Analytic Construction of Periodic Orbits in the Restricted Three-Body Problem

Mohammed A. Ghazy
Old Dominion University

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ANALYTIC CONSTRUCTION OF PERIODIC ORBITS IN THE
RESTRICTED THREE-BODY PROBLEM

by

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ABSTRACT

ANALYTIC CONSTRUCTION OF PERIODIC ORBITS IN THE RESTRICTED THREE-BODY PROBLEM

Mohammed A. Ghazy
Old Dominion University, 2010
Director: Dr. Brett Newman

This dissertation explores the analytical solution properties surrounding a nominal periodic orbit in two different planes, the plane of motion of the two primaries and a plane perpendicular to the line joining the two primaries, in the circular restricted three-body problem. Assuming motion can be maintained in the plane and motion of the third body is circular, Jacobi’s integral equation can be analytically integrated, yielding a closed-form expression for the period and path expressed with elliptic integral and elliptic function theory. In this case, the third body traverses a circular path with nonuniform speed. In a strict sense, the in-plane assumption cannot be maintained naturally. However, there may be cases where the assumption is approximately maintained over a finite time period. More importantly, the nominal solution can be used as the basis for an iterative analytical solution procedure for the three dimensional periodic trajectory where corrections are computable in closed-form. In addition, the in-plane assumption can be strictly enforced with the application of modulated thrust acceleration. In this case, the required thrust control inputs are found to be nonlinear functions in time. Total velocity increment, required to maintain the nominal orbit, for one complete period of motion of the third body is expressed as a function of the orbit characteristics.
To the soul of my father
ACKNOWLEDGMENTS

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\( \text{xyz} = \text{rotating coordinates} \)
\( T = \text{period of motion, kinetic energy} \)
\( K = \text{complete elliptic integral of first kind} \)
\( E = \text{complete elliptic integral of second kind} \)
\( F = \text{incomplete elliptic integral of first kind} \)
\( t = \text{dimensional time} \)
\( t' = \text{dimensionless time} \)
\( \mathbf{v} = \text{velocity vector in inertial frame} \)
\( \mathbf{v} = \text{velocity vector in rotating frame} \)
\( V = \text{forcing function of a gravitational force field} \)
\( h = \text{energy integral} \)
\( \Delta \mathbf{v} = \text{velocity increment} \)
\( \mathbf{\omega} = \text{angular velocity of rotating frame} \)
\( \mu = \text{mass parameter} \)
\( \epsilon = \text{perturbation parameter} \)
\( \mathbf{\rho}_1 = \text{position vector of third mass relative to first primary} \)
\( \mathbf{\rho}_2 = \text{position vector of third mass relative to second primary} \)
\( \xi \eta \zeta = \text{inertial relative coordinates} \)
\( \phi_x, \phi_y, \phi_z = \text{rotation angles relative to } x, y, \text{ and } z \text{ axis respectively} \)
\( \mathbf{\Phi} = \text{angular momentum vector, fundamental matrix} \)
\( \Phi_i = \text{component of angular momentum along } i^{th} \text{ axis, } i=\{X, Y, Z\} \)
\( \Phi_i = \text{components of angular momentum along } i^{th} \text{ axis, } i=\{x, y, z\} \)
\( \mu_1, \mu_2 = \) first and second mass parameter

\( x'_1, x'_2 = \) dimensionless positions of primaries relative to center of mass

\( \tau = \) dimensionless period, elliptic function temporal argument

\( U = \) specific forcing function, gravitational potential function

\( u, v, w = \) coordinates relative to equilibrium point

\( L_i = \) libration point, \( i=1,2,3,4,5 \)

\( \omega_z = \) out-of-plane motion frequency

\( \textbf{X} = \) state vector

\( \textbf{q} = \) generalized coordinates vector

\( \epsilon = \) small perturbations vector

\( \textbf{F} = \) force vector

\( \textbf{U} = \) principal fundamental matrix

\( \mathcal{U} = \) potential energy of third body

\( \varphi = \) state transition matrix

\( \eta = \) strained time

\( s = \) strained out-of-plane frequency

\( E_i = \) control input for \( i^{th} \) axis, \( i=x,y,z \)

\( u_i = \) control input for \( i^{th} \) axis, \( i=x,y,z, n, t, b \)

\( \theta = \) true anomaly

\( \phi = \) true anomaly in two-body case

\( \chi = \) Legendre polynomial parameter

\( G = \) generating function of Legendre polynomials
$P_m = m^{th}$ Legendre polynomial

Units

Unless otherwise stated the traditional dimensional units of the problem of three bodies are used throughout this research. For some particular cases and for validation of numerical results non-dimensional or normalized quantities will be used. Generally, the standard S.I. units are used.
CHAPTER 1
INTRODUCTION

1.1 Motivation and Definition

The three-body problem is one of the oldest problems in celestial mechanics. Though analytical solutions have been pursued for more than two hundred years, complete analytical solutions are few in the literature. Obtaining a closed-form periodic solution even for particular cases in the circular restricted three-body problem is of great importance. These analytic periodic solutions give deep insights into the qualitative behavior of the motion at infinite time. While the accuracy of numerical solutions is sufficient for a short period of time, for a long period of time this accuracy is questioned. For approximate analytical solutions representing periodic orbits, correction for such solutions up to the second or third term is found to be sufficient in many cases. Moreover, the approximate analytical solutions, taking advantage of their mathematical structure, are extendable in terms of certain parameters to investigate regions of existence of families of orbits having the same characteristics. Orbits in the neighborhood of approximate analytical solutions are of interest. In addition, the circular restricted three-body problem gives a better approximation than the two-body problem for motion prediction in the vicinity of star-planet systems or binary-planet systems. The motion of a spacecraft or an artificial satellite in the Earth-Moon system is a good example in which the two-body approximation is acceptable when the motion of the spacecraft is close to the Earth, but when the farthest point on the satellite's orbit is close to the Moon, the circular restricted three-body problem should be used.

Journal model for this dissertation is the Journal of Guidance, Control, and Dynamics.
This dissertation addresses the analytic construction of periodic orbits in the restricted three-body problem. Planar orbits in the plane of motion of the two primaries and in a plane perpendicular to the line joining the two primaries are emphasized. The expression "planar periodic orbits" will be used to denote the former while the latter will be known as "vertical periodic orbits" throughout the text. Periodic orbits about the larger primary established within the three-body problem context will provide more realistic behavior of a spacecraft than approximating the effect of the second primary as a third body effect using the two-body problem. Establishing these orbits will also enhance the theory of existence of additional local integrals in the three-body problem around the primaries, and further explain the integral of motion in a more quantitative analytical approach, useful in practical applications. Vertical orbits, if established, will be of great practical importance for permanent communication purposes between primaries. These results may also provide answers to continuity of polar orbits from the two-body to three-body problem, a problem which hasn't received much attention in the literature. The methodology employed here assumes a circular orbit in a certain plane with parametric equations containing elliptic functions instead of circular functions. Then by using the Jacobi integral equation, an approximate solution for coordinates and period is obtained. This solution will be used as the nominal solution in a correction process to obtain an improved approximation to the true solution.

1.2 Literature Review

The circular restricted three-body problem (CRTBP) describes the motion of an inertially negligible mass (third body), in the gravitational field of the two other massive bodies (primary masses), rotating in a circular path about their center of mass (bary-
center). Typical circular restricted three-body systems (CRTBS) include the Sun-Jupiter system in which the Sun and Jupiter are the two primaries and an asteroid or a comet is the third body, the Sun-Earth system in which the Sun and Earth are the two primaries and the Moon is the third body, and the Earth-Moon system in which the Earth and Moon are the two primaries and a spacecraft is the third body. Also, Jupiter with one of its moons can be considered the two primaries in a three-body system in which a spacecraft is the third body.

The three-body problem is not rigorously solvable. After finding ten integrals out of eighteen required to completely solve for the coordinates in time, Clairout said "let anyone integrate them who can." Bruns proved that for the problem of \( n \) bodies in which \( n > 2 \), there is no algebraic integral other than the ten integrals found by Clairout. In a more specific sense, Poincaré said that there is not even a uniform transcendental equation. The CRTBP is actually a single body problem after restricting the motion of two of the three masses, the two primaries, to a Keplerian circular orbit, and the only unsolvable equation is the vector second order differential equation of motion of the third body. Six algebraic integrals are needed to solve the three scalar differential equations to obtain rectangular coordinates in time, but the only integral these three equations admit is the Jacobi integral equation. However, some particular solutions restricted to specific regions in the dimensional space of the CRTBP are found.

A cornerstone of many of the obtained results originates from the advantages that exist when the problem is formulated in a rotating coordinate system moving with the primaries. Jacobi formulated a new function known as the Jacobi function to include the centripetal parts of the forces acting on the third body, due to transforming the motion
description from inertial to rotating coordinate systems, in addition to the gravitational potential function. The Jacobi function is also called the effective potential or the equivalent potential. With the help of the Jacobi function, the equations of motion of the third body are used so that an exact analytic integration is executed. This analytic integration defines the Jacobi integral equation and the Jacobi constant. Hill\textsuperscript{7} used the Jacobi integral equation to describe the properties of the lunar motion and showed that when the velocity of the third body vanishes, the Jacobi integral equation describes a family of equi-potential surfaces in the three dimensional space that bounds the motion of the third body.\textsuperscript{8-11} The topological structure of these surfaces is governed by the Jacobi integral equation and different surfaces are characterized by the integral value (Jacobi constant). Intersection of these surfaces with the plane of motion of the two primaries leads to a two-dimensional curve representation that graphically appeared earlier in the literature.\textsuperscript{12,13} Reference 8 introduced the first three dimensional representation of these surfaces utilizing the graphical capabilities of the early generations of electronic computers.

The earlier work done by Lagrange\textsuperscript{14} led to establishment of the five equilibrium solutions known as the libration points in the CRTBP, when the problem is formulated in a rotating coordinate system attached to the center of mass of the two primary bodies. Libration points represent positions at which the gravitational forces applied on the third body from the two primaries balance the centripetal forces coming from rotation of this body inside the three-body system.

Establishment of periodic orbits either numerically or analytically in the CRTBP has received a great deal of attention. Efforts to obtain periodic solutions are justifiable
for various reasons. Periodic orbits can give key insights into the qualitative behavior of
the motion of the third body in the CRTBP. Periodic solutions can be accurately
propagated forward to very large times (approaching the infinite limit) while the accuracy
of nonperiodic solutions propagated over a window of time that approaches infinity is
suspect. Additionally, periodic orbits have many applications in mission design. Periodic
solutions to the restricted problem hold special significance for several reasons.13

Certainly the analytical methods came first, but numerical approaches began even
before the existence of electronic computers. With the progress in computational power,
the previously generated orbits are recalculated with higher accuracy. Poincaré's3,5 work
on the qualitative differential equation theory with the CRTBP is an example of a
dynamical system that helped in exploring periodic orbits especially with the advent of
the surface of section principle (representation of a dynamical system as a transformation
of a surface into itself).13 Also as a result of Poincaré's work on the general problem of
dynamics, he found that the Hamiltonian function which describes the motion of the third
body can be expanded in terms of a small parameter (the mass parameter). Moulton15,16
was interested in the series solution for the coordinates in the synodic coordinate system.
A series solution of the coordinates should converge when the periodicity requirements
are satisfied. Whittaker17,18 considered the case of a particle attracted by some centers of
force and used the principle of least action to establish a criterion of periodic orbits. Most
of the work done in the analytical development of periodic orbits started from
analytically continuing the periodic orbits from the case of zero mass parameter to the
case of small mass parameter.
Birkhoff\textsuperscript{19} introduced the necessary conditions required to continue periodic orbits from the case of zero mass parameter (the two-body problem) to a small mass parameter. Implicit function theory is essentially used to express the relation between initial conditions at certain values of the mass parameter.\textsuperscript{13,16} For periodic orbits in the plane of motion of the two primaries, the third body should cross the line joining the two primaries perpendicularly to satisfy the symmetry of the equation of motion relative to this line. The initial velocity can then be expressed as a function of the initial position. Poincaré called the orbits generated in this way from two-body circular orbits the first species and those generated from two-body elliptical orbits the second species.\textsuperscript{20-24} A considerable effort in the literature was given to what are known as quasi-periodic orbits\textsuperscript{25-30} in which the solution is not completely periodic and initial and final conditions do not close. Sometimes these kinds of orbits can be used as a sufficient approximation for a finite time, but correction is necessary if it is used for more than one period. These orbits are found mainly in the vicinity of periodic orbits. Therefore, these orbits may be obtained using a series approximation such as Lindstedt's series.\textsuperscript{28} Because these orbits are considered perturbed periodic orbits, perturbation methods such as the Lindstedt-Poincaré technique are used to generate quasi-periodic orbits.\textsuperscript{27-30}

The main idea behind all the perturbation approaches is linearization about periodic orbits, so these methods are used primarily in the case of small perturbations or deviations from certain periodic orbits in which the effect of nonlinearity is not yet dominating. In that sense the expansion of the motion equation shouldn't contain any critical terms that produce small divisors, and the choice of parameters may be restricted to certain regions in dimensional space.\textsuperscript{31} Numerical corrections may be needed to
qualify approximate solutions to be used in applications needing high accuracy. The continuation of periodic orbits may be done numerically using a Predictor-Corrector Algorithm.32

A main topic associated with periodic orbits other than their continuation, approximation, and correction is stability.33-35 Stability of motion near a periodic orbit is mainly investigated using the variational equations approach; characteristic roots are usually calculated numerically along one complete period. However, in symmetrical periodic orbits calculations are done for only one half of the period. Periodic orbits are explored not only in the CRTBP but in the general planar three-body problem as well.36-38

An important class of these orbits is dedicated to studying the characteristics of motion of the Moon. The work done by Hill7,39 is considered fundamental to the theory of motion of the Moon. Though this work was dedicated to studying the motion of the Moon, Hill used the CRTBP as a basis for his work. Strömgren40 used some analysis and results by Hill and explored the existence of periodic orbits in the CRTBP. He proved that the asymptotic orbits with infinitely long period are the boundary of classes of periodic orbits. These asymptotic orbits start at one of the primaries or one of the libration points. The orbit is calculated until it crosses the line of syzygy (the line joining the two primaries) perpendicularly to be periodic and symmetric. A class of Strömgren's orbit is recalculated by Szebehely.41 The results obtained from the planar case of three bodies are generalized to the three dimensional case.

Three dimensional periodic orbits are obtained using the analytic continuation method. The initial conditions should be chosen to express symmetrical conjunction when expressed in canonical coordinates.42 Some three dimensional orbits are obtained
from bifurcating planar orbits when their amplitudes reach certain limits; most of the orbits obtained this way are related to periodic orbits at one of the collinear libration points as shown in detail later. The case of small mass parameter has received the most attention in the literature because of its direct use of continuing periodic orbits from the case of zero mass parameter. In other words, if the mass parameter is not small the problem of obtaining periodic orbits analytically is more complicated. However, as exceptions, the case of equal primary masses has a rectilinear solution, and even the case of three equal masses has a planar figure eight solution. Another motivation for using the small mass parameter case is the existence of the Earth-Moon system as a direct application. Assuming that the underlined three-body system has a small mass parameter and motion of the third body is circular about the system center of mass, which is very close to the first primary in this case, this orbit is then periodic about the first primary. Analytical correction of this approximate solution is found to be necessary before using it in any particular orbit design problem.

Great efforts in the literature have been given to generating periodic orbits obtained from infinite series expansions. Procedures typically start with a base solution and then iteratively correct this solution, often satisfying the linear dynamics, either numerically or analytically. Correction for such solutions up to the third term is found to be sufficient in many cases. Moreover, by taking advantage of the mathematical structure of the three-body problem the periodic orbits are sometimes extendable in terms of certain parameters to investigate other regions of existence of families of periodic orbits. Wintner discussed continuation of Kepler's circular orbit, the limiting case of zero mass parameter, to periodic orbits in the three-body problem. The variational motion equation
is a second order linear differential equation with periodic coefficients. For sufficiently small mass parameter, the parameter excitation in the differential equation is expressed as a function of time, and for a certain Jacobi constant this function is assumed to be analytic in the mass parameter. On the other hand, at a fixed value of the mass parameter the continued groups or families of orbits are parameterized using the Jacobi constant. Continuing orbits from the zero mass parameter case to small mass parameter cases depends on the preservation of closed orbits when the mass parameter is perturbed by a very small real value. All functions of the mass parameter, i.e., the Hamiltonian and the Jacobi integral equation, should vary smoothly and admit continuous differentiation.\textsuperscript{46} After establishing an approximate periodic orbit the following task corrects this orbit.

Farquhar\textsuperscript{47} used the solution of the linearized equation of motion about collinear equilibrium points as a base solution, and then corrected it to implement the nonlinear terms as perturbing accelerations. Other perturbing accelerations such as solar radiation pressure and gravitational force of a massive body located outside the three-body system were also included. Richardson\textsuperscript{48} proposed a generating orbit formulated in terms of circular functions similar to the solution obtained from linearized equations about a collinear equilibrium point. The generating orbit consisted of a periodic in-plane motion and an oscillating motion in the vertical direction. To have a three dimensional periodic orbit, the amplitudes of both the in-plane and out-of-plane motions were nonlinearly related while the phases of the two motions were linearly related. Ghazy and Newman\textsuperscript{49} assumed a circular orbit in a plane perpendicular to the line joining the two primaries and confirmed satisfaction of the nonlinear motion in the tangential direction only. Subsequently, their solution was corrected to implement the missing motions in the
normal and bi-normal directions. The solution originated from the Jacobi integral equation and was expressed in terms of Jacobi elliptic functions. After a lengthy but systematic correction process, relations between motion amplitudes were established in order to eliminate any secular harmonics. Ghazy and Newman\textsuperscript{24} also discussed the particular planar CRTBP case in which the mass parameter was close but not equal to zero and the motion was assumed circular about the center of mass of the first and second primaries. In the equations, the second and higher order terms of the mass parameter were neglected. Regions of validity of the approximations in the dimensional space and constraints on initial conditions were determined.

Poincaré\textsuperscript{3-5} and then Schwarzschild\textsuperscript{50} stated that in the vicinity of a periodic orbit there is another periodic orbit with possibly long periods. The benefit of this statement is that it doesn't restrict the search for periodic orbits to certain regions in the dimensional space or the phase space. Once a periodic orbit is found, even numerically, searching for periodic orbits in the vicinity of this orbit is logical. The exclusive work done by Thiele\textsuperscript{51} and the collected work of Darwin\textsuperscript{52} located and classified periodic orbits numerically. Darwin\textsuperscript{52} considered the case in which the mass of the second primary was one tenth of that of the first primary. His key point in the process of finding periodic orbits was to find the initial conditions which when used in numerically integrating the motion equation produced periodic orbits. When the third body starts moving perpendicular to the line of syzygies in two successive trials, Darwin concluded that if the second intersections were with obtuse and acute angles, respectively, then between these two orbits there is one periodic orbit which intersects the line of syzygies at a right angle for the second time. Moulton\textsuperscript{16} calculated some of these orbits in the plane of motion of the two primaries.
These orbits were not necessarily simple periodic orbits, but they may be classified as surrounding one of the primaries or one of the collinear equilibrium points. Some of these orbits were recalculated, and their accuracy was tested and approved by Szebehely.\textsuperscript{53} It is interesting to note that when these orbits were recalculated some of the values needed to be corrected. Table 1 shows the characteristics of some of Moulton's planar orbits, named \textit{a} to \textit{e}, with the values of initial conditions, Jacobi constants, and periods taken from Szebehely.\textsuperscript{53} Updated values have the subscript \textit{new}.

**Tab. 1.1 Moulton's Class of Planar Periodic Orbits**

<table>
<thead>
<tr>
<th>Orbit</th>
<th>Characteristics of Periodic Orbits</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y(0) )</td>
<td>( \dot{x}(0) )</td>
</tr>
<tr>
<td>\textit{a}</td>
<td>0.1200</td>
</tr>
<tr>
<td>\textit{b}</td>
<td>0.2030</td>
</tr>
<tr>
<td>\textit{c}</td>
<td>0.3500</td>
</tr>
<tr>
<td>\textit{d}</td>
<td>0.5554</td>
</tr>
<tr>
<td>\textit{e}</td>
<td>0.7428</td>
</tr>
</tbody>
</table>

In these orbits the third body starts moving from the \( y \) axis in the negative \( x \) axis direction, and some values of the period \( T \) and the initial velocity \( \dot{x}(0) \) need to be corrected. A measure of the accuracy of the integrated orbits is the requirement that the Jacobi constant, \( C \), maintain the same value at any time along the orbit. With the invention of the electronic computer in the 1960s, Broucke\textsuperscript{54} introduced comprehensive and organized research in calculating periodic orbits by numerical integration in the Earth-Moon system. Initial conditions, period of motion, and Jacobi constant data for each orbit were tabulated and classified into families. The CRTBP was mainly an initial value problem and the problem of finding periodic orbits became a problem of finding the initial conditions that gave periodic orbits when used in the numerical integration. The phase space in the CRTBP has an infinite number of such points.\textsuperscript{40} Figure 1 shows an author derived periodic orbit in the Earth-Moon system with a new set of initial
conditions $x(0) = -2.47780$, $y(0) = 2.90005$; coordinates and velocities are normalized using the distance between primaries and the rotation rate of the synodic (rotating) system. This orbit was obtained by a simple approach similar to that of Darwin.\textsuperscript{52} In Figure 1, the symbols $L_1$, $L_2$, $L_3$, $L_4$, and $L_5$ denote what are known as the libration (equilibrium) points.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{sample_orbit.png}
\caption{Sample Numerically Generated Planar Periodic Orbit}
\end{figure}

Complete analytical formulations of special orbits in the CRTBP are rare in the literature. Some of the analytical solutions in the CRTBP include but are not limited to the following solutions. Periodic rectilinear motion of the third body in the rotating frame, along an axis perpendicular to the plane of motion of the primaries and passing through the center of mass, when the mass parameter is taken as in the Copenhagen case (equal primaries) and the Jacobi integral is reduced to the Legendre normal form of an elliptic integral, was accomplished by MacMillan.\textsuperscript{44} This motion was later revisited in the elliptic three-body problem by Sitnikov\textsuperscript{55} and was introduced through a different
With the use of elliptic integrals, the period of motion corresponding to certain initial conditions is expressible in closed-form. Though the rectilinear motion of the third body along the vertical axis was an early trial to obtain a closed-form solution, motions in a vertical plane have not received much attention. Ghazy and Newman assumed a circular orbit in a plane perpendicular to the line joining the two primaries for any value of the mass parameter. The Jacobi integral equation was used to develop a differential equation for angular displacement variation with time which was reduced to the normal form of an elliptic integral of the first kind through which a closed-form expression for period of motion and analytic expressions for coordinates were obtainable. The assumed orbit solution was used as the base solution through an iterative correction scheme to obtain a first correction step that brought the assumed orbit closer to a true halo type orbit. For two dimensional orbits, Szebehely considered the limiting case when the mass parameter equals zero (two-body problem) and introduced the conditions required for the orbit to be periodic in both sidereal (inertial) and synodic (rotating) coordinate systems. Szebehely suggested using the zero velocity curves in the plane of motion of the two primaries as periodic orbits, mentioning the necessary conditions which the forcing function must satisfy to produce these orbits. The applicability of the criterion for identifying such orbits was restricted to certain regions in dimensional space.

Though the problem of motion of a spacecraft in the vicinity of one of the libration points of a three-body system has received great attention in the literature, investigations continue based on importance and relevance. Looking for new approximate solutions or even using the already existing ones in different applications is
the subject of many research studies up till now. Libration point solutions and their applications are considered the most important part of the CRTBP, especially relating to use of these solutions in applications in the Earth-Moon system. Five static (in the rotating system) equilibrium points where gravitational and centripetal accelerations balance were found by Lagrange.\textsuperscript{14} The libration solutions are actually special cases of the conic section solutions existing for the more general unrestricted three-body problem.\textsuperscript{13,56}

Moulton\textsuperscript{16} laid the analytical foundation for classification and solution of periodic orbits about the collinear libration points using linear analysis. Three main classes were discovered: two dimensional, horizontal orbits; one dimensional, vertical orbits; and three dimensional orbits. These orbits were extended analytically by Moulton\textsuperscript{16} and others, over many decades, to larger amplitudes. Reference 13 efficiently summarizes these efforts. At least three classes, having their origins tracable to Moulton's work, are the in-plane or Lyapunov periodic orbits, the nearly vertical or out-of-plane dominated periodic orbits, and three dimensional periodic halo orbits.\textsuperscript{58} Figure 1.2 shows Moulton's representation of nearly vertical orbits about the collinear equilibrium points.

![Figure 1.2 Moulton's Almost Vertical Orbits](image-url)
Farquhar and Kamel\textsuperscript{59} used an analytical higher order technique to study these naturally occurring but unstable halo orbits. Richardson\textsuperscript{48,60} used the linearized motion equations and their solution about collinear equilibrium points as a generating orbit to produce halo orbits through a successive analytical approximation technique applied to the full nonlinear equations of motion in which the local coordinate system origin was at one of the collinear equilibrium points. A correction of the frequency and a restriction on the amplitudes of coordinates were found to be necessary for establishing such orbits. Since 1980 this type of analytical work for the restricted problem has not been pursued.

Starting in the 1960s and continuing through the present time, numeric computation has been used to construct and investigate periodic orbits. Such differential correction techniques are the computational engine for many investigations.\textsuperscript{61} Using pure computational tools, Henon showed that halo orbits result from in-plane orbit bifurcations at critical amplitudes.\textsuperscript{62} References 63-65 follow this type of approach to identify periodic families, characterize properties, and study their relationships to one another. In 1990, Marchal addressed the more recent efforts.\textsuperscript{66} Starting in the early 1990s, researchers began to couple dynamical systems theory with numeric computation to discover new relationships and insights pertaining to restricted three-body periodic orbits.\textsuperscript{58} Manifold theory is central to this approach and relies heavily on computation using tangency concepts applied to the eigen structures of the monodromy matrix along a halo orbit,\textsuperscript{67} or Lidstedt-Poincaré type numeric constructions.\textsuperscript{68} This research has shown that in-plane periodic, out-of-plane dominated periodic, three dimensional periodic, and three dimensional quasi-periodic orbital families constitute the four dimensional center
manifold for each libration point. Reference 69 provides a semi-analytic approach for restricted problems based on statistical concepts, not necessarily for periodic solutions.

Halo orbits may be classified according to their geometrical characteristics. Being periodic the halo orbits' stability have been investigated mainly using Lyapunov characteristic numbers and Routh criteria. The unstable nature of the collinear equilibrium points necessitates the use of station keeping strategies to ensure periodicity of any of these orbits, and some control inputs must be introduced.

The station keeping strategy of a spacecraft in a libration point orbit depends mainly on two parts; the first part is the nominal path the spacecraft follows during one complete period of motion, and the second part is the control inputs or the thrust forces used to keep the spacecraft close enough to this nominal orbit. The accuracy of the nominal orbit or its closeness to a natural orbit is critical for determining the propulsion requirements in a station keeping strategy. For station keeping applications, the lack of a beginning nominal orbit and an efficient correction process significantly affects the cost in terms of extra thrust demands. Using the linearized equation solution as a nominal path necessarily requires a process of correction which is tedious and lengthy, especially when correction is extended beyond the second term of an expansion technique.

Analogous to the work done in the two-body problem by Tsien, motion in the three-body problem with the inclusion of constant thrust acceleration has been investigated. In Reference 85, the stability of collinear equilibrium points when applying continuous constant magnitude jet acceleration, directed either to one of the primaries (radial thrust) or parallel to the line joining them was investigated. Lyapunov stability analysis gave the sufficient and necessary conditions for gyroscopic stabilization.
of the two external libration points and indicated that the internal libration point could not be stabilized. In Reference 86, the stability characteristics and the number of new libration points were investigated for the case of constant radial thrust. Regions of space for existence of libration points were divided according to the magnitude of the thrust acceleration. The case of nonconstant radial low thrust was also included, in which the stability characteristics changed from the case of constant radial thrust.

Farquhar\textsuperscript{47} used the solution of the linearized equation of motion about a collinear equilibrium point as a nominal orbit. Corrections to implement nonlinear terms and other perturbing forces, such as solar radiation pressure and gravitational force of a massive body located outside the three-body system and eccentricity in the two primary orbits, were introduced using the linear superposition approach. Thrust control inputs were found to be necessary due to natural instability of the collinear points, and stability was attainable through a linear single axis control with appropriate gains. Breakwell\textsuperscript{74} used a truncated Fourier series solution to the equation of motion in the vicinity of the collinear equilibrium point, $L_2$, as a nominal orbit. Through a quadratic cost criterion involving position variation and control acceleration, three different control strategies, one for three axis control and two for single axis control, were used to negate an unstable nominal orbit.

Halo and lissajous trajectories can be used as nominal orbits in station keeping strategies in which correction maneuvers are applied at discrete time intervals.\textsuperscript{75} In some particular cases the nominal orbit is taken to be exactly one of the equilibrium solutions (libration points), and corrections to the nominal solution may not be needed, but control forces are applied to counter any perturbing forces. Lagrange multipliers are used in a
typical optimization problem to relate a cost function of the thrust magnitude to the state
equations near the libration points. An approximation is required to solve the system of
differential equations obtained from applying the necessary conditions of the
minimization process; such an approximation can neglect the Coriolis accelerations.\textsuperscript{79}
However, the influence of Coriolis terms is found to be substantial\textsuperscript{80} when a more exact
solution to the same problem is introduced. Following the same method and in a more
recent work, the nonlinear differential equations of extremals were integrated numerically
with the Lagrange multipliers calculated using the linearized equations. Then, more
accurate values were obtained by using a shooting technique.\textsuperscript{82} For the specific case of
obtaining a circular halo orbit, the station keeping requirements for the control inputs in
two different axes were found to be more costly than keeping a spacecraft exactly at the
same Lagrange point with single axis control input.\textsuperscript{81} Roithmayr\textsuperscript{87} calculated the velocity
increment required from a propulsion system to relocate a spacecraft into the Sun-Earth
$L_2$ point, balancing the perturbing gravitational force of the Moon, for every lunar period.
It was found that if the spacecraft was allowed to move control-free along the line
between the Sun and the Earth and balancing only the perpendicular motion, the
propellant expenditure was reduced to nearly half of that required to control the motion in
both directions. A circular orbit in a plane perpendicular to the line joining the two
primaries of a restricted three-body system was used as a nominal generating orbit in
Reference \textsuperscript{77}. The nominal orbit can be made to exactly satisfy the motion in all three
axes when control inputs are implemented in the equation of motion such that they negate
the instability of this orbit. Both the nominal orbit and control inputs were expressible in
explicit analytic form as functions of system parameters and initial conditions.\textsuperscript{77}
Libration points and their halo orbits are candidates for stationary behavior in long duration space missions.\textsuperscript{88} However, in other space missions, like interplanetary missions, other techniques are used. An example of a libration point mission was the third International Sun-Earth Explorer (ISEE-3) which used a halo orbit in the vicinity of the $L_1$ point in the Sun-Earth system to study the solar wind.\textsuperscript{89} Other missions followed relying on the advantages of libration points and their associated orbits. Such advantages are the fixed location relative to the primary masses which enable long term studies of their weather while enabling constant communication between them.\textsuperscript{47} For a communication satellite set at any of the outer collinear libration points in the plane of motion of the two primaries, an offset is needed to allow permanent links from the two primaries to this satellite.\textsuperscript{90} Other advantages of libration point solutions are the analytical and numerical advances that have occurred which for example allow the use of halo orbits in libration point communication satellites to continuously communicate between primaries.\textsuperscript{91,92} Reference 93 is a recent, detailed, and comprehensive treatment of libration point solution systems and their orbits, especially the halo type orbits. It covers mathematical fundamentals, dynamics, and mission design near both collinear and triangular libration points. Reference 88 introduces an intensive study of the same problem dedicated to the Earth-Moon system.

For some values of the Jacobi constant, the corresponding closed surfaces surrounding the two primary masses is the space in which the motion is admissible. Existence of periodic orbits in these closed spaces has been investigated intensively. Motion in the neighborhood of one of the two primaries has also been studied.\textsuperscript{94} Usually the two-body approximation is taken as the first acceptable assumption, and the orbit can
be continued for this case taking advantage of the small mass parameter and also when the motion of the third body is close to the first primary for sufficiently long time periods. This process is actually a continuation of the two-body orbit, as mentioned before, but continued orbits do not show the three-body problem characteristics, especially the nonlinearity of the problem. In other words, the effect of the second primary on large altitude orbits is not attainable. Darwin suggested that the generating orbit should be periodic and implemented by itself with the CRTBP physical properties so that when continued it will maintain periodicity and show the CRTBP characteristics. Furthermore, the nonlinearity in the CRTBP motion equation comes from the existence of the relative distances between the third body and the two primaries in the denominators of its right hand side. In certain subsets of the domain and under certain conditions, however, these nonlinear terms can be expanded using Legendre polynomials, and terms other than the first terms in these expansions, if included in the generating solution, will show the effect of the second primary. In the case of motion very close to one of the primaries, collision of the third body with this primary may take place and a regularized version of the motion equation should be used.

Many kinds of periodic orbits that have been explored previously in connection with the invariant manifold theory are used in part or in whole in space mission design. As the dynamical system theory allowed describing the qualitative classification of regions separated by potential barriers, the manifold theory allows quantitatively described transitions between these regions. This theory also explains transition of comets between the Sun and Jupiter and establishes the tubular transmissions between exterior and interior regions through a transition region bounded by surfaces of zero
velocity in a three-body system. Stable and unstable manifolds are used in chain like orbits to achieve missions within or outside the three-body system. The three dimensional version of tubular dynamics are thought of as the super highways in the planetary mission design space in which libration orbits play the role of transition orbits between unstable-stable or stable-unstable manifolds. Periodic orbits about the primaries are used as parking orbits.

