( f(y) \) and \( g(y) \) are, therefore, treated as known functions and the series expansions derived in chapters 3 and 4 for the related quantities of interest, \( f''(0) \) and \( g'(0) \), are used in our study of equations (5.21-5.22). The large-\( y \) behaviors of \( f(y) \) and \( g(y) \) are given by

\[
\begin{align*}
  f(y) &\sim y - B_C + \text{exponentially small terms,} \\
  g(y) &\sim y - B_C + \text{exponentially small terms,}
\end{align*}
\]

(5.27)

where \( B_C \) is the boundary layer thickness.

Equation (5.21) is integrated once and the resulting constant of integration is then determined from (5.27) and the far-field conditions given in (5.25-5.26). We find

\[
m'' + fm'' - f'm' - \beta (gn'' - g'n') + 2B_C\bar{k}(1 - \beta) = 0.
\]

(5.28)

Equations (5.22) and (5.28) can be made independent of \( \bar{k} \) by introducing the following transformations

\[
m'(y) = 2\bar{k}e(y), \quad n'(y) = 2\bar{k}l(y).
\]

(5.29)
On substituting (5.29) into equations (5.28) and (5.22), we obtain

\[ e'' + f e' - f e - \beta \{ g l' - g' l \} + B \epsilon (1 - \beta) = 0, \quad (5.30) \]

\[ l' + \epsilon \{ f l - g e \} = 1, \quad (5.31) \]

under the conditions

\[ e(0) = 0, \quad e'(\infty) = l'(\infty) = 1. \quad (5.32) \]

It is observed that, in the limiting case when there is only shear flow parallel to the wall, the full equations of motion (5.19), (5.20), (5.22) and (5.28) are satisfied by \( \psi(x, y) = y^2 \) and \( \phi(x, y) = y^2 \) where we take \( k = 1 \). This "limiting case" solution exhibits no-slip along the wall as expected; but it also indicates that the magnetic potential function satisfies \( \phi(x, 0) = \phi_y(x, 0) = 0 \). Even when there is a component of flow orthogonal to the wall, the component of magnetic potential parallel to the wall will still exhibit this behavior at the wall. Thus in the non-orthogonal flow, we have \( \phi(x, y) = x g(y) + n(y) \) where the boundary conditions on \( n(y) \) are

\[ n(0) = n'(0) = 0. \quad (5.33) \]

The boundary condition on \( l(y) \) at infinity can then be replaced by

\[ l(0) = 0. \quad (5.34) \]

The primary objective here is to find the shear component of tangential stress at the wall, \( e'(0) \). The values of \( f''(0), g'(0) \) and \( B \) must be known before equations
(5.30) and (5.31) can be integrated numerically. The simultaneous search for \( f''(0) \) and \( g'(0) \) is resolved in chapter 2 by using the transformations defined in (2.33).

We extend the existing transformations to include the functions \( e(y) \) and \( l(y) \) as follows:

\[
E(\eta) = A^{-\frac{1}{2}} e(A^{\frac{1}{2}} \eta), \quad L(\eta) = A^{-\frac{3}{2}} Bl(A^{\frac{1}{2}} \eta), \quad (5.35)
\]

where the constants \( A \) and \( B \) are defined in (2.34). In addition to (2.35-2.40), the following differential equations and boundary conditions which describe the functions \( E(\eta) \) and \( L(\eta) \) are found:

\[
E'' + \mathcal{F}E' - \mathcal{F}'E - \mathcal{G}L' + \mathcal{G}'L + A^{-\frac{3}{2}} KB_c = 0, \quad (5.36)
\]

\[
L' + \epsilon(\mathcal{F}L - \mathcal{G}E) = \frac{B}{A}, \quad (5.37)
\]

under the conditions

\[
E(0) = 0, \quad E'(\infty) = 1, \quad (5.38)
\]

\[
L(0) = 0. \quad (5.39)
\]

The numerical procedure requires that we fix \( K = A^2 - B^2 \) and \( \epsilon \), and first solve (2.36-2.37) to obtain the values of \( \mathcal{F}''(0) = C(K, \epsilon), A(K, \epsilon), B(K, \epsilon) \) and \( B_c(K, \epsilon) \).

The results enable us to integrate equations (5.36-5.37) subject to the conditions (5.38-5.39). A shooting method to satisfy the far-field condition, \( E'(\infty) = 1 \), is used to determine \( E'(0) = R(K, \epsilon) \). The corresponding solution to equation (5.30) is found from (5.35). In particular, we find that

\[
e'(0) = E'(0) = R. \quad (5.40)
\]
Table 5.1: \( e'(0) \) for various values of \((\epsilon, \beta)\)

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Figure 5.1: Streamlines for the magnetohydrodynamic non-orthogonal stagnation-point flow where $\epsilon = 1.0$, $\beta = 0.2$ and $m = -1.0$. 

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Figure 5.2: Streamlines for the magnetohydrodynamic non-orthogonal stagnation-point flow where $\epsilon = 10.0$, $\beta = 0.6$ and $m = -1.0$. 

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Chapter 6

Asymptotic Analysis for Small Conductivity, Non-Orthogonal Flow

6.1 Equations for Small Conductivity

For a non-conducting fluid in which $\epsilon$ is zero the governing equations are reduced to

$$
\begin{align*}
\frac{f'''}{f''} + f' - f'^2 + 1 - \beta(g'g'' - g'^2 + 1) &= 0, \\
ge'' &= 0, \\
e'' + fe' - f'e - \beta(g'l' - g'l) + 0.64790 &= 0, \\
l' &= 1,
\end{align*}
$$

(6.1)
where
\[
\begin{align*}
    f(0) &= f'(0) = g(0) = e(0) = l(0) = 0, \\
    f'(\infty) &= g'(\infty) = e'(\infty) = 1.
\end{align*}
\] (6.2)

The solutions for \( f, g \) and \( l \) are easily found to be
\[
f(y) = H(y), \quad g(y) = l(y) = y,
\] (6.3)

where \( H(y) \) is the Hiemenz function defined in (3.1). It, therefore, follows that the function \( e(y) \) must satisfy
\[
\begin{align*}
    e'' + He' - H'e &= -0.64790, \\
    e(0) &= 0, \quad e'(\infty) = 1.
\end{align*}
\] (6.4)

The differential equation (6.4) is discussed and solved by Dorrepaal [3] and the solution is recorded in (1.26). When \( \epsilon \) is small but nonzero, the governing equations (5.20), (5.30) and (5.31) are not satisfied by (6.3) and (1.26). The perturbation problem is singular and the asymptotics are pursued in a similar way as in the case of the stagnation-point flow.

The inner solution is generated by equations (5.19), (5.20), (5.30) and (5.31) under the conditions (6.2). Following the scaling considered in chapter 3, we define the outer functions \( F(\xi), G(\xi), E(\xi) \) and \( L(\xi) \) as such
\[
\begin{align*}
    f(y) &= \epsilon^{-\frac{1}{2}} F(\epsilon^{\frac{1}{2}} y), \\
    g(y) &= \epsilon^{-\frac{1}{2}} G(\epsilon^{\frac{1}{2}} y), \\
    e(y) &= \epsilon^{-\frac{1}{2}} E(\epsilon^{\frac{1}{2}} y), \\
    l(y) &= \epsilon^{-\frac{1}{2}} L(\epsilon^{\frac{1}{2}} y).
\end{align*}
\] (6.5)

Substitution of (6.5) and (3.83) into the governing equations yields the following equations for the outer solution:
\[
\epsilon F''' + F F'' - (F')^2 + 1 - \beta \{ GG'' - (G')^2 + 1 \} = 0,
\] (6.6)
\[ G'' + FG' - F'G = 0, \quad (6.7) \]

\[ \epsilon E'' + FE' - F'E - \beta(EL' - G'L) + 0.64790\epsilon^{\frac{1}{2}} - 0.29314 \frac{\beta}{\sqrt{1 - \beta}} \epsilon + [0.15441 - 0.0076621\beta] \frac{\beta}{1 - \beta} \epsilon^{\frac{3}{2}} + O(\epsilon^2) = 0, \quad (6.8) \]

\[ L' + FL - GE = 1, \quad (6.9) \]

where \( F'(\infty) = G'(\infty) = E'(\infty) = L'(\infty) = 1 \). When \( \xi = \epsilon^\frac{1}{2} y \) is small the values of \( F, G, E \) and \( L \) must be in accord with the values of \( f, g, e \) and \( l \) for \( y \) large; therefore, we require that the first derivatives of the corresponding inner and outer functions be balanced at each stage of the series expansion. The differential equations (5.19), (5.20), (6.6) and (6.7) with their corresponding boundary conditions are exactly the same as in the case of the stagnation-point flow and hence, the results found for \( F(\xi), f(y), G(\xi) \) and \( g(y) \) in chapter 3 apply in this problem and are of great assistance in deriving the asymptotic expansions for \( E(\xi), e(y), L(\xi) \) and \( l(y) \). Consequently, we shall, hereinafter, consider only those equations dealing with the functions \( E(\xi), e(y), L(\xi) \) and \( l(y) \) for the sole purpose of deriving a series solution for the shear component of the tangential stress at the wall, \( e'(0) \).

