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Teresa Kouri
Old Dominion University, tkouri@odu.edu

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A Reply to Heathcote's "On the Exhaustion of Mathematical Entities by Structures"

Teresa Kouri

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Mathematical *ante rem* structuralism (ARS), as characterized by Shapiro (2000), is the position that mathematical objects are places in certain abstract structures. A structure is a collection of places and relations between those places. The natural number structure, for example, is a countable set of places which validate the (second order) Peano axioms. Thus, the number 2 can be thought of as the third place in the natural number structure in the usual ordering, and, given the structure, we know that that place is the successor of one, is the successor of the successor of 0, etc. ARS proposes solutions to Frege's Caesar problem, the problem presented in Benacerraf's "What Numbers Cannot Be" and the problem in "Mathematical Truth." More recently, it has come under fire for not being able to distinguish between numerically distinct but structurally identical objects (more on this below).

Heathcote (2013) proposes a new problem for the *ante rem* structuralist, which he calls the "issue of ontic exhaustion". In this paper, I will show that Heathcote's problem is not actually a problem, but a feature, of the *ante rem* structuralist's position. Further, I will suggest that a closely related analogue to Heathcote's problem is an issue for realist philosophy of mathematics generally, and that the *ante rem* structuralist already has a solution for it.

1 Some Background on ARS and the Original Identity problem

Shapiro (2000) defines ARS as the theory that mathematical objects are places in structures. An *ante rem* structuralist is one for whom structures are the referents of mathematical discourse in and of themselves. This is in contrast to an eliminative structuralist position, which holds that talk of structures is shorthand for talk of other objects, and a non-structuralist, who holds that mathematical objects are something else entirely (e.g. forms). For the *ante rem* structuralist, the number 3 would be nothing more or less then the fourth place in the natural number structure. Thus, in ARS, when mathematicians talk of natural number systems they refer to structures, and when they talk of individual numbers they refer to places within structures. A place in a structure is, for our purposes, exactly the set of structural relations. An instantiation of a structure is a collection of objects which satisfy the structural relations (usually in a model). Mathematical objects are not instantiations of particular places in a particular structure, but rather they are the places themselves. According to Shapiro, "each mathematical object is a place in a particular structure" (p 78). Anything can instantiate any place in any structure (e.g. the number 1, this table, Julius Caesar).[1]

For Shapiro, there is an intimate relationship between the practice of mathematics and the philosophy of mathematics. He says "I propose the metaphor of a partnership or healthy marriage rather than a merger or blending a stew rather than a melting pot" (Shapiro, 2000, p 35). Philosophy of mathematics must remain faithful to the practice

[1]I will assume throughout that the concept of structure is generally unproblematic, except for the explicit problems I will be considering here.

of mathematics. Philosophy should not affect what mathematics is capable of, but only lend insight to why mathematics is so capable. If any philosophy of mathematics suggests drastic changes to the practice of mathematics, other than changes to how we think about the practice of mathematics, it should be reassessed. The constraint that philosophy of mathematics must accord with the practice of mathematics is what I will refer to as "Shapiro's faithfulness constraint."

Another important aspect of ARS is that "every [place in a structure] is characterized completely in terms of how its [potential] occupant relates to the occupants of any other [place] of the structure and any object can occupy any of its places." Mathematical objects are characterized by their structural relations.

This conception of mathematical objects, where each object is characterized exactly by its structural relationship to other objects, gives rise to what I will call the Original Identity Problem (OIP).[2] The OIP is a problem involving structurally indistinguishable places. The OIP is, roughly, that there are some mathematical objects which seem to share all of their structural properties and relations, but are distinct objects.[3] This arises, for example, with any pair of points on the Euclidean plane. There is no structural property which one has that the other does not. If Shapiro is correct, and mathematical objects are places in structures characterized entirely in terms of their structural properties (and the *ante rem* structuralist is committed to the Principle of Identity of Indiscernibles (PII), more on this below), then this seems to imply that there

[2]This terminology is not common in the literature, where this problem is normally referred to as simply the identity problem for ARS, or the problem of indistinguishable objects for ARS. I use this terminology to distinguish it from Heathcote's ontic exhaustion problem, which is also a problem of identity and indistinguishability.