Perturbing forces in the CRTBP such as attraction of any massive body outside the system and the irregular shape of any of the primaries may affect the periodic orbits around primaries, libration points and their locations, and orbits related to them. The effect of oblateness of the Earth on motion of an artificial satellite has been intensively investigated in the literature.\textsuperscript{98,99} The oblateness of the two primaries can be included as a perturbation in the potential function which affects the range of stability of the triangular equilibrium points while the collinear points remain unstable.\textsuperscript{100} The effect of the oblateness of the bigger primary alone on the locations of the libration points and their characteristic equations has been studied.\textsuperscript{101,102} Required conditions for periodic orbits about primary masses have been extended to the case of oblate primaries; both the first kind (using Keplerian circular orbits as generating orbits) and the second kind (using Keplerian elliptical orbits as generating orbits) and have been proven to exist.\textsuperscript{103} The influence of the oblateness of the larger primary on Hill's curves has been investigated; when the second primary was treated as a perfect sphere, the new triangular equilibrium points were shown to be shifted toward the spherical primary.\textsuperscript{104}

1.3 Research Contributions

To the author's knowledge this research introduces a novel idea to analytically
establish periodic orbits in the three-body problem. The circular motion of the third body in two different planes is emphasized. In the \( xy \) plane, new analytic expressions for the period of motion and coordinates as functions of time are introduced in the case of small mass parameter. Correcting this nominal orbit to include the out-of-plane motion showed that a) the out-of-plane motion is decoupled from the in-plane motion, and b) out-of-plane stability depends on perturbations produced by initial conditions. When the supposed circular motion is applied to a vertical plane the objective is to investigate methodology whereby approximate but pure analytical relationships between high inclination orbit characteristics and fundamental three-body system parameters can be developed. Richardson's work in constructing periodic orbits may be the closest work to this part of the research. A major difference between this work and Richardson's is that a nonlinear generating orbit is used. Several advantages may exist with this generating solution. From the outset, this analysis incorporates aspects of the three-body problem that are not present in Richardson's approach until higher order terms are addressed. Next, the analysis should hold for larger radius orbits located farther from the libration points. Finally, insights afforded by analysis of the generating orbit properties are unique and not present in the Richardson work. Iterating analytic corrections can lead the nominal solution to be closer to one of the halo type orbits. Station keeping methodology to strictly maintain the nominal orbit is introduced. Specific thrust formulas required during each period and an indirect minimization approach are also introduced.

1.4 Dissertation Outline

Chapter 2 contains all the basic theories needed in subsequent chapters. This chapter furnishes the theoretical and mathematical foundations for the three-body
problem. Starting from the $n$-body problem the circular restricted three-body problem is obtained. Lagrangian and Hamiltonian functions are written, equations of motion are derived, and dynamical behavior at the equilibrium points is discussed.

Chapters 3 through 7 represent the main body of the dissertation. In Chapter 3, a generating circular orbit in the plane of motion of the two primaries is derived. Motion constraints, imposed by relations between initial conditions and motion characteristic parameters, are established. In Chapter 4, the correction for the generating orbit introduced in Chapter 3 is carried out. The first step is the correction for the out-of-plane motion and the second step is the correction for the in-plane motion. For both steps the variational equations are used to solve for the correction terms and to investigate stability around the generating orbit using Floquet theory. In Chapter 5, a nominal circular orbit in a plane of motion perpendicular to the line joining the two primaries is derived. As before, motion constraints, imposed by the initial condition and described by relations with motion characteristic parameters are established. In Chapter 6, correction for the nominal orbit introduced in Chapter 5 is carried out. Iterative analytical procedures are used to solve for the corrections. An example of a true halo orbit is used to compare with the corrected solution. In Chapter 7, the nominal orbit introduced in Chapter 5 is assumed to be maintained over a long time by thrust forces applied to exactly negate the deviations that appear as a result of the natural instability of the nominal solution. Expressions for thrust forces with time, and the total velocity increment required for one complete period are introduced.

Finally, in Chapter 8 conclusions from this work and recommendations for future work are drawn.
CHAPTER 2
FUNDAMENTAL THEORIES

2.1 Introduction

In this chapter the basic concepts, definitions, and equations are reviewed starting from the n-body problem which represents the superset of the three-body problem to the most celebrated circular restricted three-body problem. The reason for starting with the n-body problem, other than being a natural sequence, is to illustrate important theories that are less apparent in the restricted problem, i.e., integrals, solvability, and singularity. Solutions for special cases in both the general and the restricted three-body problem are introduced. The equilibrium point solutions are covered in a detailed discussion since their theory is used in the subsequent chapters. Though most of the sections in this chapter can be found in many celestial mechanics texts or three-body problem publications, they are reviewed using different and simpler mathematics. Finally, sections of this chapter are emphasized according to their need in the other chapters.

2.2 N-Body Problem

The n-body problem addresses the motion of n point masses in three dimensional space under mutual gravitational attractions. Let \( m_i, i = 1,2,\ldots,n \) represent the masses of n bodies. The coordinates of these masses in an inertial coordinate system are \( X_i, Y_i, Z_i, \) \( i = 1,2,\ldots,n \). Equations of motion for these n masses are

\[
\frac{d^2 X_i}{dt^2} = -Gm_i \sum_{j=1 \atop j \neq i}^{n} m_j \frac{X_i - X_j}{r_{ij}^3} \tag{2.1a}
\]

\[
\frac{d^2 Y_i}{dt^2} = -Gm_i \sum_{j=1 \atop j \neq i}^{n} m_j \frac{Y_i - Y_j}{r_{ij}^3} \tag{2.1b}
\]
\[ m_i \frac{d^2 Z_i}{dt^2} = -Gm_i \sum_{j=1 \atop j \neq i}^n \frac{Z_i - Z_j}{r_{ij}^3} \]  

(2.1c)

where \( i, j = 1, 2, \ldots, n \), \( i \neq j \), \( G \) is the gravitational constant, and \( r_{ij} \) are the relative distances between masses defined as

\[ r_{ij} = \left\{ (X_i - X_j)^2 + (Y_i - Y_j)^2 + (Z_i - Z_j)^2 \right\}^{1/2} \]  

(2.2)

The forcing function which represents the gravitational force field of this system is

\[ V = G \sum_{i=1}^n \sum_{j=i+1}^n \frac{m_i m_j}{r_{ij}} \]  

(2.3)

The importance of the forcing function is that components of the gravitational forces applied on a point mass from the other \( n-1 \) masses are simply the gradients of the forcing function of the system, and the gravitational potential of the system is \(-V\). Equation (2.1) is now reformulated with the right hand side replaced with the partial derivatives of \( V \) with respect to coordinates \( X_i, Y_i, Z_i \) respectively to be

\[ m_i \frac{d^2 X_i}{dt^2} = \frac{\partial V}{\partial X_i} \]  

(2.4a)

\[ m_i \frac{d^2 Y_i}{dt^2} = \frac{\partial V}{\partial Y_i} \]  

(2.4b)

\[ m_i \frac{d^2 Z_i}{dt^2} = \frac{\partial V}{\partial Z_i} \]  

(2.4c)

The fact that the forcing function contains only the relative distances between point masses makes it invariant under coordinate transformation.

The function \( V \) does not contain time \( t \) explicitly, i.e., \( \partial V / \partial t = 0 \); thus, the total derivative of \( V \) with respect to time is
\[
\frac{dV}{dt} = \sum_{i=1}^{n} \left( \frac{\partial V}{\partial X_i} \frac{dX_i}{dt} + \frac{\partial V}{\partial Y_i} \frac{dY_i}{dt} + \frac{\partial V}{\partial Z_i} \frac{dZ_i}{dt} \right)
\]  

(2.5)

Consequently, if one multiplies Equations (2.4a), (2.4b), and (2.4c) by \(dX_i/dt\), \(dY_i/dt\), and \(dZ_i/dt\), respectively, then combines the three modified equations into one equation and sums the two sides of that equation over \(i\), the right hand side is a total time derivative. After integrating both sides, one obtains the energy integral \(h\) as a constant of the integration process

\[
\frac{1}{2} \sum_{i=1}^{n} m_i v_i^2 = V + h
\]  

(2.6)

where

\[
v_i = \sqrt{\left( \frac{dX_i}{dt} \right)^2 + \left( \frac{dY_i}{dt} \right)^2 + \left( \frac{dZ_i}{dt} \right)^2}
\]  

(2.7)

is the magnitude of the velocity vector of point mass \(i\). The left hand side of Equation (2.6) is nothing but the kinetic energy \(T\) of the system. Hence, Equation (2.6) can be rewritten as

\[T - V = h\]  

(2.8)

The left hand side of Equation (2.8) is the total energy of the system. Thus, Equation (2.8) states that the total energy of the system of \(n\) bodies, moving only under their gravitational forces, is constant and the system is said to be conservative.

2.2.1 Integrals of the N-Body Problem

Besides the energy integral, two other integrals can be explored via coordinate transformations. For a transformation corresponding to translating the center of coordinates to the center of mass, whose coordinates are \(X_c, Y_c, Z_c\), coordinates of the \(i^{th}\)
The relative distances between point masses remain invariant after transformation

\[ r_{ij} = \left\{ (\xi_i - \xi_j)^2 + (\eta_i - \eta_j)^2 + (\zeta_i - \zeta_j)^2 \right\}^{1/2} \]  

(2.12)

and since the forcing function depends only on the relative distances, it is also invariant after coordinate translation. From Equation (2.9)
\[
\frac{\partial \xi}{\partial X_c} = \frac{\partial \eta}{\partial Y_c} = \frac{\partial \zeta}{\partial Z_c} = -1
\]  

(2.13)

and since the forcing function does not depend on mass center coordinates one obtains

\[
\frac{\partial V}{\partial X_c} = \sum_{i=1}^{n} \frac{\partial V_i}{\partial \xi_i} = 0 \Rightarrow \sum_{i=1}^{n} \frac{\partial V_i}{\partial \xi_i} = 0
\]  

(2.14a)

\[
\frac{\partial V}{\partial Y_c} = \sum_{i=1}^{n} \frac{\partial V_i}{\partial \eta_i} = 0 \Rightarrow \sum_{i=1}^{n} \frac{\partial V_i}{\partial \eta_i} = 0
\]  

(2.14b)

\[
\frac{\partial V}{\partial Z_c} = \sum_{i=1}^{n} \frac{\partial V_i}{\partial \zeta_i} = 0 \Rightarrow \sum_{i=1}^{n} \frac{\partial V_i}{\partial \zeta_i} = 0
\]  

(2.14c)

Substituting Equations (2.9, 2.11, 2.14) into Equation (2.4) and integrating twice with respect to time, one obtains the coordinates of the center of mass as functions of time:

\[
X_c = A_1 t + A_2
\]  

(2.15a)

\[
Y_c = A_3 t + A_4
\]  

(2.15b)

\[
Z_c = A_5 t + A_6
\]  

(2.15c)

Equation (2.15) indicates that the motion of the center of mass is rectilinear with time.

Constants \( A_i, \ i = 1, 2, \ldots, 6 \) can be eliminated if the origin of the coordinate system is transferred to the center of mass.

For a transformation corresponding to a rotation of coordinates around the \( z \) axis by an angle \( \phi \), one finds

\[
r_i^{xyz} = Ar_i^{xyz}
\]  

(2.16)
where \( \mathbf{r}_i \) is the algebraic position vector of the \( i^{th} \) mass, and superscripts \( xyz \) and \( XYZ \) denote the new and initial coordinate systems respectively. Substituting Equations (2.16, 2.17) into Equations (2.2, 2.3) one finds that the relative distances between masses are invariants under the rotation of coordinates. Also, the forcing function is invariant under rotation of coordinates:

\[
\frac{\partial V}{\partial \phi_x} = \sum_{i=1}^{n} \left( \frac{\partial V}{\partial X_i} \frac{\partial X_i}{\partial \phi_x} + \frac{\partial V}{\partial Y_i} \frac{\partial Y_i}{\partial \phi_x} + \frac{\partial V}{\partial Z_i} \frac{\partial Z_i}{\partial \phi_x} \right) = 0
\] (2.18)

but from Equation (2.16)

\[
\frac{\partial X_i}{\partial \phi_x} = -\sin(\phi_x)x_i - \cos(\phi_x)y_i = -Y_i
\] (2.19a)

\[
\frac{\partial Y_i}{\partial \phi_x} = \cos(\phi_x)x_i + \sin(\phi_x)y_i = -X_i
\] (2.19b)

\[
\frac{\partial Z_i}{\partial \phi_x} = 0
\] (2.19c)

Substituting Equations (2.19) into Equation (2.18) one obtains

\[
\frac{\partial V}{\partial \phi_x} = \sum_{i=1}^{n} \left( -Y_i \frac{\partial V}{\partial X_i} + X_i \frac{\partial V}{\partial Y_i} \right) = 0
\] (2.20)

Multiplying Equation (2.4a) by \( -Y_i \) and Equation (2.4b) by \( X_i \) and collecting, then applying a summation from \( i=1 \) to \( i=n \), one obtains the following

\[
\sum_{i=1}^{n} m_i (X_i \ddot{y}_i - Y_i \ddot{x}_i) = \sum_{i=1}^{n} \left( X_i \frac{\partial V}{\partial y_i} - Y_i \frac{\partial V}{\partial x_i} \right)
\] (2.21)

Using Equation (2.20) and the fact that
\[
\frac{d}{dt} \left( X_i \dot{Y}_i - Y_i \dot{X}_i \right) = X_i \ddot{Y}_i - Y_i \ddot{X}_i
\] (2.22)

then substituting Equation (2.22) into Equation (2.21) one obtains

\[
\frac{d}{dt} \sum_{i=1}^{n} m_i \left( X_i \dot{Y}_i - Y_i \dot{X}_i \right) = 0
\] (2.23)

Equation (2.23) indicates that the angular momentum of the system in the \( Z \) direction is constant

\[
\sum_{i=1}^{n} m_i \left( X_i \dot{Y}_i - Y_i \dot{X}_i \right) = \Phi_Z
\] (2.24a)

where \( \Phi_Z \) is the component of angular momentum in the \( z \) direction. Similarly, and without further details, if transformations corresponding to rotations about the \( X \) and \( Y \) axes by angles \( \phi_x \) and \( \phi_y \), respectively, the forcing function is found to be independent of the angles of rotation. Using the same analysis as in the development of Equation (2.18) to Equation (2.24a), one finds that the angular momentums \( \Phi_x, \Phi_y \) of the system in the \( X \) and \( Y \) directions are also constant. This result is indicated as follows.

\[
\sum_{i=1}^{n} m_i \left( Y_i \dot{Z}_i - Z_i \dot{Y}_i \right) = \Phi_x
\] (2.24b)

\[
\sum_{i=1}^{n} m_i \left( Z_i \dot{X}_i - X_i \dot{Z}_i \right) = \Phi_y
\] (2.24c)

From Equations (2.24a) to (2.24c), one concludes that the total angular momentum vector \( \Phi_{xyz} \) of the system is constant assuming no external forces are applied on the point mass system, in which masses move under their mutual gravitational forces. The magnitude of the total momentum vector is

\[
\Phi = \left( \Phi_x^2 + \Phi_y^2 + \Phi_z^2 \right)^{1/2}
\] (2.25)
It is interesting to express the angular momentum in terms of the rotated coordinates. Taking the simple transformation expression for rotation about the Z axis with angle $\phi_z$, the components of angular momentum in the $x$, $y$, and $z$ axes are

$$
\Phi_x = \sum_{i=1}^{n} m_i \left( y_i \dot{z}_i - z_i \dot{y}_i \right) 
$$

$$
\Phi_y = \sum_{i=1}^{n} m_i \left( z_i \dot{x}_i - x_i \dot{z}_i \right) 
$$

$$
\Phi_z = \sum_{i=1}^{n} m_i \left( x_i \dot{y}_i - y_i \dot{x}_i \right) 
$$

Using the transformation in Equation (2.16) and differentiating with respect to time to get the time derivatives of the point mass coordinates, then substituting into Equation (2.26), one finds that except for the component along the axis about which rotation takes place, i.e., the Z axis in this case, the components of the angular momentum along the rotated coordinates are different than those along the initial coordinates. Nevertheless, total angular momentum of the system is constant, and its value is the same as that in Equation (2.25):

$$
\Phi = \left( \Phi_x^2 + \Phi_y^2 + \Phi_z^2 \right)^{1/2} = \left( \Phi_x^2 + \Phi_y^2 + \Phi_z^2 \right)^{1/2}
$$

Upon the invariance of the total angular momentum of the system under finite reorientation of coordinates, the angles of rotation are chosen so that the total angular momentum can have only one component, along a certain axis and vanishes along the other two axes. The plane perpendicular to this certain axis is the invariable plane. In a special case, for the n-body problem, the motion of the system of point masses is a planar motion in the invariable plane. If one tries to determine the invariable plane by one rotation only, for example the rotation about the $Z$ axis by angle $\phi_z$, from this
transformation one finds

\[ \Phi_z = \sum_{i=1}^{n} m_i (y_i \dot{z}_i - z_i \dot{y}_i) = \Phi_y \sin(\phi_z) + \Phi_x \cos(\phi_z) \]  
(2.28a)

\[ \Phi_y = \sum_{i=1}^{n} m_i (z_i \dot{x}_i - x_i \dot{z}_i) = \Phi_y \cos(\phi_z) - \Phi_x \sin(\phi_z) \]  
(2.28b)

\[ \Phi_z = \sum_{i=1}^{n} m_i (x_i \dot{y}_i - y_i \dot{x}_i) = \Phi_z \]  
(2.28c)

There are three possible solutions for Equation (2.28) depending on the initial coordinate system configuration. In Case 1, if the z axis is chosen to be perpendicular to the invariable plane, then

\[ \Phi_z^2 = \Phi_x^2 + \Phi_y^2 \]  
(2.29a)

\[ \tan(\phi_z) = -\frac{\Phi_x}{\Phi_y} = \frac{\Phi_y}{\Phi_x} \]  
(2.29b)

Equation (2.29) is satisfied if and only if \( \Phi_x = \Phi_y = 0 \), which means that the initial coordinate system is primarily chosen so that the Z axis is perpendicular to the invariable plane. The angle of rotation is undefined and any arbitrary value can be chosen. In Case 2, the x axis is chosen to be perpendicular to the invariable plane, then

\[ \Phi_x^2 = \Phi_x^2 + \Phi_y^2 \]  
(2.30a)

\[ \Phi_x = \Phi_x = 0 \]  
(2.30b)

\[ \Phi_y = 0 \Rightarrow \tan(\phi_{xz}) = \frac{\Phi_y}{\Phi_x} \]  
(2.30c)

In Case 3, the y axis is chosen to be perpendicular to the invariable plane, then

\[ \Phi_y^2 = \Phi_x^2 + \Phi_y^2 \]  
(2.31a)

\[ \Phi_y = \Phi_y = 0 \]  
(2.31b)
\[
\Phi_s = 0 \Rightarrow \tan(\phi_{z_1}) = -\frac{\Phi_x}{\Phi_y} \tag{2.31c}
\]

Equations (2.30) and (2.31) indicate that in order to get the invariable plane from only one rotation, one should have an inertial axis in the invariable plane and then use the rotation either by angle \( \phi_{z_2} \) or by angle \( \phi_{z_3} \). The two angles are not two independent solutions since from Equations (2.30c) and (2.31c) one notices that
\[
\tan(\phi_{z_2})\tan(\phi_{z_3}) = -1 \tag{2.32}
\]
and from the law of tangent of a difference between two angles, one obtains
\[
\phi_{z_3} = \phi_{z_2} + \frac{\pi}{2} \tag{2.33}
\]

In general, if the initial coordinate system has no axes that lie in the invariable plane, one needs two transformations corresponding to two rotations about two different axes. The first transformation is required to transform one of the axes into the invariable plane. Then the second step is a rotation about this axis so that one additional coordinate lies also in the invariable plane. This step is identical to the case discussed in the previous paragraph [Equations (2.28) to (2.33)]. Assuming an initial coordinate system, \( XYZ \), generally oriented in space, if there is a rotation about the \( Z \) axis by angle \( \phi_z \), Equation (2.28) is still used to explore different possibilities of having one of the new axes, \( x_1, y_1, z_1 \), in the invariable plane. Since the initial axes are chosen generally it is impossible to have the \( z_1 \) axis in the invariable plane. There are only two possibilities: either the \( x_1 \) axis is in the invariable plane or the \( y_1 \) axis is in the invariable plane. The angles of rotation are as shown in Equations (2.31c) and (2.30c), and they are related as indicated in Equation (2.33). The second rotation takes place about the axis that was chosen to lie in the invariable plane in the first rotation. If \( x_1 \) in the first rotation is chosen...
to lie in the invariable plane, then the transformation in the second step corresponds to a
rotation about \( x_1 \) by angle \( \phi_\xi \). In the new coordinate system \( x_2, y_2, z_2 \), the \( x_2 \) axis lies in
the invariable plane, so there are two possibilities for having another axis in the
invariable plane. If the \( y_2 \) axis is chosen to lie in the invariable plane, one obtains

\[
\tan(\phi_{x_1}) = -\sqrt{\Phi^2_X + \Phi^2_Y} \Phi^2_z
\]

(2.34a)

\[
\Phi^2_{x_2} = \Phi^2_X + \Phi^2_Y + \Phi^2_Z
\]

(2.34b)

and the \( x_2y_2 \) plane is the invariable plane. If the \( z_2 \) axis is chosen to lie in the invariable
plane, one obtains

\[
\tan(\phi_{x_2}) = \frac{\Phi_Z}{\sqrt{\Phi^2_X + \Phi^2_Y}}
\]

(2.35a)

\[
\Phi^2_{y_2} = \Phi^2_X + \Phi^2_Y + \Phi^2_Z
\]

(2.35b)

and the \( x_2z_2 \) plane is the invariable plane. One notices from Equations (2.34a) and
(2.35a) that the two angles, \( \phi_{x_1} \) and \( \phi_{x_2} \) are related as follows

\[
\tan(\phi_{x_1}) \tan(\phi_{x_2}) = -1 \Rightarrow \phi_{x_1} = \phi_{x_2} + \frac{\pi}{2}
\]

(2.36)

In summary, for the general coordinate system orientation there are four
possibilities (four angles of rotation) required to obtain a coordinate system in which one
of the axes is perpendicular to the invariable plane. However, these four possible angles
represent two independent successive rotations. Knowing components of the angular
momentum in the initial coordinate system; angles of rotations are completely
determinable using Equations (2.30c) and (2.34a).
2.2.2 Insolvability of the N-Body Problem

The equations of motion of n bodies can be reduced to 6n first order differential equations. The state vector, including coordinates and velocities, has 6n unknowns. Unfortunately, there are only ten integrals: six integrals describing the uniform motion of the center of mass, three integrals representing conservation of angular momentum, and one integral representing conservation of energy. Thus, a set of ten algebraic or integral equations exist for 6n unknowns indicating a lack of analytic solvability. However, solutions do exist for some particular configurations with simplifying assumptions as in Lagrange's solution of the three-body problem and the detailed and extensive work in the two-body problem. In most cases numerical solutions are possible when the motion is away from singular points, i.e., given the coordinates and velocities of 6n point masses at a certain initial time, values of these variables are calculated at another time.

2.2.3 Motion in a Rotating Coordinate System

It is sometimes useful to express motion of a system of point masses in a rotating coordinate system; some hidden properties of this dynamical system are then uncovered. Since the direct applications of the n-body problem are motion of celestial bodies in the solar system, it is more realistic to use a rotating coordinate system with a reference attached either to one of these bodies or a center of mass of some body. Knowing the rate of rotation of some such rotating coordinate system is another advantage. Properties of different modes of motion under assumptions regarding coordinates and velocities along the axes of the rotating system are investigated. As a matter of fact, most of the particular cases of solutions to the n-body problem are introduced in rotating coordinate systems.
Define a general transformation matrix $A$ and consider the initial coordinate system as an inertial one. Position $r$, velocity $v$, and acceleration $a$ are transformed from an inertial $XYZ$ system to a rotating $xyz$ coordinate system, whose rotation rate vector is $\omega$, as follows:\textsuperscript{56}

$$A r_{i}^{XYZ} = r_{i}^{xyz} \quad (2.37a)$$

$$A v_{i}^{XYZ} = v_{i}^{xyz} = \frac{dr_{i}^{xyz}}{dt} + \omega \times r_{i}^{xyz} \quad (2.37b)$$

$$A a_{i}^{XYZ} = a_{i}^{xyz} = \frac{d^{2}r_{i}^{xyz}}{dt^{2}} + \frac{d\omega}{dt} \times r_{i}^{xyz} + 2\omega \times \frac{dr_{i}^{xyz}}{dt} + \omega \times \omega \times r_{i}^{xyz} \quad (2.37c)$$

Reformulating Equation (2.1) into a vector form yields

$$a_{i}^{XYZ} = -G \sum_{j=1}^{n} m_{j} \frac{r_{j}^{xyz} - r_{i}^{xyz}}{r_{ij}^{3}} \quad (2.38)$$

From now on, variables without superscripts refer to variables used to describe motion in the rotating coordinate system. Multiplying Equation (2.38) by $A$ and substituting into Equation (2.37c), a vector form of the equation of motion in the rotating coordinate system is obtained:

$$\frac{d^{2}r_{i}}{dt^{2}} + \frac{d\omega}{dt} \times r_{i} + 2\omega \times \frac{dr_{i}}{dt} + \omega \times \omega \times r_{i} = -G \sum_{j=1}^{n} m_{j} \frac{r_{j} - r_{i}}{r_{ij}^{3}} \quad (2.39)$$

where

$$r_{ij} = \left\{(x_{i} - x_{j})^{2} + (y_{i} - y_{j})^{2} + (z_{i} - z_{j})^{2}\right\}^{1/2} \quad (2.40)$$

### 2.3 Three-Body Problem

The three-body problem is a subset of the n-body problem in which a set of three point masses move under their mutual gravitational forces. Many applications exist for such a model. For example, planetary mission trajectories built on the two-body problem assume that the maneuver or swingby of a spacecraft or an asteroid by a planet is
instantaneous, which is far from the real situation. In that sense, the three-body problem serves as a better approximation or more accurate first solution, yet it is mathematically more complicated. Though the three-body problem and its reduced case (the restricted problem) are well stated and defined, there is no complete general analytic solution.\textsuperscript{1-5}

The equations of motion derived in Section 2.2 for the special case of \( n=3 \) will be used in this section.

Lagrange was interested in solutions in which the three-body system configuration maintained its geometry with time. This configuration means that the resultant force applied on any point mass passes through the center of mass. For this to happen, the first assumption is a constant angular rate of rotation of the coordinate system, and the second assumption is that the position vectors of the point masses are coplanar and remain unchanged inside the rotating system.\textsuperscript{*} Equation (2.39) is now rewritten as follows:

\[
-\omega^2 \mathbf{r}_i = -G \sum_{j=1, j \neq i}^{3} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{r_j^3}
\]  

(2.41)

2.3.1 Lagrange's Equilateral Triangle Solution

When the three point masses are located at the vertices of an equilateral triangle (Equilateral Triangle Solution) and the relative distance of any two masses is \( r_{ij} = \rho \), \( i, j = 1, 2, 3, \ i \neq j \), Equation (2.41) is rearranged as follows:

\textsuperscript{*}Recalling the coordinate system configuration and the invariable plane definition from Section 2.2.1, this means that the motion of the three bodies will be in the invariable plane which is perpendicular to the angular momentum vector.
\[ \left\{ \frac{\omega^2}{G} \frac{1}{\rho^3} \sum_{j \neq i}^n m_j \right\} \mathbf{r}_i = -\frac{1}{\rho^3} \sum_{j \neq i}^n m_j \mathbf{r}_j \]  

(2.42)

But from the property of the center of mass

\[ \sum_{j=1}^n m_j \mathbf{r}_j = \sum_{j \neq i}^n m_j \mathbf{r}_j + m_i \mathbf{r}_i = 0 \]  

(2.43)

and substituting Equation (2.43) into Equation (2.42), one can find that

\[ \left\{ \frac{\omega^2}{G} \frac{1}{\rho^3} \sum_{j \neq i}^n m_j \right\} \mathbf{r}_i = 0 \]  

(2.44)

Equation (2.44) indicates that in this case the three vector equations in Equation (2.41) reduce to one identical vector equation. For nontrivial solutions, the coefficient of \( \mathbf{r}_i \) vanishes, and the angular velocity of the system is expressed as follows

\[ \omega^2 = \frac{G(m_1 + m_2 + m_3)}{\rho^3} \]  

(2.45)

### 2.3.2 Lagrange's Straight Line Solution

If the three point masses are located on the same line (Straight Line Solutions), then their position vectors have the same direction and Equation (2.41) is reduced to three scalar equations in the relative distances of the three masses and the angular velocity of the system. If masses are arranged by their subscripts (i.e., 1,2,3) and a new variable \( \chi = r_{23} / r_{12} \) is defined, one obtains the following fifth order polynomial

\[ (m_1 + m_2) \chi^5 + (3m_1 + 2m_2) \chi^4 + (3m_1 + m_2) \chi^3 \]
\[-(m_2 + 3m_3) \chi^2 - (2m_2 + 3m_3) \chi - (m_2 + m_3) \chi = 0 \]  

(2.46)

called the Lagrange Quintic Equation\(^{56,106}\) which has only one positive root. Once \( \chi \) is determined from Equation (2.46), the angular velocity of the system is calculated as
follows:

\[ \omega^2 = \frac{G(m_1 + m_2 + m_3)}{r_{12}^3(1 + \chi)^2} + \frac{m_2(1 + \chi)^2 + m_3}{m_2 + (1 + \chi)m_3} \]  

(2.47)

Since there are three different arrangements of the three masses on the straight line, there are three different solutions of Equations (2.46) and (2.47).

2.3.3 Lagrange's Conic Section Solution

In a more general case in which the angular velocity of the rotating coordinate system and position vectors of point masses in the rotating system are not constant, the only assumption is that position vectors are proportional to their initial values and each other, or

\[ \mathbf{r}_i(t) = \rho(t)\mathbf{r}_i(t_0) \]

(2.48)

Substituting Equation (2.48) into Equation (2.39), one obtains

\[ \left\{ \left( \dot{\mathbf{r}} - \rho\omega^2 \right)\mathbf{I} + \left( 2\omega\dot{\mathbf{r}} + \rho\dot{\omega} \right)\mathbf{J} \right\}\mathbf{r}_i(t_0) = \frac{1}{\rho^2} \left\{ -G \sum_{j \neq i} m_j \frac{\mathbf{r}_j(t_0) - \mathbf{r}_i(t_0)}{r_{ij}^3(t_0)} \right\} \]

(2.49)

where dots above a variable represent differentiation with respect to time. \( \mathbf{I} \) and \( \mathbf{J} \) are 2×2 matrices such that

\[ \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{J} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \]

(2.50)

The term on the right hand side of Equation (2.49) is the net specific force acting on the point mass number \( i \) at initial time \( t_0 \). If this force is assumed to be proportional to the initial position vector of this mass, with proportionality constant \( -k^2 \), Equation (2.49) is reduced to two differential equations

\[ \mathbf{r}_i^2 \omega = \Phi_i \]  

(2.51a)
\[
\dot{\rho} - \rho \omega^2 = \frac{-k^2}{\rho^2}
\]  

(2.51b)

where \( \Phi_1 \) is a constant and Equation (2.51) represents motion as in a two-body problem expressed in polar coordinates in which \( \omega = d\theta / dt \) and \( \theta \) is the angular displacement of the rotating system.

2.4 Circular Restricted Three-Body Problem

The circular restricted three-body problem (CRTBP) addresses the motion of a negligible mass, \( m_3 \), under the gravitational forces of two other huge masses \( m_1 \) and \( m_2 \) (\( m_2 \leq m_1 \)). The motion of the two primary masses is confined to the \( xy \) plane (which will be considered the invariable plane), and they are assumed to move in circular orbits about their common center of mass, while the third mass moves anywhere in the \( xyz \) coordinate system. The \( xyz \) coordinate system rotates with the primary masses so that the \( x \) axis is aligned to the line joining the two primaries with its positive direction toward the first (largest) primary, the \( z \) axis is in the direction of the angular velocity vector of the system, and the \( y \) axis completes the orthogonal right hand system. In Figure 2.1 position vector \( \mathbf{r} \) is the position vector of the third body relative to the center of mass. \( \mathbf{r}_1, \mathbf{r}_2 \) are the position vectors of the two primaries relative to the center of mass.
Figure 2.1 Geometry of Circular Restricted Three-Body Problem

\( \rho_1, \rho_2 \) are the position vectors of the third body relative to the two primaries. These vectors are expressed as follows

\[
\begin{align*}
\mathbf{r}_i &= x_i \hat{i}, \ i = 1, 2, \quad x_i > 0, x_2 < 0 \\
\mathbf{r} &= x \hat{i} + y \hat{j} + z \hat{k} \\
\mathbf{\rho}_i &= (x - x_i) \hat{i} + y \hat{j} + z \hat{k}, \ i = 1, 2
\end{align*}
\] (2.52a)

(2.52b)

(2.52c)

where \( \hat{i}, \hat{j}, \hat{k} \) are the unit vectors along the rotating coordinate system and \( x, y, z \) are the coordinates of the third body in the rotating coordinate system.