### 6.2 Solutions for Small Conductivity

The first approximations, \( e_0(y), l_0(y), E_0(\xi) \) and \( L_0(\xi) \), are found by taking \( \epsilon \) to be zero in (5.30), (5.31), (6.8) and (6.9). The required solutions of (6.8) and (6.9)
are

\[ E_0(\xi) = L_0(\xi) = \xi. \quad (6.10) \]

Appropriate forms of expansion turn out to be

\[
\begin{align*}
E(\xi) &= \xi + \epsilon^{\frac{1}{2}} E_1(\xi) + \epsilon \ln \epsilon E_2(\xi) + \epsilon \xi E_3(\xi) + O(\epsilon^{\frac{3}{2}} \ln \epsilon), \\
e(y) &= e_0(y) + \epsilon^{\frac{1}{2}} e_1(y) + \epsilon \ln \epsilon e_2(y) + \epsilon e_3(y) + O(\epsilon^{\frac{3}{2}} \ln \epsilon), \\
L(\xi) &= \xi + \epsilon^{\frac{1}{2}} L_1(\xi) + \epsilon \ln \epsilon L_2(\xi) + \epsilon L_3(\xi) + O(\epsilon^{\frac{3}{2}} \ln \epsilon), \\
l(y) &= y + \epsilon^{\frac{1}{2}} l_1(y) + \epsilon \ln \epsilon l_2(y) + \epsilon l_3(y) + O(\epsilon^{\frac{3}{2}} \ln \epsilon),
\end{align*}
\]

(6.11)

where \( e_0(y) \) is given in (1.26). The following system of equations are obtained when (6.11), (3.8) and (3.9) are substituted into equations (5.30), (5.31), (6.8) and (6.9), keeping terms of \( O(\epsilon^{\frac{3}{2}}) \):

\[
\begin{align*}
F_0 E_1' - F_1 E_0' + F_1 E_0' - F_1 E_0 - \beta(G_0 L_1' - G_0 L_0') + c &= 0, \quad (6.12) \\
L_1' + F_0 L_1 + F_1 L_0 - G_0 E_1 - G_1 E_0 &= 0, \quad (6.13)
\end{align*}
\]

\[
\begin{align*}
e_1'' + f_0 e_1' - f_0' e_1 + f_1 e_0' - f_1' e_0 - \beta(g_0 l_1') + g_1' - g_0' l_1 - g_0' l_0 - 0.29314 \frac{\beta}{\sqrt{1 - \beta}} &= 0, \quad (6.14) \\
l_1' &= 0, \quad (6.15)
\end{align*}
\]

where \( c = 0.64790 \), as defined in (1.14).

Using (3.7) and the fact that \( F_1(\xi) - \beta G_1(\xi) = -c \), equations (6.12) and (6.13) become

\[
\xi (E_1' - \beta L_1') - (E_1 - \beta L_1) = 0, \quad (6.16)
\]
and
\[ L'_{1} + \xi(L_{1} - E_{1}) - \xi(1 - \beta)G_{1} - c\xi = 0, \tag{6.17} \]

where \( G_{1}(\xi) \) is given in (3.22). The boundary conditions at infinity are
\[ E'(\infty) = L'(\infty) = 0. \]

The value of \( E(0) \) is obtained by recalling from (6.5) that
\[ E(\xi) = \varepsilon^\frac{1}{2} e(y) = \varepsilon^\frac{1}{2} e_0(y) + O(\varepsilon) \]
\[ \sim \varepsilon^\frac{1}{2} y + O(\varepsilon) \quad \text{for large } y \]
\[ = \xi + O(\varepsilon), \quad \text{for small } \xi \quad \tag{6.18} \]

and it is given by \( E(0) = 0 \). In a similar way, we find that \( L(0) = 0 \).

Equation (6.16) is easily handled if we define
\[ U(\xi) = E_{1}(\xi) - \beta L_{1}(\xi). \tag{6.19} \]

The boundary-value problem then becomes
\[ \begin{aligned}
\xi U'(\xi) - U(\xi) &= 0, \\
U'(\infty) &= 0.
\end{aligned} \tag{6.20} \]

The trivial solution is the only solution satisfying (6.20), implying that
\[ E_{1}(\xi) = \beta L_{1}(\xi). \tag{6.21} \]

On substituting (6.21) into (6.17), the differential equation is simplified to
\[ L'_{1} + (1 - \beta)\xi L_{1} - (1 - \beta)\xi G_{1} - c\xi = 0, \tag{6.22} \]
which can be solved exactly. The solution is found to be

\[
L_1(\xi) = \sqrt{\frac{\pi}{2}} \frac{c}{\sqrt{1 - \beta}} \exp\left(-\frac{1}{2} (1 - \beta) \xi^2 \right) \left\{ \frac{1}{\sqrt{2 \pi}} (1 - \sqrt{\frac{2}{1 - \beta}}) \xi^2 \right. \\
- \xi \exp\left[\frac{1}{2} (1 - \beta) \xi^2 \right] \text{erfc}\left[\frac{1}{\sqrt{2}} (1 - \beta) \frac{1}{2} \xi \right] \\
+ \int_0^\xi \text{erfc}\left[\frac{1}{\sqrt{2}} (1 - \beta) \frac{1}{2} t \right] \exp\left[\frac{1}{2} (1 - \beta) t^2 \right] dt \right\}. \tag{6.23}
\]

It is easily seen from (6.21) and (6.23) that

\[E'_1(0) = 0. \tag{6.24}\]

The first degree differential equation (6.15) together with the condition \( l_1(0) = 0 \) imply that

\[l_1(y) = 0. \tag{6.25}\]

Using (3.27), (3.28), (1.26) and (3.3), equation (6.14) is simplified to

\[e''_1(y) + H(y)e'_1(y) - H'(y)e_1(y) = p(y) \frac{\beta}{\sqrt{1 - \beta}}, \tag{6.26}\]

where \( p(y) = \frac{\sqrt{\pi}}{4} e_0 (H + yH') - e_0 (yH'' + 2H') \) + 0.29314. The boundary conditions are

\[e_1(0) = 0, \quad e'_1(\infty) = E'_1(0) = 0. \tag{6.27}\]

Equation (6.26) can be made independent of \( \beta \) by introducing the following transformation:

\[e_1(y) = \frac{\beta}{\sqrt{1 - \beta}} \tilde{e}_1(y), \tag{6.28}\]

where \( \tilde{e}_1(y) \) does not depend on \( \beta \). The resulting equation,

\[\tilde{e}_1''(y) + H(y)\tilde{e}_1'(y) - H'(y)\tilde{e}_1(y) = p(y), \tag{6.29}\]
is then integrated numerically using a shooting method to satisfy the condition at infinity. We obtain

$$e_1'(0) = -0.037087 \frac{\beta}{\sqrt{1-\beta}}.$$  \hfill (6.30)

The behaviors of the functions $E_1(\xi)$ and $e_1(y)$ for small $\xi$ and large $y$ respectively make important contributions to the later terms of the series. The small-$\xi$ expansion of $E_1(\xi)$,

$$E_1(\xi) \sim \frac{1}{2} \beta c \xi^2 - \frac{1}{3} \sqrt{\frac{\pi}{2}} c \beta \xi^3 + O(\xi^5),$$  \hfill (6.31)

(6.11) and (6.5) suggest that for large $y$, 

$$\begin{align*}
    e'(y) &= E'(\epsilon^\frac{1}{3} y) = 1 + \epsilon^\frac{1}{3} E'_1(\epsilon^\frac{1}{3} y) + O(\ln \epsilon) \\
    &\sim 1 + \epsilon^\frac{1}{3} \{c \epsilon x^3 y - \sqrt{\frac{\pi}{2}} c \beta \epsilon y^2 + O(\epsilon^4)\} + O(\ln \epsilon) \\
    &= 1 + \beta c y \epsilon + O(\ln \epsilon). \hfill (6.32)
\end{align*}
$$

This clearly indicates that $e_3'(y)$ when $y$ is large has a term $\beta c y$. The large behavior of $e_1(y)$ given by $e_1(y) \sim [\sqrt{\frac{2}{4}} \epsilon^2 - 0.29314] \frac{\beta}{\sqrt{1-\beta}}$ implies that for small $\xi$, we have

$$\begin{align*}
    E(\xi) &= e^{\frac{1}{3}} e(e^{-\frac{1}{3}} \xi) = e^{\frac{1}{3}} \{e_0(e^{-\frac{1}{3}} \xi) + e^\frac{1}{3} e_1(e^{-\frac{1}{3}} \xi) + O(\ln \epsilon)\} \\
    &\sim \xi + \left[\frac{\sqrt{2\pi}}{4} \epsilon^2 - 0.29314\right] \frac{\beta}{\sqrt{1-\beta}} \epsilon + O(\beta^\frac{3}{2} \ln \epsilon). \hfill (6.33)
\end{align*}
$$

The value of $E_3(0)$ is now easily seen from (6.33) to be

$$E_3(0) = \left[\frac{\sqrt{2\pi}}{4} \epsilon^2 - 0.29314\right] \frac{\beta}{\sqrt{1-\beta}}.$$  \hfill (6.34)

The equations which determine the next batch of functions are:

$$e_3'' + f_0 e_3' - f_0' e_3 + f_1 e_1' + f_2 e_0' - f_2' e_0 - f_1' e_1 - \beta (g_0 l_3' + g_2 l_0')$$
\[ g_1 l_1' - g_2 l_3' - g_2' l_0 - g_1' l_1 + [0.15441 - 0.0076621 \beta] \frac{\beta}{1 - \beta} = 0, \quad (6.35) \]

\[ l_3' + f_0 l_0 - g_0 e_0 = 0, \quad (6.36) \]

\[ F_0 E_3' - F_0' E_3 + E_0' + F_2 E_0' + F_1 E_1' - F_1' E_0 = F_1 E_1 - \beta \{ G_0 L_3' + \]

\[ G_1 L_1' + G_2 L_0' - G_2' L_0 - G_1 L_1 - G_0 L_3 \} - 0.29314 \frac{\beta}{\sqrt{1 - \beta}} = 0, \quad (6.37) \]

\[ L_3' + F_0 L_3 + F_3 L_0 + F_1 L_1 - G_2 E_0 - G_0 E_3 - G_1 E_1 = 0, \quad (6.38) \]

\[ e_2'' + f_0 e_2' - f_0' e_2 - \beta \{ g_0 l_2' - g_0' l_2 \} = 0, \quad (6.39) \]

\[ l_2' = 0. \quad (6.40) \]

After making the appropriate substitutions, equations (6.37) and (6.38) become

\[ \xi (E_3' - \beta L_3') - (E_3 - \beta L_3) + (F_2 - \beta G_2) - \xi (F_2' - \beta G_2') + \beta (1 - \beta)(G_1 L_1 - G_1 L_1') - c \beta L_1' - 0.29314 \frac{\beta}{\sqrt{1 - \beta}} = 0, \quad (6.41) \]

\[ L_5' + \xi L_3 + \xi (F_2 - G_2) - \xi E_3 - c L_1 = 0. \quad (6.42) \]