[3]Just what it means to be a structural relation has occupied much of the discussion which followed the introduction of the OIP. In particular, it is often asked whether the identity relation is a structural relation. Several authors (see, for example, Ladyman and Leitgeb (2008)) have postulated different accounts of distinguishability to solve the OIP. Because the application of this solution to Heathcote's problem does not rely on any of these accounts of distinguishability, or any of the subtleties of the debate about what counts as a structural relation, I will put these issues to one side.

is really one point on the plane. Clearly, this cannot be the case. There are infinitely many such points. This is what the mathematics says, and so in order to abide by Shapiro's faithfulness constraint, ARS must say as well. If ARS predicts there really is only one point on the Euclidean plane, then either ARS is mistaken and mathematical objects are not characterized solely in terms of their structural properties as places in structures, or something else must go. The OIP is more prolific than I have described so far: any structure which admits of a non-trivial automorphism will have structurally indistinguishable places. In section 4.2, I will explain the structuralist solution to this problem. For now, this brief characterization suffices to explain Heathcote's problem.

2 Heathcote

Heathcote (2013) presents a different problem about identity for ARS. Heathcote's thought is that the OIP affects not just the places within structures, but also the structures themselves. In Heathcote's words, the issue is that "mathematical entities can plausibly be the same across different mathematical structures, ...[and] the intended mathematical structure [may be] insufficient to determine all of the properties that may be properly thought of as arithmetic [or complex, or etc.]" (Heathcote, 2013, p 172). I call the possibility that mathematical objects can be the same across structures the first part of the ontic issue, and the possibility that mathematical structures underdetermine the relevant properties the second part of the ontic issue. The two examples Heathcote uses are helpful, and so I will use them here as well.

First, Heathcote notes that there is a difference between the field of complex numbers, and the algebra of complex numbers. The field of complex numbers is just the system characterized by the field axioms for the complex numbers, with some specification that it is algebraically closed.[4] In the field of complex numbers, there is no distinction between real and imaginary numbers, though we will be able to specify the rational ones. The algebra of complex numbers, on the other hand, is the result of taking the algebraic closure of the real numbers. The algebra of complex numbers generates the usual interpretation of the complex numbers in the Argand plane, and generates the single non-trivial automorphism of each complex number onto its complex conjugate. The algebra is what is normally referred to by $\mathbb{C}$ (Heathcote, 2013, p 169).

When discussing complex numbers, philosophers normally restrict themselves to the algebra of complex numbers. It is the algebra of complex numbers which admits of the complex conjugate automorphism which makes the two roots of -1 indistinguishable.[5] The field of complex numbers, on the other hand, admits of many more non-trivial automorphisms. In the structure of the field of complex numbers, then, many more places will be indistinguishable. Heathcote's challenge to the *ante rem* structuralist is to identify which of these structures is *the* complex number structure; ARS needs to provide a referent for the term "the complex numbers." According to him, the *ante rem* structuralist must answer that there are in fact two types of complex numbers, those in the field structure and those in the algebra structure. However, he further claims that this is counter-intuitive. What we usually mean by the term "complex numbers" is specific and unique. The field axioms do not characterize a unique system (they generate both the field and the algebra, according to Heathcote), and so the field

[4]Heathcote is not clear about whether the axioms for the complex field can be satisfied for more than one system (up to isomorphism). If there is only exactly one system which satisfies them, then there might be more to the complex field than he suggests, since on first glance the axioms specified do not seem to pick out a unique system. If there is more than one system which satisfies the axioms of the complex field, then Heathcote's ontic issue might be more widespread then he lets on (it might affect all realist philosophers of mathematics, for example). This would be because any philosopher of mathematics would have to answer which of the distinct systems is *the* complex field.