2.4.1 Normalization

Before establishing the equations of motion of the third body, it is useful to introduce the following non-dimensional quantities
\[ \mu_i = \frac{m_i}{m_1 + m_2} \]  \hfill (2.53a)

\[ \mu_2 = \frac{m_2}{m_1 + m_2} \]  \hfill (2.53b)

where \( \mu_1 \) and \( \mu_2 \) are non-dimensional parameters known as the mass parameters. In Equation (2.53), the total mass of the system is used for mass normalization. The system parameter that is used for normalizing distances is the distance between the two primaries, \( r_{12} = x_1 - x_2 \), since it is considered constant in the circular restricted three-body problem. Thus, one obtains

\[ x'_1 = \frac{x_1}{r_{12}}, \quad x'_2 = \frac{x_2}{r_{12}} \]  \hfill (2.54)

It is interesting to note from Equations (2.53) and (2.54) that

\[ \mu_1 + \mu_2 = 1 \]  \hfill (2.55a)

\[ x'_1 + x'_2 = 1 \]  \hfill (2.55b)

but from the center of mass property

\[ \mathbf{r}_{cm} \sum_{i=1}^{3} m_i = \sum_{i=1}^{3} m_i \mathbf{r}_i \]  \hfill (2.56)

where \( \mathbf{r}_{cm} \) is the position vector of the center of mass which is equal to zero since the center of mass is the origin of the coordinate system. Dividing Equation (2.56) by the total mass of the system and the distance between the two primaries, one finds that

\[ x'_1 = \frac{m_2}{m_1 + m_2} = \mu_2 \]  \hfill (2.57a)

\[ -x'_2 = -\frac{m_1}{m_1 + m_2} = \mu_1 \]  \hfill (2.57b)
The angular velocity, $\omega$, of the rotating system is calculated from the Keplerian motion of any of the primary masses. Since for either primary the centripetal force due to rotation in a circular orbit is balanced by the gravitational force from the other primary, one has

$$\omega^2 r_{12} = \frac{G(m_1 + m_2)}{r_{12}^2} \quad (2.58a)$$

$$\omega^2 = \frac{G(m_1 + m_2)}{r_{12}^3} \quad (2.58b)$$

Being constant, the angular velocity $\omega$ is used for time normalization. Assume that $t$ is the dimensional time and $t'$ is the dimensionless time, then

$$t' = t\omega \quad (2.59)$$

The dimensional period of the motion of the two primaries is $T = 2\pi / \omega$. The dimensionless period is thus

$$\tau = \omega T = 2\pi \quad (2.60)$$

From now on one parameter only is used in the normalized equations (whenever normalization is needed, this happens mainly for numerical purposes) of the CRTBP. This parameter is $\mu = \mu_2 = x'_2$; and hence, $\mu_1 = -x'_2 = 1 - \mu$.

### 2.4.2 Equation of Motion of the Third Body

The specific Lagrangian function, $L$, of the third body in an inertial coordinate system whose unit vectors are $\hat{i}, \hat{j}, \hat{k}$ is

$$L = \frac{1}{2} \|\dot{r}\|^2 + U \quad (2.61)$$

where $U$ is the specific forcing function of the third body defined as follows
\[ U = \frac{Gm_1}{\rho_1} + \frac{Gm_2}{\rho_2} \]  

(2.62)

and \( \mathbf{r}, \dot{\mathbf{r}} \) are the position and velocity vectors of the third body in the inertial frame

\[
\mathbf{r} = X \hat{i} + Y \hat{j} + Z \hat{k}, \quad \| \dot{\mathbf{r}} \| = (\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2)^{1/2}
\]  

(2.63)

where \( X, Y, Z \) are the coordinates of the third body.

\[ \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} \cos \omega t & -\sin \omega t & 0 \\ \sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \]  

(2.64)

and by differentiating Equation (2.64) with respect to time one obtains the relation between velocity components in both coordinate systems.
\[
\begin{bmatrix}
\dot{X} \\
\dot{Y} \\
\dot{Z}
\end{bmatrix} = \omega
\begin{bmatrix}
-\sin \omega t & -\cos \omega t & 0 \\
\cos \omega t & -\sin \omega t & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{bmatrix}
\]
(2.65)

Substituting from Equation (2.65) into the second part of Equation (2.63), one has

\[
\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 + 2\omega(x\dot{y} - y\dot{x}) + \omega^2(x^2 + y^2)
\]
(2.66)

Equation (2.66) indicates that as a result of transformation into a rotating coordinate system, velocity in the inertial frame is decomposed into three different quantities: the first is the velocity along the rotating coordinates, the second contains products of velocity and position components in the rotating system, and the third depends on position in the rotating system.

For an appropriate formulation one may temporarily replace the coordinates \(x, y,\) and \(z\) in Equation (2.66) by a set of generalized coordinates \(q_1, q_2,\) and \(q_3\) respectively, then substitute into Equation (2.61) to transform the Lagrangian function of the third body to the rotating coordinate system

\[
L = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 + 2\omega \left(q_1\dot{q}_2 - q_2\dot{q}_1\right) + \omega^2 \left(q_1^2 + q_2^2\right)\right) + \frac{Gm_1}{\rho_1} + \frac{Gm_2}{\rho_2}
\]
(2.67)

where

\[
\rho_1 = \left((q_1 - x_1)^2 + q_2^2 + q_3^2\right)^{1/2}
\]
(2.68a)

\[
\rho_2 = \left((q_1 - x_2)^2 + q_2^2 + q_3^2\right)^{1/2}
\]
(2.68b)

Defining the conjugate momenta \(p_i, \ i = 1, 2, 3,\) associated with the generalized coordinates \(q_i, \ i = 1, 2, 3,\) the Hamiltonian function corresponding to the motion of the third body is
\[ H = \sum_{i=1}^{3} p_i \dot{q}_i - L \]  
(2.69a)

\[ H = \frac{1}{2} \left( p_1^2 + p_2^2 + p_3^2 \right) + \omega (p_1 q_2 - p_2 q_1) - \frac{G m_1}{\rho_1} - \frac{G m_2}{\rho_2} \]  
(2.69b)

and Hamilton's canonical equations \( \frac{dq}{dt} = \frac{\partial H}{\partial p_1}, \frac{dp}{dt} = -\frac{\partial H}{\partial q_1}, i = 1, 2, 3 \) are

\[ \frac{dq_1}{dt} = p_1 + \omega q_2 \]  
\[ \frac{dp_1}{dt} = \omega p_2 \]  
(2.70a)

\[ \frac{dq_2}{dt} = p_2 - \omega q_1 \]  
\[ \frac{dp_2}{dt} = -\omega p_1 \]  
(2.70b)

\[ \frac{dq_3}{dt} = p_3 \]  
\[ \frac{dp_3}{dt} = \frac{\partial U}{\partial q_3} \]  
(2.70c)

Differentiating the left hand side of Equation (2.70) with respect to time again and substituting from the right hand side of Equation (2.70) and from Equations (2.67)-(2.69), the scalar equations of motion of the third body in the rotating coordinate system are

\[ \ddot{x} - 2 \omega \dot{y} = J_x \]  
(2.71a)

\[ \ddot{y} + 2 \omega \dot{x} = J_y \]  
(2.71b)

\[ \ddot{z} = J_z \]  
(2.71c)

where \( J \) is the Jacobi function of the system defined as follows

\[ J = \frac{1}{2} \omega^2 (x^2 + y^2) + \frac{G m_1}{\rho_1} + \frac{G m_2}{\rho_2} \]  
(2.72)

The right hand side of Equation (2.71) represents the partial derivatives of the Jacobi function with respect to third body coordinates in the rotating frame. These partial derivatives are
\[ J_x = \omega^2 x - \frac{Gm_1}{\rho_1^3} (x-x_1) - \frac{Gm_2}{\rho_2^3} (x-x_2) \]  

\[ J_y = \omega^2 y - \frac{Gm_1}{\rho_1^3} y - \frac{Gm_2}{\rho_2^3} y \]  

\[ J_z = -\frac{Gm_1}{\rho_1^3} z - \frac{Gm_2}{\rho_2^3} z \]

By multiplying through the first, second, and third parts of Equation (2.71) by \( \dot{x}, \dot{y}, \) and \( \dot{z}, \) respectively, then collecting the three parts and integrating, one obtains the Jacobi integral equation

\[ \nu^2 = 2J - C \]  

where \( \nu \) is the magnitude of the total velocity vector in the rotating coordinate system and \( C \) is the Jacobi constant. The Jacobi integral equation plays an important role in defining and calculating the zero velocity surfaces (or curves) in the restricted problem of three bodies, as shown latter.

### 2.4.3 Dimensionless Equation of Motion

Dividing Equations (2.67) and (2.69) by \( \omega^2 r_{12}^2, \) the dimensionless Lagrangian and Hamiltonian functions corresponding to the motion of the third body in the rotating coordinate system are

\[ L = \frac{1}{2} \left\{ \dot{q}_1^2 + \dot{q}_2^2 - 2(q_1 \dot{q}_2 - q_2 \dot{q}_1) \right\} + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2} \]  

\[ H = \frac{1}{2} (p_1^2 + p_2^2 + p_3^2) - (p_2 q_1 - p_1 q_2) - \frac{1-\mu}{\rho_1} - \frac{\mu}{\rho_2} \]

Dividing Equation (2.68) by \( r_{12}, \) one also obtains

\[ \rho_1 = \left( (x-\mu)^2 + y^2 + z^2 \right)^{1/2} \]
\[ \rho_2 = \left( (x - \mu + 1)^2 + y^2 + z^2 \right)^{1/2} \]  

(2.76b)

From Equation (2.75), and the dimensionless Hamilton's canonical equations are

\[
\frac{dq_1}{dt'} = p_1 + q_2 \quad \frac{dp_1}{dt'} = p_2 + \frac{\partial U}{\partial q_1} \tag{2.77a}
\]

\[
\frac{dq_2}{dt'} = p_2 - q_1 \quad \frac{dp_2}{dt'} = -p_1 + \frac{\partial U}{\partial q_2} \tag{2.77b}
\]

\[
\frac{dq_3}{dt'} = p_3 \quad \frac{dp_3}{dt'} = \frac{\partial U}{\partial q_3} \tag{2.77c}
\]

and the equations of motion are

\[
\ddot{x} - 2\dot{y} = \frac{\partial J}{\partial x} \tag{2.78a}
\]

\[
\ddot{y} + 2\dot{x} = \frac{\partial J}{\partial y} \tag{2.78b}
\]

\[
\ddot{z} = \frac{\partial J}{\partial z} \tag{2.78c}
\]

and

\[
J = \frac{1}{2} (x^2 + y^2) + \frac{1 - \mu}{\rho_1} + \frac{\mu}{\rho_2} \tag{2.79}
\]

is the dimensionless Jacobi function. For simplicity, variables without a prime may be used to denote dimensionless variables later in the text.

### 2.4.4 Equilibrium Solutions

If the velocities and accelerations in Equation (2.78) are set to zero, the left hand side will be zero. Equation (2.78c) is reduced as follows

\[
0 = - \left\{ \frac{(1 - \mu)}{\rho_1^3} + \frac{\mu}{\rho_2^3} \right\} z \tag{2.80}
\]
The term in parentheses is equal to zero only when

\[ \mu = -\left(\frac{\rho_2}{\rho_1}\right)^3 (1 - \mu) \]  

(2.81)

which contradicts the definition of the range of the mass parameter, \( 0 \leq \mu \leq 0.5 \). In other words, there is no value for the mass parameter that satisfies Equation (2.81). The only solution that satisfies Equation (2.80) is \( z = 0 \), which indicates that an equilibrium solution, if it exists, is located in the \( xy \) plane.

Equations (2.78b) and (2.78c) are

\[ 0 = x - \frac{(1 - \mu)}{\rho_1} (x - \mu) - \frac{\mu}{\rho_2} (x - \mu + 1) \]  

(2.82a)

\[ 0 = y - \frac{(1 - \mu)}{\rho_1} y - \frac{\mu}{\rho_2} y \]  

(2.82b)

A remarkable solution for Equation (2.82) is when \( \rho_1 = \rho_2 = 1 \). In this case, a solution is obtainable when solving the two parts of Equation (2.76) simultaneously. If when replacing \( y \) with \(-y\), Equation (2.76) remains unchanged, this indicates that there are two solutions symmetric with respect to the \( x \) axis. In these solutions the three masses constitute the vertices of two equilateral triangles. These two solutions are known as equilateral triangular equilibrium points, \( L_4 \) and \( L_5 \), first introduced by Lagrange (defined in Section 2.3).

\[ L_4 \equiv (x = \mu - \frac{1}{2}, y = \frac{\sqrt{3}}{2}), \quad L_5 \equiv (x = \mu - \frac{1}{2}, y = -\frac{\sqrt{3}}{2}) \]  

(2.83)
Noticing that the second of Equation (2.82) is satisfied when \( y = 0 \), the two coupled equations are reduced to the first part only. Substituting \( \rho_1 \) and \( \rho_2 \) from Equation (2.76) into Equation (2.82) yields a quintuple equation in \( x \) as follows

\[
Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F = 0
\]  

(2.84)

where the coefficients \( A, B, C, D, E, F \) are functions of the mass parameter \( \mu \). Solutions in this case are known as the collinear equilibrium points (defined in Section 2.3) and expressions for these coefficients depend on the location of an equilibrium point relative to the center of mass. These coefficients determine the signs of \( \rho_1, \rho_2 \) in Equation (2.76). For the collinear equilibrium point, \( L_3 \), these coefficients are

\[
\begin{align*}
A &= 1 \\
B &= 2 - 4\mu \\
C &= 6\mu^2 - 6\mu + 1 \\
D &= -4\mu^3 + 6\mu^2 - 2\mu - 1 \\
E &= \mu^4 - 2\mu^3 + \mu^2 + 4\mu - 2 \\
F &= -3\mu^2 + 3\mu - 1
\end{align*}
\]  

(2.85)

Having the value of the mass parameter determined, all the above coefficients are calculated, and the roots are obtainable numerically. According to the theory of equations, no closed-form solution is known for Equation (2.84). However, a quick approximate solution is obtainable directly from the first of Equation (2.82) when \( y = z = 0 \), and the mass parameter is very small so that \( \mu \ll x, \mu \ll 1 \). In this case the mass of the second primary is negligible compared to that of the first primary and two of the equilibrium points coalesce. The three equilibrium collinear points in this case are

\[
\begin{align*}
L_1 &= (x \approx -1, y = 0) \\
L_2 &= (x \approx -1, y = 0) \\
L_3 &= (x \approx 1, y = 0)
\end{align*}
\]  

(2.86)
Equation (2.86) gives rough estimates of the locations of the collinear equilibrium points. Noteworthy when \( \mu = 0 \), Equations (2.83) and (2.86) indicate that the five equilibrium points are located on a unit circle whose origin is the first primary.

Generally, for any value of the mass parameter satisfying the inequality \( 0 \leq \mu \leq 0.5 \), the numerical solution to Equation (2.84) gives one real root, representing one collinear equilibrium point. There are three collinear equilibrium points according to the permutations of the signs of \( p_1 \), \( p_2 \). The five equilibrium points in the Earth-Moon system, whose mass parameter is \( \mu = 1/82.25 \), are shown in Figure 2.3 where

\[
\begin{align*}
L_1 &= (-0.8369, 0) \\
L_2 &= (-1.1557, 0) \\
L_3 &= (1.0051, 0) \\
L_4 &= (-0.4878, 0.8660) \\
L_5 &= (-0.4878, -0.8660)
\end{align*}
\]

Figure 2.3 Equilibrium Points in the Earth-Moon System
2.4.5 Variational Equations

One primary way to explore the behavior of the third body in the vicinity of an equilibrium point is to linearize the equation of motion about this equilibrium point. The behavior of the linear model can qualitatively give insights into the behavior of the nonlinear dynamics. Assume that \( x_i, y_i, z_i \) are the coordinates of an equilibrium point, and apply the following transformation to the equation of motion:

\[
\begin{align*}
    x &= x_i + u \\
    y &= y_i + v \\
    z &= z_i + w
\end{align*}
\] (2.88a-b-c)

where \( u, v, w \) are the linear coordinates relative to the equilibrium point. The right hand side of the equation of motion is linearized using Taylor series expansions, neglecting the second order and higher partial derivatives of the Jacobi function with respect to \( x, y, z \).

The new set of equations is known as the variational equations:

\[
\begin{align*}
    \ddot{u} - 2\dot{v} &= J_{xx}^l u + J_{xy}^l v + J_{xz}^l w \\
    \ddot{v} + 2\dot{u} &= J_{yx}^l u + J_{yy}^l v + J_{yz}^l w \\
    \ddot{w} &= J_{zx}^l u + J_{zy}^l v + J_{zz}^l w
\end{align*}
\] (2.89a-b-c)

The subscripts on the right hand side of Equation (2.89) indicate partial derivatives, while the superscript indicates that these derivatives are evaluated at the equilibrium point.

Since for all the equilibrium points, \( z = 0 \), one finds that the mixed partial derivatives \( J_{xz} = J_{yx} = 0 \). This result means that the out-of-plane motion is independent or decoupled from the in-plane motion; in other words, the effect of variation of \( w \) in the \( z \) direction on the variations \( u, v \) in the \( x, y \) directions respectively is negligible, at least in
the linear model. The third part of Equation (2.89c) represents a simple harmonic motion in the z direction whose standard solution is

\[ w = A_z \cos \omega_z t + B_z \sin \omega_z t \tag{2.90} \]

where \( A_z, B_z \) are constant coefficients depending on the initial conditions, and the out-of-plane frequency \( \omega_z \) is calculated from the relation \( \omega_z^2 = -J_{zz} \). The final form of the out-of-plane motion with initial conditions \( w_0, \dot{w}_0 \) is

\[ w = w_0 \cos \omega_z t + \frac{\dot{w}_0}{\omega_z} \sin \omega_z t \tag{2.91} \]

Equations (2.89a) and (2.89b) are coupled and constitute a system of two second order homogeneous linear differential equations. Define the state vector

\[ \mathbf{X} = [u \ v \ \dot{u} \ \dot{v}]^T \tag{2.92} \]

where superscript T stands for transpose. The state equations for the in-plane motion can be written in the following vector form

\[ \dot{\mathbf{X}} = \mathbf{A} \mathbf{X} \tag{2.93} \]

where

\[ \mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ J_{xx} & J_{xy} & 0 & 2 \\ J_{yx} & J_{yy} & -2 & 0 \end{bmatrix} \tag{2.94} \]

The characteristic equation of the above matrix is

\[ \lambda^4 + (4 - J_{xx} - J_{yy}) \lambda^2 + (J_{xx} J_{yy} - J_{xy}^2) = 0 \tag{2.95} \]

where \( \lambda \) is an eigenvalue. The roots of the above equation depend on the partial derivatives \( J_{xx}, J_{yy}, J_{xy} \). It is found that for all the collinear equilibrium points
For better analysis Equation (2.95) is reformulated according to Szebehely as follows

\[ \lambda^2 + 2\beta_1 \lambda - \beta_2^2 = 0 \]  

(2.96)

where

\[ \Lambda = \lambda^2, \quad \beta_1 = 2 - \frac{J_{xx} + J_{yy}}{2}, \quad \beta_2 = \sqrt{-J_{xx}J_{yy}} \]  

(2.97)

The roots of Equation (2.96) are

\[ \lambda_1 = -\beta_1 + \sqrt{\beta_1^2 + \beta_2^2} > 0, \quad \Lambda_2 = -\beta_1 - \sqrt{\beta_1^2 + \beta_2^2} < 0 \]  

(2.98)

and the eigenvalues are

\[ \lambda_1 = +\sqrt{\Lambda_1} > 0 \quad \text{(real)}, \quad \lambda_2 = +\sqrt{\Lambda_2} \quad \text{(imaginary)} \]  

(2.99)

The solutions for \( u, v \) take the form

\[ u = A_x e^{\lambda_1 t} + B_x e^{-\lambda_1 t} + C_x \cos s_3 t + D_x \sin s_3 t \]  

(2.100a)

\[ v = A_y e^{\lambda_2 t} + B_y e^{-\lambda_2 t} + C_y \cos s_3 t + D_y \sin s_3 t \]  

(2.100b)

where \( s_3 = -i\lambda_3 \). The first term in each part of the right hand side of Equation (2.100) represents an aperiodic increase, while the second term in each part represents aperiodic decay. The initial conditions \( u_0, v_0, \dot{u}_0, \dot{v}_0 \) are chosen so that the coefficients of these two terms vanish, i.e.,

\[ C_x = u_0, \quad D_x = \frac{\dot{u}_0}{s_3} \]  

(2.101a)

\[ C_y = v_0, \quad D_y = \frac{\dot{v}_0}{s_3} \]  

(2.101b)
The above coefficients are coupled since the original in-plane equations of motion are coupled. Substituting Equation (2.100) into Equations (2.89a) and (2.89b), one obtains

\[ C_y = \left( \frac{s_3^2 + J_{xx}}{2s_3} \right) D_x, \quad D_y = -\left( \frac{s_3^2 + J_{xx}}{2s_3} \right) C_x \] (2.102)

Moreover, the above coefficients represent the amplitudes of the motion of the third body in the xy-plane in the vicinity of the collinear equilibrium point. These coefficients determine the shape and size of the center manifold around that equilibrium point. If one sets \( v_0 = 0 \) and \( u_0 = 0 \), the resulting orbit is an ellipse with a semi-major axis \( a = \dot{v}_0 / s_3 \), a semi-minor axis \( b = u_0 \) and an eccentricity \( e = (1 / u_0 s_3) \sqrt{s_3^2 u_0^2 - \dot{v}_0^2} \). If the ratio \( s_3 u_0 / \dot{v}_0 \) is maintained constant while \( u_0 \) and \( v_0 \) vary individually, the shape of the orbit is the same, but its size is changed. Since \( s_3 \) is the frequency or the mean motion of the third body, the period of the motion of the third body on the elliptic orbit is \( T = 2\pi / s_3 \). Figure 2.4 shows a periodic orbit in the vicinity of the collinear equilibrium point \( L_1 \) plotted using the linear model from Equation (2.100).

Since the mean motion \( s_3 \) is a function of the mass parameter \( \mu \), i.e., the characteristics of the three-body system the ratio \( s_3 u_0 / \dot{v}_0 \) relates the initial conditions to the physics of the three-body system. One should expect that a range of initial conditions gives periodic orbits \( 0 \leq s_3 u_0 / \dot{v}_0 \leq 1 \). The equalities represent two limiting cases; in the first limiting case \( s_3 u_0 / \dot{v}_0 \approx 0 \), in which either \( u_0 \) is very small or \( \dot{v}_0 \) is very large and the third body cannot maintain periodicity. In the second limiting case \( s_3 u_0 / \dot{v}_0 \approx 1 \), and the orbit is a circular orbit.
Orbits calculated using the linearized equations are compared graphically to those calculated using numerical integration of the nonlinear equations. One should expect that the linearized equations give good approximation to the nonlinear equations as long as

\[ L_1 \]

Figure 2.4 Elliptic Orbit About First Collinear Equilibrium Point \( L_1 \)

the overall motion is sufficiently close to the equilibrium point when \( t \to \infty \). Should the position in the \( x \) direction in the vicinity of an equilibrium point be analytic, it is expanded as follows

\[ x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3) \]  \hspace{1cm} (2.103)

where \( O(\varepsilon^3) \) means of order \( \varepsilon^3 \), and \( \varepsilon \) is a small parameter \( \varepsilon \ll 1 \). If only a two term expansion is considered then \( \varepsilon x_1 \) is the same as \( u \) in Equation (2.88); this means that \( u_0 / x_0 = O(\varepsilon) \), and \( u_0 \) is sufficiently small. Here, the condition on the position in the \( x \) direction is sufficient to describe the validity of the linear model since, as explained,
the amplitudes of motion in the two directions are linearly dependent. Figures 2.5 and 2.6 show the linear and nonlinear solutions calculated for different initial condition $u_0$, $v_0$. In Figure 2.5, for $u(0) = 0.0001$, the nonlinear solution is spatially close to the linear

Figure 2.5 Linear and Nonlinear Solutions for $u(0) = 0.0001$
Figure 2.6 Linear and Nonlinear Solutions for \( u(0) = 0.1 \)

The difficulty in the numerical integration lies in finding the appropriate initial conditions that give a periodic orbit, an issue which has received much attention in the last two centuries. The solution of the linearized equation can be used as a first guess for initial conditions in any numerical integration process though it is approximate and not as accurate as the numerical solution of the nonlinear equation.

2.5 Nonlinear Dynamics

At this point, one major question is "To what degree does the linearized equation
represent the actual behavior of the full nonlinear dynamical system in the vicinity of a
collinear equilibrium point?" To exploit the answer to this important question, a physical
approach is used. In this section the motion in the neighborhood of the equilibrium point
is described more qualitatively. Let the nonlinear equations be written as follows
\[
\dot{X} = AX + \varepsilon(X) \tag{2.104}
\]
where \(X\) and \(A\) are as defined in Equation (2.92) and (2.94) respectively. The vector
function \(\varepsilon(X)\) is represented by a power series starting from the second order terms of the
states of the vector \(X\), where the nonlinear terms \(\varepsilon(X)\) satisfy the following relation
\[
\lim_{|X| \to 0} \frac{\varepsilon_i}{|X|^i} = 0, \quad i = 1,2,3,4 \tag{2.105}
\]
When the equilibrium point is isolated, i.e., there is no other equilibrium point in the
neighborhood of the underlined equilibrium point, \(^{107}\) the linearized equation of motion
approximates the behavior of the nonlinear equation. An exception is when the
eigenvalues of the Jacobian matrix, \(A\), have pure imaginary values and an equilibrium
point is a center; in such a critical case the linearized equation does not completely
represent the full nonlinear dynamics. \(^{107}\)

### 2.5.1 Nonlinear Conservative System

In conservative systems, the existence of integrals of the motion or integral curves
enable study of the motion without having to confine attention to the neighborhood of an
equilibrium point. \(^{108}\) The reason for the existence of the integral curves is that the right
hand side of the nonlinear differential equation of motion is a function of the position of
the moving particle. A formulation of a potential function is possible so that if the right
hand side of the equation of motion is expressed as gradients of this function a total
derivative is obtained. This process allows a first integral to be obtainable through integrating both sides of the equation of motion. The constant of integration is determinable from the initial conditions and is invariant along the orbit, i.e., on a certain trajectory of the moving particle, the state vector $X = [q \dot{q}]^T$, $q = [u \ v]^T$ satisfies the algebraic equation

$$C(X) = \text{constant} \quad (2.106)$$

Equation (2.106) represents a family of curves or surfaces parameterized by the constant on the right hand side for motion with either two degrees of freedom or three degrees of freedom.

Generally, conservative systems possess a conservation of energy that is formulated as follows

$$\frac{1}{2} \| \dot{q} \|^2 + \mathcal{U}(q) = h \quad (2.107)$$

where $\mathcal{U}(q)$ is the potential energy and $h$ is the total energy which is constant. By reformulating Equation (2.107) one obtains

$$\frac{1}{2} \| \dot{q} \|^2 = h - \mathcal{U}(q) \quad (2.108)$$

The regions for possible motion of the third body satisfy the inequality

$$h - V(q) > 0 \quad (2.109)$$

The forces acting on the third body are expressed as the gradient of the potential function.

$$\mathbf{f}(q) = \left[ \frac{\partial V}{\partial q} \right]^T \quad (2.110)$$
At the equilibrium points, where coordinates $q = q_i$, the force acting on the third body vanishes, meaning that $(\partial V / \partial q)_{q=q_i} = 0$, which represents an extremum point for the potential energy. Depending on whatever the potential energy at the equilibrium point is, a suprema or infema, the motion of the third body in the vicinity of the equilibrium point is broadly said to be unstable or stable respectively.\textsuperscript{109,110}

Assuming that the function $V(q)$ is analytic at the equilibrium point, it can be expanded using Taylor's theory as follows.

$$V(q) = V(q_i) + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{\partial V}{\partial q} \right)_{q=q_i} \left( q - q_i \right)^n$$

The product $(\partial V / \partial q)_{q=q_i}$ is a tensor product represented by an $n \times n$ matrix, where all the partial derivatives are evaluated at the equilibrium point. The partial derivative of the first order $(\partial V / \partial q)_{q=q_i}$ vanishes according to the definition of the equilibrium point; this means that Equation (2.111) contains no linear terms in the coordinates of the third body. If the value of the potential energy converges to the first term on the right hand side, the trajectory converges to an isolated point. At any other constant value of the potential, Equation (2.111) represents a hyper surface. In that sense, from conservation of energy, the potential energy is constant only for zero velocity and that hyper surface is known as the zero velocity surface or equipotential surface that determines the boundaries of regions of possible motion of the third body.\textsuperscript{8,10} The trajectory of the third body cannot pass through these surfaces unless the total energy is increased to a higher level.

### 2.5.2 Surfaces of Zero Velocity

Recall the Jacobi integral equation and the Jacobi constant
\[ x^2 + y^2 + z^2 = 2J(x, y, z) - C \]  \hspace{1cm} (2.112)

which is reformulated to match the form of the conservation of energy as follows.

\[ \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = J(x, y, z) - C/2 \]  \hspace{1cm} (2.113)

The fact that the left hand side is positive definite leads to the following condition on the motion of the third body.

\[ J(x, y, z) - C/2 > 0 \]  \hspace{1cm} (2.114)

The equality gives the surfaces of zero velocity. Figure 2.7 shows the zero velocity curves in the Earth-Moon system. Definitive properties of such a surface depend on the properties of the Jacobi function. By definition, the angular velocity of the rotating system is considered constant in this analysis, and the centrifugal terms are functions of position only. Thus, the Jacobi function is a function only of position.

![Zero Velocity Curves in the Earth-Moon System](image-url)
Rewrite the Jacobi integral equation as follows.

\[ G(x, y, z, x', y', z') = \frac{1}{2} (x'^2 + y'^2 + z'^2) - J(x, y, z) + C/2 \]  

(2.115)

The function \( G(X) \), where \( X \) is the three dimensional state vector, represents manifolds in phase space where singularities occur when

[ \frac{\partial G}{\partial X} = 0 \]  

(2.116)

leading to the following results

\[ \dot{x}_0 = \dot{y}_0 = \dot{z}_0 = 0 \]
\[ J_0 = C/2 \]
\[ J_{x_0} = J_{y_0} = J_{z_0} = 0 \]  

(2.117)

where subscripts in the first and second lines of Equation (2.117) represent evaluation at the singular points, and the subscripts in the third line represent partial derivatives evaluated at the singular points. The first line of Equation (2.117) means that the singular points are also equilibrium points, i.e., if the third body is placed at any one of these points its position remains fixed relative to the rotating coordinate system. The third line of Equation (2.117) gives three nonlinear algebraic equations for the coordinates of an equilibrium point. Solutions to these equations were developed in detail in Section 2.4.4.

Since it is analytic at the equilibrium points, the Jacobi function is expanded using Taylor theory as follows

\[ J(q) = J(q_0) + \left\{ \frac{\partial J}{\partial q} \right\}_{q=q_0} (q - q_0) + \frac{1}{2} (q - q_0)^T \left[ \frac{\partial^2 J}{\partial q \partial q} \right]_{q=q_0} (q - q_0) + \ldots \]  

(2.118)

where \( q = [x \ y \ z]^T \). The first term on the right hand side of Equation (2.118) can be calculated at any equilibrium point; the second term vanishes since the equilibrium points
are singular points, while the higher order terms survive. For the sake of simplicity and since all the equilibrium points are located in the \( xy \) plane, Equation (2.118) is reformulated in two dimensional space

\[
J(x, y) = J(x_0, y_0) + \left\{ \frac{\partial J}{\partial x} \right\}_{x = x_0, y = y_0} \{x - x_0\} + \left\{ \frac{\partial J}{\partial y} \right\}_{x = x_0, y = y_0} \{y - y_0\} + \frac{1}{2} \begin{bmatrix} x - x_0 & y - y_0 \end{bmatrix} \begin{bmatrix} J_{xx} & J_{xy} \\ J_{yx} & J_{yy} \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}^T + \text{HOT}
\]  

(2.119)

where HOT denotes higher order terms. Let \( J(x_0, y_0) = J_0 \) and reformulate Equation (2.118) in terms of coordinates \( u, v \) relative to an equilibrium point.

\[
J(u, v) = J_0 + \frac{1}{2} \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} J_{uu} & J_{uv} \\ J_{vu} & J_{vv} \end{bmatrix}_{u = u_0, v = v_0} \begin{bmatrix} u \\ v \end{bmatrix}^T + \text{HOT}
\]  

(2.120)

When the value of the Jacobi function equals \( J_0 \), the trajectory is an isolated point. However, when the third body is very close but not exactly at the equilibrium point, the effect of the second term is significant. The manifold in this case is a curve in the \( xy \) plane with its center at the equilibrium point. Substituting from Equation (2.120) into Equation (2.115) for the equality case, the following equation represents the curves of zero velocity in the \( xy \) plane.

\[
\frac{1}{2} \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} J_{uu} & J_{uv} \\ J_{vu} & J_{vv} \end{bmatrix}_{u = u_0, v = v_0} \begin{bmatrix} u \\ v \end{bmatrix}^T + \text{HOT} = \frac{C}{2} - J_0
\]  

(2.121)

As indicated in the preceding sections, at the collinear equilibrium points \( J_{uv} = J_{vu} = 0 \) and \( J_{uu} > 0, J_{vv} < 0 \); this result is obtainable directly from the \( xy \) system noting that \( du/dx = 1 \), \( dv/dy = 1 \). Hence, the determinant of the Hessian matrix of the
function \( J(u, v) \) at any collinear equilibrium point is negative, and the collinear equilibrium points are saddle points. For this to be obvious, Figures 2.8a-2.8c show three dimensional plots of the Jacobi function in the vicinity of the collinear equilibrium points in the Earth-Moon system.

Figure 2.8a Jacobi Function at Point \( L_1 \) in the Earth-Moon System
Figure 2.8b Jacobi Function at Point $L_2$ in the Earth-Moon System

Figure 2.8c Jacobi Function at Point $L_3$ in the Earth-Moon System
Further, Equation (2.121) is rewritten as

\[ J_x \dot{u}^2 - |J_{yy}| \dot{v}^2 + \text{HOT} = C - 2J_0 \]  \hspace{1cm} (2.122)

When neglecting the higher order terms and substituting \(2J_0 = C_0\), where \(C_0\) is the value of the Jacobi constant at a collinear equilibrium point, Equation (2.122) is reformulated to give a standard form of a conic section

\[ \frac{u^2}{a^2} - \frac{v^2}{b^2} = 1 \]  \hspace{1cm} (2.123a)

where

\[ a^2 = \frac{(C - C_0)}{J_{xx}}, \quad b^2 = \frac{(C - C_0)}{|J_{yy}|} \]  \hspace{1cm} (2.123b)

Equation (2.123b) is valid if and only if \(C > C_0\) for \(a^2\) and \(b^2\) to be real values. In this case the semi-major axis of the hyperbola is parallel to the \(x\) axis. Figures 2.9a to 2.9c show the curves of zero velocity in the vicinity of the collinear equilibrium points in the Earth-Moon system. A family of curves that match Equation (2.123) are represented by level curves on the right and left of a collinear point. Obviously, these curves are hyperbolas with semi-major axes parallel to the \(x\) axis, and values of the Jacobi constant for these curves are larger than \(C_0\). According to the analysis of the regions of possible motion, the motion of the third body in this case is either on the left of the left branch of the hyperbola or on the right of the right branch of the hyperbola; thus, the third body will not intercept the collinear equilibrium point in such a trajectory. On the other hand, it will move either toward the first primary \(m_1\) or the second primary \(m_2\).