The following transformations are used to uncouple the above differential equations:

\[ \beta W(\xi) = E_3 - \beta L_3, \quad \beta U(\xi) = F_2 - \beta G_2, \quad (6.43) \]

where the function \( U(\xi) \) is discussed in §3.2. The resulting differential equations and boundary conditions are

\[ \begin{align*}
\xi W' - W &= \xi U' - U - (1 - \beta)(G_1 L_1 - G_1 L_1') + c L_1' + \frac{0.29314}{\sqrt{1 - \beta}}, \\
W(0) &= E_3(0) = -0.030089 \frac{\beta}{\sqrt{1 - \beta}}.
\end{align*} \quad (6.44) \]
\[
L_3 + (1 - \beta)\xi L_3 = \beta\xi(W - U) + cL_1 + (1 - \beta)\xi G_2,
\]
\[
L_3(0) = 0.
\]
The small-\(\xi\) expansions of \(W(\xi)\) and \(L_3(\xi)\) are given by
\[
W(\xi) \sim \frac{-0.030089}{\sqrt{1 - \beta}} + c^2\xi \ln \xi + \alpha(\beta)\xi - 1.05222\sqrt{1 - \beta}\xi^2 + O(\xi^3 \ln \xi), \quad (6.46)
\]
\[
L_3(\xi) \sim -0.14657\xi^2 + \frac{\beta}{1 - \beta} + \frac{1}{3}\beta c^2\xi^3 \ln \xi + O(\xi^3), \quad (6.47)
\]
where the value of \(\alpha(\beta)\) is determined numerically. The small-\(\xi\) expansion of \(E_3'(\xi)\) can then be derived from (6.46-6.47) and it is given by
\[
E_3'(\xi) = \beta[W''(\xi) + L_3'(\xi)] \sim \beta(c^2 + \alpha(\beta)) + \beta c^2 \ln \xi + O(\xi). \quad (6.48)
\]
The inner function \(l_3(y)\) is determined analytically from the equation (6.36) together with the initial condition, \(l_3(0) = 0\), and it is given by
\[
l_3(y) = \int_0^y \eta[e_0(\eta) - H(\eta)]d\eta, \quad (6.49)
\]
where \(e_0(\eta)\) and \(H(\eta)\) are given in (6.4) and (3.1) respectively.

Equation (6.35) can then be simplified to
\[
e_3'' + He_3' - H'e_3 = \beta\{f_{2,1}'e_0 - f_{2,1}e_0' + y''(e_0 - H)
+ g_{2,1} - yg_{2,1}' - 0.15441 - \int_0^y \eta[e_0(\eta) - H(\eta)]d\eta\}
+ \frac{\beta^2}{1 - \beta}\left(\frac{\sqrt{2\pi}c}{4}\right)[(H + yH')e_1' - (yH'' + 2H')e_1]
- (f_{2,2}'e_0 - f_{2,2}e_0') + f_{2,1}'e_0 - f_{2,1}e_0' - 0.14675), \quad (6.50)
\]
where \(e_1(y)\) is described by (6.29) and the functions \(g_{2,1}(y)\), \(f_{2,1}(y)\) and \(f_{2,2}(y)\) are discussed in §3.2.
The behavior of $e_3(y)$, when $y$ is large, can now be studied from (6.50) since the right-side functions are all known. For large $y$, equation (6.50) gives

$$e''_3 + (y - c)e'_3 - e_3 \sim \frac{1}{2} \beta c y^2 + \text{constant}, \quad (6.51)$$

which can only be satisfied if

$$e'_3(y) \sim \beta cy + \beta c^2 \ln (y - c) + \beta (c^2 + \bar{\alpha} (\beta)) \quad \text{as} \quad y \to \infty, \quad (6.52)$$

where the value of $\bar{\alpha} (\beta)$ is determined from the matching condition. The large-$y$ expansion (6.52) can be represented in terms of $\ln y$ if the logarithmic term is rewritten as

$$\ln (y - c) = \ln y + \ln \left(1 - \frac{c}{y}\right) \sim \ln y - \frac{c}{y} - \frac{1}{2} \frac{c^2}{y^2} + O\left(\frac{1}{y^3}\right) \quad \text{as} \quad y \to \infty.$$

We have

$$e'_3(y) \sim \beta cy + \beta c^2 \ln y + \beta (c^2 + \bar{\alpha} (\beta)) + O(y^{-1}) \quad \text{as} \quad y \to \infty. \quad (6.53)$$

The first term of (6.53) is in agreement with (6.32) and the second term establishes the need for the logarithmic term, $O(\epsilon \ln \epsilon)$, in the expansions. On comparing (6.53) with (6.48), we deduce that $\alpha = \bar{\alpha}$ in order to balance the inner and outer solutions.

When the logarithmic term in (6.48) is replaced by its equivalent, $\frac{1}{2} \ln \epsilon + \ln y$, we see that an additional contribution

$$e'_2(\infty) = \frac{1}{2} \beta c^2, \quad (6.54)$$

is required for matching the inner and outer solutions. Similarly it can be shown that $\ell'_2(\infty) = 0$ which is consistent with equation (6.40).
The function \( l_2(y) \) is easily determined from (6.40) together with the initial condition \( l_2(0) = 0 \) to be

\[
l_2(y) = 0. \tag{6.55}
\]

Using (6.55), equation (6.39) reduces to

\[
e'' + H e' - H'e = 0, \tag{6.56}
\]

with the boundary conditions being given by \( e_2(0) = 0 \) and (6.54). It is interesting to note that the homogeneous solution of (1.24) found by Dorrepaal [3], as recorded in (1.26), is proportional to the solution of (6.56). The solution of (6.56) is therefore of the form

\[
e_2(y) = \gamma_1(\beta)H''(y) \int_0^y w(t) dt, \tag{6.57}
\]

where \( \gamma_1(\beta) \) is determined by the condition at infinity and \( w(t) \) is given in (1.27).

The values of \( \gamma_1(\beta) \) and \( e'_2(0) \) are then found to be

\[
\begin{align*}
\gamma_1(\beta) &= \frac{1}{2}\beta c^2 a_0 (D - a_0 c), \\
e'_2(0) &= \frac{1}{2}\beta c^2 (D - a_0 c) = 0.12760/\beta,
\end{align*}
\tag{6.58}
\]

where \( a_0 = 1.23259 \) and \( D = 1.40654 \).

The next stage of approximation deals with \( O(\varepsilon) \) terms of the inner and outer solutions. The objective is to obtain the value of \( e'_3(0) \) from integrating equations (6.44) and (6.50) numerically. The differential equation (6.44) is first integrated to determine the value of \( \alpha(\beta) \) and the result is then utilized in obtaining the value of \( e'_3(y) \) at \( y = 0 \). The small-\( \xi \) expansion of \( W(\xi) \), given in (6.46), is used to
maneuver around the logarithmic singularity of $W'(\xi)$ at $\xi = 0$. Having replaced the initial condition with the required condition at infinity, $W'(\infty) = 0$, in (6.44), the integration procedures are carried out from $\xi = 0.01$ to determine $\alpha(\beta)$ for selected values of $\beta$. The following relation,

$$\alpha(\beta) = 0.68808 + \frac{1}{2}c^2 \ln(1 - \beta),$$

(6.59)

is then obtained. We are now ready to inspect the function $e_3(y)$ closely. A simple study of equation (6.50) together with the large-$y$ behavior of $e_3(y)$ suggest that $e_3(y)$ is of the form

$$e_3(y) = e_{3,1}(y) \beta + e_{3,2}(y) \frac{\beta^2}{1 - \beta} + e_{3,3}(y) \beta \ln(1 - \beta),$$

(6.60)

where the functions $e_{3,1}(y)$, $e_{3,2}(y)$ and $e_{3,3}(y)$ are defined as follows:

\[
\begin{align*}
\begin{cases}
 e''_{3,1} + He'_{3,1} - H'e_{3,1} &= f'_{2,1}e_0 - f_{2,1}e'_0 + y^2(\eta_0 - H) \\
 + g_{2,1}(y) - f''_0 \eta(\eta_0 - H)d\eta - yg'_{2,1}(y) - 0.15441, \\
 e_{3,1}(0) &= 0, \quad e'_{3,1}(y) \sim cy + c^2 \ln y + c^2 + 0.68808,
\end{cases}
\end{align*}
\]

(6.61)

\[
\begin{align*}
\begin{cases}
 e''_{3,2} + He'_{3,2} - H'e_{3,2} &= f'_{2,1}e_0 - f_{2,1}e'_0 - (f_{2,2}e'_0 - f'_{2,2}e_0) \\
 + \sqrt{\frac{2}{2}} c\{(H + yH')e'_1(y) - (yH'' + 2H')e_1(y)\} - 0.14675, \\
 e_{3,2}(0) &= 0, \quad e_{3,2}(\infty) = 0,
\end{cases}
\end{align*}
\]

(6.62)

\[
\begin{align*}
\begin{cases}
 e''_{3,3} + He'_{3,3} - H'e_{3,3} &= 0, \\
 e_{3,3}(0) &= 0, \quad e_{3,3}(\infty) = \frac{1}{2}c^2.
\end{cases}
\end{align*}
\]

(6.63)
The closed-form solution for \( e_{3,3}(y) \) so obtained is

\[
e_{3,3}(y) = \frac{1}{2} a_0 c^2 (D - a_0 c) H''(y) \int_0^y w(t) dt,
\]

(6.64)

where \( w(t) \) is defined in (1.27). This implies that

\[
e'_{3,3}(0) = \frac{1}{2} c^2 (D - a_0 c) \simeq 0.12760.
\]

(6.65)