[5]Presumably, if there is any sense in which the complex field contains i and $-i$, they will be indistinguishable as well. However, the complex field will likely not admit of the complex conjugate automorphism.

axioms do not give a unique referent to "the complex numbers". Heathcote takes the meaning of a term to be tied to its referent, and so claims, "what we mean by the term *complex number* is not tied to, and exhausted by, the field axioms" (Heathcote, 2013, p 172). The *ante rem* structuralist, on the other hand, takes the complex numbers to be characterized entirely by the field axioms. Because of this, on Heathcote's view, ARS cannot give a satisfactory account of meaning of the term "the complex numbers." This is an example of the first part of Heathcote's ontic issue: complex numbers can be (are?) the same across both structures, and ARS cannot account for that.

The second example Heathcote gives is similar. He claims the natural number structure is underspecified by the Peano axioms.[6] The Peano axioms give us no information about what base the numbers are in. We could have one structure in base ten, one in base two, one in base 60, etc. All of these natural number structures satisfy the Peano axioms. However, in each structure, different things will be true of the different places in question. For example, the third place in the base two structure will be related differently to the tenth place in the base two structure than the third place in the base ten structure will be to the tenth place in the base ten structure. Heathcote uses the technique of casting out 9s to show that this is the case. He claims that, in the base ten structure, the third place can be "cast out" with respect to the tenth, but that this is not so in the base two structure. Casting out nines is a technique sometimes made use of to check arithmetical calculations. Essentially, it relies on the fact, that in base 10, a number and the sum of its digits are equivalent mod 9. The third place in a base ten structure will be equivalent to the sum of its digits mod the tenth place, but the third

[6] It is quite important for Shapiro's ARS that we have use of second order Peano arithmetic rather than first order (see footnote 9). It is not clear whether Heathcote intends to say that the natural number structure is underspecified by the first or second order Peano axioms. I will give the characterization of his argument using the term "Peano axioms," but the reader should note that part of the solution to Heathcote's ontic issue requires that it is the second order axioms which are being used by Shapiro.

place in a base two structure will not be. Heathcote asks, which of these structures is *the* natural number structure? Again, Heathcote claims that the *ante rem* structuralist must say that there is no unique natural number structure (no unique referent for "the natural number structure"), and again he thinks that this points to the fact that the meaning of "*natural number*" is not exhausted by the Peano axioms. Thus, what it is to be a natural number cannot just be what it is to be a place in a structure which is characterized by the Peano axioms. This is the second part of Heathcote's ontic issue: the natural number structure, as given by some variation of the Peano axioms, does not determine a relevant arithmetic property, namely a base. If Heathcote is right, Shapiro and the *ante rem* structuralists are wrong.

3 Heathcote's Ontic Issue is a Non-Issue

The solution to Heathcote's problem is similar to the solution to the OIP. First, I will show that it is not a bug of the *ante rem* structuralist program that it admits of more than one complex or natural number structure. Second I will give a possible explanation, parallel to the solutions for places in structures, of how mathematicians still manage to use the terms "the natural numbers" and "the complex numbers" referentially.

On the face of it, the *ante rem* structuralist can easily dismiss Heathcote's claims. As to the question of whether the complex algebra or the complex field really defines the complex numbers, she can simply say "whichever the mathematician chooses." If there is really a second set of complex number axioms mathematicians make use of, then there had best be a corresponding structure philosophers can account for. In this sense, the structuralist is ahead of the game. If the philosopher of mathematics cannot account for the mathematical frameworks actually used by mathematicians, then she will have violated Shapiro's faithfulness constraint. In order to abide by the faithfulness principle, the philosopher of mathematics *must* have a structure for the complex algebra, complex field, and natural numbers in all bases. In this sense, Heathcote's realization that the *ante rem* structuralist can account for such structures is a point in the structuralist's favor. On the face of it, then, Heathcote's concerns are a feature, not a bug, of ARS, and are necessary in order to remain faithful (though, of course, the problem of the definite description in "the complex numbers" remains).