Alternatively, if \(C < C_0\) Equation (2.123) is rewritten as follows
\[
\frac{v^2}{a^2} - \frac{u^2}{b^2} = 1 \tag{2.124a}
\]

where

\[
a^2 = \frac{(C_0 - C)}{|J_{yy}|}, \quad b^2 = \frac{(C_0 - C)}{J_{xx}} \tag{2.124b}
\]

Equation (2.124) represents a hyperbola whose semi-major axis, \(a\), is parallel to the \(y\) axis. The distance between the two vertices of the hyperbola equals \(2a = 2\sqrt{(C_0 - C)/|J_{yy}|}\). Since values of \(C_0\) and \(|J_{yy}|\) are calculated at the collinear point, the only parameter that determines the distance between the two vertices is the Jacobi constant of the trajectory of the third body. Motion in such a case is permissible between the two branches of the hyperbola, and that region includes the collinear equilibrium point. As shown in Figures 2.9a to 2.9c, a family of these hyperbolas whose branches are above and below the equilibrium point is present. The smaller the Jacobi constant, the wider the distance between the two vertices of the hyperbola and the larger the area in which the third body moves. This feature is noteworthy because communication from a third body or a spacecraft moving about the collinear equilibrium point to the two primaries is possible and continuous here.
Figure 2.9a Zero Velocity Contours at Point $L_1$ in the Earth-Moon System

Figure 2.9b Zero Velocity Contours at Point $L_2$ in the Earth-Moon System
So far, the results obtained from this physical analysis concern the linearized equation approach. This shows the accessibility of the regions of possible motion around the collinear equilibrium points. An elliptic orbit with its semi-major axis parallel to the \( y \) axis is feasible from the physical analysis and explains the shape of the elliptic orbit obtained by solving the variational equations. It is noteworthy to recognize that the zero velocity curves are not orbits but bound the orbit of the third body and determine the space attainable by the third body.
CHAPTER 3

PLANAR PERIODIC ORBIT

3.1 Introduction

In this chapter, an analytic solution for the notional motion of the third body in a circular orbit in the plane of motion of the two primaries is introduced for small, non-zero mass parameters and when the motion is in the vicinity of one of the two primaries. In this case, the Jacobi function allows implementation of Legendre polynomials, and the Jacobi integral equation is reduced to the Legendre normal form of an elliptic integral. A closed-form expression for the period of motion is formulated, and expressions for coordinates as functions of time are also introduced. The speed of the third body is found to be nonuniform along the path. The path consists of a circular orbit offset from the orbited primary and centered about the system mass center. The analytical form of the solution illuminates how the basic parameters of the system influence the motion. Existence conditions, particular case assumptions, and properties of the elliptic integral identity constraints on the required initial conditions are discussed. The accuracy of the notional motion will be investigated so that this solution can be used as a generating orbit for purposes of numerical or analytical continuation of periodic orbits in these types of three-body systems.

3.2 Planar Circular Orbits

When the motion of the third body is maintained in the plane of the primaries, i.e., $\ddot{z} \approx 0$, the $z$ component governing equation can be eliminated when the initial conditions $z(t_0) = z_0, \dot{z}(t_0) = \dot{z}_0$ are appropriately chosen. In this case, the other two motion components represent a system of two coupled nonlinear differential equations
describing the motion of the third body in the $xy$ plane. The CRTBP in this case is known as the planar circular restricted three-body problem (PCRTBP). Figure 3.1 shows the circular motion of the third body in the $xy$ plane, where $a$ denotes the radius of the notional circular orbit circumscribing the first primary and $\theta$ denotes a form of true anomaly or the angle measured counterclockwise from the positive end of the $x$ axis (when viewed from above the motion plane). This circular path is centered at the $xyz$ frame origin, offset from the first primary. The new equations of motion are

$$\ddot{x} - 2\omega \dot{y} = \omega^2 x - \frac{Gm_1}{\rho_1^3}(x-x_1) - \frac{Gm_2}{\rho_2^3}(x-x_2) \quad (3.1a)$$

$$\ddot{y} + 2\omega \dot{x} = \omega^2 y - \frac{Gm_1}{\rho_1^3} y - \frac{Gm_2}{\rho_2^3} y \quad (3.1b)$$

Coupling in Equation (3.1) occurs through the gravitational acceleration and Coriolis acceleration. The nonlinearity results from the gravitational acceleration, particularly through the distances between the third body and the two primaries.

$$\rho_1 = \{(x-x_1)^2 + y^2\}^{1/2} \quad (3.2a)$$

$$\rho_2 = \{(x-x_2)^2 + y^2\}^{1/2} \quad (3.2b)$$

For circular motion in the $xy$ plane, the coordinates of the third body are expressed as parametric functions of the true anomaly, $\theta$, which is an unknown function of time. From the geometry in Figure 3.1, the parametric equations for the coordinates are

$$x(t) = a \cos\{\theta(t)\} \quad (3.3a)$$

$$y(t) = a \sin\{\theta(t)\} \quad (3.3b)$$

$$z(t) = 0 \quad (3.3c)$$

Substituting Equation (3.3) into Equation (3.2) reveals
The Jacobi function can be reformulated as follows.

\[
J = \frac{1}{2} \omega^2 (x^2 + y^2) + \frac{Gm_1}{\rho_1} \left(1 + \frac{m_2}{m_1} \frac{\rho_2}{\rho_1} \right) + \left(3.5\right)
\]

When the second primary mass is very small compared to the first primary mass, \(m_2 \ll m_1\), and the motion of the third body is in the vicinity of the first primary, \(\rho_1 \ll \rho_2\), for a sufficiently long time, one expects that

\[
\frac{m_2}{m_1} \frac{\rho_1}{\rho_2} = \varepsilon^2 \quad \left(3.6\right)
\]

where \(\varepsilon = o(1)\) is a small parameter and \(o\) means of order less than one. Dividing both sides of Equation (3.5) by \(\omega \cdot r_{12}^3\) yields

\[
\frac{m_2}{m_1} \frac{\rho_1}{\rho_2} = \varepsilon^2
\]
\[ J' = \frac{1}{2} (x'^2 + y'^2) + \frac{1}{\rho_1'} (1 - \mu + \frac{\rho_1'}{\rho_2'}) \mu \]  

where \( \mu \) is the mass parameter and \( J' = J / \omega^2 r_{12}^2 \), \( x' = x / r_{12} \), \( y' = y / r_{12} \), \( \rho_1' = \rho_1 / r_{12} \), and \( \rho_2' = \rho_2 / r_{12} \). Equation (3.7) now has a single mass parameter which is assumed to be very small, compared to unity, for this particular case.

From Equation (3.4) it is noted that the derivatives \( d\rho_1 / d\theta \), \( d\rho_2 / d\theta \) vanish when \( \theta = i\pi \), \( (i = 0, 1, 2\ldots) \), i.e., when the third body crosses the line of syzygy. Substituting \( \theta = 0, \pi \) into Equation (3.4), and using the identity \( r_{12} = x_1 - x_2 \), the two extrema of \( \rho_1 \) and \( \rho_2 \) are

\[ \begin{align*}
\theta = 0 & \rightarrow \rho_{1\min} = a - x_1, \quad \rho_{2\max} = r_{12} + a - x_1 \\
\theta = \pi & \rightarrow \rho_{1\max} = a + x_1, \quad \rho_{2\min} = r_{12} - a - x_1
\end{align*} \]  

(3.8a)

(3.8b)

where \( x_1 < a < -(x_1 + x_2) / 2 \) is required to maintain closeness to the first primary. Using the binomial expansion

\[ \begin{align*}
\left\{ \frac{\rho_1}{\rho_2} \right\}_{\min} &= \frac{a-x_1}{r_{12}} - \frac{(a-x_1)^2}{r_{12}} + \frac{(a-x_1)^3}{r_{12}} - \ldots \\
\left\{ \frac{\rho_1}{\rho_2} \right\}_{\max} &= \frac{a+x_1}{r_{12}} + \frac{(a+x_1)^2}{r_{12}} + \frac{(a+x_1)^3}{r_{12}} + \ldots
\end{align*} \]  

(3.9a)

(3.9b)

From the center of mass property, \( x_1 = \mu r_{12} \), and \( x_2 = (\mu - 1) r_{12} \). The lower and upper bounds of the coefficient \( \rho_1' / \rho_2' \) in Equation (3.7) are determined to first order from the following inequality

\[ (a' - \mu) - (a' - \mu)^2 + \ldots \leq \frac{\rho_1'}{\rho_2'} \leq (a' + \mu) + (a' + \mu)^2 + \ldots \]  

(3.10)
where $a' = a / r_{12}$. The range between the upper and lower bound of the coefficient $\rho'_1 / \rho'_2$ equals $2\mu + f(a''; \mu')$, where $f(a''; \mu')$ is a function of the higher order values of both the orbit radius and the mass parameter. The maximum value of the coefficient $\rho'_1 / \rho'_2$ occurs when the third body crosses the $x$ axis between the two primary masses. At this location the third body undergoes a maximum gravitational force coming from the second mass. In contrast, the minimum of the coefficient $\rho'_1 / \rho'_2$ occurs when the third body crosses the $x$ axis on the side that is far from the second primary, the location at which the third body is subject to a minimum gravitational force coming from the second primary. When substituting $(\rho'_1 / \rho'_2)_{\text{max}}$ in Equation (3.7), the normalized Jacobi constant is

$$J' = \frac{1}{2} (x'^2 + y'^2) + \frac{1 - \mu}{\rho'_1} (1 + \frac{a' + \mu}{1 - (a' + \mu)} \frac{\mu}{1 - \mu})$$  \hspace{1cm} (3.11)

Then for this planar circular motion to satisfy the Equation (3.6) restriction, the radius of the circular orbit should satisfy the inequality $a' < (1 - 2\mu)/2$, given that $\mu < 0.5$. From this restriction on the value of the orbit radius, the assumption that the motion is sufficiently close to the first primary is quantitatively described.

According to the previous analysis, the Jacobi function in Equation (3.5) or Equation (3.7) can be approximated as follows.

$$J = \frac{1}{2} \omega^2 (x^2 + y^2) + \frac{Gm}{\rho_i}$$  \hspace{1cm} (3.12a)

$$J' = \frac{1}{2} (x'^2 + y'^2) + \frac{1 - \mu}{\rho'_1}$$  \hspace{1cm} (3.12b)

Equation (3.12b) still includes a first order term in the small mass parameter $\mu$. This model is different from the two-body approach in which the first order terms of the mass
parameter are eliminated through the approximation \(1 - \mu \approx 1\). Substituting Equations (3.3), (3.4), and (3.12a) into the Jacobi integral equation, one obtains

\[
v^2 = \omega^2 a^2 + \frac{2Gm}{a} [1 - 2 \left( \frac{x_1}{a} \right) \cos \theta + \left( \frac{x_1}{a} \right)^2 ]^{-1/2} - C
\]  

(3.13)

Define a new function \(G(\theta; \chi)\), where \(\chi = x_i/a\), so that

\[
G(\theta; \chi) = [1 - 2 \chi \cos \theta + \chi^2 ]^{-1/2} = \sum_{m=0}^{\infty} \chi^m P_m(\cos \theta)
\]  

(3.14)

where \(P_m(\cos \theta)\) is the Legendre polynomials defined as

\[
P_m(\cos \theta) = \sum_{l=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \frac{(-1)^l (2m-2l)!}{2^m l! (m-l)! (m-2l)!} (\cos \theta)^{m-2l}
\]  

(3.15)

and the expression \(\left\lfloor \frac{m}{2} \right\rfloor\) means

\[
\left\lfloor \frac{m}{2} \right\rfloor = \begin{cases} 
\frac{m}{2} & \text{if } m \text{ is even} \\
\frac{m-1}{2} & \text{if } m \text{ is odd}
\end{cases}
\]  

(3.16)

Equation (3.14) indicates that the generating function \(G(\theta; \chi)\) has the Legendre polynomials \(P_m(\cos \theta)\) as the coefficients in a Taylor series expansion of \(G(\theta, \chi)\) about \(\chi = 0\) (\(x_i = 0\)). The first three Legendre polynomials are

\[
P_0 = 1, P_1 = \cos \theta, P_2 = \frac{1}{2} (3 \cos^2 \theta - 1)
\]  

(3.17)

Extracting the terms up to \(m = 2\) in Equation (3.14) and substituting the first three coefficients of \(P_m(\cos \theta)\), \(m = 0, 1, 2\) from Equation (3.17), one finds that

\[
G(\theta, \chi) = 1 + \chi \cos \theta + \frac{1}{2} \chi^2 (3 \cos^2 \theta - 1) + \sum_{m=3}^{\infty} P_m(\cos \theta) \chi^m
\]  

(3.18)
From the properties of Legendre polynomials, note \(|P_m(\cos\theta)| \leq 1\). Hence, the coefficients of powers of \(x\) in Equation (3.18) are bounded. Then the necessary condition for convergence in the expansion of the function \(G(\theta, x)\) is that \(|x| < 1\) or \(x_l/a < 1\) (a detailed derivation is found in Appendix A). This condition restricts the lower bound of the orbit radius and the domain of the orbit radius is the open set \((x_l, r_2/2 - x_l)\). Physically, the lower bound of the orbit radius should not be less than the equatorial radius of the first primary \(R_{mi}\) plus the shift from the center of mass \(x_1\); thus, the actual domain of the orbit radius is \((R_{mi} + x_1, r_2/2 - x_l)\). The function \(G(\theta, x)\) is simply the ratio \(a/\rho_1\), expanded as shown in Equation (3.18), and can be geometrically explained as a result of the shift \(x_1\) of the first primary from the center of mass. This ratio should converge to unity when the first primary is considered the center of mass, as in a two-body problem. As a result of convergence of the series in Equation (3.18), terms containing \((x_l/a)^m\) for \(m \geq 2\) can be neglected.

Substituting from Equation (3.18) into Equation (3.13) yields

\[
v^2 = \omega^2 a^2 + \frac{2Gm_1}{a} \left[ 1 + \left\{ \frac{x_1}{a} \right\} \cos\theta + \ldots \right] - C \quad (3.19)
\]

and from the assumed motion in Equation (3.3), the value of \(v\) is substituted as a function of the angular velocity \(v = a\dot{\theta}\). Therefore, Equation (3.19) is a first order nonlinear differential equation for the variation of angle \(\theta\) with time.

\[
\dot{\theta}^2 = \left\{ \omega^2 + \frac{2Gm_1}{a^3} - \frac{C}{a^2} \right\} + \frac{2Gm_1x_1}{a^4} \cos\theta \quad (3.20)
\]

Define a new constant \(C_1\) such that
and by substituting Equation (3.21) into Equation (3.20), the Jacobi integral is rewritten as

$$\dot{\vartheta}^2 = C_1 + \frac{2Gm}{a^3} \cos \theta$$

(3.22)

By evaluating Equation (3.22) at the initial time $t = t_0$, $C_1$ is determined from initial condition $\theta_0$, $\dot{\theta}_0$ as follows.

$$C_1 = \dot{\theta}_0^2 - \frac{2Gm x_1}{a^4} \cos \theta_0$$

(3.23)

Substituting Equation (3.23) into Equation (3.22), the differential equation becomes

$$\dot{\vartheta}^2 = \dot{\theta}_0^2 - \frac{2Gm x_1}{a^4} (\cos \theta_0 - \cos \theta)$$

(3.24)

or expressing the right hand side of Equation (3.24) in terms of half angles of $\theta$ and $\theta_0$

$$\dot{\vartheta}^2 = \dot{\theta}_0^2 \left(1 - \frac{4Gm x_1}{a^4} \frac{\sin^2 \frac{\theta}{2} - \sin^2 \frac{\theta_0}{2}}{2}\right)$$

(3.25)

In a more appropriate format, Equation (3.25) is rewritten as

$$\dot{\vartheta} = \dot{\theta}_0 \left(1 - k^2 \sin^2 \frac{\theta}{2}\right)^{1/2}$$

(3.26)

where

$$\dot{\theta} = \dot{\theta}_0 \left(1 - \frac{4Gm x_1}{a^4} \sin \frac{\theta}{2}\right)$$

(3.27a)

$$k^2 = \frac{4Gm x_1}{a^4} \sin^2 \frac{\theta}{2}$$

(3.27b)
and \( \dot{\theta}' = \dot{\theta}_0 / \omega \). To transform Equation (3.26) to the normal form for the Legendre elliptic integral of the first kind, the transformation \( \phi = \theta / 2 \) is introduced, where \( \phi \) is a dummy variable. Substituting this transformation into Equation (3.26) and integrating gives

\[
\int_0^\phi \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}} = \frac{\dot{\theta}_0}{2} \int_{t_0}^{t} dt
\]

where \( k \) is the modulus of the elliptic integral with \( 0 \leq k^2 \leq 1 \). The upper limit of the integral on the left hand side can be set as \( \phi = \phi_0 + \alpha \) where \( \alpha \) is any interval over the angular displacement \( \phi \). Equation (3.28) is thus rewritten as

\[
\int_0^\alpha \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}} = \frac{\dot{\theta}_0}{2} \int_{t_0}^{t} dt
\]

When \( \alpha = \pi \), the integration is carried out over one complete period of the motion, since \( \phi : \phi_0 \to \phi_0 + \pi, \ \theta : \theta_0 \to \theta_0 + 2\pi, \ t = t_0 \to t_0 + T \). In this case Equation (3.29) is integrated to yield

\[
F(k, \pi) = \frac{\dot{\theta}_0}{2} T
\]

where \( F(k, \pi) \) is the incomplete elliptic integral of the first kind and \( T \) is the period of motion. Using the recursive formula of the elliptic integral of the first kind, Equation (3.30) can be reformulated in terms of the complete elliptic integral of the first kind \( K(k) \) to give the period of the motion as

\[
T = \frac{4}{\dot{\theta}_0} K(k)
\]
For integrating from the initial conditions to a general angular position \( \phi = \theta/2 \) at a general time \( t \), Equation (3.28) is rewritten as

\[
\int_{\theta_0/2}^{\theta/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \frac{\dot{\theta}_0}{2} \int_{t_0}^{t} dt
\]  

(3.32a)

\[
F\left(\frac{\theta}{2}, k\right) = F\left(\frac{\theta_0}{2}, k\right) + \frac{\dot{\theta}_0}{2} (t - t_0)
\]

(3.32b)

In Equation (3.32b), the function \( F(\psi, k) \) is the incomplete elliptic integral evaluated at an angle \( \psi \) for a modulus \( k \). Utilizing the theory of theta functions, the angular position \( \theta \) is expressed as a function of time by

\[
\sin\left\{ \frac{1}{2} \theta(t) \right\} = \text{sn}(F\left(\frac{\theta_0}{2}, k\right) + \frac{\dot{\theta}_0}{2} (t - t_0), k)
\]

(3.33)

where \( \text{sn}(\tau, k) \) is the \( \text{sn} \) Jacobi elliptic function evaluated at an argument \( \tau \). The function \( \text{sn}(\tau, k) \) gives the amplitude \( \phi \) which is the inverse of the elliptic integral \( F(\phi, k) \) of the integration, where \( 0 \leq \phi \leq \pi/2 \).

Analytic expressions for the parameterizing variable \( \theta(t) \) and its time derivative are

\[
\sin\{\theta(t)\} = 2\text{sn}(F\left(\frac{\theta_0}{2}, k\right) + \frac{\dot{\theta}_0}{2} (t - t_0), k)\text{cn}(F\left(\frac{\theta_0}{2}, k\right) + \frac{\dot{\theta}_0}{2} (t - t_0), k)
\]

(3.34a)

\[
\cos\{\theta(t)\} = 1 - 2\text{sn}^2(F\left(\frac{\theta_0}{2}, k\right) + \frac{\dot{\theta}_0}{2} (t - t_0), k)
\]

(3.34b)

\[
\dot{\theta}^2(t) = \dot{\theta}_0^2 \text{dn}^2(F\left(\frac{\theta_0}{2}, k\right) + \frac{\dot{\theta}_0}{2} (t - t_0), k)
\]

(3.34c)

where \( \text{cn}(\tau, k) \) and \( \text{dn}(\tau, k) \) are the \( \text{cn} \) and \( \text{dn} \) Jacobi elliptic functions. Equation (3.34) represents a closed-form integral of the Jacobi integral equation when the motion of the third body is described parametrically by Equation (3.3).
3.3 Initial Conditions and Motion Constraints

Equation (3.27b) gives the modulus $k$ as a function of the mass parameter of the three-body system, the initial angular position, the initial angular velocity, and the radius of circular orbit. Once the specific three-body system is determined, the modulus is only a function of the characteristics of the circular orbit (orbit radius and initial conditions). The inequality of the modulus $0 \leq k^2 \leq 1$ enables exploring possible restrictions on the initial conditions which satisfy the proposed motion. Substituting Equation (3.27b) into the modulus inequality, one has the new inequality

$$0 \leq 4Gm_s \frac{x_1}{a^4} \sin^2 \left\{ \frac{\theta}{2} \right\} \leq 4Gm_s \frac{x_1}{a^4} \left( 1 - \sin^2 \left\{ \frac{\theta}{2} \right\} \right) \leq \dot{\theta}_0^2 \quad (3.35)$$

The left hand side of the inequality in Equation (3.35) leaves no restriction on the characteristics of the motion, while the right hand side describes the lower bound of the initial angular velocity once the initial angular position and orbit radius are determined. When a strict equality holds, the right hand side of Equation (3.35) corresponds to the limiting case when the modulus approaches unity. For any other value of the modulus, Equation (3.27b) can be rewritten as follows.

$$\dot{\theta}_0^2 = 4Gm_s \frac{x_1}{a^4} \left( \frac{1}{k^2} - \sin^2 \left\{ \frac{\theta}{2} \right\} \right) \quad (3.36)$$

In Equation (3.36) the initial angular velocity depends on the initial angular position, circular orbit radius, and the modulus of the elliptic integral. For small values of the modulus, the dependency of the initial angular velocity on the initial angular position is not significant. When the value of the modulus approaches zero this effect may be neglected altogether since $\sin^2 \left\{ \theta_0 / 2 \right\}$ is bounded and of order of one. In this case Equation (3.36) is approximated to
\( \dot{\theta}_0^2 \approx \frac{4Gm x_1}{a^4} \frac{1}{k^2} \)  

(3.37)

Figure 3.2 shows the initial angular velocity, from Equation (3.36), as a function of the initial angular position and radius of the orbit for a family of modulus values. For certain families of orbits having the same modulus, a periodic nature is noticed for the dependency of the initial angular velocity on the initial angular position at specific orbit radii. Also, the initial angular velocity is inversely proportional to the square of orbit radius at certain initial angular positions.

Figure 3.2 Initial Angular Velocity vs. Initial Angular Position and Orbit Radius

The initial velocity of the third body can be obtained from Equation (3.36) as follows.

\[ v_x = \frac{2}{k} \sqrt{\frac{x_1}{a}} \sqrt{1 - k^2 \sin^2 \left( \frac{\theta_0}{2} \right)} \sqrt{\frac{Gm}{a}} \]  

(3.38)

Equations (3.21) and (3.22) define constant \( C_i \); combining these two equations one has
Equation (3.39) is nothing but the truncated Jacobi integral equation, evaluated at the initial conditions \( t = t_0, \ \theta(t_0) = \theta_0, \) and \( \dot{\theta}(t_0) = \dot{\theta}_0. \) Evaluation of the Jacobi integral equation at the initial conditions enables calculating the Jacobi constant, which has the same value at any other time or any other location on the path. For zero velocity, the Jacobi constant is a function of the orbit radius and initial angular position.

\[
C = a^2 \omega^2 + \frac{2Gm}{a} \left(1 + \frac{x_1}{a} \cos \theta_0 \right)
\]  

Equation (3.40)

Figures 3.3 and 3.4 show, at zero velocity, the variation of the Jacobi constant, as in Equation (3.40), with \(a, \theta_0\) and \(x_0, y_0\), respectively.

Figure 3.3 Jacobi Constant and Polar Initial Conditions at Zero Velocity
Equation (3.40) can also be reformulated into a fourth order polynomial in the orbit radius, \( a \), with coefficients of powers of \( a \) containing the initial angular position, \( \theta_0 \), and the Jacobi constant \( C \). This polynomial has the following form

\[
\sum_{i=0}^{4} A_i a^i = 0
\]  

where

\[
A_0 = 2Gm_x \cos(\theta_0) / \omega^2, A_1 = 2Gm_x / \omega^2, A_2 = -C / \omega^2, A_3 = 0, A_4 = 1
\]  

Figure 3.5 shows two separate three dimensional surfaces representing solutions of Equation (3.41). In Figure 3.5, for the same values of the Jacobi constant \( C \) and the initial angle \( \theta_0 \), there are two real positive roots for Equation (3.41); the first root \((a_1 : a / r_{12} < 1)\) represents a circular orbit about the first primary, while the second root \((a_2 : a / r_{12} > 1)\) represents a circular orbit about the two primaries. Also, Figure 3.5
depicts the boundaries for possible circular orbits in the PCRTBP. Circular motion is possible when radius, $a$, is either smaller than $a_1$ or larger than $a_2$ (as explained in detail later). The lower surface in Figure 3.5 gives the ultimate upper boundary values for a circular orbit about the first primary for specific initial conditions, while the upper surface gives the ultimate lower boundary values for a circular orbit which revolves about the two primaries for prescribed initial conditions. As the Jacobi constant decreases, the two surfaces approach each other. At the minimum value of the Jacobi constant, the two surfaces contact each other when $\theta_0 = 0$ or $2\pi$. Between the values $\theta_0 = 0$ and $2\pi$, a gap between the upper and lower surfaces occurs, its maximum gap occurring at $\theta_0 = \pi$.

Figure 3.5 Boundaries of Circular Orbit Radii

Nonuniformity of the solution for Equation (3.41) at the minimum value of the Jacobi constant comes from the singularity in the PCRTBP when the third body intercepts the second primary (it is known that the singularity when the third body
intercepts the first primary, or when the third body moves in a circular orbit with radius
$a = x_1$, is to be avoided through the introduced solution). Since the circular motion is
assumed to be about the center of mass, which is close to the first primary, for small mass
parameters, a singularity is experienced when the orbit of the third body approaches the
second primary, i.e., when the circular orbit radius approaches the distance between the
two primaries. This behavior can be extracted from Equation (3.40), when calculating the
value of $a$ that makes the partial derivative $dC/da$ vanish. The two positive values that
satisfy this condition are found to be $a^* = r_{12} (1 + (-1)^i \varepsilon_i)$, $\varepsilon_i < 1$, $i = 1, 2$. Since the
second partial derivative $d^2C/da^2$ is positive for all $a$, the Jacobi constant is a minimum
at the two values $a^*_i$, $i = 1, 2$. Figure 3.5 shows that singularity in the solution occurs at
the minimum Jacobi constant, $C = C_{\min}$. At that point the orbit of the third body
approaches the second primary either from the side near the first primary or from the side
far from the first primary. When neglecting the effect of the second primary (as in a two-
body problem$^{13}$), Equation (3.41) is reduced to a third order polynomial in $a$ and the
minimum Jacobi constant is $C_{\min} = 3\omega^2 r_{12}^2$ which occurs at two equal roots,
$a^*_i = r_{12}$, $i = 1, 2$. In this case the two surfaces in Figure 3.5 contact along one continuous
line at $C = C_{\min} = 3\omega^2 r_{12}^2$. Singularities at the two primaries are considered in the three-
body problem, while in the two-body problem a singularity is only present at the first primary.

Considering the lower zero velocity surface in Figure 3.5 which is characterized
by orbit radius $a_i < 1$, when slightly changing the orbit radius from $a_i$ to $a = a_i + \Delta a_i$ or
$a = a_i - \Delta a_i$ in such a manner that $\Delta a_i > 0$ and $a < 1$, and when both changes occur at the
same angle $\theta_0$, and Jacobi constant $C_1$, it is found that the velocity $v_1$ changes from zero (at the surface) to $v$ such that either $v^2 < -(\Delta a_1)^2$, or $v^2 > (\Delta a_1)^2$, respectively. Thus, the third body cannot move in a circular orbit with a radius value above the lower zero velocity surface. A similar analysis considering the upper zero velocity surface in Figure 3.5 indicates that when the orbit radius changes from $a_2$ to $a = a_2 + \Delta a_2$ or $a = a_2 - \Delta a_2$, $\Delta a_2 > 0$ and $a > 1$, at the same angle $\theta_0$ and Jacobi constant $C_1$, the velocity $v_2$ changes from zero, at the surface, to $v$ such that either $v^2 > (\Delta a_2)^2$, or $v^2 < -(\Delta a_2)^2$, respectively. Thus, the third body cannot move in a circular orbit with a radius value below the upper zero velocity surface in Figure 3.5. The preceding analysis indicates that the third body cannot move in the notional circular orbit with a radius value taken from the region between the two zero velocity surfaces in Figure 3.5. Together, values of orbit radius either below the lower zero velocity surface or above the upper zero velocity surface in Figure 3.5 constitute the open set $(a_1, a_2)$ of values of possible circular orbit radii (a detailed derivation is found in Appendix B).

The constant term $A_0$ in the polynomial in Equation (3.41) is a periodic function in the initial angular position. This relation explains the periodic nature of the dependency of the orbit radius on the initial angular position. In order to explore this dependency, it is interesting to view the polar plot in Figure 3.6, in which the orbit radius is drawn on the radial axis and initial angular position is drawn on the transverse axis, for a group of values of the Jacobi constant. Notice that at the same value of the Jacobi constant the orbit radius is a maximum at $\theta_0 = 0$ and a minimum at $\theta_0 = \pi$. This behavior can be investigated using Equation (3.40) when finding the angles at which the
partial derivative $\partial a / \partial \theta_0$ vanishes ($\theta'_0 = 0$ and $\pi$) and then substituting these two angles into Equation (3.40) and checking the sign of the second partial derivative $\partial^2 a / \partial \theta^2_0$. The dependency of the orbit radius on the initial angular position can be explained as follows. For the same value of the Jacobi constant, the third body starts at a point radially farther from the center of mass when the initial angular position varies from $\theta_0 = 0$, i.e., when the effect of the second primary decreases. When the third body starts at a location on the line of syzygy, specifically on the negative $x$ axis, the effect of the second primary is maximum, and the corresponding orbit radius is closer to the mass center.

![Figure 3.6 Orbit Radius and Initial Angle Polar Phase Space at Zero Velocity]

**Figure 3.6 Orbit Radius and Initial Angle Polar Phase Space at Zero Velocity**

### 3.4 Initial Relative Velocity

Since the motion is assumed to be in the vicinity of the first primary, it is interesting, from a practical point of view, to explore the expression of the velocity of the third body relative to the first primary. In the rotating coordinate system, the first primary
is fixed on the rotating $x$ axis at a distance $x_1$ from the center of mass. When transforming from a synodic (rotating) $xyz$ to a sidereal (inertial) $XYZ$ coordinate system, as indicated by Equation (3.43)

$$r_{xyz} = A r^{XYZ}$$

(3.43a)

where

$$A = \begin{bmatrix} \cos{\omega \tau} & \sin{\omega \tau} & 0 \\ -\sin{\omega \tau} & \cos{\omega \tau} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(3.43b)

is the transformation matrix, and $\tau = t - t_0$; the first primary appears to move in a circular orbit relative to the center of mass with the same angular velocity of the synodic system. The instantaneous velocity vector $V_i = \omega x_i \hat{j}$ is parallel to the rotating $y$ axis. Figure 3.7 shows the PCRTBP in both the sidereal and synodic coordinate systems.

Figure 3.7 CRTBP in Sidereal and Synodic Coordinate Systems
Generally, $V$ is the velocity vector of the third body in the sidereal coordinate system, obtained using the following formula

$$V = v + \omega \times r$$  \hspace{1cm} (3.44)

where $v$ is the velocity vector of the third body in the synodic coordinate system. The magnitude of $v$ is defined by Equation (3.38), and its direction is the tangential at any point along the orbit. Applying the assumed circular motion characteristics through Equation (3.44), one finds

$$V = \begin{bmatrix} -v \sin \{\theta + \omega \tau\} - \omega (x \sin \{\omega \tau\} + y \cos \{\omega \tau\}) \\ v \cos \{\theta + \omega \tau\} + \omega (x \cos \{\omega \tau\} - y \sin \{\omega \tau\}) \\ 0 \end{bmatrix}$$  \hspace{1cm} (3.45)

The velocity vector of the third body relative to the first primary in the sidereal coordinate system, $V_r = V - V_1$, is

$$V_r = \begin{bmatrix} -v \sin \{\theta + \omega \tau\} - \omega [(x - x_1) \sin \{\omega \tau\} + y \cos \{\omega \tau\}] \\ v \cos \{\theta + \omega \tau\} + \omega [(x - x_1) \cos \{\omega \tau\} - y \sin \{\omega \tau\}] \\ 0 \end{bmatrix}$$  \hspace{1cm} (3.46)

When $t = t_0 \Rightarrow \tau = 0$, $\theta(t_0) = \theta_0 = 0$, $x(t_0) = x_0 = a$, $y(t_0) = y_0 = 0$, $v(t_0) = v_0$, the third body motion starts at point $P_0$ perpendicular to the $x$ axis in the positive direction of the $y$ axis. The magnitude of the velocity vector $V_{r_0}$ is

$$V_{r_0} = \frac{2}{k} \sqrt{\frac{x_1}{a}} \sqrt{\frac{Gm_1}{a}} + \omega(a - x_1)$$  \hspace{1cm} (3.47)

Figure 3.8 shows the initial relative velocity magnitude in the sidereal coordinate system, for a range of values of the modulus and the orbit radius, in this particular case. It is instructive to compare the velocity of the small body relative to a finite mass, as in a two-body problem, expressed in the sidereal coordinate system as
Figure 3.8 Initial Relative Velocity vs. Orbit Radius and Elliptic Modulus

\[ V_{tb} = \sqrt{\frac{Gm_1}{(a - x_i)}} \]  

(3.48)

where the subscript \( tb \) denotes the two-body problem. Using a binomial expansion, Equation (3.48) is rewritten as follows

\[ V_{tb} = \sqrt{\frac{Gm_1}{a}} \sum_{i=0}^{\infty} \frac{(2i)!}{(2^i i!)^2} \kappa^i \]  

(3.49)

Equation (3.49) indicates that the velocity of the third mass is a function of the orbit radius, and it is constant along the circular path. However, the velocity of the third body in Equation (3.47) depends on two parameters: orbit radius and the modulus of the elliptic integral. Figure 3.9 shows the velocity of the third body as calculated from Equations (3.47) and (3.49).