To complete this stage of approximation, differential equations (6.61) and (6.62) are integrated numerically and the following results are obtained:

\[
\begin{aligned}
e'_{3,1}(0) &= 0.93321, \\
e'_{3,2}(0) &= -0.0065952.
\end{aligned}
\]

(6.66)

The value of \( e'(0) \) is therefore completely known and it is given by

\[
e'(0) = 0.93321 \beta - 0.0065952 \frac{\beta^2}{1 - \beta} + \frac{1}{2} c^2 (D - a_0 c) \beta \ln (1 - \beta).
\]

(6.67)

The inner solution is now completely determined to \( O(\epsilon) \). The quantity of interest is the shear component of tangential stress at the wall, \( e'(0) \), and it is given by

\[
\begin{aligned}
e'(0) &= 1.40654 - 0.037087 \beta \left( \frac{\epsilon}{1 - \beta} \right)^{\frac{1}{2}} + 0.12760 \beta \epsilon \ln \epsilon + (0.93321 \beta \\
&\quad - 0.0065952 \frac{\beta^2}{1 - \beta} + 0.12760 \beta \ln (1 - \beta)) \epsilon + O(\epsilon^{\frac{3}{2}} \ln \epsilon).
\end{aligned}
\]

(6.68)

Table 6.1 compares the asymptotic predictions with the numerical results provided in table 5.1.
Table 6.1: Comparison of $e'(0)$ actual (numerical) with $e'(0)$ asymptotic.

| $\epsilon$ | $\beta$ | Actual(Num.) | Asymptotic | $|\text{Error}|$ | $|\%\text{-Error}|$ |
|------------|---------|--------------|------------|----------------|-----------------|
| 0.001      | 0       | 1.40654      | 1.40654    | 0              | 0               |
| 0.001      | 0.2     | 1.40627      | 1.40629    | $1.79 \times 10^{-5}$ | $1.28 \times 10^{-3}$ |
| 0.001      | 0.4     | 1.40590      | 1.40593    | $3.16 \times 10^{-5}$ | $2.25 \times 10^{-3}$ |
| 0.001      | 0.6     | 1.40535      | 1.40539    | $3.94 \times 10^{-5}$ | $2.81 \times 10^{-3}$ |
| 0.001      | 0.8     | 1.40426      | 1.40430    | $3.76 \times 10^{-5}$ | $2.68 \times 10^{-3}$ |
| 0.001      | 0.99    | 1.39389      | 1.39376    | $1.38 \times 10^{-4}$ | $9.87 \times 10^{-3}$ |
| 0.01       | 0       | 1.40654      | 1.40654    | 0              | 0               |
| 0.01       | 0.2     | 1.40589      | 1.40635    | $4.60 \times 10^{-4}$ | $3.27 \times 10^{-2}$ |
| 0.01       | 0.4     | 1.40491      | 1.40573    | $8.24 \times 10^{-4}$ | $5.86 \times 10^{-2}$ |
| 0.01       | 0.6     | 1.40329      | 1.40434    | $1.05 \times 10^{-3}$ | $7.49 \times 10^{-2}$ |
| 0.01       | 0.8     | 1.39979      | 1.40082    | $1.03 \times 10^{-3}$ | $7.35 \times 10^{-2}$ |
| 0.01       | 0.99    | 1.36463      | 1.36097    | $3.66 \times 10^{-3}$ | $2.68 \times 10^{-1}$ |
| 0.1        | 0       | 1.40654      | 1.40654    | 0              | 0               |
| 0.1        | 0.2     | 1.40606      | 1.41611    | $1.00 \times 10^{-2}$ | $7.15 \times 10^{-1}$ |
| 0.1        | 0.4     | 1.40481      | 1.42328    | $1.85 \times 10^{-2}$ | 1.31             |
| 0.1        | 0.6     | 1.40180      | 1.42617    | $2.44 \times 10^{-2}$ | 1.74             |
| 0.1        | 0.8     | 1.39306      | 1.41818    | $2.51 \times 10^{-2}$ | 1.80             |
| 0.1        | 0.99    | 1.29427      | 1.23092    | $6.33 \times 10^{-2}$ | 4.89             |

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Chapter 7

Asymptotic Analysis for Large Conductivity, Non-Orthogonal Flow

7.1 Equations for Large Conductivity

As in the previous case where the procedures relied heavily on the results found in the corresponding orthogonal problem, we treat the case where the electrical conductivity is large in the same way. The inner and outer functions derived in chapter 4 are now utilized in determining the inner and outer solutions of the non-orthogonal problem. The appropriate change of variables, given in (4.1), are
extended in order to include the functions \( e(y) \) and \( l(y) \) as follows:

\[
\begin{align*}
  f(y) &= (1 - \beta)^{-\frac{1}{2}} P[(1 - \beta)^{\frac{1}{2}} y], \quad g(y) = (1 - \beta)^{-\frac{1}{2}} Q[(1 - \beta)^{\frac{1}{2}} y], \\
  e(y) &= (1 - \beta)^{-\frac{1}{2}} S[(1 - \beta)^{\frac{1}{2}} y], \quad l(y) = (1 - \beta)^{-\frac{1}{2}} R[(1 - \beta)^{\frac{1}{2}} y],
\end{align*}
\]

and

\[
\lambda = \frac{\epsilon}{1 - \beta}. \tag{7.2}
\]

The transformations are then substituted into the governing equations (5.19), (5.20), (5.30) and (5.31) to obtain, in addition to (4.2-4.3), the following system of differential equations:

\[
\begin{align*}
  (1 - \beta) S'' + P S' - S P' - \beta (Q R' - R Q') + Bc(1 - \beta)^{\frac{3}{2}} &= 0, \tag{7.3} \\
  R' + \lambda (PR - QS) &= 1, \tag{7.4}
\end{align*}
\]

where the series expansion of the boundary layer thickness, \( Bc \), is given in (4.94). On substituting (4.94) into (7.3), equation (7.3) when \( \lambda \) is large becomes:

\[
\begin{align*}
  (1 - \beta) S'' + P S' - S P' - \beta (Q R' - R Q') + c(1 - \beta) +1.64326 / 3 A^{-1} + O(\lambda^{-\frac{3}{2}} \ln \lambda) &= 0, \tag{7.5}
\end{align*}
\]

where \( c = 0.64790 \). The boundary conditions are

\[
S(0) = R(0) = 0 \quad \text{and} \quad S'(\infty) = 1. \tag{7.6}
\]

For a perfectly conducting fluid, the parameter \( \lambda \) is infinite and the equation (7.4) reduces to

\[
PR - QS = 0. \tag{7.7}
\]
From (4.7) we deduce that

\[ S(\theta) = R(\theta), \]  

(7.8)

where \( \theta = (1 - \beta)\frac{1}{2} y \). Equation (7.5) is then reduced to

\[ S'' + HS' - SH' = -c. \]  

(7.9)

It follows from (1.26) that \( S'(0) = D = 1.40654 \). If \( \lambda \) is finite the solution described by (7.9) is incorrect in the vicinity of the wall. It implies that \( R'(0) = 1.40654 \) whereas equation (7.4) together with the corresponding boundary conditions imply that \( R'(0) = 1 \). The reason lies in the fact that equation (4.7) is no longer valid when \( \lambda \) is finite, as discussed in §4.1. This implies that equation (7.9) is only true when \( \lambda \) is infinite. It is therefore necessary to re-scale the appropriate equations so that the terms become of comparable magnitude. The transformations (4.8) are extended to introduce the inner functions \( t(\xi) \) and \( r(\xi) \) as follows:

\[ S(\theta) = \lambda^{-\frac{1}{2}} t(\xi), \quad R(\theta) = \lambda^{-\frac{1}{2}} r(\xi), \]  

(7.10)

where \( \xi = \lambda^{\frac{1}{2}} \theta \). It follows that \( S'(\theta) = t'(\xi) \) and \( R'(\theta) = r'(\xi) \); subsequently, it is convenient to carry out the matching of terms \( S'(\theta), R(\theta) \) with \( t(\xi) \) and \( r(\xi) \) respectively by balancing their corresponding first derivatives.

The inner solution is generated by the following equations:

\[ (1 - \beta)\lambda t'' + pt' - tp' - \beta(qr' - rq') + c(1 - \beta)\lambda^{\frac{3}{2}} \]

\[ + 1.64326\beta \lambda^{-\frac{1}{2}} + O(\lambda^{-\frac{3}{2}} \ln \lambda) = 0, \]  

(7.11)
\( r' + pr - qt = 1, \quad (7.12) \)

subject to the boundary conditions

\( r(0) = q(0) = 0. \quad (7.13) \)

### 7.2 Solutions for Large Conductivity

The leading terms of the outer expansion, \( S_0(\theta) \) and \( R_0(\theta) \), are identical and given by (1.26). The functions \( t_0(\xi) \) and \( r_0(\xi) \), corresponding to the first approximations of the inner solution, are determined by solving the following linear ordinary differential equations:

\[ t''_0(\xi) = 0, \quad (7.14) \]

\[ r'_0 + pr_0 - qo_t_0 = 1. \quad (7.15) \]

The solution of (7.14) under the initial condition \( t_0(0) = 0 \) is given by

\[ t_0(\xi) = a\xi, \quad (7.16) \]

where \( a \) is a constant. The constant \( a \) is determined by analyzing the large-\( \xi \) behavior of \( t'_0(\xi) \) via the small-\( \theta \) expansion of \( S'_0(\theta) \). The form of the outer function \( S'_0(\theta) \) for small \( \theta \) shows that

\[ S'_0(\theta) \sim D - c\theta + \frac{a_0}{6} D\theta^3 + O(\theta^4), \quad (7.17) \]

which requires that for \( \xi \) large

\[ t'(\xi) = S'(\lambda^{-\frac{1}{3}}\xi) \sim D - c\xi\lambda^{-\frac{1}{3}} + \frac{a_0}{6} D\xi^3\lambda^{-1} + O(\lambda^{-\frac{1}{3}}). \quad (7.18) \]

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The first term indicates that for $\xi$ large

$$t_0(\xi) \sim D,$$  \hspace{1cm} (7.19)

and the second establishes the need for a $\lambda^{-\frac{1}{2}}$ term in the expansions while the third contributes to the asymptotic form of $O(\lambda^{-1})$ term of the inner solution. The matching condition requires that $a = D$, which implies that

$$t'_0(0) = D.$$  \hspace{1cm} (7.20)

Equation (7.15) can now be solved exactly and is given by

$$r_0(\xi) = \exp\left(-\frac{1}{6}a_0\xi^3\right) \int_0^\xi \left\{1 + kD\eta^2 M\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{6}a_0\eta^3\right)\right\} \exp\left(\frac{1}{6}a_0\eta^3\right)d\eta, \hspace{1cm} (7.21)$$

where $M(a, b, z)$ is a confluent hypergeometric function and the constant $k$ is defined in (4.24).