Another matter to sort out is to deduce whether the meaning of the term "complex number," "natural number' or, the example I will use, "the number 2," are under determined by structural considerations.[7] Both parts of Heathcote's ontic issue imply that they are. I will argue that they are not. It is certainly the case that Heathcote thinks that the meaning of "the number 2" is under determined by ARS. He claims that, for example, the number 2 is much more (that it means much more than) than just a place in the (a?) natural number structure. For Heathcote, "it is also an integer, a real number, a complex number, a quaternion, an octonion, a number in $\mathbb{Z} = p\mathbb{Z}$ for all p prime... an even number, a prime number, an Eisenstein prime, and so on." (Heathcote, 2013, p 171). If being the third place in the (a?) natural number structure is exhaustive of what it means to be the number 2, then clearly none of these other features are part of what it means to be the number 2. For Heathcote, this is unintuitive at best, and reason to dismiss ARS at worst.

In the case of the first set of properties (those that correspond to the first part of the ontic issue, i.e. being an integer, real, etc.) there is an immediate problem. Pre-theoretically, it seems intuitive that the natural 2 is the same as the rational 2 and the same as the real 2. However, whether the natural 2 is the same as the real 2 is actually quite a philosophically contentious claim. Many philosophers working in the

[7]The arguments which follow can be adapted to show that the meanings of "the complex numbers" and "the natural numbers" are also not under determined by ARS.

foundations of mathematics would not agree that they are the same. In particular, it seems Frege and Russell would disagree strongly.[8] In the debates about neo-logicism these days, there are philosophers who think they must be the same (for example, Neil Tennant) and those who do not (for example, Bob Hale and Crispin Wright). For Hale and Wright, the question of whether natural 2 is the same as real 2 turns out to be a something akin to a Caesar problem. More importantly for Heathcote, it seems that it would even be possible for a platonist to hold the view that natural 2 is different from real 2, as seems to hold that the mathematical philosophy we should pick over ARS is something like a platonistic conception of mathematical objects. The pre-theoretic intuition, I think, needs a lot more support than what is given by Heathcote. In the end, whether the natural 2 is the same as the real 2 is quite contentious, and Heathcote has not made a sufficient case for it. If Hale, Wright, Frege, Russell and our potential platonist are right, then the first part of the ontic issue is solved, as mathematical object could not occur across structures.

There is a further problem Heathcote sees. He claims that the natural number structure will under determine the meaning of the number 2 because the structure does not provide us with any notion of a base. In Heathcote's words, "determining a base has some surprising consequences" (Heathcote, 2013, p 172). Suppose we have two different natural number systems at hand, one in base ten and one in base two. Heathcote claims that the corresponding places in these structures have different properties (recall his example of casting out 9s), and that the structuralist needs to be able to say which of these properties is part of the meaning of "the number 2." Unsurprisingly, Heathcote claims that the structuralist cannot accomplish this task. Selecting one of the bases as

[8] For Frege, "it is not possible to extend the domain of the cardinal numbers to that of the real numbers; they are simply completely separate domains" (*Grundgesetze*, §157). Cardinal numbers are extensions of concepts, while real numbers are measures of magnitudes. For Russell, real and natural numbers occur at different places in the type hierarchy, and thus cannot be identical.

the base which gives rise to the non-arithmetical properties of "the number 2" omits the others. The problem here is that this reasoning must also include what Heathcote calls the ascending properties of the number 2 (those properties which correspond to the first part of the ontic issue: being an integer, being real, etc.). In a sense, and parallel to Heathcote's reasoning, determining a type has some surprising consequences. The real two, for example, has a square root. It is not clear this is so for the natural two. Heathcote thinks he does not need to select a type (natural, ration, real, etc.), but that the structuralist needs to select a base. There are two responses for the *ante rem* structuralist here. The first is to suggest Heathcote is asking ARS to do something incoherent by forcing the structuralist to choose a base, but not a number type. The second is to claim that at the very least, ARS is consistent in that it does not pick either a base or a type.