The coordinates of the third body in the synodic coordinate system can be obtained when substituting Equations (3.34a) and (3.34b) into Equation (3.3).
3.5 Periodic Orbits in the Sidereal and Synodic Systems

\[
x(t) = a\left(1 - 2\sin^2 \left\{ F\left( \frac{\theta_0}{2}, k \right) + \frac{\dot{\theta}_0}{2} (t - t_0), k \right\} \right)
\]

\[
y(t) = 2a \sin \left\{ F\left( \frac{\theta_0}{2}, k \right) + \frac{\dot{\theta}_0}{2} (t - t_0), k \right\} \cosh \left\{ F\left( \frac{\theta_0}{2}, k \right) + \frac{\dot{\theta}_0}{2} (t - t_0), k \right\}
\]

Equation (3.50) indicates that the coordinates are periodic functions of time and three other parameters, i.e., the orbit radius, the initial angular velocity, and the modulus of the elliptic integral. Since Equation (3.36) gives the initial angular velocity as a function of the orbit radius and the modulus, only two parameters are needed and once specified the coordinates in Equation (3.50) are only functions of time. Using the transformation in Equation (3.43a), the coordinates of the third body are obtainable in the sidereal coordinate system.

\[
X = x \cos \{ \omega t \} - y \sin \{ \omega t \}
\]
\[ Y = x \sin\{\omega t\} + y \cos\{\omega t\} \quad (3.51b) \]

The right hand side of Equation (3.51) is not periodic in general because of the existence of time explicitly inside the circular functions even though the coefficients for \( x \) and \( y \) are periodic. The sidereal trajectory is periodic only when the period of motion of the third body in the synodic coordinate system is a multiple of the period of rotation of the synodic system relative to the sidereal system, i.e.,

\[ T = n \left( \frac{2\pi}{\omega} \right) \quad (3.52) \]

In the special case when \( \theta_0 = 0 \) and when substituting from Equations (3.27a), (3.27b), and (3.31) into Equation (3.52), one obtains the following relation between the orbit radius and the modulus.

\[ a^2 = \frac{\pi \sqrt{Gm_1x_1}}{\omega k K(k)} n \quad (3.53) \]

Figure 3.10 shows the relation between \( a \) and \( k \) for different values of \( n \).

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**Figure 3.10 Orbit Radius vs. Elliptic Modulus for Different Period Ratios**
CHAPTER 4
CORRECTING THE PLANAR PERIODIC ORBIT

4.1 Introduction

In this chapter, the approximate planar periodic solution to the circular restricted three-body problem when the mass parameter is small and motion of third body is in the vicinity of the first primary, which is introduced in Chapter 3, is subject to an analytic iterative correction process. The correction will be done for both out-of-plane (vertical) and in-plane motion. The present iterative scheme is based on adding small terms with orders of magnitude less than the base solution. The correction process steps are summarized as follows. The correct solution is written in a perturbation-like expansion form and then substituted into the original nonlinear equation of motion to give an expanded version of this equation. The nominal solution is considered as a base solution and is extracted from the expanded equation of motion. Generally, this procedure results in a system of variational like differential equations in the correction terms with variable coefficients and forcing terms. If the variational equation consists of linear differential equations with periodic coefficients it can be solved qualitatively using Floquet theory and quantitatively using perturbation methods. Approximating the Jacobi elliptic functions as circular functions, the out-of-plane motion is solved independently of the in-plane motion. A Lindstedt-Poincaré technique (Method of Strained Parameters) is used to eliminate secular perturbation producing terms and obtain a uniformly valid perturbation solution. Finally, the correction for the out-of-plane motion is found to be decoupled from the in-plane motion and only exists for out-of-plane initial excitation conditions.
4.2 Correction Process

The circular orbit developed in Chapter 3 can be used as a base solution in the first step of this iterative correction process. Coordinates in this base solution are expressed as functions of time as follows

\[ x_b(t) = a - 2a \text{sn}^2 \left( \frac{\theta_0}{2}, k \right) + \frac{\dot{\theta}_0}{2} (t - t_0), k \] (4.1a)

\[ y_b(t) = 2a \text{sn} \left( \frac{\theta_0}{2}, k \right) + \frac{\dot{\theta}_0}{2} (t - t_0), k \text{cn} \left( \frac{\theta_0}{2}, k \right) + \frac{\dot{\theta}_0}{2} (t - t_0), k \] (4.1b)

\[ z_b(t) = 0 \] (4.1c)

where the subscript \( b \) denotes the base solution. The coordinates in this orbit have the following initial conditions at \( t = t_0 \): \( x(t_0) = x_0, \ y(t_0) = y_0, \ z(t_0) = z_0 \). Let the coordinates in the corrected solution take the form

\[ x(t) = x_b(t) + x_{c1}(t) + x_{c2}(t) + \ldots \] (4.2a)

\[ y(t) = y_b(t) + y_{c1}(t) + y_{c2}(t) + \ldots \] (4.2b)

\[ z(t) = z_b(t) + z_{c1}(t) + z_{c2}(t) + \ldots \] (4.2c)

where the subscripts \( c_i, i = 1, 2, 3, \ldots \) denote correction terms. Let \( q \) denote a magnitude of a coordinate \( x(t), \ y(t), \) or \( z(t) \), then \( q_{ci+1} < q_{ci} \), where \( q_{ci} \) represents the \( i^{th} \) correction term, and \( \sum_{i=1}^{\infty} q_{ci} < q_b \) and \( q_b \) is a base solution term. Table 1 represents the mechanics of the correction process.
Table 4.1 Iterative Correction Procedures

\[ Z \rightarrow x(t), y(t) \text{ (Base Solution)} \]
\[ x(t) \rightarrow x(t), y(t) \text{ (First Correction)} \]
\[ x(t) \rightarrow x(t), y(t) \text{ (Second Correction)} \]

Substituting Equation (4.2), with the right hand side including only the base and the first term correction, into Equation (2.71) motion relations one obtains

\[
\ddot{x}_b + \dot{x}_{cl} - 2\omega (\dot{y}_b + \dot{y}_{cl}) = \omega^2 (x_b + x_{cl}) - \left[ \frac{Gm_1}{\rho_1^3} (x - x_1) \right]_b - \left[ \frac{Gm_2}{\rho_2^3} (x - x_2) \right]_b \\
- \frac{Gm_1}{\rho_1^5} \left[ \frac{1}{\rho_1^3} - \frac{3(x - x_1)^2}{\rho_1^5} \right] x_{cl} - \frac{3y(x - x_1)}{\rho_1^5} y_{cl} - \frac{3z(x - x_1)}{\rho_1^5} z_{cl} \\
- \frac{Gm_2}{\rho_2^5} \left[ \frac{1}{\rho_2^3} - \frac{3(x - x_2)^2}{\rho_2^5} \right] x_{cl} - \frac{3y(x - x_2)}{\rho_2^5} y_{cl} - \frac{3z(x - x_2)}{\rho_2^5} z_{cl} \\
+ \text{HOT}
\]

(4.3a)
\[ \dot{y}_b + \dot{y}_{c1} + 2\omega(\dot{x}_b + \dot{x}_{c1}) = \omega^2(y_b + y_{c1}) - \left[ \frac{Gm_1 y}{\rho_1^3} \right]_b - \left[ \frac{Gm_2 y}{\rho_2^3} \right]_b \\
- Gm_1 \left[ - \frac{3(x - x_1) y}{\rho_1^5} \right]_{c1} x_{c1} + \left[ \frac{1}{\rho_1^3} - \frac{3y^2}{\rho_1^5} \right] y_{c1} - \left[ \frac{3yz}{\rho_1^5} \right] z_{c1} \right] \\
- Gm_2 \left[ - \frac{3(x - x_2) y}{\rho_2^5} \right]_{c1} x_{c1} + \left[ \frac{1}{\rho_2^3} - \frac{3y^2}{\rho_2^5} \right] y_{c1} - \left[ \frac{3yz}{\rho_2^5} \right] z_{c1} \right] + \text{HOT} \quad (4.3b) \]

\[ \ddot{z}_b + \ddot{z}_{c1} = - \left[ \frac{Gm_1 z}{\rho_1^3} \right]_b - \left[ \frac{Gm_2 z}{\rho_2^3} \right]_b \\
- Gm_1 \left[ - \frac{3(x - x_1) z}{\rho_1^5} \right]_{c1} x_{c1} - \left[ \frac{1}{\rho_1^3} - \frac{3z^2}{\rho_1^5} \right] y_{c1} + \left[ \frac{1}{\rho_1^3} - \frac{3z^2}{\rho_1^5} \right] z_{c1} \right] \\
- Gm_2 \left[ - \frac{3(x - x_2) z}{\rho_2^5} \right]_{c1} x_{c1} - \left[ \frac{1}{\rho_2^3} - \frac{3z^2}{\rho_2^5} \right] y_{c1} + \left[ \frac{1}{\rho_2^3} - \frac{3z^2}{\rho_2^5} \right] z_{c1} \right] + \text{HOT} \quad (4.3c) \]

The right hand side of the equation of motion, which contains partial derivatives of the Jacobi function with respect to coordinates, is expanded using Taylor series concepts about the base solution, where HOT means higher order terms in the correction variables (second order or higher order terms). Any term in Equation (4.3) with the subscript \( b \) indicates that this term is evaluated using the base solution. Extracting the base solution in Equation (4.3) and enforcing \( z_b(t) = 0 \), one obtains a system of differential equations for the first corrections

\[ \ddot{x}_{c1} - 2\omega \dot{x}_{c1} = \left\{ \omega^2 - Gm_1 \left[ \frac{1}{\rho_1^3} - \frac{3(x - x_1)^2}{\rho_1^5} \right] - Gm_2 \left[ \frac{1}{\rho_2^3} - \frac{3(x - x_2)^2}{\rho_2^5} \right] \right\} x_{c1} \]
\[ + \left\{ Gm_1 \left[ \frac{3y(x - x_1)}{\rho_1^5} \right] + Gm_2 \left[ \frac{3y(x - x_2)}{\rho_2^5} \right] \right\} y_{c1} + \text{HOT} + E_x \quad (4.4a) \]
\[ \ddot{y}_{cl} + 2\omega \dot{x}_{cl} = \left[ Gm_1 \left( \frac{3(x-x_1)y}{\rho_1^5} \right) + Gm_2 \left( \frac{3(x-x_2)y}{\rho_2^5} \right) \right] x_{cl} + \left[ \omega^2 - Gm_1 \left( \frac{1}{\rho_1^3} - \frac{3y^2}{\rho_1^5} \right) - Gm_2 \left( \frac{1}{\rho_2^3} - \frac{3y^2}{\rho_2^5} \right) \right] y_{cl} + \text{HOT} + E_y \]  
(4.4b)

\[ \ddot{z}_{cl} = -\left[ Gm_1 \left( \frac{1}{\rho_1^3} \right) + Gm_2 \left( \frac{1}{\rho_2^3} \right) \right] z_{cl} + \text{HOT} + E_z \]  
(4.4c)

where \( E_x, E_y, \) and \( E_z \) are the control inputs required to negate variations due to applying the nominal solution in the \( x, y, \) and \( z \) directions, respectively.

\[ E_x = \omega^2 x_b - \left[ Gm_1 \left( x-x_1 \right) \right] - \left[ Gm_2 \left( x-x_2 \right) \right] - \ddot{x}_b + 2\omega \dot{y}_b \]  
(4.4d)

\[ E_y = \omega^2 y_b - \left[ Gm_1 \left( \frac{y}{\rho_1^3} \right) \right] - \left[ Gm_2 \left( \frac{y}{\rho_2^3} \right) \right] - \dot{y}_b - 2\omega \dot{x}_b \]  
(4.4e)

\[ E_z = -\left[ Gm_1 \left( \frac{z}{\rho_1^3} \right) \right] - \left[ Gm_2 \left( \frac{z}{\rho_2^3} \right) \right] - \ddot{z}_b \]  
(4.4f)

4.3 Vertical Correction

In this section the out-of-plane motion which is not considered in the approximate base solution is corrected. The perturbation method is used to quantitatively solve for the small order correction terms. Floquet theory and period advance mapping are used to explore stability of the vertical motion.

Equation (4.4f) shows that \( E_z \) vanishes when \( z_b = \dot{z}_b = \ddot{z}_b = 0 \), so that no external input is needed to maintain the component of the base solution in the vertical direction.
To take advantage of the assumptions of the nominal solution Equation (4.4c) can be reformulated in the following form.

$$
\ddot{z}_{c1} = -\left(\frac{Gm_2}{\rho_1^3}\right)_b \left[1 + \frac{m_2}{m_1} \left(\frac{\rho_1}{\rho_2}\right)^3\right] z_{c1}
$$

(4.5)

To determine the order of magnitude of the term \((m_2/m_1)(\rho_1/\rho_2)^3\), Equation (4.5) can be normalized using the parameters of the CRTBP discussed in Chapter 2, or

$$
\ddot{z}_{c1}' = -\left(\frac{1-\mu}{\rho_1^3}\right) \left[1 + \frac{\mu}{1-\mu} \left(\frac{\rho_1'}{\rho_2'}\right)^3\right] z_{c1}'
$$

(4.6)

where \(\mu = m_2/(m_1 + m_2)\) is the mass parameter and \(\dot{z}_{c1}' = \dot{z}_{c1}/\omega r_{12}, \quad \rho_1' = \rho_1/r_{12}, \quad \rho_2' = \rho_2/r_{12}, \quad z_{c1}' = z_{c1}/r_{12}\). From Chapter 3, the maximum value of the quantity \((\rho_1' / \rho_2')\) is \((a' + \mu)/(1-a' - \mu)\) with \(a' = a/r_{12}\) denoting the non-dimensional orbit radius base. The sufficient conditions for \((m_2/m_1)(\rho_1/\rho_2)^3\) in Equation (4.5), to be on the order of less than one are

$$
\mu < 1/2
$$

(4.7a)

and

$$
a' < (1-2\mu)/2
$$

(4.7b)

These two inequalities determine boundaries for the base solution to be valid for the iteration process such that the corrected solution accuracy is unaffected by neglecting second order and higher terms. From the above analysis, the qualitative statements regarding the small mass parameter and motion in the vicinity of the first primary are
quantitatively described. As a result of the above analysis, the term containing 
\((m_2/m_1)(\rho_1/\rho_2)^3\) can be neglected and Equation (4.5) is simplified to

\[
\ddot{z}_{cl} = -\left[\frac{Gm_1}{\rho_1^3}\right] z_{cl} \tag{4.8}
\]

4.3.1 Free Oscillation Solution \((\mu = 0)\)

Equation (3.4) showed that when the second primary was neglected, the origin of the synodic system was transferred to the first primary, i.e., \(x_1 \to 0\), the distance, \(\rho_1\), is approximated as the orbit radius, and Equation (4.8) represents a second order linear ordinary differential equation with constant coefficients. The general solution of this equation takes the form

\[
z_{cl}(t) = A_{z_{cl}} e^{i\omega_n t} + \overline{A}_{z_{cl}} e^{-i\omega_n t} \tag{4.9a}
\]

where \(A_{z_{cl}}, \Lambda_{z_{cl}}\) are complex amplitudes and complex characteristic roots and \(\overline{A}_{z_{cl}}\), \(\overline{\Lambda}_{z_{cl}}\) are their conjugates. If the initial conditions are \(z_{cl}(t_0) = z_{cl}(0) = z_{cl_0}\), and \(\dot{z}_{cl}(t_0) = \dot{z}_{cl}(0) = \dot{z}_{cl_0}\), Equation (4.9a) can be rewritten as follows

\[
z_{cl}(t) = z_{cl}(0) \cos\{\omega_n t\} + \frac{\dot{z}_{cl}(0)}{\omega_n} \sin\{\omega_n t\} \tag{4.9b}
\]

where \(\omega_n = \sqrt{Gm_1/a^3}\) is the natural frequency of the out-of-plane motion of the third body. This frequency is equal to the mean motion of a small body as it orbits the Earth in the two-body problem. If the initial conditions of the out-of-plane motion are chosen so that \(z_{cl}(0) = \dot{z}_{cl}(0) = 0\), the lead correction term in the \(z\) direction vanishes.
4.3.2 Parametric Excitation Solution ($\mu \neq 0$)

After applying the base solution, $\rho_l$ can generally be written as follows

$$\rho_l = a \left( 1 - 2 \left( \frac{x_1}{a} \right) \cos(\theta) + \left( \frac{x_1}{a} \right)^2 \right)^{1/2} \quad (4.10)$$

Thus,

$$\frac{1}{\rho_l} = \frac{1}{a} \sum_{m=0}^{\infty} \left( \frac{x_1}{a} \right)^m P_m(\cos(\theta)) \quad (4.11)$$

where $P_m(\cos(\theta))$ are the Legendre polynomials with arguments $\cos(\theta)$. Substituting Equation (4.11) into Equation (4.8), one obtains

$$\ddot{z}_{cl} + \frac{Gm}{a^2} \left( 1 + 3 \left( \frac{x_1}{a} \right) \cos(\theta) + \ldots \right) z_{cl} = 0 \quad (4.12)$$

Equation (4.12) is a second order linear differential equation with periodic coefficients.

When normalizing Equation (4.12) by dividing by $\omega_n^2 r_1$ (which is different from the normalization used in Equation (4.6)) and reformulating, one obtains

$$\ddot{z}_{cl} + (1 + f_{z_{cl}}(t)) \dot{z}_{cl} = 0 \quad (4.13a)$$

where

$$f_{z_{cl}}(t) = 3 \left( \frac{x_1}{a} \right) \cos(\theta) + \frac{3}{2} \left( \frac{x_1}{a} \right)^2 \left( 5 \cos^2(\theta) - 3 \right) + \ldots \quad (4.13b)$$

is a $T$ periodic function. Equation (4.13a) represents a simple harmonic oscillator with a natural frequency, $\omega_n$, but with an internal (included in the coefficient) excitation of period $T$. 
The solution for the out-of-plane correction with $\mu \neq 0$ using a perturbation method depends on the structure of a small parameter multiplied by the internal time-dependent coefficient term. When this small parameter is set to zero the differential equation has no internal excitation, and the period and final solution are a limiting case (see previous section with $\mu = 0$). Furthermore, when the small parameter is set to zero there is only one solution typically possessing marginal stability, and the motion in this case is governed by selecting appropriate initial conditions. When initial conditions are chosen to eliminate coefficients for the unbounded terms, the solution becomes periodic. On the other hand, when the small parameter takes any small arbitrary finite value (of order less than one), there are an infinite set of small higher order functions added to the limiting solution (more details are found in Reference 21). The motion in this case can be stable or unstable, in addition to marginally stable, depending on certain factors.

If $\varepsilon = x_i/a$ is a small parameter, Equation (4.13a) can be rewritten as follows

$$z_{el}' + \left(1 + \varepsilon f_1(t) + \varepsilon^2 f_2(t) + \ldots\right)z_{el}' = 0$$

(4.14a)

where

$$f_1(t) = 3\cos\{\theta\}$$

(4.14b)

$$f_2(t) = \frac{3}{2}\left(5\cos^2\{\theta\} - 3\right)$$

(4.14c)

Using the straightforward expansion method, the variable $z_{el}'$ can be restructured as follows.

$$z_{el}' = \sum_{i=0}^{\infty} \varepsilon^i z_{el}'$$

(4.15)
Substituting Equation (4.15) into Equation (4.14a) and equating coefficients of equal powers, one obtains

\[ \ddot{z}_{c10} + \dot{z}_{c10} = 0 \]  (4.16a)

\[ \ddot{z}_{c11} + \dot{z}_{c11} = -f_1(t)\dot{z}_{c10} \]  (4.16b)

\[ \ddot{z}_{c12} + \dot{z}_{c12} = -f_1(t)\dot{z}_{c11} - f_2(t)\dot{z}_{c10} \]  (4.16c)

Equation (4.15) can also be used to calculate the initial conditions (positions and velocities) for all orders; from this, one finds that

\[ z_{c10}(0) = z_{c10}, \quad \dot{z}_{c10}(0) = \dot{z}_{c10} \]  (4.16d)

\[ z_{c11}(0) = z_{c11}, \quad \dot{z}_{c11}(0) = \dot{z}_{c11} = 0 \]  (4.16e)

\[ z_{c12}(0) = z_{c12}, \quad \dot{z}_{c12}(0) = \dot{z}_{c12} = 0 \]  (4.16f)

The solution to Equation (4.16a) represents the zero\textsuperscript{th} order solution; a free oscillation without internal excitation in the out-of-plane motion. This solution was introduced in Equation (4.9b) in Section 4.3.1 and is rewritten after normalization as

\[ z'_{c10}(t) = z'_{c10} \cos \{t\} + z'_{c10} \sin \{t\} \]  (4.17)

Observe that the internal parametric forcing in the original equation, (Equation (4.13a)), is converted either to external forcing terms or nonhomogeneity in Equations (4.16b) and (4.16c). The first order solution has a homogeneous part identical to the zero\textsuperscript{th} order solution and vanishes depending on its zero initial conditions, as described in Equation (4.16e). The forcing function, \( f_1(t) \), is expressed explicitly in time using
Equations (4.14b) and (3.34b). When substituting Equation (3.34b) into Equation (4.14b) and using the trigonometric two term expansion\textsuperscript{112, 113} of the elliptic function, one obtains

\begin{equation}
 f_i(t) = 3\left\{1 - 2\sin^2 \{\tau\} + k^2 \sin\{\tau\}\cos\{\tau\}\right\} \left[\tau - \sin\{\tau\}\cos\{\tau\}\right] + \ldots \end{equation} (4.18a)

\begin{equation}
 \tau = F\left(\frac{\theta_0}{2}, k\right) + \frac{\theta_0}{2}\left(t - t_0\right) \end{equation} (4.18b)

For small values of the modulus, i.e., \( k \ll 1 \), the elliptic function \( \text{sn}(\tau, k) \) is approximated by the circular function \( \sin(\tau) \). With this approximation, the solution for the out-of-plane motion through the first term expansion is

\begin{align}
 z_{c1}' &= z_{c10}'(0)\cos(\tau') + \dot{z}_{c10}'(0)\sin(\tau) + \varepsilon A_{z_{c11}} \cos\left((\dot{\theta}_0 + 1)t\right) + B_{z_{c11}} \cos\left((\dot{\theta}_0 - 1)t\right) \\
 &\hspace{1cm} + C_{z_{c11}} \sin\left((\dot{\theta}_0 + 1)t\right) + D_{z_{c11}} \sin\left((\dot{\theta}_0 - 1)t\right) + O(\varepsilon^2) \end{align} (4.19)

where

\begin{align}
 A_{z_{c11}} &= -\frac{3}{2} \frac{z_{c10}'}{\left[1 - (\dot{\theta}_0 - 1)^2\right]} \end{align} (4.20a)

\begin{align}
 B_{z_{c11}} &= -\frac{3}{2} \frac{z_{c10}'}{\left[1 - (\dot{\theta}_0 + 1)^2\right]} \end{align} (4.20b)

\begin{align}
 C_{z_{c11}} &= -\frac{3}{2} \frac{\dot{z}_{c10}'}{\left[1 - (\dot{\theta}_0 + 1)^2\right]} \end{align} (4.20c)

\begin{align}
 D_{z_{c11}} &= \frac{3}{2} \frac{\dot{z}_{c10}'}{\left[1 - (\dot{\theta}_0 - 1)^2\right]} \end{align} (4.20d)

The perturbation solution in Equation (4.19) is nonuniformly valid due either to singularity or small divisors, when the initial angular velocity has values \( \dot{\theta}_0 = 0, 2, \) or \(-2\). If additional terms in the expansion of the elliptic \( \text{sn} \) function are included, additional
singularities occur when $|\dot{\theta}_0| = n$, where $n$ is an integer greater than two. From Equation (3.36), the initial conditions are related to the elliptic modulus and for initial angular position $\theta_0 = 0$, the relation between orbit radius and elliptic modulus at the singularity is

$$\frac{2}{k} \sqrt{\frac{x_1}{a}} = \frac{\dot{\theta}_0}{\omega_n}, \quad \frac{\dot{\theta}_0}{\omega_n} = 0 \text{ or } 2$$

(4.21)

Figure 4.1 shows the relation between orbit radius and elliptic modulus from Equation (4.21) for $\dot{\theta}_0 = 2$ in the Earth-Moon system. Recalling that the range of orbit radius values are chosen so that $x_1 < a < (r_{12} - 2x_1)/2$, the minimum value for the orbit radius

![Figure 4.1](image-url)  

**Figure 4.1** $k$ vs. $a$ at Singularity in the Perturbation Solution

$a = x_1$ corresponds to the upper limit of the elliptic modulus $k = 1$. For any value within the required range of the orbit radius, if the corresponding value of the elliptic modulus is
chosen not to be located on the curve in Figure 4.1, the perturbation solution is valid because \( \dot{\theta}_0' \neq 2 \).

If the point in the \( a-k \) plane is located on the curve containing the singularity, an additional expansion of the independent time variable must be introduced through transforming time to another variable. A technique such as the Lindstedt-Poincaré procedure\(^{111}\) (Method of Strained Parameters) is used here, introducing a new independent variable \( \eta \) such that

\[
\eta = st, \quad s = 1 + \varepsilon_1 + \varepsilon^2 s_2 + \ldots \tag{4.22}
\]

where \( s \) is a dimensionless frequency expanded about the natural frequency (which is normalized to be unity). Substituting Equations (4.15) and (4.22) into Equation (4.14a) and equating coefficients, one obtains

\[
\frac{d^2z_{el0}'}{d\eta^2} + z_{el0}' = 0 \tag{4.23a}
\]

\[
\frac{d^2z_{el1}'}{d\eta^2} + z_{el1}' = -2s_1 \frac{d^2z_{el0}'}{d\eta^2} - f_1(\eta)z_{el0}' \tag{4.23b}
\]

\[
\frac{d^2z_{el2}'}{d\eta^2} + z_{el2}' = -f_2(\eta)z_{el0}' - (s_1^2 + 2s_2) \frac{d^2z_{el0}'}{d\eta^2} - f_1(\eta)z_{el1}' - 2s_1 \frac{d^2z_{el1}'}{d\eta^2} \tag{4.23c}
\]

with initial conditions

\[
z_{el0}'(t_0) = z_{el0}', \quad \frac{dz_{el0}'}{d\eta}(t_0) = \dot{z}_{el0}' \tag{4.23d}
\]

\[
z_{el1}'(t_0) = 0, \quad \frac{dz_{el1}'}{d\eta}(t_0) = 0 \tag{4.23e}
\]
\[ z'_{e10}(t_0) = 0, \quad \frac{dz'_{e10}}{d\eta}(t_0) = 0 \]  

(4.23f)

The zero\textsuperscript{th} order solution has the same form as Equation (4.17). After transforming the initial conditions using Equation (4.22) it is found that \[ z'_{e10}(\eta_0) = z'_{e10}(t_0), \]

\[ \frac{dz'_{e10}(\eta_0)}{d\eta} = z'_{e10}(t_0). \]

For the case in which the initial velocity vanishes, i.e., \[ z'_{e10}(t_0) = 0, \quad z'_{e10}(t_0) = z'_{e10}, \]

the zero\textsuperscript{th} order solution is

\[ z'_{e10}(\eta) = z'_{e10} \cos \{\eta\} \]  

(4.24)

The function \( f_i(t) \) can be transformed using the inverse to the transformation in Equation (4.22) as follows

\[ t = \eta/s = (1 - \varepsilon_1 + \varepsilon^2(s_1^2 - s_2))\eta \]  

(4.25)

The expansion of the right hand side of Equation (4.23b) includes some secular producing terms with dependency on circular forcing functions whose frequency is the same as the natural frequency. A value of \( s_1 = 3/4 \) is necessary to remove these secular producing terms. Approximating the elliptic function as a circular function, Equation (4.23b) can be rewritten as follows

\[ \frac{d^2z'_{e11}}{d\eta^2} + z'_{e11} = -\frac{3}{2} z'_{e10} \cos \{3\eta\} \]

(4.26)

The particular solution to Equation (4.26) is

\[ z'_{e11}(\eta) = \frac{3}{16} z'_{e10} \cos \{3\eta\} \]  

(4.27)
When substituting Equations (4.24) and (4.27) into Equation (4.15), one obtains a uniformly valid perturbation solution of the first correction term in the out-of-plane motion.

\[ z'_{cl}(\eta) = z'_{cl0} \cos \{\eta\} + \varepsilon \left( \frac{3}{16} z'_{cl0} \cos \{3\eta\} \right) + O(\varepsilon^2) \]  

(4.28)

Transforming the independent variable back from \( \eta \) to \( t \), and using Equation (4.22), Equation (4.28) can be rewritten as follows

\[ z'_{cl}(t) = z'_{cl0} \cos \left( \left( 1 + \frac{3}{4} \varepsilon + O(\varepsilon^2) \right) t \right) + \varepsilon \left( \frac{3}{16} z'_{cl0} \cos \{3(1 + \frac{3}{4} \varepsilon + O(\varepsilon^2))t\} \right) + O(\varepsilon^2) \]  

(4.29)

Finally when the small parameter is replaced with its physical definition, \( \varepsilon = x_i / a \), one obtains

\[ z'_{cl}(t) = z'_{cl0}(t) \cos \left( \left( 1 + \frac{3 x_i}{4 a} \right) t \right) + \frac{3 x_i}{16 a} \cos \left( 3 \left( 1 + \frac{3 x_i}{4 a} \right) t \right) \]  

(4.30a)

Equation (4.30a) shows that the out-of-plane motion depends on the perturbation in initial position and orbit radius. The larger the perturbation in the initial position, the larger the amplitude in the out-of-plane motion. The orbit radius is included inside the fraction \( x_i / a \), indicating that its effect on the out-of-plane motion is coupled with \( x_i \).

The period of the first harmonic, \( T \), is three times that of the second harmonic so that in Figure 4.2 the solution is plotted for one complete period of the first harmonic. The range of orbit radius values is chosen as in Figure 4.1 such that the condition \( x_i < a < (r_{12} - 2x_i) / 2 \) is satisfied.
If only an initial velocity excitation is allowed, i.e., \( \dot{z}_{cl0} \neq 0, z_{cl0} = 0 \), a value of \( s_k = -3/4 \) is necessary to remove the secular producing terms from the perturbation solution. The first term correction solution in this case is

\[
\dot{z}_{cl}'(t) = \dot{z}_{cl0}' \sin \left( \left\{ \frac{1 - \frac{3}{4} \frac{x_1}{a}}{4} \right\} t + \frac{3}{16} \frac{x_1}{a} \sin \left\{ 3 \left( 1 - \frac{3}{4} \frac{x_1}{a} \right) t \right\} \right)
\] (4.30b)

Equations (4.30a) and (4.30b) indicate that when the three-body problem is approximated as a two-body problem, i.e., \( x_1 \rightarrow 0 \), the perturbation solution is the same as Equation (4.9b) which describes the free oscillation solution.

![Figure 4.2 Out-of-Plane Motion vs. Time and Orbit Radius](image)

**Figure 4.2 Out-of-Plane Motion vs. Time and Orbit Radius**

### 4.3.3 Stability Analysis

The equivalent first order system to Equation (4.13a) is obtained as follows

\[
\dot{u} = Au
\] (4.31)
where \( u = [u_1 \ u_2]^T \), \( u_1 = z'_{cl} \) and \( u_2 = z''_{cl} \), is the state vector, and \( A \) is the matrix of coefficients.

\[
A = \begin{pmatrix}
0 & 1 \\
-(1 + f_{z_{cl}}(t)) & 0
\end{pmatrix}
\]  
(4.32)

The fundamental matrix, \( U \), of the previous system can generally be written as follows

\[
U = \begin{pmatrix}
z'_{cl}^{1} & z'_{cl}^{2} \\
z''_{cl}^{1} & z''_{cl}^{2}
\end{pmatrix}
\]  
(4.33)

where \( z'_{cl}^{i}, \ i=1,2 \) are two linearly independent solutions of Equation (4.31), with \( i \) denoting the ordinate number of a solution. Should the matrix, \( U \), be a principal fundamental matrix, it satisfies the condition \( U(0) = I \) where \( I \) is a \( 2 \times 2 \) identity matrix. In other words,

\[
z'_{cl}^{1}(t_0) = 1, \quad z'_{cl}^{2}(t_0) = 0 \\
z'_{cl}^{1}(t_0) = 0, \quad z'_{cl}^{2}(t_0) = 1
\]  
(4.34)

and from the periodicity theorem\textsuperscript{114} the matrix \( U(t) \) has a similar value after one complete period:

\[
U(t + T) = U(t)B
\]  
(4.35)

where \( B \) is a constant \( 2 \times 2 \) matrix. Evaluating Equation (4.35) at \( t = 0 \) and using the property that \( U \) is a principal fundamental matrix, one obtains

\[
B = U(T)
\]  
(4.36)

Matrix \( B \) has the following property
\[
\text{det} \mathbf{B} = e^{\int_0^T \text{tr} \mathbf{A}(\xi) d\xi}
\] (4.37)

where \(\text{det}\) denotes the determinant of the matrix and \(\text{tr}\) denotes the trace of the matrix.