Studies of the leading terms of both inner and outer solutions suggest that the expansions proceed in powers of $\lambda^{-\frac{1}{2}}$ and further studies indicate that logarithmic terms are also needed to improve the matching. The appropriate expansions are of the form

$$t(\xi) = t_0 + \lambda^{-\frac{1}{2}}t_1 + \lambda^{-\frac{3}{2}}t_2 + \lambda^{-1}\ln\lambda t_L + \lambda^{-1}t_1 + O(\lambda^{-\frac{3}{2}}\ln\lambda),$$

$$S(\theta) = S_0 + \lambda^{-\frac{1}{2}}S_1 + \lambda^{-\frac{3}{2}}S_2 + \lambda^{-1}\ln\lambda S_L + \lambda^{-1}S_1 + O(\lambda^{-\frac{3}{2}}\ln\lambda),$$

and similarly for $R(\theta)$ and $r(\xi)$.

On substituting the corresponding expansions into (7.4-7.5) and (7.11-7.12), the following system of equations are obtained:

$$(1 - \beta)S''_\frac{1}{2} + P_0 S'_\frac{1}{2} - P'_0 S_\frac{1}{2} + P_\frac{1}{2} S'_0 - S'_0 P_\frac{1}{2}$$
\[
\begin{align*}
- \beta(Q_{\frac{1}{3}}R_0' + Q_0R_{\frac{1}{3}}' - R_{\frac{1}{3}}Q_0' - R_0Q_{\frac{1}{3}}') = 0, & \quad (7.23) \\
P_3R_0 + P_0R_{\frac{1}{3}} - Q_0S_{\frac{1}{3}} - Q_{\frac{1}{3}}S_0 = 0, & \quad (7.24)
\end{align*}
\]

\[
(1 - \beta)S''_0 + P_0S'_{\frac{1}{3}} - P_0'P_{\frac{1}{3}} + P_3S'_{\frac{1}{3}} - S_0P_{\frac{1}{3}}' + P_3S_{\frac{1}{3}}' + P_3S'_{\frac{1}{3}} \\
- \beta(Q_{\frac{1}{3}}R_0' + Q_0R_{\frac{1}{3}}' - R_{\frac{1}{3}}Q_0' - R_0Q_{\frac{1}{3}}') = 0, & \quad (7.25)
\]

\[
P_0R_{\frac{3}{2}} + P_3R_0 + P_3R_{\frac{1}{3}} - Q_0S_{\frac{3}{2}} - Q_{\frac{3}{2}}S_0 - Q_{\frac{1}{3}}S_{\frac{1}{3}} = 0, & \quad (7.26)
\]

\[
(1 - \beta)S''_0 + P_0S'_{\frac{1}{3}} - P_0'P_{\frac{1}{3}} + P_3S'_{\frac{1}{3}} - S_0P_{\frac{1}{3}}' + P_3S_{\frac{1}{3}}' + P_3S'_{\frac{1}{3}} \\
- Q_0S_{\frac{1}{3}} - Q_1S_0 - Q_{\frac{1}{3}}S_{\frac{3}{2}} - Q_{\frac{3}{2}}S_{\frac{1}{3}} = 1, & \quad (7.28)
\]

\[
(1 - \beta)S''_0 + P_0S'_{\frac{3}{2}} + P_0S'_{\frac{1}{3}} - S_0P_{\frac{3}{2}}' - S_0P_{\frac{1}{3}}' \\
- \beta(Q_LR_0' + Q_0R_{\frac{1}{3}}' - R_0Q_{\frac{1}{3}}' - R_LQ_{\frac{1}{3}}') = 0, & \quad (7.29)
\]

\[
R_LP_0 + R_0P_L - Q_LS_0 - Q_0S_L = 0, & \quad (7.30)
\]

\[
t''_{\frac{3}{2}} + c = 0, & \quad (7.31)
\]

\[
r_{\frac{1}{3}}' + p_0r_{\frac{1}{3}} + p_1r_0 - q_0t_{\frac{1}{3}} - q_{\frac{1}{3}}t_0 = 0, & \quad (7.32)
\]

\[
t''_{\frac{3}{2}} = 0, & \quad (7.33)
\]

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\[ r'_{\frac{3}{3}} + p_{0}r_{\frac{3}{3}} + p_{3}r_{0} + p_{3}r_{\frac{3}{3}} - q_{3}t_{0} - q_{0}t_{\frac{3}{3}} - q_{3}t_{\frac{3}{3}} = 0, \quad (7.34) \]

\[ (1 - \beta)t''_{1} + p_{0}t'_{0} - p'_{0}t_{0} - \beta(q_{0}r''_{0} - r_{0}q''_{0}) = 0, \quad (7.35) \]

\[ r'_{1} + p_{1}r_{0} + p_{0}r_{1} + p_{3}r_{\frac{3}{3}} - q_{1}t_{0} - q_{0}t_{1} - q_{3}t_{\frac{3}{3}} - q_{3}t_{\frac{3}{3}} = 0, \quad (7.36) \]

\[ t''_{L} = 0, \quad (7.37) \]

\[ r'_{L} + p_{0}r_{L} + p_{L}r_{0} - q_{0}t_{L} - q_{L}t_{0} = 0. \quad (7.38) \]

Using the results given in (4.7) and (4.40), equation (7.24) is simplified to

\[ H(\theta)[R_{\frac{3}{3}}(\theta) - S_{\frac{3}{3}}(\theta)] = 0, \quad (7.39) \]

which clearly implies that

\[ R_{\frac{3}{3}}(\theta) = S_{\frac{3}{3}}(\theta). \quad (7.40) \]

Equation (7.23) is then reduced to

\[ S''_{\frac{3}{3}} + HS'_{\frac{3}{3}} - H'S_{\frac{3}{3}} = 0, \quad (7.41) \]

subject to the boundary conditions \( S_{\frac{3}{3}}(0) = S'_{\frac{3}{3}}(\infty) = 0 \). The only solution satisfying (7.41) under the given conditions is the trivial solution. We therefore have

\[ S_{\frac{3}{3}}(\theta) = R_{\frac{3}{3}}(\theta) = 0. \quad (7.42) \]

The asymptotic form of \( t_{\frac{3}{3}}(\xi) \) for large \( \xi \) as suggested by the expansion (7.18) is not affected by the result found in (7.42); hence, the large-\( \xi \) behavior of \( O(\lambda^{-\frac{1}{2}}) \) term is given by

\[ t_{\frac{3}{3}}(\xi) \sim -c\xi. \quad (7.43) \]

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The inner function, $t_\frac{1}{3}(\xi)$, can now be determined from (7.31) under the conditions $t_\frac{1}{3}(0) = 0$ and (7.43). The solution is found to be

$$t_\frac{1}{3}(\xi) = -\frac{1}{2}c\xi^2,$$  \hspace{1cm} (7.44)

which indicates that $t_\frac{1}{3}(\xi)$ makes no contribution to the series solution of $t'(0)$.

The function $r_\frac{1}{3}(\xi)$ is then expressed in an integral form by

$$r_\frac{1}{3}(\xi) = \exp\left(-\frac{1}{6}a_0\xi^3\right)\int_0^\xi \{ -\frac{1}{2}k\eta^2 M(-\frac{1}{3}, \frac{4}{3}, \frac{1}{6}a_0\eta^3) 
+ Dq_\frac{1}{3}(\eta) + \frac{1}{6}\eta^2 r_0(\eta) \} \eta \exp\left(\frac{1}{6}a_0\eta^3\right)d\eta, \hspace{1cm} (7.45)$$

where $g_\frac{1}{3}(\xi)$ is described by the equation (4.45) and $r_0(\xi)$ is given in (7.21).

We next focus our attention on determining the $O(\lambda^{-\frac{5}{3}})$ terms of the outer solution. From equation (7.26) we deduce that $S_\frac{1}{3}(\theta) = R_\frac{1}{3}(\theta)$ which helps reducing equation (7.25) to

$$S_\frac{2}{3} + HS_\frac{1}{3} - H'S_\frac{2}{3} = 0, \hspace{1cm} (7.46)$$

subject to the boundary conditions $S_\frac{1}{3}(0) = S_\frac{1}{3}(\infty) = 0$. The resulting differential equation (7.46) is identical to the equation (7.41) which describes $S_\frac{1}{3}(\theta)$, sharing the same boundary conditions. The trivial solution is therefore the only solution satisfying (7.46) and its boundary conditions. It follows then that

$$S_\frac{1}{3}(\theta) = R_\frac{1}{3}(\theta) = 0. \hspace{1cm} (7.47)$$

Equations (7.29-7.30) yield same results for the outer functions $S_L(\theta)$ and $R_L(\theta)$. We have

$$S_L(\theta) = R_L(\theta) = 0. \hspace{1cm} (7.48)$$
The large-£ behavior of $t(\xi)$, as recorded in (7.18), is therefore accurate to the $O(\lambda^{-1} \ln \lambda)$. The behavior of $i'_3(\xi)$ for large $\xi$ is seen from (7.18) to be

$$i'_3(\xi) \sim 0. \quad (7.49)$$

The inner function, $t_\frac{3}{3}(\xi)$, under the conditions $t_\frac{3}{3}(0) = 0$ and (7.49), is easily found from (7.33) to be

$$t_\frac{3}{3}(\xi) = 0. \quad (7.50)$$

It follows from (7.34) that

$$r_\frac{3}{3}(\xi) = \exp \left( -\frac{1}{6}a_0 \xi^3 \right) \int_0^\xi \left\{ \frac{1}{6} \eta^2 r_\frac{1}{3} - Dq_\frac{1}{3} + \frac{1}{2} c\eta q_\frac{1}{3} \right\} \eta d\eta, \quad (7.51)$$

where $q_\frac{1}{3}(\xi)$ and $q_\frac{3}{3}(\xi)$ are described by (4.45) and (4.46) respectively and $r_\frac{3}{3}(\xi)$ is given in (7.45).