The first response on the *ante rem* structuralist's behalf is that Heathcote is requesting that ARS make true incompatible claims. Heathcote holds that an identity like "21 in base eight is 35 in base ten" is a legitimate identity. He also holds that "the real number 2 is the natural number 2" is a legitimate identity. However, he seems to think that the structuralist must select exactly one base to be part of the meaning of a natural number, and yet must also be able to account for a number being both real and natural. If the structuralist must select exactly one base to be constitutive of the meaning of "the number 2", then she ought to also be able to select exactly one type of number (e.g. real or natural). If Heathcote wants "the number 2" to be both natural and real, then he must admit that it is also both "in" base ten and "in" base eight. He cannot have one without the other. Thus, it seems his demands of ARS may be incoherent. If the structuralist must make a choice about whether numbers are by definition "in" base x or "in" base y, then Heathcote must allow her to make a choice about whether such numbers are natural or real. He cannot consistently require that numbers are both natural and real, but only in one base. So ARS can dodge this argument by pointing out that Heathcote is asking ARS to make true incompatible things.

The second part of the ARS response is more conciliatory. At the very least, it might be said, ARS is consistent. Yes, ARS cannot explain the pre-theoretic intuition that “the number 2” ought to be by definition natural and real. However, ARS also makes no claims about “the number 2” being in one base rather than another. The situation seems to be this: “the number 2” is either real, natural, in base ten and in base eight, or it is exactly one of those. It cannot be real and natural and in only base ten, or real and natural and in only base eight, on pain of inconsistency. Because ARS does not imply that the number 2 is both real and natural, in order to remain consistent, it must also not imply that the number 2 is in base ten and in base eight. ARS does not imply that 2 is in both bases. So, though what Heathcote is asking for may be inconsistent, by answering both questions the same way, ARS remains consistent throughout. These arguments put a pin in the first part of the ontic issue: mathematical objects are not identical across structures.

In the case of the second set of properties, Heathcote claims that ARS cannot give an appropriate meaning to “the number 2” because it under-determines the relevant properties of 2. This is the second part of the ontic issue. 2 is also prime, even, etc. According to Heathcote, the natural number structure does not capture this, yet he holds they are part of the meaning of “2”. However, he is wrong. Most of these criteria are going to be captured by the third place in *any* natural number structure of *any* base. The majority of the descending properties Heathcote points to are all arithmetic properties, and any arithmetic property that a place has in the base ten natural number structure will be shared by the corresponding place in the base two natural number structure. The arithmetic properties of a number are consequence of the second order Peano Axioms, and since any natural number structure in any base satisfies those axioms, all corresponding places in structures will share them.[9] The fact of the matter is that because of the nature of what it is to be the third place in a natural number structure involves being indivisible by anything but the unit and the third place (being prime), being divisible by the third place (being even), etc. Whichever natural number structure the mathematician is speaking of, the 2-place will have all of the arithmetical "descending" properties.

Interestingly, Heathcote also suggests that being Einsteinian prime is a descending property of the number 2. An Einsteinian prime is a number which is irreducible in a ring-theoretic sense in a complex system. However, this requires the claim that the real 2, natural 2 and complex 2 are in fact one in the same (which Heathcote holds to be true). Above I suggested that this is a much harder claim to make than he takes it to be, and is far from clearly true. Thus, rather than being an arithmetic property, I hold that being an Einsteinian prime is a complex property, and not a property of the natural 2 at all. In this way, being Einsteinian prime is not part of what it means to be natural 2, and not a "descending" property of 2. The reasoning effectively solves the second part of Heathcote's ontic issue: the consequence of second order Peano arithmetic do not under-determine the arithmetical properties.

One last thing to notice is that it is not at all clear that the notion of a digit is an arithmetical property. Though this criticism of Heathcote's argument applies only to his concerns with the natural number structures, and not the complex number structures,

[9]This is an example of why it is so important for Shapiro that we work with second order Peano arithmetic. If we were to take as the arithmetic properties of numbers all and only those properties which could be derived from the first order Peano axioms, then we would have issues with incompleteness and non-standard models. For example, given the first order axioms, and in light of Gödel's incompleteness theorems, there may be (arithmetic) properties of the natural numbers which are not derivable from those axioms. We have a similar issue with non-standard models of the first order axioms: it would not be clear whether the properties of the non-standard numbers ought to be considered arithmetical properties. Thanks to an anonymous reviewer for pointing this out.