From Equation (4.14b), \(\text{tr} \mathbf{A} = 0\), Equation (4.37), and using Equations (4.33), (4.36), it is obvious that

\[
\text{det} \mathbf{B} = z_{cl}^{(i)}(T)z_{c2}^{(i)}(T) - z_{cl}^{(i)}(T)z_{c2}^{(i)}(T) = 1
\] (4.38)

The eigenvalues \(\lambda_i, \ i = 1,2\) of matrix \(\mathbf{B}\) are calculated from the characteristic equation.

\[
\lambda^2 - (z_{cl}^{(i)}(T) + z_{c2}^{(i)}(T))\lambda + z_{cl}^{(i)}(T)z_{c2}^{(i)}(T) - z_{cl}^{(i)}(T)z_{c2}^{(i)}(T) = 0
\] (4.39)

Let

\[
\sigma = \frac{1}{2}(z_{cl}^{(i)}(T) + z_{c2}^{(i)}(T))
\] (4.40)

and substituting Equations (4.38) and (4.40) into Equation (4.39), one obtains

\[
\lambda^2 - 2\sigma\lambda + 1 = 0
\] (4.41)

Thus,

\[
\lambda_1 = \sigma + \sqrt{\sigma^2 - 1}
\] (4.42a)

\[
\lambda_2 = \sigma - \sqrt{\sigma^2 - 1}
\] (4.42b)

The two roots have the following properties.

\[\lambda_1\lambda_2 = 1 \text{ and } \lambda_1 + \lambda_2 = 2\sigma\] (4.43)

Define the Floquet exponents \(\nu_1, \nu_2\) such that

\[
\lambda_i = e^{\nu_i T}, \ i = 1,2
\] (4.44)
and substitute into Equation (4.43); one obtains the following conditions on the Floquet exponents

\[ \nu_1 + \nu_2 = 0, \; \sigma = \cosh(\nu_1 T) \] (4.45)

The stability of the solution to Equation (4.12) depends mainly on the values of the Floquet exponents. That is, the solution depends on \( \sigma \) which in turn depends on values of the independent solutions at the end of one complete period. When \( \sigma = \pm 1 \), the matrix \( B \) is simply \( \pm I \), and Equation (4.35) simplifies to

\[ U(t + T) = \pm U(t) \] (4.46)

When the internal forcing period is a rational multiple of the natural period, there is resonance. Solutions at these values represent boundaries between stable and unstable regions.\textsuperscript{114}

The theory of period advance mapping, also referred to as the \textit{Poincaré Map}, is now considered to further explore the solution. The fundamental matrix represents the solution space to Equation (4.31). Generally, a solution is written as

\[ u(t) = U(t)c \] (4.47)

where \( c \) is a constant \( 2 \times 1 \) vector, determined from the initial conditions \( u(t_0) \), assuming a nonsingular fundamental matrix. Equation (4.47) is rewritten as

\[ u(t) = U(t)U^{-1}(t_0)u(t_0) \] (4.48)

Defining the state transition matrix\textsuperscript{114,115} \( \varphi(t, t_0) = U(t)U^{-1}(t_0) \), Equation (4.48) is written as
\[ u(t) = \varphi(t, t_0)u(t_0) \]  

(4.49)

When investigating the solution behavior after one complete period, i.e., \( t \rightarrow t + T \), Equation (4.49) is rewritten as

\[ u(t + T) = \varphi(t + T, t_0)u(t_0) \]  

(4.50)

The matrix \( \varphi(t + T, t_0) \) can be proved to be topologically conjugate to \( \varphi(T, t_0) \), i.e.,

\[ \varphi(t + T, t_0) = \varphi(t, t_0)\varphi(T, t_0)\varphi^{-1}(t, t_0) \]  

(4.51)

The detailed proof of Equation (4.51) is found in Reference 116. Behavior of the solution is then determined through the matrix \( \varphi(T, t_0) \) which is also known as the Poincaré map.

Equations (4.30a) and (4.30b) approximate the solution space for the problem in Equation (4.31) since they are two independent solutions. Using the renaming convention from Equation (4.33), the fundamental matrix has the following properties

\[
U(t_0) = 
\begin{pmatrix}
\dot{z}'_{c_0} \left( \frac{1 + \frac{3}{16} \varepsilon}{4} \right) & 0 \\
0 & \tilde{z}'_{c_0} \left( \frac{1 - \frac{3}{4} \varepsilon}{4} \left( 1 + \frac{9}{16} \varepsilon \right) \right)
\end{pmatrix}
\]  

(4.52a)

This fundamental matrix is not a principal matrix since \( \det U \neq 1 \). The period of the forcing function is \( T = 4K \), where \( K \) is the complete elliptic integral of the first kind. Thus, the fundamental matrix can be evaluated at the end of one complete period:

\[
U = 
\begin{pmatrix}
\dot{z}'_{c_1} (4K) & \dot{z}'_{c_2} (4K) \\
\ddot{z}'_{c_1} (4K) & \ddot{z}'_{c_2} (4K)
\end{pmatrix}
\]  

(4.52b)

From Equations (4.52a) and (4.52b), the period advance mapping \( \varphi(T, t_0) \) can be calculated; its eigenvalues determine the stability of the solution. Equation (4.52)
indicates that the stability of the solution depends on the initial conditions in the vertical direction and the orbit radius.

Using the perturbation solution to approximately evaluate the period advance mapping is a new approach. The analytic relationship for the out-of-plane motion reveals parametric information concerning the amplitude and frequency of the vertical motion.

### 4.4 In-Plane Correction

The correction for the out-of-plane motion indicates that it is still decoupled from the in-plane motion and depends on the out-of-plane initial excitation. Thus, for now it is assumed that the correction for the in-plane motion is still independent for the out-of-plane motion. More appropriately, the first two parts of Equation (4.3) are rewritten, after neglecting the higher order terms, as follows.

\[
\dot{x}_{cl} - 2\omega \dot{y}_{cl} = \left\{ \frac{\omega^2 - \frac{Gm_i}{\rho_i^3}}{1 + \frac{m_2}{m_1} \left\{ \frac{\rho_1}{\rho_2} \right\}^3} + \frac{3Gm_i(x-x_i)^2}{\rho_i^5} \left[ 1 + \frac{m_2}{m_1} \left\{ \frac{\rho_1}{\rho_2} \right\}^5 \left\{ \frac{x-x_2}{x-x_1} \right\}^2 \right] \right\} x_{cl} \\
+ \left[ \frac{3Gm_y(y(x-x_i))}{\rho_i^5} \left[ 1 + \frac{m_2}{m_1} \left\{ \frac{\rho_1}{\rho_2} \right\}^5 \left\{ \frac{x-x_2}{x-x_1} \right\} \right] \right] y_{cl} + E_x
\]

\[ (4.53a) \]

\[
\dot{y}_{cl} + 2\omega \dot{x}_{cl} = \left\{ \frac{3Gm_y(x-x_i)}{\rho_i^5} \left[ 1 + \frac{m_2}{m_1} \left\{ \frac{\rho_1}{\rho_2} \right\}^5 \left\{ \frac{x-x_2}{x-x_1} \right\} \right] \right\} y_{cl} \\
+ \left\{ \frac{\omega^2 - \frac{Gm_i}{\rho_i^3}}{1 + \frac{m_2}{m_1} \left\{ \frac{\rho_1}{\rho_2} \right\}^3} + \frac{3Gm_y^2}{\rho_i^5} \left[ 1 + \frac{m_2}{m_1} \left\{ \frac{\rho_1}{\rho_2} \right\}^5 \right] \right\} y_{cl} + E_y
\]

\[ (4.53b) \]

Equation (4.53) can be simplified based on the assumptions that the third body moves close to the first primary for a sufficiently long time so that \( \rho_1 < \rho_2 \) and the three-body
The system has a very small mass parameter $m_2 << m_1$. The quantities $\rho_1 / \rho_2$ and $m_2 / m_1$ are of an order less than one and can be neglected compared to unity. Fortunately these two quantities are found multiplied in three terms in each part of Equation (4.53) and their product can be eliminated compared to unity. Moreover, the quantity $x - x_2 / x - x_1$ can be thought of as a critical term (small divisor producing term).

The advantage of the existence of this quantity multiplied by the other small order quantities can be explored as follows.

\[
\left(\frac{x - x_2}{x - x_1}\right)^2 = \frac{\rho_2^2 - y^2}{\rho_1^2 - y^2} = \frac{\rho_2^2}{\rho_1^2} \frac{1 - (y/\rho_2)^2}{1 - (y/\rho_1)^2}
\]

\[
= \left(\frac{\rho_2}{\rho_1}\right)^2 \left\{1 + \left(\frac{y}{\rho_1}\right)^2 \left[1 - \left(\frac{\rho_2}{\rho_1}\right)^2\right] - \left(\frac{y}{\rho_1}\right)^2 \left(\frac{y}{\rho_2}\right)^2 + ... \right\}
\]

Note from Equation (3.2) $y / \rho_1 < 1$ and $y / \rho_2 < 1$; thus

\[
\left(\frac{x - x_2}{x - x_1}\right) \approx \left(\frac{\rho_2}{\rho_1}\right)
\]

As a result

\[
\left\{\frac{\rho_1}{\rho_2}\right\} \left\{\frac{x - x_2}{x - x_1}\right\} \approx 1
\]

The above analysis indicates that some of the higher order terms are small enough in magnitude to be neglected if they do not include any critical terms. Applying this result to Equation (4.53) one obtains

\[
\ddot{x}_{c_1} - 2\omega \dot{y}_{c_1} = \left\{\omega^2 - \frac{Gm_2}{\rho_1^3} + \frac{3Gm_3(x - x_1)^2}{\rho_1^5}\right\} x_{c_1} + \left\{\frac{3Gm_4(y(x - x_1))}{\rho_1^5}\right\} y_{c_1} + E_x
\]
\[
\dot{y}_{cl} + 2\omega \dot{x}_{cl} = \left\{\frac{3Gm_b y(x-x_i)}{\rho_i^3} \right\}_b x_{cl} + \left\{\frac{\omega^2 - Gm_i}{\rho_i^3} + \frac{3Gm_y y^2}{\rho_i^3} \right\}_b y_{cl} + E_y
\]

(4.55b)

and \(E_x, E_y\) are also approximated:

\[
E_x = \omega^2 x_b - \left[ \frac{Gm_i}{\rho_i^3} (x-x_i) \right]_b - \dot{x}_b + 2\omega \dot{y}_b
\]

(4.55c)

\[
E_y = \omega^2 y_b - \left[ \frac{Gm_i}{\rho_i^3} y \right]_b - \dot{y}_b - 2\omega \dot{x}_b
\]

(4.55d)

The gravitational attraction of the second primary is not explicit in the coefficients of the homogeneous part of Equation (4.55). Still, the effect of the second primary is implicitly represented by the existence of \(x_i\). That effect completely disappears in the case of the two-body problem when \(x_i = 0\) and the coordinate system originates at the first primary. The effects of the nonhomogeneous parts of Equations (4.55a) and (4.55b) will be discussed in detail later.

The system in Equation (4.55) can be represented in state space form after defining the new set of variables \(q_1 = x_{cl}, q_2 = y_{cl}, q_3 = \dot{x}_{cl}, \) and \(q_4 = \dot{y}_{cl}\), as follows

\[
\dot{q} = Aq + E
\]

(4.56a)

where

\[
q = [q_1 \quad q_2 \quad q_3 \quad q_4]^T
\]

(4.56b)

\[
A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ f & \Omega \end{pmatrix}, \quad f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 2\omega \\ -2\omega & 0 \end{pmatrix}
\]

(4.56c)

\[
E = [0 \quad 0 \quad E_x \quad E_y]^T
\]

(4.56d)
In Equation (4.56c), the submatrix 0 is a two by two zero matrix, and the submatrix I is the two by two identity matrix and

\[ f_{11} = \left\{ \omega^2 - \frac{Gm}{\rho_1} + \frac{3Gm(x - x_1)^2}{\rho_1^3} \right\} \]  
(4.57a)

\[ f_{12} = f_{21} = \left\{ \frac{3Gm y(x - x_1)}{\rho_1^3} \right\} \]  
(4.57b)

\[ f_{22} = \left\{ \omega^2 - \frac{Gm}{\rho_1^2} + \frac{3Gm y^2}{\rho_1^4} \right\} \]  
(4.57c)

By recalling that \( \omega^2 = \frac{G(m_1 + m_2)}{r_{12}^3} \), \( m_1 x_1 + m_2 x_2 = 0 \), \( x_1 - x_2 = r_{12} \), the rotating system frequency can be expressed as

\[ \omega^2 = \frac{Gm_1}{r_{12}^3} \left\{ 1 + \frac{x_1}{r_{12}} + \left( \frac{x_1}{r_{12}} \right)^2 + \ldots \right\} = \frac{Gm_1}{r_{12}^3} \left\{ 1 + \sum_{i=1}^{\infty} \left( \frac{x_1}{r_{12}} \right)^i \right\} \]  
(4.58)

because \( \rho_1 / r_{12} \) is very small according to the assumptions of the nominal solution. Thus, the quantity \( (\rho_1 / r_{12})^3 \) is neglected compared to unity in Equation (4.56). Physically, this means that the frequency of motion when a spacecraft traverses a two-body circular orbit is much higher than the rotation rate of the rotating system in the CRTBP. The distance from the third body to the first primary as in Equation (3.4a),

\[ \rho_1 = a \left\{ 1 - 2 \left( \frac{x_1}{a} \right) \cos \theta + \left( \frac{x_1}{a} \right)^2 \right\}^{1/2} \]  
(4.59a)

is rewritten so that

\[ \frac{1}{\rho_1} = \frac{1}{a} \sum_{i=0}^{\infty} P_i(\cos \theta) \left( \frac{x_1}{a} \right)^i \]  
(4.59b)
where \( P_i(\cos \theta) \) is the \( i \)th Legendre polynomial. Substituting Equations (4.58) and (4.59b) into Equation (4.57), one obtains

\[
f_{11} = \frac{Gm_i}{a^3} \left\{ 2 - 3\sin^2 \theta + \frac{x_1}{a} \cos \theta (6 - 15\sin^2 \theta) + \ldots \right\}_b \quad (4.60a)
\]

\[
f_{12} = \frac{Gm_i}{a^3} \left\{ \sin \theta \cos \theta + \frac{x_1}{a} \sin \theta (5\cos^2 \theta - 1) + \ldots \right\}_b \quad (4.60b)
\]

\[
f_{21} = f_{12} \quad (4.60c)
\]

\[
f_{22} = \frac{Gm_i}{a^3} \left\{ -1 + 3\sin^2 \theta + \frac{x_1}{a} \cos \theta (15\sin^2 \theta - 3) + \ldots \right\}_b \quad (4.60d)
\]

Though the angle, \( \theta \), is not explicitly obtained as a function of time in the nominal solution, the circular functions \( \sin \theta \) and \( \cos \theta \) are known with time and can be substituted from Equation (3.34).

### 4.4.1 Reduction to Zero Mass Parameter

The system in Equation (4.56) has two second order linear nonhomogeneous differential equations. For additional analysis, and before discussing how to solve Equation (4.56), consider the two-body approach. Since the nominal solution is actually produced as a slight deviation from the two-body solution, when setting the mass parameter to zero, i.e., completely neglecting the effect of the second primary, the Jacobi function in this case can be written as follows:

\[
J = \frac{1}{2} \omega^2 r^2 + \frac{Gm_i}{r} \quad (4.61)
\]
where \( r = \rho = \sqrt{x^2 + y^2} \) is the orbit radius. Figure 4.3 shows that \( x_1 = 0, x_2 = -r_{12}, \) and \( \hat{r}, \hat{\theta} \) are the unit vectors along the radial and transverse directions, respectively.

![Figure 4.3 Two-Body Circular Planar Orbit](image)

The planar equation of motion of the third body in this case is

\[
\begin{align*}
\ddot{x} - 2\omega \dot{y} &= \frac{x}{r} J_r \\
\ddot{y} + 2\omega \dot{x} &= \frac{y}{r} J_r
\end{align*}
\]

where \( J_r = (\omega^2 r - Gm_1/r^2) \). Equation (4.62) can be transformed to polar coordinates by first multiplying Equation (4.62a) by \( \cos \theta \) and Equation (4.62b) by \( \sin \theta \), then by
collecting the two parts, one finds the result represents the motion equation in the radial direction.

\[
(\ddot{x}\cos\theta + 2\dot{x}\dot{\sin}\theta) + (\dot{y}\sin\theta - 2\dot{y}\dot{\cos}\theta) = \frac{1}{r}(x\cos\theta + y\sin\theta)
\]  

(4.63a)

Then, multiplying Equation (4.62a) by \(-\sin\theta\) and Equation (4.62b) by \(\cos\theta\) and collecting the two parts to represent the motion equation in the transverse direction, one finds

\[
(-\ddot{x}\sin\theta + 2\dot{x}\dot{\cos}\theta) + (\dot{y}\cos\theta + 2\dot{y}\dot{\sin}\theta) = \frac{1}{r}(-x\sin\theta + y\cos\theta)
\]  

(4.63b)

When substituting the nominal solution from Equation (3.3) into Equation (4.63) and replacing \(r\) by the orbit radius, \(a\), which is constant, one obtains

\[
\dot{\theta}^2 + 2\omega\dot{\theta} + \frac{1}{a}J = 0
\]  

(4.64a)

\[
\dot{\theta} = 0
\]  

(4.64b)

Equation (4.64b) indicates that there is no acceleration in the direction of motion (constant \(\dot{\theta}\)) at any time along the orbit. The velocity of the third body in the nominal orbit should be constant to satisfy Equation (4.64b). On the other hand, Equation (4.64a) should be satisfied which restricts the velocity to be a function of the orbit radius and the characteristic of the three-body system.

\[
\dot{\theta} = -\omega \pm \sqrt{\frac{Gm}{a^3}}
\]  

(4.65a)

Figure 4.4 shows the relation between the angular velocity and the orbit radius from Equation (4.65a).
Equation (4.65a) can be written in a normalized form as follows

\[
\frac{\dot{\theta}}{\omega} = -1 \pm \left( \frac{a}{r_{12}} \right)^{-3/2}
\]  

(4.65b)

Equation (4.65b) indicates that the angular velocity approaches infinity when the orbit radius approaches zero for both direct and retrograde orbits. Also, when \( a = r_{12} \), the angular velocity in the direct orbit goes to zero, this actually represents the orbit of the second primary, while in the retrograde orbit, \( \dot{\theta} = -2\omega \). For values of the orbit radius \( a > r_{12} \), a third body which starts the motion in a direct sense, has an instantaneous direct motion, but the overall orbit is a retrograde orbit. In this region, all circular orbits are retrograde orbits. Equation (4.65b) can also be written in a different normalized form as follows.

\[ a \]
\[
\frac{\dot{\theta}}{\omega} = -1 \pm \frac{n}{\omega} \tag{4.65c}
\]

where \( n = \sqrt{\frac{Gm_2}{a^3}} \) is the mean motion of the third body as if it rotates in an inertial two-body circular orbit.

In summary, the nominal circular orbit completely satisfies the motion equation in a three-body system with zero mass parameter if the initial conditions are chosen according to Equation (4.64a), and the third body moves with a constant angular velocity. When the two-body case is considered in Equation (4.56), the nominal solution completely satisfies the motion equation, and the error in both directions vanish, i.e., \( E_x = E_y = 0 \).

Furthermore, the angle \( \phi \) of the total acceleration with the \( x \) axis is calculated as follows

\[
\tan \phi = \frac{\dot{y}}{\dot{x}} \tag{4.66}
\]

and from Equation (4.63b)

\[
\tan \theta = \frac{\ddot{y} + 2\alpha \dot{x} - \frac{y}{r} J_r}{\ddot{x} - 2\alpha \dot{y} - \frac{x}{r} J_r} \tag{4.67}
\]

Note that when the Coriolis and gravitational accelerations cancel, the two angles \( \phi \) and \( \theta \) are related by \( \phi = \theta + j\pi \), where \( j = 0, 1, 2 \ldots \) which means that \( \tan \phi = \tan \theta \). Substituting this relation into Equations (4.66) and (4.67), one obtains

\[
\alpha x^2 + (x\dot{y} - \dot{x}y) = \alpha x_0^2 + (x_0\dot{y}_0 - \dot{x}_0 y_0) \tag{4.68}
\]
Equation (4.68) represents conservation of angular momentum in the rotating coordinate system, which can be rewritten as follows

\[ \mathbf{\bar{r}} \mathbf{\bar{o}} \mathbf{r} + \mathbf{r} \times \mathbf{\dot{r}} = \mathbf{\bar{r}}_0 \mathbf{\bar{o}}_0 \mathbf{r}_0 + \mathbf{r}_0 \times \mathbf{\dot{r}}_0 \]  

(4.69a)

where \( \mathbf{\bar{r}} \) and \( \mathbf{\bar{o}}_0 \) are the skew-symmetric matrices

\[ \mathbf{\bar{r}} = \begin{pmatrix} 0 & 0 & 0 & y \\ 0 & 0 & -x & 0 \\ -y & x & 0 & 0 \end{pmatrix}, \quad \mathbf{\bar{o}}_0 = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]  

(4.69b)

and \( \mathbf{\bar{r}}_0 \) and \( \mathbf{\bar{o}}_0 \) are their initial conditions. Substituting the nominal solution from Equation (3.3) into the left hand side of Equation (4.68) one obtains

\[ \omega \mathbf{r}^2 + (x \dot{y} - \dot{x} y) = \mathbf{r}^2 (\omega + \dot{\theta}) \]  

(4.70)

Thus, the nominal solution satisfies the conservation of angular momentum if both the orbit radius, \( r \), and the angular velocity, \( \dot{\theta} \), are constants (a detailed derivation is found in Appendix C).

### 4.4.2 Solution Using Variation Of Parameters

The state vector \( \mathbf{q}(t) \) in Equation (4.56a) takes the form

\[ \mathbf{q}(t) = \mathbf{\Phi}(t) \mathbf{C}(t) \]  

(4.71)

where \( \mathbf{\Phi}(t) \) is a 4 by 4 variable matrix and \( \mathbf{C}(t) \) is a 4 by 1 variable vector. Assuming that the matrix \( \mathbf{\Phi}(t) \) is a fundamental matrix, i.e., satisfies the associated homogenous matrix equation

\[ \mathbf{\Phi}(t) = \mathbf{A}(t) \mathbf{\Phi}(t) \]  

(4.72)
and substituting Equations (4.71) and (4.72) into Equation (4.56a), one obtains

\[ q(t) = \varphi(t; t_0)q(t_0) + \int_{t_0}^{t} \varphi(t; \tau)E(\tau)d\tau \] (4.73a)

where

\[ \varphi(t; t_0) = \Phi(t)\Phi^{-1}(t_0), \varphi(t; \tau) = \Phi(t)\Phi^{-1}(\tau) \] (4.73b)

The matrix \( \varphi(t; t_0) \) was defined before as the state transition matrix, and \( \varphi(T; t_0) \) is again known as the Poincaré map. The first step in obtaining a solution depends on finding the fundamental matrix solution which is the basis for all possible solutions of the associated homogeneous equation. The solution for this system is not straightforward; it actually depends on the nature of the matrix of coefficients, \( A \).

**4.4.3 Floquet-Lyapunov Theorem**

Apparently, the matrix \( A \) is a periodic matrix, i.e., \( A(t) = A(t+T) \) where \( T \) is the minimum value of a period which is that indicated in Equation (3.31). The periodic coefficients are expressed in terms of Jacobi elliptic functions which can be expanded in circular functions. The expansion may be approximated based on the value of the modulus \( k \). Equation (3.27b), which relates the elliptic modulus to the initial conditions and the characteristics of the three-body system, can be used to investigate the range of values of the modulus. When the initial angular position is \( \theta(t_0) = 0 \), the modulus can be written as follows

\[ k = \pm 2 \left( \frac{n}{\theta_0} \right) \sqrt{\frac{r_1}{a}} \] (4.74)
Noticing that the initial angular velocity deviates slightly from the two-body angular velocity \( n / \dot{\theta}_0 \approx 1 \), for values of the orbit radius \( a >> 4x_1 \), the modulus is very small compared to unity and the Jacobi elliptic functions are approximated as follows\(^\text{112}\)

\[
\begin{align*}
\text{sn}(\tau, k) &\approx \sin \tau - k^2 \cos \tau (\tau - \sin \tau \cos \tau) / 4 \\
\text{cn}(\tau, k) &\approx \cos \tau + k^2 \sin \tau (\tau - \sin \tau \cos \tau) / 4 \\
\text{dn}(\tau, k) &\approx 1 - (k^2 \sin^2 \tau) / 2
\end{align*}
\]

and when \( a \approx 4x_1 \) and the modulus \( k \approx 1 \), the functions are approximately

\[
\begin{align*}
\text{sn}(\tau, k) &\approx \tanh \tau + k'^2 \sech^2 \tau (\sinh \tau \cosh \tau - \tau) / 4 \\
\text{cn}(\tau, k) &\approx \sech \tau - k'^2 \tanh \tau \sech \tau (\sinh \tau \cosh \tau - \tau) / 4 \\
\text{dn}(\tau, k) &\approx \sech \tau + k'^2 \tanh \tau \sech \tau (\sinh \tau \cosh \tau + \tau) / 4
\end{align*}
\]

where \( k'^2 = 1 - k^2 \). In the case of small values of the modulus, the periodic coefficients in Equation (4.60) can be written as follows

\[
\begin{align*}
f_{11} &= \frac{Gm_1}{2a^3} \left\{ 1 + 3 \cos \dot{\theta}_b t \right\}_b \\
f_{12} &= \frac{3Gm_1}{2a^3} \left\{ \sin \dot{\theta}_b t \right\}_b \\
f_{21} &= f_{12} \\
f_{22} &= \frac{Gm_1}{2a^3} \left\{ 1 - 3 \cos \dot{\theta}_b t \right\}_b
\end{align*}
\]
If this approximation is used, the period of matrix $A(t)$ becomes $2\pi / \dot{\theta}_0$, and the determinant $\det A(t) = -2n^4$ is constant. More importantly, the trace of this matrix $\text{tr}A(t)$ is always zero. Recalling that the determinant of the fundamental matrix is calculated at any time using the identity

$$\det \Phi(t) = \det \Phi(t_0) e^{\int^{t_0}_t \text{tr}(A) dx}$$

Equation (4.78) indicates that the determinant of the fundamental matrix is constant and doesn't vanish at any time; the fundamental matrix solution is nonsingular. This property allows writing the fundamental matrix after one period as follows

$$\Phi(t + T) = \Gamma \Phi(t) \tag{4.79}$$

where $\Gamma$ is a constant matrix.

Introduce the following transformation

$$\Phi(t) = P v(t) \tag{4.80}$$

where $P$ is a constant matrix and $v(t)$ satisfies Equation (4.72). Substituting Equation (4.80) into Equation (4.79), one obtains

$$v(t + T) = P^{-1} \Gamma P v(t) \tag{4.81}$$

The matrix $P^{-1} \Gamma P$ can be written in diagonal form where the diagonal elements are the eigenvalues of the matrix $\Gamma$. Equation (4.81) can be written using the Jordan form of a diagonal matrix as follows

$$v(t + T) = e^{\delta T} v(t) \tag{4.82}$$
where $\delta$ is a constant matrix. Equation (4.82) is nothing but the Floquet-Lyapunov theorem applied to an arbitrary fundamental matrix $\mathbf{v}(t)$. Multiplying Equation (4.82) by $e^{-\delta(t+T)}$ one finds that the matrix $e^{-\delta t}\mathbf{v}(t)$ is a periodic matrix with period $T$ and can be written as $\mathbf{v}(t)$ which is also a $T$-periodic matrix.

\[ \mathbf{v}(t) = e^{\delta T} \mathbf{v}(t) \]  

(4.83)

Substituting Equation (4.83) into Equation (4.80), the fundamental matrix can be written as

\[ \Phi(t) = Pe^{\delta T} \mathbf{v}(t) \]  

(4.84)

The fundamental matrix for specific initial conditions can be calculated numerically after one complete period of motion, and the matrix, $\Gamma$, can be calculated from Equation (4.79) as follows.

\[ \Gamma = \Phi(T)\Phi^{-1}(0) \]  

(4.85)

If $\Phi(0) = I$, the matrix $\Gamma = \Phi(T)$ is known as the monodromy matrix whose eigenvalues $\lambda_i$, $i=1,2,3,4$, determine the stability of motion, i.e., when $|\lambda_i| < 1$, the motion is stable for all time and when $|\lambda_i| > 1$, the motion is unstable for large time. A proof of boundness of the solution of a linear system of differential equations with periodic coefficients is found in Reference 117 (this proof is reproduced in greater detail in Appendix D).
CHAPTER 5

VERTICAL CIRCULAR ORBIT

5.1 Introduction

In this chapter Jacobi's integral equation, governing the motion of the third body in the circular restricted three-body problem, is integrated again assuming certain characteristics for the motion of the third body, even though these characteristics may be only approximately satisfied in practice. This procedure is similar to that used in the rectilinear oscillation theory. Not only can an analytical formulation of the period of motion be obtained, but also a closed-form expression for the orbital path is available. Motion in two of the three axes can be solved for functionally. The projected motion on the corresponding plane is circular with nonuniform speed. The period and projected path are expressed in terms of elliptic integrals and functions. The governing characteristics do not permit motion along the third axis. In this chapter a description of the equations of motion and Jacobi's integral for the circular restricted problem of three bodies are reviewed. A suppositional circular solution for the third body motion is analytically derived, and the properties of the proposed orbit are discussed. Natural constraints imposed on the third body motion and initial condition are investigated. Also, the accuracy of the supposed conditions and analytical solution are analyzed.

5.2 Suppositional Circular Motion in a Vertical Plane

Figure 5.1 illustrates a suppositional circular motion path for the third body in the \( y'z' \) plane, which is offset from the \( yz \) plane by the constant distance \( d_x \). This motion is not strictly permitted by the governing motion equations. However, the motion solves Jacobi's integral equation exactly, solves the tangential equation of motion exactly, and
approximately solves the radial and cylindrical motion equations in bounded-averaged and banded senses. The analysis for Jacobi's integral equation is contained in this section while the equations of motion analysis is given in Section 5.4.