The next stage of approximation deals with determining the inner functions $t_L(\xi)$ and $t_1(\xi)$. Equations (7.27) and (7.28) are first examined to derive the small-$\theta$ behavior of $S_1(\theta)$. The equations uncouple if we define

$$\begin{cases}
\beta Y(\theta) = S_1(\theta) - \beta R_1(\theta), \\
X(\theta) = S_1(\theta) - R_1(\theta).
\end{cases} \quad (7.52)$$

The system of differential equations becomes

$$HX = S'_0 + S_0 V - 1, \quad (7.53)$$

$$Y'' + HY' - H'Y = X'' - S'_0 W + S_0 W' - 1.64326, \quad (7.54)$$
where $W(\theta)$ and $V(\theta)$ are defined in (4.56). The expansions derived in (4.59) and (4.60) for small $\theta$ are of great assistance in deriving a similar expansion for $Y(\theta)$ and $X(\theta)$. We then obtain the corresponding small-$\theta$ expansion for the function of interest, $S_1(\theta)$. We obtain

$$S_1'(\theta) \sim \frac{\beta}{1 - \beta} \{ \tau_0 + (\frac{8}{3} D - 4) \ln \theta + O(\theta) \}, \quad (7.55)$$

where $\tau_0$ is a constant.

The outer function $S'(\theta)$ for small $\theta$ requires that for $\xi$ large,

$$t'(\xi) = S'_{\lambda^\frac{1}{3}}(\xi) = S_0'(\lambda^{-\frac{1}{3}} \xi) + \lambda^{-1} S_1'(\lambda^{-\frac{1}{3}} \xi) + O(\lambda^{-\frac{2}{3}} \ln \lambda)$$

$$\sim D - c_{\xi} \lambda^{-\frac{1}{3}} - \frac{1}{6} a_0 D \xi^3 \lambda^{-1} + \frac{\beta}{1 - \beta} \{ \tau_0 + (\frac{8}{3} D - 4) \ln \xi \}$$

$$- \frac{1}{3} (\frac{8}{3} D - 4) \ln \lambda + O(\lambda^{-\frac{1}{3}}) \lambda^{-1} + O(\lambda^{-\frac{2}{3}} \ln \lambda)$$

$$= D - c_{\xi} \lambda^{-\frac{1}{3}} - \frac{1}{3} (\frac{8}{3} D - 4) (\frac{\beta}{1 - \beta}) \lambda^{-1} \ln \lambda + \lambda^{-1} \{ \frac{1}{6} a_0 D \xi^3 \}$$

$$+ \{ \frac{\beta}{1 - \beta} \} \{ \tau_0 + (\frac{8}{3} D - 4) \ln \xi \} \} + O(\lambda^{-\frac{2}{3}} \ln \lambda). \quad (7.56)$$

The third term in (7.56) proves that the logarithmic terms must be present in the expansions and it also gives the behavior of $t'_L(\xi)$ at infinity to be

$$t'_L(\infty) = - \frac{1}{3} (\frac{8}{3} D - 4) (\frac{\beta}{1 - \beta}). \quad (7.57)$$

The function $t_L(\xi)$ is determined exactly from equation (7.37) together with the boundary conditions $t_L(0) = 0$ and (7.57) with the result being given by

$$t_L(\xi) = - \frac{1}{3} (\frac{8}{3} D - 4) (\frac{\beta}{1 - \beta}) \xi. \quad (7.58)$$
The asymptotic form of $t_1' (\xi)$ for large $\xi$ is obtained from equation (7.35) and it is given by

$$t_1' (\xi) \sim \frac{1}{6} a_0 D \xi^3 + \frac{\beta}{1 - \beta} \{ \tau_1 + \left( \frac{8}{3} D - 4 \right) \ln \xi + O(\xi^{-3}) \}, \quad (7.59)$$

where $\tau_1$ is a constant. The resulting expansion is then compared with (7.56) to determine the matching condition for this stage of approximation. We deduce that $\tau_0 = \tau_1$ if the inner and outer solutions to balance.

The function $R_1(\theta)$,

$$R_1 = S_1 + \frac{1}{H} \{ 1 - S_0' - S_0(P_1 - Q_1) \}, \quad (7.60)$$

obtained from (7.28), is used to rewrite (7.27) as

$$S_1'' + H S_1' - H' S_1 = \{ S_0 W' - S_0' W - S_0'' \\ - S_0 V' - S_0' V - 1.64326 \frac{2H'}{H} (1 - S_0' - S_0 V) \}, \quad (7.61)$$

where $\bar{S}_1 = \frac{1 - \theta}{\beta} S_1$. The boundary conditions are given by

$$\bar{S}_1(0) = \bar{S}_1'(\infty) = 0. \quad (7.62)$$

The integration procedure is carried out starting from $\theta = 0.05$ with the intention of satisfying the far-field condition. This enables us to determine the constant $\tau_0$. We find

$$\tau_0 = \tau_1 = 1.55276. \quad (7.63)$$

It remains then to obtain the value of $t_1'(0)$ from (7.35). Making the appropriate
substitutions, equation (7.35) becomes

\[ t''_1 = \frac{1}{2} a_0 D\xi^2 + \left( \frac{\beta}{1 - \beta} \right) \left( \frac{1}{2} a_0 D\xi^2 + q_0 r'_0 - r_0 q'_0 \right), \quad (7.64) \]

where \( q_0 \) and \( r_0 \) are given in (4.20) and (7.21) respectively. Equation (7.64) clearly shows that \( t_1(\xi) \) is of the form

\[ t_1(\xi) = t_{1,1}(\xi) + \left( \frac{\beta}{1 - \beta} \right) t_{1,2}(\xi). \quad (7.65) \]

The corresponding differential equations for the functions \( t_{1,1}(\xi) \) and \( t_{1,2}(\xi) \) are given by

\[
\begin{align*}
& t''_{1,1}(\xi) = \frac{1}{2} a_0 D\xi^2, \\
& t_{1,1}(0) = 0, \quad t'_{1,1}(\xi) \sim \frac{1}{6} a_0 D\xi^3 \quad \text{as} \; \xi \to \infty, \\
& t''_{1,2} = \frac{1}{2} a_0 D\xi^2 + q_0 r'_0 - r_0 q'_0, \\
& t_{1,2}(0) = 0, \quad t'_{1,2}(\xi) \sim 1.55276 + \left( \frac{3}{2} D - 4 \right) \ln \xi \quad \text{as} \; \xi \to \infty.
\end{align*}
\]

The function \( t_{1,1}(\xi) \) is easily found to be

\[ t_{1,1}(\xi) = \frac{1}{24} a_0 D\xi^4, \quad (7.68) \]

which implies that \( t'_{1,1}(0) = 0 \). The general solution of (7.67) is given by

\[ t_{1,2}(\xi) = d_1 + d_2 \xi + t_{(1,2)p}(\xi), \quad (7.69) \]

where \( d_1 \) and \( d_2 \) are constants and \( t_{(1,2)p}(\xi) \) denotes a particular solution of (7.67). The constant \( d_1 \) is identically zero and \( d_2 \) can be determined numerically from the relation

\[ d_2 = t'_{1,2}(0) = \lim_{\xi \to \infty} \left\{ t'_{1,2}(\xi) - t'_{(1,2)p}(\xi) \right\}, \quad (7.70) \]
if we let $t_{(1,2)\mu}(0) = t'_{(1,2)\mu}(0) = 0$. We obtain

$$t'_{1,2}(0) = 0.0070369. \quad (7.71)$$

The value of $t'_1(0)$ is hence given by

$$t'_1(0) = 0.0070369 \left( \frac{\beta}{1 - \beta} \right). \quad (7.72)$$

The inner solution is now completely determined to $O(\lambda^{-1})$. The quantity of interest is the shear component of tangential stress at the wall, $e'(0)$. From (7.1) and (7.10), we have

\[
e'(0) = 1.40654 - 0.083072 \frac{\beta}{\varepsilon} \ln \left( \frac{\varepsilon}{1 - \beta} \right) + 0.0070369 \frac{\beta}{\varepsilon} + O[\varepsilon^{-\frac{1}{3}} \ln \left( \frac{\varepsilon}{1 - \beta} \right)]. \tag{7.73}
\]

Table 7.1 compares the asymptotic predictions with the numerical results provided in table 5.1.
Table 7.1: Comparison of $e'(0)$ actual (numerical) with $e'(0)$ asymptotic.

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Chapter 8

Conclusion

8.1 Discussion on the Flow Near the Wall

Following Dorrepaal's analysis of the electrically inert version of the flow, discussed in chapter 1, we proceed to investigate the behavior of our flow near the rigid wall by deriving the small-$y$ expansions of $f(y)$ and $m(y)$. The expansions are given by

\[
\begin{align*}
  f(y) &= \frac{1}{2} f''(0)y^2 + \frac{1}{6} \{ \beta [1 - g'(0)^2] - 1 \} y^3 + \frac{1}{360} f''(0)^2 y^5 + O(y^6), \\
  m(y) &= \tilde{k}e'(0)y^3 - \frac{1}{3} B_\varepsilon \tilde{k}(1 - \beta) y^3 + \frac{1}{60} \tilde{k} f''(0)e'(0)y^5 + O(y^6),
\end{align*}
\]

where $\tilde{k}$, as described in (1.15), depends on the angle of incidence. On substituting the resulting expansions (8.1) into (5.18), we obtain

\[
\psi(x, y) = \frac{1}{2} f''(0)y^2 \{ x + \frac{2\tilde{k}e'(0)}{f''(0)} - \frac{2}{3 f''(0)} B_\varepsilon \tilde{k}(1 - \beta) y + \frac{1}{3 f''(0)} \} \\
\cdot [\beta (1 - g'(0)^2) - 1] xy + \frac{1}{30} \tilde{k}e'(0)y^3 + \frac{1}{60} f''(0)xy^3 + O(y^4)].
\]

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Figure 8.1: The dividing streamline $\psi = 0$ far from the wall is a straight line, $l$, with slope $-\frac{1}{k}$. The line $l$, whose slope is denoted by $m_s$, is tangent to the dividing streamline at $X = 0$.