I think it is nonetheless important to mention. Much of Heathcote's argument against the natural 2 being given an appropriate meaning by the structuralist notion of a place in a structure rests on the fact that it is, first and foremost, a digit in a base ten system. His argument based on casting out nines requires that we consider 2 as both a digit and a place in a structure. However, the notion of a digit can only be expressed relative to a particular base. Once we have a particular base in mind, the structuralist has in hand all the details of this structure, and can explain casting out nines. There is a version of casting out nines which can be expressed in regardless of the base,[10] but unfortunately for Heathcote, the usual second order Peano axioms (or, rather, the usual axioms with an arbitrary base b) can prove that it works. This version of casting out nines is a property which ARS can easily account for.

If digits are not arithmetical properties/characteristics, then one odd feature of Heathcote's view is that he holds that there are some non-arithmetic properties which count as part of the meaning of mathematical terms, but other which do not. Why, for example, do digits matter and not "being the number of square roots of -1," or even worse, "being Frege's favorite number?" I'm not sure he has an answer for this. Certainly, if we have to either have all non-arithmetic properties of numbers be part of their meanings, or have none of them, Heathcote's argument falls apart. He cannot allow that the property "being Frege's favorite number" be any part of the meaning of a number. If we must have all or none, we must go for none, and his arguments about Einsteinian primes and digits fall apart. If he wants only some non-arithmetic properties of numbers to be part of their meanings, then he owes us an account of which ones and why.

[10]The explanation of casting out nines rests of the fact that in base ten, nine is one less than the base. This can generalize. Assume that we are working on base b, and we would like to cast out $b-1$s for some number n. Then we will be able to show that $n \equiv$the sum of the base-b digits of $n (mod\, b-1)$, which is what is required to show that casting out $b-1$ works for any base b.

In sum, Heathcote wants a philosophy of mathematics to provide a single object to be the referent of "the natural numbers" and a single object to be the reference of "the complex numbers." As I have shown, ARS needs there to be more than one complex number structure and more than one natural number structure or it will violate Shapiro's faithfulness constraint. Moreover, many of Heathcote's arguments are generally flawed, as they rely in part on the unsubstantiated claim that part of what it is to be 2 is to be both real and natural. Many philosophers would argue that being a real number is no part of being a natural number, and so the meaning of "the number 2" must be specified to either the meaning of "the natural number 2" or "the real number 2." This means the first part of Heathcote's ontic issue is solved. His "descending" properties either suffer the same fate or are arithmetic and can be account for by ARS, thus solving the second part. Thus, Heathcote's concerns about the structuralist meaning of "the number 2" either can be addressed by ARS or should not be.

4 An Adaptation of Heathcote's Ontic Issue, and the ARS Solution

What I have just suggested is that Heathcote's ontic issue is no problem at all. There may, however, be another problem in the vicinity. This can be thought of as the problem of how mathematicians actually refer to the complex numbers and the natural numbers. Heathcote suggests that this version of the problem, in addition to his ontic exhaustion issue, when he requests that ARS provide a unique and specific referent to the terms "the complex numbers" and "the natural numbers." Even if we have solved the ontic exhaustion problem, we may still have no tools for reference. This type of reference problem will turn out to be a problem for all realist philosophers of mathematics (from now on, referred to as "everyone"). Moreover, the same solution which applies for the reference problem associated with the OIP will be a solution here. Thus, not only will this adapted version of Heathcote's problem be a problem for everyone (if it is a problem), but the *ante rem* structuralist can solve it.

Consider the following argument. Heathcote would like an explanation of what the referent of "the complex numbers" is. When mathematicians speak of the complex numbers, when they use the term "the complex numbers," it seems like they are referring to a unique (possibly plural) entity. If this is the case, then the structuralist must explain what structure it is to which they are referring. This problem would amount to the claim that the structuralist cannot provide a single, unique referent to the mathematician's utterance of "the complex numbers." Call this the adapted Heathcote problem, or AHP for short. There are two things to notice about this version of the problem. First, it will be a problem for everyone. Second, the structuralist has a ready and available solution.