Under the supposition, coordinates of the third body are equal to

\[ x(t) = d_x \]  
(5.1a)

\[ y(t) = a \sin(\theta(t)) \]  
(5.1b)

\[ z(t) = a \cos(\theta(t)) \]  
(5.1c)

where \( a \) denotes the constant radius of the circular path, and angle \( \theta(t) \) measured from the positive end of the \( z' \) axis parameterizes the location along the path as an undetermined function of time, not necessarily linear. From the geometry in Figure 5.1, or substituting Equation (5.1) into Equation (2.76), this path maintains constant separation between the two primaries and the third body.

\[ \rho_1 = \left( (d_x - x_1)^2 + a^2 \right)^{1/2} \]  
(5.2a)

\[ \rho_2 = \left( (d_x - x_2)^2 + a^2 \right)^{1/2} \]  
(5.2b)
In Figure 5.1 point P represents the location of the third body on the supposed orbit. Assuming point P₀ is the initial position of the third body, the six initial conditions at \( t = t₀ \) are

\[
[x₀, y₀, z₀, \dot{x}_₀, \dot{y}_₀, \dot{z}_₀] = (d_x, a \sin{\theta}_₀, a \cos{\theta}_₀, \\
0, a \dot{\theta}_₀ \cos{\theta}_₀, -a \dot{\theta}_₀ \sin{\theta}_₀)
\]

(5.3)

where \( \theta(t₀) = \theta₀ \) and \( \dot{\theta}(t₀) = \dot{\theta}_₀ \) denote the initial angular position and angular velocity of the third body. Equation (5.3) indicates the two independent constants \( \theta₀ \) and \( \dot{\theta}_₀ \) are all that is needed to describe the initial state of the \( y'z' \) planar circular motion, assuming the radius and plane location \( a \) and \( d_x \) are prespecified.
Along this circular path, the rate at which $\theta(t)$ changes with time is not constant. Although the assumed orbital path for the third body is a circle, the speed at which the body travels along that path is nonuniform or accelerated. To determine the governing differential relation for the angular position, substitute the time derivatives of $x(t)$, $y(t)$, and $z(t)$ into Jacobi's integral result in Equation (2.74).

$$\dot{\theta}^2(t) = \omega^2 \sin^2 \{\theta(t)\} + \frac{1}{a^2} \left\{ \omega^2 d_x^2 + 2G \left( \frac{m_1}{\rho_1} + \frac{m_2}{\rho_2} \right) - C \right\}$$

(5.4)

Note in Equation (5.4) the second term in the right hand side is a constant that is renamed $C_0$.

$$C_0 = \frac{1}{a^2} \left\{ \omega^2 d_x^2 + 2G \left( \frac{m_1}{\rho_1} + \frac{m_2}{\rho_2} \right) - C \right\}$$

(5.5)

Constant $C_0$ is determinable from the initial conditions $\theta_0$ and $\dot{\theta}_0$ existing at $P_0$.

$$C_0 = \dot{\theta}_0^2 - \omega^2 \sin^2 \{\theta_0\}$$

(5.6)

Equation (5.4) thus becomes

$$\dot{\theta}^2(t) = \omega^2 \left( \sin^2 \{\theta(t)\} - \sin^2 \{\theta_0\} \right) + \dot{\theta}_0^2$$

(5.7)

Now, an analytical solution for the period of the circular path is sought from Equation (5.7). Define a new angle $\phi(t)$ for transformation purposes.

$$\phi(t) = \frac{\pi}{2} - \theta(t)$$

(5.8)

Modify Equation (5.7), using the transformation variable $\phi(t)$, to a standard Legendre elliptic differential form of the first kind.$^{112,113}$
\[-\frac{d\phi(t)}{dt} = \frac{\omega}{k} \left(1 - k^2 \sin^2(\phi(t))\right)^{1/2}\]  

(5.9)

where $k$ is the modulus of the elliptic form.

\[k = \left(\cos^2(\theta_0) + \frac{\dot{\theta}_0^2}{\omega^2}\right)^{-1/2}\]  

(5.10)

A complete elliptic integral of Equation (5.9) can be formed to obtain the analytical expression for the period $T$ of the circular path. Integrating Equation (5.9) over a general half path ($t : t_0 \rightarrow t_0 + T/2, \theta(t) : \theta_0 \rightarrow \theta_0 + \pi$), or

\[\int_{t_0}^{t_0+\pi/2} dt = -\frac{k}{\omega} \int_{\pi/2 - \theta_0}^{\pi/2} \frac{d\phi}{\left(1 - k^2 \sin^2(\phi)\right)^{1/2}} \]  

(5.11)

yields the period $T$.

\[T = \frac{4k}{\omega} K(k)\]  

(5.12)

In Equation (5.12), $K(k)$ is the complete elliptic integral of the first kind.\textsuperscript{112,113}

For the suppositional circular motion, the period is a nonlinear function of $\theta_0, \dot{\theta}_0$, and $\omega$ is independent of $a$ and $d_x$. Nonlinear dependence on initial angular position and velocity is due to trigonometric, power, multiplication, and complete elliptic integral operations. Figure 5.2 shows the non-dimensional periodicity for various non-dimensional initial conditions. As expected, orbital period $T$ and initial angular velocity $\dot{\theta}_0$ are inversely proportional. Also note, for the same initial rate, larger initial positions can amplify the period. The low end cut off points for $\dot{\theta}_0$ in Figure 5.2 are from dynamical constraints discussed in Section 5.3.
Next, an analytical solution for the parameterization of the circular path is sought from Equation (5.7). Returning to Equation (5.9), perform an integral over a general path segment \((t : t_0 \rightarrow t, \theta(t) : \theta_0 \rightarrow \theta)\)

\[
\int_{t_0}^{t} dt = -\frac{k}{\omega} \int_{\frac{\pi}{2}}^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2(\phi)}}
\]

(5.13)

\[
\frac{\omega}{k} (t - t_0) = -F(\phi(t), k) + F\left(\frac{\pi}{2} - \theta_0, k\right)
\]

(5.14)

Figure 5.2 Periodicity Behavior of Suppositional Circular Motion
In Equation (5.14), function $F(\psi, k)$ is the incomplete elliptic integral of the first kind evaluated at $\psi$. By utilizing the theory of theta functions, Equation (5.14) can be inverted to obtain the transform angle $\phi(t)$ as a function of time:

$$\sin\{\phi(t)\} = \text{sn}\left(-\frac{\omega}{k}(t-t_0) + F\left(\frac{\pi}{2} - \theta_0, k\right), k\right)$$  \hspace{1cm} (5.15)

Where $\text{sn}(\tau, k)$ denotes the elliptic sine or $\text{sn}$ function evaluated at $\tau$ as precisely the inverse of $F(\phi(t), k)$. After transforming back to the original coordinate, analytical expressions for the parameterizing variable $\theta(t)$ for the circular path and its derivative are

$$\cos\{\theta(t)\} = \text{sn}\left(-\frac{\omega}{k}(t-t_0) + F\left(\frac{\pi}{2} - \theta_0, k\right), k\right)$$  \hspace{1cm} (5.16a)

$$\sin\{\theta(t)\} = \text{cn}\left(-\frac{\omega}{k}(t-t_0) + F\left(\frac{\pi}{2} - \theta_0, k\right), k\right)$$  \hspace{1cm} (5.16b)

$$\frac{d\theta(t)}{dt} = \frac{\omega}{k} \text{dn}\left(-\frac{\omega}{k}(t-t_0) + F\left(\frac{\pi}{2} - \theta_0, k\right), k\right)$$  \hspace{1cm} (5.16c)

In Equation (5.16), $\text{cn}(\tau, k)$ and $\text{dn}(\tau, k)$ are the elliptic $\text{cn}$ and $\text{dn}$ functions evaluated at $\tau$.

In effect, Equations (5.14)-(5.16) represent a closed-form integral of Jacobi's integral equation under the imposed suppositional conditions. These results can be referred to as an "integral of the suppositional motion." Figures 5.3-5.5 show various orbital trajectory characteristics of the suppositional motion, derived from the generally applicable Equation (5.16), across a family of modulus values for the Earth-Moon system (specified $m_1, m_2, r_{12}$), chosen only to portray graphical information for a commonly analyzed three-body system. Figure 5.3 depicts the normalized angular position against normalized time for half an orbit, while Figure 5.4 shows the corresponding angular rate.
response. Clearly, the rate of change of $\theta(t)$ with $t$ is nonuniform and strongly depends on the value of $k$. As the third body passes through the $y'$ axis, maximal angular rate occurs while a minimum occurs on the $z'$ axis. For the limiting case $k : k \to 1$, the largest variation in angular rate is experienced, and the third body approaches a state of rest in the rotating coordinate system each time it passes by the $z'$ axis. For smaller values of $k$, the $\dot{\theta}(t)$ variation around the orbit is lessened, but the averaged and peak $\dot{\theta}(t)$ values increase. For $k < 0.5$, the third body follows essentially uniform speed circular motion. The limiting case $k : k \to 0$ corresponds to precisely constant but infinite speed circular motion.

Finally, Figure 5.5 combines the information of Figures 5.3-5.4 in a $\theta(t)$ vs. $\dot{\theta}(t)$ phase space polar plot where the radial coordinate is normalized angular velocity and the transverse coordinate is normalized in the angular position. First, the suppositional motion in Equation (5.1) was assumed periodic, and the closed curves in Figure 5.5 demonstrate this trait. The varying radius phase plane curves are the signature of nonuniform angular velocity. For $k : k \to 1$, the third body state trajectory in the phase plane consists of two circles forming a "lazy figure eight" shape. Note the radius of the phase plane trajectory approaches zero for $\left(\theta(t) - \theta_0\right)/2\pi = 0$ and 0.5 ($z'$ axis passage) while for $\left(\theta(t) - \theta_0\right)/2\pi = 0.25$ and 0.75 ($y'$ axis passage) maximum radius values occur. As $k$ is reduced, the phase space closed trajectory transitions from a pinched oval shape to a flattened oval shape. For $k : k \to 0$, the trajectory approaches a constant but infinite radius oval (i.e., circular).
The suppositional motion outlined here and displayed in Figure 5.1 is classified as retrograde motion since the relative angular momentum vector points along the negative $x$ axis or since the motion from the $z'$ axis to the $y'$ axis follows a left hand rule ($z'$ cross $y'$ produces positive $x$). A complete set of identical results exists for direct motion. If the circular motion is in the opposite sense, from the $z'$ axis to the $-y'$ axis, and the 

\[ m_1 = 5.97 \times 10^{24} \text{ kg} \]
\[ m_2 = 7.35 \times 10^{22} \text{ kg} \]
\[ r_{12} = 3.84 \times 10^5 \text{ km} \]

Figure 5.3 Angular Position Behavior of Suppositional Circular Motion
Figure 5.4 Angular Velocity Behavior of Suppositional Circular Motion

Figure 5.5 Phase Space Polar Trajectory of Suppositional Circular Motion
direction of the angular position variable \( \theta(t) \) is reversed along with the supposition \( x(t) = \dot{d}_x, y(t) = -a \sin \theta, z = a \cos \theta \). Equation (5.7) is easily derived. Applying the transformation \( \phi(t) = \theta(t) - \pi/2 \) leads to Equation (5.9), and all results therein.

### 5.3 Initial Conditions and Motion Constraints

Certain restrictions on the initial condition pair \( \theta_0 \) and \( \dot{\theta}_0 \) exist within the suppositional motion theory. Equation (5.10) gives the value of the modulus of the elliptic integral in terms of the initial states of the motion. The value of the modulus \( k \) is constrained by the following mathematical inequality.

\[
k^2 < 1 \quad (5.17)
\]

Substituting Equation (5.10) into Equation (5.17) leads to the following inequality

\[
\omega^2 \sin^2 \theta_0 < \dot{\theta}_0^2 \quad (5.18)
\]

By using Equations (5.5-5.6), this inequality becomes

\[
C_0 > 0 \quad (5.19)
\]

\[
C < \omega^2 d_x^2 + 2G \left( \frac{m_1}{\rho_1} + \frac{m_2}{\rho_2} \right) \quad (5.20)
\]

The right hand side of Equation (5.20) is a function of both \( a \) and \( d_x \), and it represents an upper limit on the Jacobi constant, \( C_u \), corresponding to certain combinations of \( a \) and \( d_x \). This condition is not only a mathematical relation but also a physical constraint on the behavior of the dynamical system in the dimensional space.

Figure 5.6 shows the upper limit (the right hand side of Equation (5.20)) on the non-dimensional Jacobi constant as a function of normalized orbit radius for a family of
normalized $y'z'$ plane locations. The CRTBP parameters are selected for the Earth-Moon system and are merely used as an example. For certain values for the orbit size and motion plane, possible values for the Jacobi constant must lie below the corresponding curve in Figure 5.6. From another perspective, whatever the initial conditions are, they must yield a Jacobi constant which falls within the admissible region in Figure 5.6. The admissible region is characterized by all $\theta_0$ and $\dot{\theta}_0$, or $C_0$, which satisfy Equations (5.19)-(5.20). For example, for a given initial angle and a particular CRTBP characterization, the initial rate must exceed a threshold for suppositional circular motion to exist. For the center plane ($d_x=0$) and small orbits ($a<r_{12}$) a wide range of potential $C$ levels exist, while for off center planes the allowable $C$ range is significantly less. For large orbits ($a>r_{12}$), all planes in Figure 5.6 yield approximately the same potential $C$ range. These trends are a consequence of the manner in which $a$ and $d_x$ influence the right hand side of Equation (5.20). All curves in Figure 5.6 appear to intersect precisely at the point $(a,C) = (r_{12}, 2a^2r_{12}^2)$, but this appearance is an artifact of highly unbalanced primary masses ($m_2 << m_1$) for the Earth-Moon case.

Equation (5.16) gives the totality of the motion in this analysis. Recalling that for general motion in a plane perpendicular to the $x$ axis, the phase space is described by the elements of the vector $[y, z, \dot{y}, \dot{z}]$. The motion is completely determined by the initial conditions $[y_0, z_0, \dot{y}_0, \dot{z}_0]$ as the equations of motion are numerically integrated from this point. Under the suppositional theory, the equivalent vector is $[d_x, a, \theta_0, \dot{\theta}_0]$, but the Jacobi integral equation (Equation (5.4)) evaluated at $t = t_0$ represents a restriction on the initial conditions. This observation means the four initial conditions are not completely
arbitrarily chosen. Once three of the initial conditions are chosen, the fourth one is calculated from the Jacobi integral equation for a given value of \( C \). In other words, the Jacobi integral equation is reformulated as follows

\[
f(d_x, a, \theta_0, \dot{\theta}_0) = C
\]  

(5.21)

\[d_{x/r_{12}} = \begin{cases} 0 & \text{if } d_x = 0 \\ \pm 0.2 & \text{if } d_x = \pm 0.2 \\ \pm 0.4 & \text{if } d_x = \pm 0.4 \end{cases}
\]

\[m_1 = 5.97 \times 10^{24} \text{ kg}
\]

\[m_2 = 7.35 \times 10^{22} \text{ kg}
\]

\[r_{12} = 3.84 \times 10^5 \text{ km}
\]

Figure 5.6 Upper Limit for Jacobi Constant for Suppositional Circular Motion

Equation (5.21) represents an algebraic relation between the four initial conditions, which could be utilized as follows. Parameters \( d_x \) and \( a \) could be specified, leaving, for a particular \( C \), a relation for \( \dot{\theta}_0 \) in terms of \( \theta_0 \), or vice versa. Additionally, if \( k \) was specified, Equation (5.10) and (5.21) would leave two equations for the two dependent parameters \( \theta_0, \dot{\theta}_0 \).
With a completely different perspective, Equation (5.4) sets a basis for the principle of accessible and forbidden regions of motion for the third body within the theoretic supposition since the magnitude square of the angular velocity of the third body must be equal to or greater than zero. For zero \( \dot{\theta}(t) \) the suppositionally modified Jacobi integral equation constitutes curves of zero velocity or what is known as equi-potential curves. The governing relation is

\[
\sin^2 \{\theta\} + \frac{1}{\omega^2 a^2} \left[ \omega^2 a^2 + 2G \left( \frac{m_1}{\rho_1} + \frac{m_2}{\rho_2} \right) - C \right] = 0 \tag{5.22}
\]

These curves are parameterized by the Jacobi constant which can be determined by the initial conditions as previously indicated. For a certain \( C \), Equation (5.22) is used to solve for \( \theta \) as a function of \( a \).

Results from this effort are contained in Figure 5.7, which shows a family of level curves parameterized by non-dimensional \( C \) for the center plane location passing through the CRTBP mass center for the example Earth-Moon system. The curves of zero velocity consist of two types depending on the value of \( C \). For lower values of \( C \), a pair of opposing "trough" shaped curves that open vertically exist. These curves tend to constrain the accessible region of motion from the top and bottom or along the \( z' \) axis. In Figure 5.7, the center, left, and right regions correspond to real \( \dot{\theta} \), while the top and bottom regions yield imaginary \( \dot{\theta} \). For higher values of \( C \), a pair of nearly vertical curves offset from the center exists along with an oval shaped curve near the center. These curves disallow motion in the intermediate regions (imaginary \( \dot{\theta} \)), but leave large regions to the left and right along with the smaller enclosed regions near the center for allowable motion (real \( \dot{\theta} \)). The accessible region corresponding to real \( \dot{\theta} \) for \( C = 4\omega^2 r_{12}^2 \) is hash.
marked. Note, this region lies within the accessible region for the \( C = 3.4\omega^2 r_{12}^2 \) curve. Thus, as \( C \) increases the accessible region of motion decreases in size. To see the effect of another \( y'z' \) plane location, Figure 5.8 shows the level curves for a plane passing through the collinear libration point \( L_1 \) for the Earth-Moon system. Overall, the topological structure in Figure 5.8 is similar to Figure 5.7. The major difference is that at \( L_1 \), the regions of admissible motion, for the same potential level, are smaller than for the center plane case.

The suppositional theory is exclusively circular motion. Thus, the main information extracted from Figure 5.7 is the maximum allowable orbit size \( a \) for a given \( C \) and \( d_x \). For example, the largest accessible radius for \( C = 4\omega^2 r_{12}^2 \) and \( d_x = 0 \) is \( a = 0.497 r_{12} \). Constant \( C \) and the maximum allowable \( a \) tend to be inversely proportional. In Figure 5.7 the largest radius orbit allowed is always tangent to the zero velocity curves at \( y' = 0 \) or where the curves cross the \( z' \) axis. To show this, interpret Equation (5.22) as the implicit function \( a = f(\theta) \). The condition \( da/d\theta = 0 \) can be used to determine critical values \( \theta = \theta_c \) where \( a = a_{\min}, a = a_{\max} \). For computation, use the explicit function \( \theta = f'(a) \), and invert the resulting expression for \( d\theta/da \) yielding
Figure 5.7 Zero Velocity Curves in $y'z'$ Plane Located at Center of Mass

\[
\frac{da}{d\theta} = -\frac{\omega^2 a^2 \sin \theta \cos \theta}{\omega^2 a \sin^2 \theta + G \left( m_1 \frac{d}{da} \left( \frac{1}{\rho_1} \right) + m_2 \frac{d}{da} \left( \frac{1}{\rho_2} \right) \right)} \tag{5.23}
\]

Applying the extremum condition to Equation (5.23) results in

\[
\frac{da}{d\theta} = 0 \Rightarrow \theta = 0, \pm \pi \quad \text{or} \quad \theta = \pm \frac{\pi}{2} \tag{5.24}
\]

Determination of extremum type requires examination of \( \frac{d^2 a}{d\theta^2} \). Differentiation of Equation (5.23) followed with evaluation at \( \theta = \theta_a \) gives an expression for the second derivative.

\[
\left. \frac{d^2 a}{d\theta^2} \right|_{\theta = \theta_a} = -\frac{\omega^2 a^2 \left( \cos^2 \theta_a - \sin^2 \theta_a \right)}{\omega^2 a \sin^2 \theta_a + G \left( m_1 \frac{d}{da} \left( \frac{1}{\rho_1} \right) + m_2 \frac{d}{da} \left( \frac{1}{\rho_2} \right) \right)} \tag{5.25}
\]
At \( \theta_* = 0, \pm \pi \), the second derivative simplifies to

\[
\left. \frac{d^2 a}{d\theta^2} \right|_{\theta_* = 0, \pm \pi} = \frac{\omega^2 a}{G \left\{ \frac{m_1}{\rho_1^3} + \frac{m_2}{\rho_2^3} \right\}} > 0
\] (5.26)

revealing \( a = a_{\text{min}} \). At \( \theta_* = \pm \pi / 2 \), the second derivative simplifies to

\[
\left. \frac{d^2 a}{d\theta^2} \right|_{\theta_* = \pm \pi / 2} = \frac{\omega^2 a}{G \left( m_1 \left( \frac{1}{r_{12}^3} - \frac{1}{\rho_1^3} \right) + m_2 \left( \frac{1}{r_{12}^3} - \frac{1}{\rho_2^3} \right) \right)}
\] (5.27)

Both \( a_{\text{min}} \) and \( a_{\text{max}} \) can potentially occur here, depending on the value of \( a \). Both \( \theta_* \) cases are observable in Figure 5.7; however, only the case of \( \theta_* = 0, \pm \pi \) has relevant meaning.
To determine \( a_{\text{min}} \) corresponding to \( \theta_0 = 0, \pm \pi \), substitute the critical \( \theta \) value in Equation (5.22) and solve for \( a \) from

\[
\omega^2 d_x^2 + 2G \left( \frac{m_1 + m_2}{\rho_1 + \rho_2} \right) - C = 0
\]

Note \( a_{\text{min}} \) becomes an upper limit, \( a_u \), on the allowable orbit radius (\( a_u = a_{\text{min}} \)). For computational advantages, Equation (5.28) can be converted to the polynomial equation

\[
\frac{1}{2} x^2 \left( \delta_1^2 + \alpha^2 \right) \left( \delta_2^2 + \alpha^2 \right) \left( \frac{1}{8} x^2 \left( \delta_1^2 + \alpha^2 \right) \left( \delta_2^2 + \alpha^2 \right) - \mu_1^2 \left( \delta_1^2 + \alpha^2 \right) - \mu_2^2 \left( \delta_2^2 + \alpha^2 \right) \right) + \left( \mu_1^2 \left( \delta_1^2 + \alpha^2 \right) - \mu_2^2 \left( \delta_2^2 + \alpha^2 \right) \right)^2 = 0
\]

where

\[
\delta_1 = \frac{d_x}{r_{12}} + \mu_1 \quad \mu_1 = \frac{m_1}{m_1 + m_2}
\]

\[
\delta_2 = \frac{d_x}{r_{12}} - \mu_2 \quad \mu_2 = \frac{m_2}{m_1 + m_2}
\]

\[
x = \frac{C}{\omega^2 r_{12}^2} - \frac{d_x^2}{r_{12}^2} \quad \alpha = \frac{a}{r_{12}}
\]

In contrast, substituting \( \theta_0 = \pm \pi / 2 \) in Equation (5.22) yields an expression from which \( a_{\text{min}} \) or \( a_{\text{max}} \) can be found depending on the sign of \( d^2 a / d\theta^2 \) from Equation (5.27).

\[
\omega^2 a^2 + \omega^2 d_x^2 + 2G \left( \frac{m_1 + m_2}{\rho_1 + \rho_2} \right) - C = 0
\]

An equivalent polynomial equation is
Here $a_{\text{max}}$ becomes the lower limit, $a_l$, and $a_{\text{min}}$ is the upper limit, $a_u$, on the allowable orbit radius ($a_l = a_{\text{max}}, a_u = a_{\text{min}}$).

Figure 5.9 shows the upper limit on the non-dimensional orbit radius as a function of the normalized Jacobi constant for a family of $y'z'$ plane locations, for the example Earth-Moon system. For certain values for the Jacobian potential level and motion plane, possible values for the orbit radius must lie below the corresponding curve in Figure 5.9. For both center plane ($d_x = 0$) and off center planes ($d_x \neq 0$), and low potential levels ($C < 2\omega^2 r_{1z}^2$), a wide range of potential circular orbits exist, while for high potential levels ($C > 2\omega^2 r_{1z}^2$), the allowable $a$ range is significantly less. In this latter region, conditions for existence of center plane orbits are always satisfied, but off center plane orbits may or may not exist depending on the value of $C$. Inspection of Figures 5.6 and 5.9 reveals identical trends, since the limiting case in Equation (5.20) and Equation (5.28) are equivalent. An implication from this equivalence is the nonnegative velocity magnitude square constraint (accessible motion space bounded by zero velocity curves) is one and the same with the less than unity modulus constraint (admissible function space bounded by elliptic integral existence).

An interesting study is to assess consistency of the suppositional zero velocity curves from Equation (5.22) with exact zero velocity curves computed from Equation (2.74) with $v = 0$. After computing the exact level curves and comparing them with the suppositional level curves shown in Figure 5.7 for the center plane passing through the
CRTBP mass center, for the example Earth-Moon system, no differences were found. Perfect correlation between the exact and suppositional curves exists. Further, the constraints imposed on the suppositional planar orbit (not the true orbit which repeatedly penetrates this plane) by the two sets of curves are also in precise agreement. Tab. 5.1 shows a comparison of allowable circular orbit radii from both the suppositional curves and the exact curves, for the specific cases shown in Figure 5.7. For example, at $C = 4\omega^2 r_1^2$ the largest circular orbit that would lie within the boundaries would have a radius of $a = 0.4965 r_1$. This orbit would touch the suppositional and exact boundaries at the point $(y/r_1, z/r_1) = (0, 0.4965)$.

To show how this perfect correlation exists, consider Equation (2.74) with $\nu = 0$ and $x = d_x$.

$$
\omega^2 y^2 + \omega^2 d_x^2 + 2G \frac{m_1 + m_2}{\rho_1 \rho_2} - C = 0
$$

(5.33)

Within $\rho_1$ and $\rho_2$, the $y^2 + z^2$ term can be replaced by $r^2$ where $r$ denotes the radius to a point lying on the exact zero velocity curve in the $y'z'$ plane. Equation (5.33) represents the implicit function $r = f(y)$. The condition $dr/d\nu = 0$ yields critical values $y = y_*$. where $r = r_{\min}$, $r = r_{\max}$. For computation, use the explicit function $y = f'(r)$ and invert $dy/dr$ giving

$$
\frac{dy}{dr} = \frac{\omega^2 y}{Gr \left( \frac{m_1}{\rho_1^3} + \frac{m_2}{\rho_2^3} \right)}
$$

(5.34)
\[
\frac{dr}{dy} = 0 \Rightarrow y_* = 0 \quad (5.35)
\]

Computing a second derivative and evaluating at \( y = y_* \) gives

\[
\left. \frac{d^2r}{dy^2} \right|_{y=y_*} = \frac{\omega^2}{Gr \left( \frac{m_1}{\rho_1^3} + \frac{m_2}{\rho_2^3} \right)} > 0 \quad (5.36)
\]

\[
\left. \frac{d^2r}{dy^2} \right|_{y=y_*} > 0 \Rightarrow r = r_{\text{min}} \quad (5.37)
\]

Note, \( r_{\text{min}} \) becomes an upper limit, \( r_u \), on the allowable orbit radius \( (r_u = r_{\text{min}}) \). The equation determining the upper limit \( r_u \) is obtained by substituting \( y = y_* \) in Equation (5.33). The resulting expression matches Equation (5.28), hence the perfect correlation.

**Table 5.1 Quantitative Constraints of Suppositional and Exact Zero Velocity Curves**

<table>
<thead>
<tr>
<th>Normalized Jacobi Constant</th>
<th>Normalized Orbit Radius Upper Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C/(\omega r_{\text{zo}})^2 )</td>
<td>Suppositional</td>
</tr>
<tr>
<td>2.2</td>
<td>0.9054</td>
</tr>
<tr>
<td>2.8</td>
<td>0.7106</td>
</tr>
<tr>
<td>3.4</td>
<td>0.5846</td>
</tr>
<tr>
<td>4</td>
<td>0.4965</td>
</tr>
</tbody>
</table>

**5.4 Suppositional Motion Accuracy**

Equation (5.16) represents an "exact" integration of Jacobi's integral equation, under the supposition that \( x(t) = d_x \) and \( \dot{x}(t) = 0 \) are strictly maintained. Unfortunately, this condition is not met. Recall the equations of motion in Equation (2.71) with the right hand sides fully expanded.
Applying the suppositional motion from Equation (5.1) to Equation (5.38) yields

\[ -2\omega^2 d = \omega^2 d_x - \frac{G m_1 (d_x - x_1)}{((d_x - x_1)^2 + a^2)^{3/2}} - \frac{G m_2 (d_x - x_2)}{((d_x - x_2)^2 + a^2)^{3/2}} \]  

(5.39a)

\[ a (\dot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) = \omega^2 a \sin \theta - \frac{G m_1 a \sin \theta}{((d_x - x_1)^2 + a^2)^{3/2}} - \frac{G m_2 a \sin \theta}{((d_x - x_2)^2 + a^2)^{3/2}} \]  

(5.39b)

\[ -a (\dot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta) = -\frac{G m_1 a \cos \theta}{((d_x - x_1)^2 + a^2)^{3/2}} - \frac{G m_2 a \cos \theta}{((d_x - x_2)^2 + a^2)^{3/2}} \]  

(5.39c)
Multiplying Equation (5.39b) by $\cos\{\theta\}$ and Equation (5.39c) by $\sin\{\theta\}$ and combining leads precisely to Equations (5.6)-(5.7) and the corresponding analytic solution outlined in Section 5.2. This preciseness does not imply that either Equation (5.39b) or (5.39c) individually are satisfied; rather, it implies a combination of the two equations (Jacobi's integral equation) is satisfied. However, one equation of motion is precisely satisfied.

Transform Equations (5.39b)-(5.39c) to radial and tangential components.

\[ a\dot{\theta}^2 + \omega^2 a \sin^2\{\theta\} = G \left( \frac{m_1}{\rho_1^3} + \frac{m_2}{\rho_2^3} \right) a \]  \hspace{1cm} (5.39d)

\[ a\ddot{\theta} - \omega^2 a \sin\{\theta\} \cos\{\theta\} = 0 \]  \hspace{1cm} (5.39e)

Substituting Equations (5.16a) and (5.16b) and the derivative of Equation (5.16c) into Equation (5.39e) shows the tangential motion equation is exactly solved by the suppositional motion. Thus, the accuracy to which the assumed circular trajectory satisfies the motion equations reduces to the degree to which Equation (5.39a) and (5.39d) are satisfied.

Equation (5.39a), with the left hand side being a function of time and the right hand side being constant, cannot equate over a finite segment of time. The suppositional motion is incorrect in the strictest mathematical sense. Nevertheless, Equation (5.39a) can be interpreted to be satisfied in the following approximate sense. The right hand side of Equation (5.39a), \( \frac{\partial J}{\partial a}\bigg|_{\text{Supposition}} = J_x(a, d_x) \), is a function of \( d_x \) and \( a \). Expanding this function with respect to variable \( a \) about the point \( a = 0 \) yields
\[ J_x(d_x, a) = J_x(d_x, a)_{a=0} + \frac{\partial J_x(d_x, a)}{\partial a} \Bigg|_{a=0} a + \frac{1}{2} \frac{\partial^2 J_x(d_x, a)}{\partial a^2} \Bigg|_{a=0} a^2 + ... \]

\[ = \left\{ \omega^2 d_x - G \left( \frac{m_1(d_x - x_1)}{|d_x - x_1|^3} + \frac{m_2(d_x - x_2)}{|d_x - x_2|^3} \right) \right\} + (0)a + \frac{1}{2} \left\{ 3G \left( \frac{m_1(d_x - x_1)}{|d_x - x_1|^5} + \frac{m_2(d_x - x_2)}{|d_x - x_2|^5} \right) \right\} a^2 + ... \]  

(5.40)

If the y'z' plane is selected to pass through any of the three collinear libration points, \( L_1, L_2, L_3 \), the first term in the expansion of the right hand side of Equation (5.39a) becomes zero \( (J_x(d_x, 0) = 0 \text{ at } d_x = d_{L_i}(\mu), d_{L_2}(\mu), d_{L_3}(\mu), \mu = \mu_2) \). Thus, the right hand side of Equation (5.39a) is approximately zero assuming \( a \ll (d_x - x_i)^2 \) for \( i = 1, 2 \) (zero through second order in \( a \)). The left hand side represents a finite zero mean oscillating perturbation, consisting of an elliptic \( \text{sn}(\tau, k) \) and \( \text{dn}(\tau, k) \) product (see Equation (5.16)). Thus, Equation (5.39a) is satisfied, in an averaged sense, for each whole orbit completed by the third body. Further, the maximum error at any specific time in Equation (5.39a) is bounded. Planes other than the collinear libration planes can be considered; however, Equation (5.39a) will not be correct even in the averaged sense.

To show the averaged property, integrate the left hand side of Equation (5.39a) over a general whole orbit \( (t : t_0 \rightarrow t_0 + T, \theta(t) : \theta_0 \rightarrow \theta_0 + 2\pi) \), or

\[ [-2\omega \dot{\theta}]_{avg} = -\frac{2\alpha \chi}{T} \int_{t_0}^{t_0+T} \dot{\theta}(t) \cos{\theta(t)} dt = \frac{2\alpha \chi}{T} \sin{\theta(t)} \Bigg|_{t_0}^{t_0+T} = \frac{2\alpha \chi}{T} \sin{\theta} \Bigg|_{t_0}^{t_0+2\pi} \]

(5.41)

For the bounded property, first differentiate the left hand side of Equation (5.39a), leading to
\[
\frac{d[-2\omega \dot{y}]}{dt} = -2\omega a \frac{d}{dt} \left( \theta(t) \cos \{\theta(t)\} \right) = -2\omega a \frac{d}{dt} \left( \frac{\omega}{k} \text{dn}(\tau(t),k)\text{sn}(\tau(t),k) \right)
\]

\[
= \frac{2\omega^2 a}{k^2} \text{cn}(\tau(t),k)(1 - 2k^2\text{sn}^2(\tau(t),k))
\]

where

\[
\tau(t) = -\frac{\omega}{k} (t - t_0) + F\left(\frac{\pi}{2} - \theta_0, k\right)
\]

Applying the condition for perturbation extremums results in

\[
\frac{d[-2\omega \dot{y}]}{dt} = 0 \Rightarrow \text{cn}(\tau(t),k) = 0 \text{ or } \text{sn}(\tau(t),k) = \frac{1}{\sqrt{2}k}
\]

The \text{cn}(\tau(t),k) condition leads to minima and maxima when \( k \leq 1/\sqrt{2} \), whereas the condition on \text{sn}(\tau(t),k) corresponds to minima and maxima for \( k \geq 1/\sqrt{2} \). Thus, upper and lower bounds on the perturbation are

\[
[-2\omega \dot{y}]_u = \frac{2\omega^2 a}{k} \text{dn}(\tau_*,k)\text{sn}(\tau_*,k)
\]

\[
[-2\omega \dot{y}]_l = -\frac{2\omega^2 a}{k} \text{dn}(\tau_*,k)\text{sn}(\tau_*,k)
\]

where

\[
\tau_* = \begin{cases} 
F(\pi/2, k) = K(k) & \text{for } k \leq 1/\sqrt{2} \\
F(\sin^{-1}(1/\sqrt{2}k), k) & \text{for } k \geq 1/\sqrt{2}
\end{cases}
\]

The peak error in Equation (5.39a) is proportional to the CRTBP rotation rate squared and the third body orbit radius and inversely proportional to the modulus. Note, the product \( \text{dn}(\tau_*,k)\text{sn}(\tau_*,k) \) will vary between 0.5 and 1 depending on the value of \( k \).

Now focus attention on Equation (5.39d), where again the left hand side is time dependent, and the right hand side is time independent. Thus, the suppositional motion is
again rigorously incorrect, but can be considered, in certain regions of the dimensional space, approximately correct in the following sense. The left hand side of Equation (5.39d), negative radial acceleration \(-a_r\), is greater than zero and is limited from above and below. In certain regions of the \(xyz\) space, the right hand side of Equation (5.39d) can be shown to lie between the left hand side limits. Thus, Equation (5.39d) is also satisfied, in a banded sense, for all time assuming the third body orbit falls within the defined region.