The stream function, $\psi \sim \gamma y[x + k y]$ as $y \to \infty$, indicates that the dividing streamline $\psi = 0$ far from the wall is a straight line, $l$, with slope

$$m = -\frac{1}{k}.$$  \hspace{1cm} (8.3)

If we denote the point where $l$ intersects the wall as shown in figure 8.1 by $x = 0$, then it can easily be seen from (8.2) that the dividing streamline $\psi = 0$ meets the wall at

$$x = -d = -\frac{2\bar{k}e'(0)}{f''(0)}.$$

A simple translation,

$$X = x + \frac{2\bar{k}e'(0)}{f''(0)},$$

would then be appropriate when studying the behavior of the dividing streamline $\psi = 0$ at the wall. The resulting expansion for the stream function in terms of the
new horizontal coordinate \( X \) is found to be

\[
\psi(X, y) = \frac{\tilde{k}}{3f''(0)} \Lambda y^2 \cdot \{ y + \Omega_1 X - \Omega_2 X y + O(y^2) \}. \tag{8.6}
\]

where \( \Lambda \) and the positive constants \( \Omega_1 \) and \( \Omega_2 \) are given by

\[
\begin{align*}
\Lambda &= (1 - \beta)\left( e'(0) - B Cf''(0) \right) + \beta e'(0)g'(0)^2, \\
\Omega_1 &= \frac{1.5f''(0)^2}{k[e'(0)(1-\beta(1-g'(0)^2)) - B Cf''(0)(1-\beta)]}, \\
\Omega_2 &= \frac{f''(0)[1-\beta(1-g'(0)^2)]}{2k[e'(0)(1-\beta(1-g'(0)^2)) - B Cf''(0)(1-\beta)]}.
\end{align*} \tag{8.7}
\]

It follows then that the behavior of the dividing streamline \( \psi = 0 \) at the wall can be analyzed from the following equation:

\[
y + \Omega_1 X - \Omega_2 X y + O(y^2) = 0. \tag{8.8}
\]

The slope of the dividing streamline at the wall, \( m_s \), is hence given by

\[
m_s = \frac{dy}{dX} \big|_{(0,0)} = -\frac{1.5f''(0)^2}{\tilde{k}[(1 - \beta)(e'(0) - B Cf''(0)) + \beta e'(0)g'(0)^2]}, \tag{8.9}
\]

which implies that the slope ratio \( \frac{m_s}{m} \),

\[
m_r = \frac{m_s}{m} = \frac{1.5f''(0)^2}{(1 - \beta)(e'(0) - B Cf''(0)) + \beta e'(0)g'(0)^2}, \tag{8.10}
\]

is independent of the angle of incidence. This is in agreement with the result found by Dorrepaal [3] for the electrically inert version of the flow (\( \beta = 0 \)), as recorded in (1.38). Furthermore we conclude that the dividing streamline always meets the wall concave down since

\[
\frac{d^2y}{dX^2} \big|_{(0,0)} = -2\Omega_1 \Omega_2, \tag{8.11}
\]

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Another interesting result is the distance, $h_0$, from the point of attachment of the dividing streamline to the line $l$. Its value is also independent of $\tilde{k}$ and it is given by

$$h_0 = |d \cdot m| = \frac{2e'(0)}{f''(0)}.$$  \hfill (8.12)

Furthermore the distance, $h$, from the wall to the point of intersection of $l$ and $l_*$, is also independent of $\tilde{k}$ and it is found to be

$$h = \frac{m_r}{m_r - 1} h_0 = \frac{3e'(0)f''(0)}{1.5f''(0)^2 + (1 - \beta)(B Cf''(0) - e'(0)) - \beta e'(0)g'(0)^2}. \hfill (8.13)$$

Tables 8.1 and 8.2 contain values of $m_r$ and $h_0$ respectively for various combinations of $(\epsilon, \beta)$.

### 8.2 Discussion of Results

Tables (4.1) and (4.2) confirm that the large-$\epsilon$ expansions obtained for $f''(0)$ and $g'(0)$ in (4.91-4.92) are useful for $\epsilon \geq 10^2$, giving percentage errors of $O(10^{-1})$. A similar claim can be made for the small-$\epsilon$ expansions obtained in (3.61-3.62) provided $\epsilon \leq 10^{-2}$. A simple study of the large-$\epsilon$ expansions (4.91-4.92) indicates that the functions $f''(0)$ and $g'(0)$ approach their exact solutions as $\beta$ approaches its limiting values. The best agreements for the large-$\epsilon$ expansions are therefore obtained when $\beta$ or $1 - \beta$ is small. In the small-$\epsilon$ expansions, given in (3.61-3.62), the best agreement is obtained when $\beta$ is small. The expansions are, however, clearly not valid as $\beta \to 1$ and the growth in the error is evident as $\beta$ increases.
Table 8.1: Slope ratio, $m_r = \frac{m_s}{m}$, for various values of $(\epsilon, \beta)$

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<th>0.8</th>
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Table 8.2: Distance, $h_0$, for various values of $(\epsilon, \beta)$

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One of the interesting predictions from (4.91-4.92) is that $f''(0)$ and $g'(0)$ both tend to 0 as $\beta \to 1$. This parallels a discovery by Greenspan & Carrier [11] in the Blasius problem and later verified by Stewartson & Wilson [15] that both the fluid velocity field and the total magnetic field vanish identically as the applied magnetic field is increased. This plugging or freezing of the flow by the induced magnetic field occurs when the ratio of magnetic energy density to kinetic energy density within the fluid reaches unity.

The expansions obtained for $f''(0)$ and $g'(0)$ are reminiscent of those obtained by Glauert [17] for magnetohydrodynamic Blasius flow. There are significant differences, however, between the two problems. When $\epsilon$ is small, the Blasius problem has terms of $O(\epsilon \ln \epsilon)$ entering the expansion for $f''(0)$. Logarithmic terms eventually infiltrate the small-$\epsilon$ expansions in stagnation-point flow as well; but they are not in evidence at $O(\epsilon)$. When $\epsilon$ is large, the Blasius expansions proceed in powers of $\epsilon^{-1}$ (ignoring logarithms for the moment) while the stagnation-point expansions involve powers of $\epsilon^{-\frac{1}{3}}$.

Another major difference between the two problems is lack of uniqueness in the Blasius problem for certain parameter values as discussed in chapter 2. Our numerical treatment of the stagnation-point problem indicates that solutions exist for all values of $\epsilon$ and $\beta < 1$, and that these solutions are always unique.

The reason for this difference possibly lies in the fact that the governing equations (2.36-2.37) for the stagnation-point problem contain the parameter $K$ explicitly while the corresponding Blasius equations do not. Thus, in the stagnation-point
problem, once $\epsilon$ is chosen, the fixing of $K$ determines $F''(0)=C$ and subsequently the values of $A(K)$, $B(K)$ and $\beta(K)$. For all $\epsilon$, the function $\beta(K)$ is strictly monotone decreasing and onto the interval $(0,1)$, a fact which guarantees unique solutions for $0 < \beta < 1$. On the other hand, in the Blasius problem, once $\epsilon < 1$ is fixed, the value of $C$ is chosen arbitrarily and then used to generate the corresponding values of $A(C)$, $B(C)$ and $\beta(C)$. The function $\beta(C)$ is not one-to-one over the values $0 < C < +\infty$ and this fact leads to the aforementioned non-uniqueness.

In the case of non-orthogonal stagnation-point flow, the results obtained from the asymptotic expansions (6.68) and (7.73) are seen from tables (6.1) and (7.1) to be in excellent agreement with the numerical results. As in the orthogonal stagnation-point flow the best agreements found from the large-$\epsilon$ expansion are obtained when $\beta$ approaches one of its limiting values while the best agreement obtained from the small-$\epsilon$ expansion is achieved when $\beta$ is close to zero. The appropriate asymptotic expansion for large or small values of $\epsilon$ gives very reliable results for $\epsilon'(0)$ provided that $\epsilon > 10^2$ or $\epsilon < 10^{-2}$ respectively.

Our numerical treatment of the non-orthogonal stagnation-point problem shows that this flow, like its orthogonal counterpart, possesses unique solutions for all values of $\epsilon$ and $\beta \in (0,1)$. The reason becomes clear when one studies the governing equations for the non-orthogonal stagnation-point flow. The equations of motion consist of four ordinary differential equations, two of which describe the stagnation-point problem exactly and one is a first degree ordinary differential equation with its
initial condition known while the equation describing the function $E(\eta)$ is a second degree linear ordinary differential equation. The latter two equations, for a given $\epsilon$ and a fixed $K$, depend on the values of $C(K)$, $A(K)$, $B(K)$, $B_\epsilon(K)$, and $\beta(K)$ obtained uniquely from the orthogonal problem. A shooting method satisfying the far-field condition then determines $E'(0) = R(\epsilon, K)$. Therefore the function $\beta(K)$ here remains unchanged and it is strictly monotone decreasing and onto the interval $(0, 1)$ as in the orthogonal problem. This leads to a unique solution, $R$, for all $\epsilon$ and $\beta \in (0, 1)$, having fixed the parameter $K$.