Any philosophy of mathematics needs to have an account of what the complex field and complex algebra are. Without such an account, the philosophy would not be properly accounting for the mathematical practice and would violate Shapiro's faithfulness constraint. Notice that this is all it takes to engender the AHP. As soon as a philosophy has an account of both the complex field and the complex algebra, it will need an explanation of which one is the referent of "the complex numbers." Because any adequate philosophy will need an account of both, Heathcote will always be able to claim that what we mean by "the complex numbers" is not tied to whatever account is given. If the philosophy of mathematics accounts for both systems (which it must to be faithful), then it will not by default have a unique referent for "the complex numbers." Thus, any philosophy of mathematics which accounts for both systems will be subject to the AHP. Just as the structuralist must be able to provide the structure to which the mathematician is referring when she utters "the complex numbers," every other philosopher of mathematics must be able to provide a unique referent.

Thus, if the AHP holds water, it is a problem for everyone. In the next two sections, I will show that the *ante rem* structuralist already has a solution at hand. In fact, it is the same solution as the solution to the reference problem associated with the OIP.

4.1 The OIP

As suggested earlier, the OIP is as follows. If all there is to being a mathematical object is being some place in some structure, and all there is to being some place in some structure is satisfying some structural relations, then there is a serious underdetermination problem at hand. This problem was first proposed by Burgess (1999), and expanded by Keränen (2001). Since, there has been much discussion about possible solutions and further problems. The AHP can be thought of as such a further problem.

The OIP amounts to the following. Any time a structure admits of a non-trivial automorphism, that automorphism will preserve all structural relations. Thus, for some automorphism $\mathcal{A}$, if $\mathcal{A}$ maps structural place x to structural place y, then x and y will satisfy exactly the same structural relations. For the purposes of the *ante rem* structuralist, this makes x and y completely indistinguishable. The *ante rem* structuralist simply cannot tell the two objects apart. Further, if she holds that Leibniz's Principle of the Identity of Indistinguishables (PII) is true, then she is forced to claim that there are not actually two objects, x and y, but rather because they are indistinguishable, there is only one.[11]

The first step in solving this problem is to dismiss PII. PII seems to apply to normal, physical objects. However, when it comes to abstract objects, or abnormal

[11]Here, we must exclude identity as part of the range of relations in PII, otherwise the problem is solved trivially. It is normally thought that PII ranged over exactly the structural relations, where identity is not thought to be a structural relation.

physical objects (e.g. photons), it is less clear that it is true. So far, so good. PII does not hold in ARS and so we can safely claim that $i \neq -i$.[12] However, we still need some sort of tool for referring to these objects. This is the reference problem associated with the OIP. If the two square roots of -1 are indistinguishable, how do mathematicians go about referring to them? Shapiro has an answer to this question. In Shapiro (2008), the solution is to make use of parameters. Shapiro suggests that when we are speaking of indistinguishable objects, it suffices to introduce a parameter into the discourse which refers to one of them, though we may not know which one, and opt to speak about that. This move is similar to the rule for existential elimination in first order logic. As long as the parameter (which, beneficially, functions a bit like a name and a bit like a variable) has not been used in the past, when we have $\exists x \phi(x)$, we can simply say "let b be one of them, so $\phi(b)$." In the same way, the *ante rem* structuralist, when using the complex roots, can say

> in this structure [the complex numbers], there is at least one square root of -1: $\exists x(x^2 = -1)$. So they let i be one such square root, and go on from there.(Shapiro, 2008, p 300)

According to Shapiro, this explains how mathematicians are able to refer to indistinguishable objects.[13]

[12]The first solution proposed to the AHP does not require a dismissal of PII, so if this solution to the OIP is unsatisfactory, there is still a way out.

[13]A second solution, from Kouri (2010), has similar results, but uses different tools. Rather than introducing parameters, we might also make use of some sort of choice function. It is possible that mathematicians are implicitly using a choice function in their reasoning when they refer to indistinguishable objects. In effect, this solution just makes explicit the feeling of "picking one arbitrarily" that Shapiro seeks to capture with parameters. This second solution likely comes to the same thing as Shapiro's.

4.2 The ARS Solution to AHP

First, I must note one important difference between the OIP and the AHP. Structures are not indistinguishable in the same way as the places within them. There are several differences between each of the candidate structures for the natural numbers. For example, we can clearly distinguish between the complex algebra and complex field structures. The original question for Heathcote was which of these structures captured *everything* it meant to be a given complex number. As I answered above, in some sense, none of them do; we need all of them to appropriately characterize mathematics. The question now is how to refer to them.

This difference means that in certain cases it will be very clear which structure a mathematician is speaking of. For example, if he is adding or multiplying numbers in base ten, then we can be pretty certain he is referring to the base ten natural number structure. The interesting problem, I think, comes when the mathematician in question refers to a more general structure. This is a case where the mathematician does not mention or use any defining characteristic which would uniquely specify the structure to which she was referring. Suppose our mathematician decides to discuss arithmetic in general, or the complex numbers with no reference to a particular subfield. What then? Is she implicitly speaking about a natural number structure which has a particular base? Or about a complex number structure which has a particular subfield (unbeknown to her)? Or is she speaking of something else? It seems to me there are two possible answers to this question. Either there is a general natural number structure and a general complex number structure, and she is speaking of them, or we can follow Shapiro's reasoning about parameters, and use those tools here.

The first solution would require us to have a structure specified solely by the second order Peano axioms (or those axioms with an arbitrary base b), and one specified solely by the complex field axioms.[14] This seems to accord well with how structuralists discuss the issue. Then, the mathematician discussing one system in a general manner will be referring to this "generalized" structure. This solution makes sense of when mathematicians talk about frameworks generally (i.e. arithmetic without reference to a base) and more specifically (i.e. with reference to a base) since there are structures for each situation. Again, none of these structures are *the* natural number structure, but this is a feature not a bug.

The second solution, though more complicated, might be in a better position of being generalized. Suppose we follow the reasoning Shapiro ascribes to the mathematician using the two complex roots in Shapiro (2008). This mathematician essentially uses parameters to stand for the object to which she is referring. Though in the natural numbers in base x are essentially distinguishable from the natural numbers in base y, when x and y are different (even though neither of them are *the* natural numbers simpliciter), it is possible that there are indeed indistinguishable structures to be had. The same goes for the complex algebra and the complex field. Should this turn out to be the case, Shapiro's solution of interpreting mathematician's talk by parameterization will do the trick.[15] Suppose $\mathcal{A}$ and $\mathcal{B}$ are indistinguishable structures. If a mathematician were discussing one of those structures, then it would suffice to introduce a parameter into the discourse and opt to talk with that.

Ultimately, this adapted version of Heathcote's problem can be solved in two ways. First, there may be a multitude of natural number structures (at least one for each possible base, and a general one characterized by the Peano axioms alone) and referring to the natural numbers requires spelling out the context in which the utterance in such a way that we have enough details to know exactly which one the utterer is referring

[14]There is no need to reject PII to use this solution.

[15]This reasoning also applies to the choice functions in Kouri (2010).

to. Second, in a context independent manner, we use parameters to specify a referent to any utterance of "the natural numbers" or "the complex numbers." The AHP is a problem for all philosophies of mathematics (if it is a problem for any of them), and ARS can solve it.

5 Conclusion

Heathcote presented what he took to be a new identity problem for the *ante rem* structuralist. What I have shown is that the problematic structures Heathcote points to are not problematic at all, but rather required to be a faithful account of mathematical practice. This realization renders Heathcote's problem inert. Moreover, I have suggested that even if Heathcote's problem is adapted to be a problem of reference rather than ontic exhaustion, then it is a problem for all philosophers of mathematics (possibly even the mathematicians). Even in light of this, the *ante rem* structuralist already has a solution available: the very same solution to the problem Heathcote claims is parallel to his problem.[16]

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