For fixed values of $\epsilon$ and $\beta$, the slope of the dividing streamline $\psi(x, y) = 0$ at the wall divided by its slope at infinity is the same for all corresponding non-orthogonal stagnation-point flows. Far from the wall the dividing streamline is a straight line with slope $-\frac{1}{k}$. The distance between this line and the point of attachment of the dividing streamline is also independent of the angle of incidence. Figures 8.2-8.4 confirm our mathematical analysis of non-orthogonal flows near the wall. The maximum value of the slope ratio, $\frac{m_\ast}{m}(\epsilon, \beta)$, and the minimum value of the distance, $h_0(\epsilon, \beta)$, are the corresponding values obtained from the electrically inert version of the flow ($\beta = 0$), namely that $(\frac{m_\ast}{m})_{\text{max}} = 3.74851$ and $(h_0)_{\text{min}} = 2.28226$.

For a non-conducting fluid ($\epsilon = 0$), the values of $\frac{m_\ast}{m}$ and $h_0$ are found to be independent of $\beta$ and given by

$$\begin{align*}
\frac{m_\ast}{m} &= 3.74851, \\
h_0 &= 2.28226.
\end{align*}$$

(8.14)
For fluids which are perfectly conducting ($\varepsilon = \infty$), the slope-ratio constant is exactly the same; however, the distance $h_0$ varies with $\beta$ and it is given by

$$h_0 = 2.28226(1 - \beta)^{-\frac{1}{2}}.$$  \hfill (8.15)

Tables (8.1) and (8.2) support the analytical results found in (8.14-8.15).

Using the asymptotic expansions obtained for $f''(0)$, $g'(0)$, $B_C$ and $e'(0)$, the first two terms of the asymptotic expansions for $\frac{m_s}{m}$ and $h_0$ are derived to be

$$\frac{m_s}{m} \sim 3.74851 - 3.04388\beta(\frac{\varepsilon}{1 - \beta})^{\frac{1}{2}},$$
$$h_0 \sim 2.28226 + 2.71970\beta(\frac{\varepsilon}{1 - \beta})^{\frac{1}{2}},$$  \hfill (8.16)

for small values of $\varepsilon$ and

$$\frac{m_s}{m} \sim 3.74851 - 9.66919\frac{\beta}{[\varepsilon^2(1 - \beta)]^{\frac{1}{2}}},$$
$$h_0 \sim (1 - \beta)^{-\frac{1}{2}}\{2.28226 + 2.16347\frac{\beta}{\varepsilon}\ln(\frac{\varepsilon}{1 - \beta})\},$$  \hfill (8.17)

for large values of $\varepsilon$.

Tables 8.3 and 8.4 compare the asymptotic predictions with the numerical results provided in tables 8.1 and 8.2.

The class of non-orthogonal flows which attracted much attention is the flows for which their corresponding slope ratios, $\frac{m_s}{m}$, are less than unity. To illustrate the behavior of the dividing streamline, $\psi(x, y) = 0$, for such cases, we examine the case where $\varepsilon = 1.0$ and $\beta = 0.99$. The slope at infinity $m$, for simplicity, is selected to be $-1.0$. The slope ratio for this class of flows is obtained to be $0.55446$.

Our analysis of the flow near the wall indicates that all corresponding dividing streamlines, $\psi(X, y) = 0$ where $X$ is given in (8.5), regardless of the values of $\varepsilon$ and $\beta$. 

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\( \beta \), are concave down at the wall. It is therefore clear, as shown in figure 8.5, that in order for the function \( \psi(X, y) = 0 \) to reach the prescribed slope, \( m = -1.0 \), far from the wall the function must change concavity at least twice over its domain. This clearly suggests the presence of points of inflection. Our numerical analysis of such flows confirms this phenomena. The effects of points of inflection on the stream functions \( \psi(x, y) = \text{constant} \) for the case where \( \epsilon = 1.0, \beta = 0.99 \) and \( m = -1.0 \) are nicely captured in figure 8.6.
Table 8.3: Comparison of $\frac{\text{num}}{\text{act}}$ actual (numerical) with two-term asymptotic expansion of $\frac{\text{num}}{\text{act}}$.

| $\epsilon$ | $\beta$ | Actual (Num.) | Asymptotic | Error | $|\%\text{-Error}|$ |
|------------|---------|---------------|------------|-------|----------------|
| 0.001      | 0       | 3.74851       | 3.74851    | 0     | 0              |
| 0.001      | 0.2     | 3.72800       | 3.72699    | $1.00 \times 10^{-3}$ | $2.70 \times 10^{-2}$ |
| 0.001      | 0.4     | 3.70124       | 3.69881    | $2.43 \times 10^{-3}$ | $6.58 \times 10^{-2}$ |
| 0.001      | 0.6     | 3.66205       | 3.65720    | $4.85 \times 10^{-3}$ | $1.32 \times 10^{-1}$ |
| 0.001      | 0.8     | 3.58730       | 3.57633    | $1.10 \times 10^{-2}$ | $3.06 \times 10^{-1}$ |
| 0.001      | 0.99    | 2.98098       | 2.79558    | $1.85 \times 10^{-1}$ | $6.22$ |
| 1000.0     | 0       | 3.74851       | 3.74851    | 0     | 0              |
| 1000.0     | 0.2     | 3.72525       | 3.72768    | $2.43 \times 10^{-3}$ | $6.53 \times 10^{-2}$ |
| 1000.0     | 0.4     | 3.69719       | 3.70266    | $5.46 \times 10^{-3}$ | $1.48 \times 10^{-1}$ |
| 1000.0     | 0.6     | 3.66039       | 3.66977    | $9.39 \times 10^{-3}$ | $2.56 \times 10^{-1}$ |
| 1000.0     | 0.8     | 3.60151       | 3.61624    | $1.47 \times 10^{-2}$ | $4.09 \times 10^{-1}$ |
| 1000.0     | 0.99    | 3.30538       | 3.30420    | $1.17 \times 10^{-3}$ | $3.57 \times 10^{-2}$ |

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Table 8.4: Comparison of \( h_0 \) actual (numerical) with two-term asymptotic expansion of \( h_0 \).

<table>
<thead>
<tr>
<th>( \epsilon )</th>
<th>( \beta )</th>
<th>Actual (Num.)</th>
<th>Asymptotic</th>
<th>Error</th>
<th>%-Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>0</td>
<td>2.28226</td>
<td>2.28226</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.001</td>
<td>0.2</td>
<td>2.30061</td>
<td>2.30149</td>
<td>( 8.87 \times 10^{-4} )</td>
<td>( 3.86 \times 10^{-2} )</td>
</tr>
<tr>
<td>0.001</td>
<td>0.4</td>
<td>2.32497</td>
<td>2.32667</td>
<td>( 1.71 \times 10^{-3} )</td>
<td>( 7.33 \times 10^{-2} )</td>
</tr>
<tr>
<td>0.001</td>
<td>0.6</td>
<td>2.36154</td>
<td>2.36385</td>
<td>( 2.31 \times 10^{-3} )</td>
<td>( 9.80 \times 10^{-2} )</td>
</tr>
<tr>
<td>0.001</td>
<td>0.8</td>
<td>2.43416</td>
<td>2.43611</td>
<td>( 1.96 \times 10^{-3} )</td>
<td>( 8.04 \times 10^{-2} )</td>
</tr>
<tr>
<td>0.001</td>
<td>0.99</td>
<td>3.19194</td>
<td>3.13371</td>
<td>( 5.82 \times 10^{-2} )</td>
<td>1.82</td>
</tr>
<tr>
<td>1000.0</td>
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<td>2.28226</td>
<td>2.28226</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1000.0</td>
<td>0.2</td>
<td>2.55303</td>
<td>2.55510</td>
<td>( 2.07 \times 10^{-3} )</td>
<td>( 8.10 \times 10^{-2} )</td>
</tr>
<tr>
<td>1000.0</td>
<td>0.4</td>
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<td>2.95468</td>
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<td>( 1.64 \times 10^{-1} )</td>
</tr>
<tr>
<td>1000.0</td>
<td>0.6</td>
<td>3.61563</td>
<td>3.62463</td>
<td>( 9.01 \times 10^{-3} )</td>
<td>( 2.49 \times 10^{-1} )</td>
</tr>
<tr>
<td>1000.0</td>
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<td>5.11893</td>
<td>5.13626</td>
<td>( 1.73 \times 10^{-2} )</td>
<td>( 3.38 \times 10^{-1} )</td>
</tr>
<tr>
<td>1000.0</td>
<td>0.99</td>
<td>22.96894</td>
<td>23.06921</td>
<td>( 1.00 \times 10^{-1} )</td>
<td>( 4.37 \times 10^{-1} )</td>
</tr>
</tbody>
</table>
Figure 8.2: The dividing streamline $\psi = 0$ for the non-orthogonal flow where $\epsilon = 1.0$, $\beta = 0.2$. The ratio of the slopes $\frac{m_s}{m}$ and the distance $h_0$ are independent of the angle of incidence.
Figure 8.3: The dividing streamline $\psi = 0$ for various angles of incidence corresponding to the non-orthogonal flow where $\epsilon = 1.0$ and $\beta = 0.2$. The distance, $d_{\alpha} \cdot m_{\alpha}$, is shown to be the same (independent of the angle of incidence) for all three cases $\alpha = 1$, 2 and 5.
Figure 8.4: The dividing streamline $\psi = 0$ for various angles of incidence corresponding to the non-orthogonal flow where $\epsilon = 10.0$ and $\beta = 0.6$. The distance, $d_\alpha \cdot m_\alpha$, is independent of the angle of incidence and it is given by $h_0 = 3.68166$. 

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Figure 8.5: The dividing streamline $\psi = 0$ for the case $\epsilon = 1.0$ and $\beta = 0.99$. The slope ratio $\frac{m_s}{m} \simeq 0.55446$ is as predicted by the analysis.
Figure 8.6: Streamlines for the magnetohydrodynamic non-orthogonal stagnation-point flow where $\epsilon = 1.0$, $\beta = 0.99$ and $m = -1.0$. 

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Bibliography


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