To show the limiting property of the left hand side of Equation (5.39d), substitute the suppositional motion solution yielding

\[
[-a_r] = a \dot{\theta}^2(t) + \omega^2 a \sin^2 \{\theta(t)\} = \frac{\omega^2 a}{k^2} \left( \text{dn}^2(\tau(t),k) + k^2 \text{cn}^2(\tau(t),k) \right) \tag{5.45a}
\]

Differentiate the left hand side, leading to

\[
\frac{d[-a_r]}{dt} = \frac{\omega^2 a}{k^2} \frac{d}{dt} \left( \text{dn}^2(\tau(t),k) + k^2 \text{cn}^2(\tau(t),k) \right) \tag{5.45b}
\]

\[
= \frac{4\omega^2 a}{k} \text{sn}(\tau(t),k) \text{cn}(\tau(t),k) \text{dn}(\tau(t),k)
\]

Applying the condition for extremums provides

\[
\frac{d[-a_r]}{dt} = 0 \Rightarrow \text{sn}(\tau(t),k) = 0 \text{ or } \text{cn}(\tau(t),k) = 0 \tag{5.45c}
\]

The \(\text{sn}(\tau(t),k)\) condition leads to maxima, whereas the condition on \(\text{cn}(\tau(t),k)\) corresponds to minima. Thus, upper and lower limits on the negative radial acceleration are

\[
[-a_r]_u = \omega^2 a \left( \frac{1}{k^2} + 1 \right) \tag{5.46a}
\]
\[-a_r]_l = \omega^2 a \left( \frac{1}{k^2} - 1 \right) \tag{5.46b}

To determine the region where the right hand side of Equation (5.39d) is equal to or greater than the left hand side lower limit, the following necessary condition must hold.

\[\omega^2 a \left( \frac{1}{k^2} - 1 \right) \leq G \left( \frac{m_1}{\rho_1^3} + \frac{m_3}{\rho_2^3} \right) a \tag{5.47}\]

Utilizing Equations (5.5) and (5.6) and (5.10), the necessary condition becomes

\[\frac{1}{a} \left( \omega^2 \frac{d_2^2}{d_x^2} + 2G \left( \frac{m_1}{\rho_1} + \frac{m_3}{\rho_2} \right) - C \right) \leq G \left( \frac{m_1}{\rho_1^3} + \frac{m_3}{\rho_2^3} \right) a \tag{5.48}\]

For positive Jacobi constants, a sufficient condition ensuring Equation (5.48) is

\[\frac{1}{a} \left( \omega^2 \frac{d_2^2}{d_x^2} + 2G \left( \frac{m_1}{\rho_1} + \frac{m_3}{\rho_2} \right) \right) \leq G \left( \frac{m_1}{\rho_1^3} + \frac{m_3}{\rho_2^3} \right) a \tag{5.49}\]

or

\[\frac{2}{a} \left( \frac{1}{2} \omega^2 \frac{d_2^2}{d_x^2} + U(d_x,a) \right) \leq -U_r(d_x,a) \tag{5.50}\]

where \(U(x,r)\) denotes gravitational potential with dependence on \(x\) and the \(y'z'\) plane radius \(r\), and \(U_r(x,r) = \partial U(x,r) / \partial r\). Equation (5.50) implies the orbit radius must be below a certain threshold to have the right hand side of Equation (5.39d) above the left hand side lower limit. Suppositional \(y'z'\) planes close to the center plane \((d_x = 0)\) will increase this threshold. Regions where the right hand side of Equation (5.39d) is equal to or less than the left hand side upper limit are determined by the necessary condition.
\[ G \left( \frac{m_1 + m_2}{\rho_1^3 + \rho_2^3} \right) a \leq \omega^2 a \left( \frac{1}{k^2} + 1 \right) \] \hspace{1cm} (5.51)

Utilizing Equations (5.5, 5.6) and (5.10), the necessary condition becomes

\[ G \left( \frac{m_1 + m_2}{\rho_1^3 + \rho_2^3} \right) a \leq \omega^2 a \left[ 2 + \frac{1}{\omega^2 a^2} \left( \omega^2 d_x^2 + 2G \left( \frac{m_1 + m_2}{\rho_1 + \rho_2} \right) - C \right) \right] \] \hspace{1cm} (5.52)

For \( C \) values maintaining Equation (5.20), a sufficient condition ensuring Equation (5.52) is

\[ G \left( \frac{m_1 + m_2}{\rho_1^3 + \rho_2^3} \right) a \leq 2\omega^2 a \] \hspace{1cm} (5.53)

or

\[ -U_r(d_x, a) \leq 2\omega^2 a \] \hspace{1cm} (5.54)

Equation (5.54) implies the orbit radius must be above a certain threshold to have the right hand side of Equation (5.39d) below the left hand side upper limit.

Overall, the suppositional circular solution can be described as strictly correct in only one axis and approximately correct (bounded-averaged and banded) in the other axes. Of course in numerical propagation, error in Equation (5.39a) or (5.39d) will spill over to Equation (5.39e).
CHAPTER 6
CORRECTING THE VERTICAL CIRCULAR ORBIT

6.1 Introduction

In practice, the supposed planar circular motion solution, applicable to any restricted three-body system, may only be accurate over a short arc of the true three-dimensional orbit. Fortunately, the mathematical structure of Equation (5.38) allows for an iterative analytical approximate procedure to correct the suppositional motion results. The iterative procedure is used to improve the accuracy of the predicted three-dimensional motion. Thus, this study introduces a new approximate analytical foundation for an existing class of three-dimensional highly inclined quasi-periodic orbits that may be used for applications in any restricted three-body system. The analytical correction methodology is similar to that introduced in Chapter 4 used in correcting the nominal planar orbit introduced in Chapter 3. In Section 6.2 an iterative analytical procedure is offered to provide corrections to the base solution. In Section 6.3 an example of high inclination halo orbit is introduced, in which the nominal orbit is proved to come closer to a numerically calculated halo orbit.

6.2 Correction Process

Table 6.1 outlines the iterative perturbation like procedure. Under a motion supposition for the $x$ axis, the motion in the remaining axes $y$ and $z$ are solved for as in Section 5.2 using Equations (5.38b)-(5.38c). These results constitute the base solution $x_b(t), y_b(t),$ and $z_b(t).$ Now, the base solutions in the $y$ and $z$ axes are used to solve for a correction to the $x$ axis base solution $x_b(t)$ from Equation (5.38a). The total solution is expanded as the base plus first correction, and certain approximations are invoked in
solving for the correction, including all corrections are small quantities and other sizing restrictions on the $y$ and $z$ axis base solutions. Next, the $x$ axis first correction is used in Equations (5.38b)-(5.38c) to solve for a first correction to the $y$ and $z$ axis dynamics $y_b(t)$ and $z_b(t)$. A similar solution procedure is used here and then iterated between $x$, $y_c(t)$, and $z_c(t)$. The correction procedure outlined here is also general and does not rely on any specific three-body system.

<table>
<thead>
<tr>
<th>Table 6.1 Iterative Analytical Solution</th>
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\[
x(t) \approx x_b
\]

**Base Solution:**
\[
y(t) = y_b(t), \\
z(t) = z_b(t)
\]

**First Correction:**
\[
x(t) = x(t) + x_c(t), \\
y(t) = y(t) + y_c(t), \\
z(t) = z(t) + z_c(t)
\]

**Second Correction:**
\[
x(t) = x(t) + x_c(t) + x_{c2}(t), \\
y(t) = y(t) + y_c(t) + y_{c2}(t), \\
z(t) = z(t) + z_c(t) + z_{c2}(t)
\]
Base solution results are taken from Section 5.2 and include

\[ x_b(t) = d_x \quad (6.1a) \]

\[ y_b(t) = a \sin \{ \theta(t) \} \quad (6.1b) \]

\[ z_b(t) = a \cos \{ \theta(t) \} \quad (6.1c) \]

where \( \sin \{ \theta(t) \} \) and \( \cos \{ \theta(t) \} \) are given in Equation (5.16). A correction to \( x_b(t) \) is now sought. Substitute

\[ x(t) = x_b + x_{cl}(t) \quad (6.2a) \]

\[ y(t) = y_b(t) \quad (6.2b) \]

\[ z(t) = z_b(t) \quad (6.2c) \]

into Equation (5.38a), which yields

\[
\ddot{x}_{cl}(t) - 2\omega \dot{y}_b(t) = \omega^2 \left( x_b + x_{cl}(t) \right) - \frac{Gm_1 \left( x_b + x_{cl}(t) - x_1 \right)}{\left( x_b + x_{cl}(t) - x_1 \right)^2 + y_b(t)^2 + z_b(t)^2}^{1/2} \]

\[
- \frac{Gm_2 \left( x_b + x_{cl}(t) - x_2 \right)}{\left( x_b + x_{cl}(t) - x_2 \right)^2 + y_b(t)^2 + z_b(t)^2}^{1/2} \]

By expanding the nonlinear gravitational terms about the base solution in Equation (6.3a), one obtains

\[
\ddot{x}_{cl}(t) - 2\omega \dot{y}_b(t) = \omega^2 \left( x_b + x_{cl}(t) \right) \]

\[
- Gm_1 \left[ \frac{x_b - x_1}{\rho_1^5} \left\{ \frac{1}{\rho_1^5} - \frac{3(x_b - x_1)^2}{\rho_1^5} \right\} x_{cl}(t) + \ldots \right] \quad (6.3b) \]

\[
- Gm_2 \left[ \frac{x_b - x_2}{\rho_2^5} \left\{ \frac{1}{\rho_2^5} - \frac{3(x_b - x_2)^2}{\rho_2^5} \right\} x_{cl}(t) + \ldots \right] \]
To proceed analytically, select $d_x = d_i(\mu)$ for $i = 1, 2, 3$, cancel out the embedded (approximate bounded-averaged) base solution ($a \ll (d_x - x_i)^2$ for $i = 1, 2$), and delete high order terms in $x_{cl}(t)$ ($|x_{cl}(t)| \ll \rho_i$ for $i = 1, 2$).

\[ \ddot{x}_{cl}(t) + \left[ -\omega^2 + G \left( m_i \left( \frac{1}{\rho_i^3} - \frac{3(x_b - x_i)^2}{\rho_i^5} \right) + m_2 \left( \frac{1}{\rho_2^3} - \frac{3(x_b - x_2)^2}{\rho_2^5} \right) \right) \right] x_{cl}(t) = 2\omega \dot{y}_b(t) \]  

(6.3c)

\[ \ddot{x}_{cl}(t) + G \left( m_i \left( \frac{1}{\rho_i^3} - \frac{3(x_b - x_i)^2}{\rho_i^5} \frac{1}{\rho_i^3} \right) + m_2 \left( \frac{1}{\rho_2^3} - \frac{3(x_b - x_2)^2}{\rho_2^5} \frac{1}{\rho_2^3} \right) \right] x_{cl}(t) = 2\omega \dot{y}_b(t) \]  

(6.3d)

Depending on the sign of the gravitational coefficient, Equation (6.3d) represents a stable-unstable forced second order linear time invariant dynamic system. Numerical evaluation of this coefficient over parameters $d_x$ and $a$ confirms very small isolated regions exist, not necessarily at the libration planes, where the Equation (6.3d) system is stable. However, the space over parameters $d_x$ and $a$ is dominated by unstable cases, which is consistent with $L_1$, $L_2$, and $L_3$ stability analysis. Only the unstable case will be explored further, which is recast as

\[ \ddot{x}_{cl}(t) - \lambda^2 x_{cl}(t) = 2\omega \dot{y}_b(t) \]  

(6.4a)

\[ \lambda^2 = \omega^2 - G \left( m_i \left( \frac{1}{\rho_i^3} - \frac{3(x_b - x_i)^2}{\rho_i^5} \right) + m_2 \left( \frac{1}{\rho_2^3} - \frac{3(x_b - x_2)^2}{\rho_2^5} \right) \right) \]  

> 0  

(6.4b)

and the forcing function in Equation (6.4) can be rewritten in terms of Jacobi elliptic functions as follows.

\[ 2\omega \dot{y}_b(t) = \frac{2\omega^2 a}{k} \text{sn}\{\tau(t), k\} \text{dn}\{\tau(t), k\} \]  

(6.5a)

However, the Jacobi elliptic functions can be reformulated in terms of the nome expansion and the forcing becomes
\[ 2\omega \dot{y}_p(t) = \frac{2\omega^2 a}{k} \left[ \frac{2\pi}{kK(k)} \sum_{i=0}^{\infty} \frac{q^{i+\frac{1}{2}}}{1 - q^{2i+1}} \sin \left\{ \frac{(2i+1)\pi}{2K(k)} \tau(t) \right\} \right] \]
\[ \cdot \left[ \frac{\pi}{2K(k)} + \frac{2\pi}{K(k)} \sum_{j=0}^{\infty} \frac{q^j}{1 + q^{2j}} \cos \left\{ \frac{2j\pi}{2K(k)} \tau(t) \right\} \right] \]  
\[ = c_Q \left[ \sum_{i=0}^{\infty} Q_i \sin \{ \omega_i \tau(t) \} \right] \]
\[ + \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} Q_{ij} \sin \{ \omega_i \tau(t) \} \cos \{ \omega_j \tau(t) \} \]  

where

\[ c_Q = \frac{2\pi^2 \omega^2 a}{k^2 K^2(k)} \]
\[ Q_i = \frac{q^{i+\frac{1}{2}}}{1 - q^{2i+1}}, \quad Q_{ij} = 4 \frac{q^{i+\frac{1}{2}}}{1 - q^{2i+1}} \frac{q^j}{1 + q^{2j}} \]  
\[ \omega_i = \frac{(2i+1)\pi}{2K(k)}, \quad \omega_j = \frac{2j\pi}{2K(k)} \]
\[ q = e^{-\frac{\pi K(k')}{K(k)}}, \quad k' = (1 - k^2)^{1/2} \]  

The homogeneous solution to Equation (6.4a) is

\[ x_{c_{1H}}(t) = A_1 e^{\lambda_1 t} + A_2 e^{-\lambda_2 t} \]  

while the nonhomogeneous solution is

\[ x_{c_{1NH}}(t) = \sum_{i=0}^{\infty} B_{1i} \sin \{ \omega_i \tau(t) \} + B_{2i} \cos \{ \omega_i \tau(t) \} \]
\[ + \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} B_{1ij} \sin \{ \omega_i \tau(t) \} \cos \{ \omega_j \tau(t) \} + B_{2ij} \sin \{ \omega_j \tau(t) \} \cos \{ \omega_i \tau(t) \} \]  

where
\[ B_{ui} = -\frac{c_0 Q_i}{\omega_{si}^2 \left( \frac{\omega}{k} \right)^2 + \lambda_{c1}^2} \]

\[ B_{2i} = 0 \]

\[ B_{12ij} = -\frac{c_0 Q_q \left[ \left( \omega_{si}^2 + \omega_{c1}^2 \right) \left( \frac{\omega}{k} \right)^2 + \lambda_{c1}^2 \right]}{\left[ \left( \omega_{si}^2 + \omega_{c1}^2 \right) \left( \frac{\omega}{k} \right)^2 + \lambda_{c1}^2 \right] - 2 \omega_{si} \omega_{c1} \left( \frac{\omega}{k} \right)^2} \]  

\[ B_{21ij} = -\frac{c_0 Q_q \left[ 2 \omega_{si} \omega_{c1} \left( \frac{\omega}{k} \right)^2 \right]}{\left[ \left( \omega_{si}^2 + \omega_{c1}^2 \right) \left( \frac{\omega}{k} \right)^2 + \lambda_{c1}^2 \right] - 2 \omega_{si} \omega_{c1} \left( \frac{\omega}{k} \right)^2} \]  

(6.6c)

Applying the initial conditions \( x_{cl}(t_0) = x_{cl0}, \dot{x}_{cl}(t_0) = \dot{x}_{cl0} \), allow \( A_1, A_2 \) to be solved for

\[ A_1 = \frac{1}{2\lambda_{c1}} \{ \lambda_{c1} e^{-\lambda_{c1} t} (x_{cl0} - \beta_x) + e^{-\lambda_{c1} t} (\dot{x}_{cl0} - \beta_x) \} \]

\[ A_2 = \frac{1}{2\lambda_{c1}} \{ \lambda_{c1} e^{\lambda_{c1} t} (x_{cl0} - \beta_x) - e^{\lambda_{c1} t} (\dot{x}_{cl0} - \beta_x) \} \]

(6.7a)

where

\[ \beta_x = \sum_{i=0}^{\infty} B_{ui} \sin \{ \omega_{gi} \tau(t_0) \} \]

\[ + \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} B_{12ij} \sin \{ \omega_{gi} \tau(t_0) \} \cos \{ \omega_{c1} \tau(t_0) \} + B_{21ij} \sin \{ \omega_{gi} \tau(t_0) \} \cos \{ \omega_{c1} \tau(t_0) \} \]

\[ \beta_x = \sum_{i=0}^{\infty} -B_{ui} \omega_{gi} \left( \frac{\omega}{k} \right) \cos \{ \omega_{gi} \tau(t_0) \} \]

\[ + \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} -B_{12ij} \left( \omega_{gi} \cos \{ \omega_{gi} \tau(t_0) \} \cos \{ \omega_{c1} \tau(t_0) \} - \omega_{c1} \sin \{ \omega_{gi} \tau(t_0) \} \sin \{ \omega_{c1} \tau(t_0) \} \right) \]

\[ -B_{21ij} \left( \omega_{gi} \sin \{ \omega_{gi} \tau(t_0) \} \sin \{ \omega_{c1} \tau(t_0) \} + \omega_{c1} \cos \{ \omega_{gi} \tau(t_0) \} \cos \{ \omega_{c1} \tau(t_0) \} \right) \]

(6.7b)

After collecting results, the complete solution for the \( x \) axis first correction is
For a general set of initial conditions, and when $x_{cl}(t)$ is added to $x_b$ (see Equation (6.2)), the third body will move off the supposition plane with a combined multi-frequency oscillatory and aperiodic nature, at least initially while the variation from the plane is not excessive, according to Equation (6.8). A special class of Equation (6.8) solutions, unstable periodic orbits, is also possible for certain initial condition sets. Examples would be libration point halo orbits. To generate this class of motion, homogeneous coefficients $A_1, A_2$ must be nulled. An initial condition set satisfying this requirement from Equation (6.7a) is

$$x_{cl_b} = \beta_x$$  

(6.9a)

$$\dot{x}_{cl_b} = \beta_x$$  

(6.9b)

Next, corrections to $y_b(t)$ and $z_b(t)$ are addressed. A solution structure of

$$x(t) = x_b + x_{cl}(t)$$  

(6.10a)

$$y(t) = y_b(t) + y_{cl}(t)$$  

(6.10b)

$$z(t) = z_b(t) + z_{cl}(t)$$  

(6.10c)

is assumed and substituted into Equations (5.38b)-(5.38c).
\[ \ddot{y}_b(t) + \ddot{y}_{cl}(t) + 2\omega \dot{x}_{cl}(t) = \omega^2 [y_b(t) + y_{cl}(t)] - \frac{Gm_1(y_b(t) + y_{cl}(t))}{[x_b + x_{cl}(t) - x_1]^2 + (y_b(t) + y_{cl}(t))^2 + (z_b(t) + z_{cl}(t))^2]^{3/2}} \]

\[ \ddot{y}_{cl}(t) + \ddot{z}_{cl}(t) = -\frac{Gm_1(z_b(t) + z_{cl}(t))}{[(x_b + x_{cl}(t) - x_1)^2 + (y_b(t) + y_{cl}(t))^2 + (z_b(t) + z_{cl}(t))^2]^{3/2}} \]

Gravitational expansion of these relations about the base solution provides

\[ \ddot{y}_b(t) + \ddot{y}_{cl}(t) + 2\omega \dot{x}_{cl}(t) = \omega^2 [y_b(t) + y_{cl}(t)] - Gm_1 \left[ \frac{y_b(t)}{\rho_1^3} - \frac{3(x_b - x_1)y_b(t)}{\rho_1^5} x_{cl}(t) + \left\{ \frac{1}{\rho_1^3} - \frac{3y_b(t)^2}{\rho_1^5} \right\} y_{cl}(t) - \frac{3y_b(t)z_b(t)}{\rho_1^5} z_{cl}(t) + ... \right] \]  

\[ \ddot{y}_{cl}(t) + \ddot{z}_{cl}(t) = -Gm_1 \left[ \frac{z_b(t)}{\rho_1^3} - \frac{3(x_b - x_1)z_b(t)}{\rho_1^5} x_{cl}(t) + \left\{ \frac{1}{\rho_1^3} - \frac{3z_b(t)^2}{\rho_1^5} \right\} z_{cl}(t) + ... \right] \]

To proceed analytically, cancel out the embedded (approximate banded) base solution and delete high order terms in \( x_{cl}(t), y_{cl}(t), z_{cl}(t) \) where \( \|x_{cl}(t)\|, \|y_{cl}(t)\|, \|z_{cl}(t)\| \ll \rho_i \) for \( i = 1, 2 \).
Equation (6.13) represents two coupled forced second order linear time varying dynamic systems. Taking the y and z axes base solutions as small with respect to the third body relative position magnitudes \(|y_b(t)|, |z_b(t)| \ll \rho_i^{5/2}\) for \(i = 1, 2\). Equation (6.13) simplifies to uncoupled time invariant systems.

\[
\ddot{y}_{cl}(t) + \left[-\omega^2 + G \left\{m_1 \left(1 - \frac{3y_b^2(t)}{\rho_1^3} \right) + m_2 \left(1 - \frac{3y_b^2(t)}{\rho_2^3} \right) \right\}\right] y_{cl}(t) = -2\omega \dot{x}_{cl}(t) \tag{6.14a}
\]

\[
\ddot{z}_{cl}(t) + \left[G \left\{m_1 \left(\frac{1}{\rho_1^3} - \frac{3z_b^2(t)}{\rho_1^3} \right) + m_2 \left(\frac{1}{\rho_2^3} - \frac{3z_b^2(t)}{\rho_2^3} \right) \right\}\right] z_{cl}(t) = 0 \tag{6.14b}
\]

Substituting the definition of the angular velocity \(\omega\) yields

\[
\ddot{y}_{cl}(t) + G \left\{m_1 \left(\frac{1}{\rho_1^3} - \frac{1}{n_{12}^3} \right) + m_2 \left(\frac{1}{\rho_2^3} - \frac{1}{n_{12}^3} \right) \right\} y_{cl}(t) = -2\omega \dot{x}_{cl}(t) \tag{6.15a}
\]
Depending on the sign of the gravitational coefficient in Equation (6.15a), the $y$ axis system can be stable or unstable. A condition ensuring the coefficient is positive is $\omega^2 y < -\partial U/\partial y$, or the gravitational acceleration in the $y$ direction is greater than the centripetal acceleration component from a $y$ displacement (assuming $y > 0$). Small radius orbits located between the primary and secondary bodies tend to be $y$ axis stable, such as small $L_1$ halo orbits. Small radius orbits lying well outside the CRTBP system (small $L_2$, $L_3$ halo orbits), or large radius orbits located anywhere along the $x$ axis, tend to be unstable. The $z$ axis system in Equation (6.15b) is always stable. Further note the $z$ axis system is Coriolis unforced.

Only the stable $y$ axis case will be explored further. Equations (6.15a) and (6.15b) are recast as

\[
\ddot{y}_{c_1}(t) + G \left( \frac{m_1}{\rho_1^3} + \frac{m_2}{\rho_2^3} \right) z_{c_1}(t) = 0 \tag{6.15b}
\]

The forcing function in Equation (6.16a), making use of the $x$ axis first correction in Equation (6.8), is
\[ -2\omega \dot{x}_{c1}(t) = -2\omega \left\{ -A_1 \lambda_{c1} e^{\lambda_{c1} t} - A_2 \lambda_{c1} e^{-\lambda_{c1} t} + \sum_{i=0}^{\infty} -B_i \omega_i \left( \frac{\omega}{k} \right) \cos \{ \omega_i \tau(t) \} \right. \\
+ \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} -B_{12ij} \left( \frac{\omega}{k} \right) \left[ \omega_i \cos \{ \omega_i \tau(t) \} \cos \{ \omega_j \tau(t) \} - \omega_i \sin \{ \omega_i \tau(t) \} \sin \{ \omega_j \tau(t) \} \right] \\
- B_{21ji} \left( \frac{\omega}{k} \right) \left[ -\omega_i \sin \{ \omega_j \tau(t) \} \sin \{ \omega_i \tau(t) \} + \omega_i \cos \{ \omega_j \tau(t) \} \cos \{ \omega_i \tau(t) \} \right] \left( 6.17a \right) \]
\[ = c_A \left\{ A_1 e^{\lambda_{c1} t} - A_2 e^{-\lambda_{c1} t} \right\} + c_{RST} \left[ \sum_{i=0}^{\infty} R_i \cos \{ \omega_i \tau(t) \} \right] \\
+ \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} S_{ij} \sin \{ \omega_i \tau(t) \} \sin \{ \omega_j \tau(t) \} + T_{ij} \cos \{ \omega_i \tau(t) \} \cos \{ \omega_j \tau(t) \} \right] \]

where

\[ c_A = -2\omega \lambda_{c1}, \quad c_{RST} = \frac{2\omega^2}{k} \]

\[ R_i = \omega_i B_i, \quad S_{ij} = -\omega_i B_{12ij} - \omega_j B_{21ji}, \quad T_{ij} = \omega_i B_{21ji} + \omega_j B_{21ji} \quad \left( 6.17b \right) \]

The homogeneous solution to Equation (6.16a) is

\[ y_{c1h}(t) = C_1 \sin(\omega_{c1} t) + C_2 \cos(\omega_{c1} t) \quad \left( 6.18a \right) \]

while the nonhomogeneous solution is

\[ y_{c1nh}(t) = D_1 e^{\lambda_{c1} t} + D_2 e^{-\lambda_{c1} t} + \sum_{i=0}^{\infty} E_{1i} \sin(\omega_i \tau(t)) + E_{2i} \cos(\omega_i \tau(t)) \]

\[ + \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} E_{1ij} \sin(\omega_j \tau(t)) \sin(\omega_i \tau(t)) + E_{2ij} \cos(\omega_i \tau(t)) \cos(\omega_j \tau(t)) \]

where
\[ D_1 = \frac{c_{A_1} A_1}{\lambda_{c_1}^2 + \omega^2} \]
\[ D_2 = -\frac{c_{A_2} A_2}{\lambda_{c_1}^2 + \omega^2} \]
\[ E_{uu} = 0 \]
\[ E_{2i} = -\frac{c_{RST} R_i}{-\omega_{st}^2 \left( \frac{\omega}{k} \right)^2 + \omega_{c_1}^2} \]

\[ E_{11i} = \frac{c_{RST} \left[ -\left( \omega_{st}^2 + \omega_{g}^2 \right) \left( \frac{\omega}{k} \right)^2 + \omega_{c_1}^2 \right] S_{ij} - c_{RST} \left[ 2 \omega_s \omega_{g_i} \left( \frac{\omega}{k} \right)^2 \right] T_{ij}}{\left[ -\left( \omega_{st}^2 + \omega_{g}^2 \right) \left( \frac{\omega}{k} \right)^2 + \omega_{c_1}^2 \right] - \left[ 2 \omega_s \omega_{g_i} \left( \frac{\omega}{k} \right)^2 \right]^2} \] (6.18d)

\[ E_{22i} = \frac{-c_{RST} \left[ 2 \omega_s \omega_{g_i} \left( \frac{\omega}{k} \right)^2 \right] S_{ij} + c_{RST} \left[ -\left( \omega_{st}^2 + \omega_{g}^2 \right) \left( \frac{\omega}{k} \right)^2 + \omega_{c_1}^2 \right] T_{ij}}{\left[ -\left( \omega_{st}^2 + \omega_{g}^2 \right) \left( \frac{\omega}{k} \right)^2 + \omega_{c_1}^2 \right] - \left[ 2 \omega_s \omega_{g_i} \left( \frac{\omega}{k} \right)^2 \right]^2} \] (6.18e)

Applying the initial conditions \( y_{c_1}(t_0) = y_{c_b}, \dot{y}_{c_1}(t_0) = \dot{y}_{c_b} \) allow \( C_1, C_2 \) to be solved for

\[ C_1 = \frac{1}{\omega_{c_1}} \left[ \omega_{c_1} \sin(\omega_{c_1} t_0) (y_{c_b} - \delta_y - \epsilon_y) + \cos(\omega_{c_1} t_0) (\dot{y}_{c_b} - \delta_y - \epsilon_y) \right] \] (6.19a)

\[ C_2 = \frac{1}{\omega_{c_1}} \left[ \omega_{c_1} \cos(\omega_{c_1} t_0) (y_{c_b} - \delta_y - \epsilon_y) - \sin(\omega_{c_1} t_0) (\dot{y}_{c_b} - \delta_y - \epsilon_y) \right] \] (6.19b)

where

\[ \delta_y = D_1 e^{\lambda_{c_b} t_0} + D_2 e^{-\lambda_{c_b} t_0} \] \[ \delta_y = D_1 \lambda_{c_b} e^{\lambda_{c_b} t_0} - D_2 \lambda_{c_b} e^{-\lambda_{c_b} t_0} \] (6.19c)
\[ e_y = \sum_{i=0}^{\infty} E_{yi} \cos \{\omega_y \tau(t_0)\} \]
\[ + \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} E_{yi j} \sin \{\omega_y \tau(t_0)\} \sin \{\omega_j \tau(t_0)\} + E_{22j} \cos \{\omega_y \tau(t_0)\} \cos \{\omega_j \tau(t_0)\} \]
\[ e_y = \sum_{i=0}^{\infty} E_{yi} \omega_{yi} \left( \frac{\omega}{k} \right) \sin \{\omega_y \tau(t_0)\} \]
\[ + \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} -E_{yi j} \left[ \omega_{yi} \cos \{\omega_y \tau(t_0)\} \sin \{\omega_j \tau(t_0)\} + \omega_j \sin \{\omega_y \tau(t_0)\} \cos \{\omega_j \tau(t_0)\} \right] \]
\[ + E_{22j} \left( \frac{\omega}{k} \right) \left[ \omega_{yi} \sin \{\omega_y \tau(t_0)\} \cos \{\omega_j \tau(t_0)\} + \omega_j \cos \{\omega_y \tau(t_0)\} \sin \{\omega_j \tau(t_0)\} \right] \]

After collecting results, the complete solution for the \( y \) axis first correction is

\[ y_{cl}(t) = C_1 \sin(\omega_{cl} t) + C_2 \cos(\omega_{cl} t) + D_1 e^{\lambda_1 t} + D_2 e^{\lambda_2 t} \]
\[ + \sum_{i=0}^{\infty} E_{yi} \cos \{\omega_y \tau(t)\} \]
\[ + \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} E_{yi j} \sin \{\omega_y \tau(t)\} \sin \{\omega_j \tau(t)\} + E_{22j} \cos \{\omega_y \tau(t)\} \cos \{\omega_j \tau(t)\} \]

For a general set of initial conditions, and when \( y_{cl}(t) \) is added to \( y_b(t) \) (see Equation (6.10)), the third body will move off the supposition circle with a combined multi-frequency oscillatory and aperiodic nature, at least initially while the variation from the circle is not excessive, according to Equation (6.20). A special class of Equation (6.20) solutions, unstable periodic orbits, is again possible for certain values of initial conditions. This class of motion occurs when the nonhomogeneous coefficients \( D_1, D_2 \) are zero. This condition is achieved by Equation (6.9). Finally, a further special class of unstable periodic orbits with no homogeneous frequency content occurs when \( C_1, C_2 \) are zero. This condition is achieved when

\[ y_{cb} = \dot{e}_y, \quad \dot{y}_{cb} = \ddot{e}_y \]
Being unforced in the Coriolis sense, the z axis first correction solution is much simpler, having only the homogeneous component. The solution takes the form

\[ z_{cl}(t) = F_1 \sin(\omega_{cl} t) + F_2 \cos(\omega_{cl} t) \]  

(6.22a)

where

\[ F_1 = \frac{1}{\omega_{cl}} \left[ \omega_{cl} \sin(\omega_{cl} t_0)(z_{cl0}) + \cos(\omega_{cl} t_0)(\dot{z}_{cl0}) \right] \]

(6.22b)

\[ F_2 = \frac{1}{\omega_{cl}} \left[ \omega_{cl} \cos(\omega_{cl} t_0)(z_{cl0}) - \sin(\omega_{cl} t_0)(\dot{z}_{cl0}) \right] \]

and the initial conditions are \( z_{cl}(t_0) = z_{cl0} \), \( \dot{z}_{cl}(t_0) = \dot{z}_{cl0} \). For a given set of initial conditions, and when \( z_{cl}(t) \) is added to \( z_b(t) \) (see Equation (6.10)), the third body will again deviate from the supposition circle but with a single-frequency oscillatory nature, assuming the initial conditions are not excessively large, according to Equation (6.22). This homogeneous motion is the only motion allowed in the z axis first correction under the stated assumptions.

This iterative procedure can be extended in a systematic fashion, but a solution through the first correction will suffice here. The overall analytical approximate solution for the third body motion is thus

\[ x(t) = x_b + x_{cl}(t) \]
\[ y(t) = y_b(t) + y_{cl}(t) \]
\[ z(t) = z_b(t) + z_{cl}(t) \]  

(6.23)

where \( x_b, y_b(t), z_b(t) \) are listed in Equation (6.1), and \( x_{cl}(t), y_{cl}(t), z_{cl}(t) \) are listed in Equations (6.8), (6.20), and (6.22).
6.3 $L_1$ Halo Orbit Example

The objective of this section is to investigate methodology whereby approximate but pure analytical relationships between high inclination halo type orbit characteristics and fundamental three-body system parameters can be developed. A periodic $L_1$ halo orbit for an artificial CRTBP system with $\mu_2 = \mu = 0.04$ is used as a test case for the suppositional motion theory and iterated analytical solution procedure of Sections 5.2 and 6.2. Reference 121 contains several suitable halo orbits as candidates. In particular, a small $0.17r_{12} \times 0.076r_{12}$ sized orbit that undergoes an even smaller variation ($0.046r_{12}$) in the $x$ axis is selected for further analysis. At $t = t_0$, initial conditions for this exact numerical based orbit are given below and correspond to the third body intersecting the $xz$ plane in the upper quadrants.

\[
\begin{align*}
  x_{h0} &= -0.723268 r_{12} & \dot{x}_{h0} &= 0 \\
  y_{h0} &= 0 & \dot{y}_{h0} &= 0.198019 \omega \ r_{12} \\
  z_{h0} &= 0.04 r_{12} & \dot{z}_{h0} &= 0
\end{align*}
\] (6.24)

The halo orbit geometry is symmetric about the $xz$ plane. The Jacobi constant for this orbit is $C_h = 3.329168 \omega^2 r_{12}^2$, and the period is $T_h = 2.603/\omega$.

Comparison of the base solution from Section 6.2 (suppositional circular motion from Section 5.2) with the exact halo orbit, computed from nonlinear simulation, is considered first. Some arbitrariness exists in mapping the exact three dimensional orbit initial conditions to the suppositional circular two dimensional orbit initial conditions. Initial conditions at $t = t_0$ for the suppositional motion are chosen here as
\[ d_x = -0.74090984286 r_{12}(d_{i1}) \quad a = z_{h0} \]
\[ \theta_0 = 0 \quad \dot{\theta}_0 = \dot{y}_{h0} / z_{h0} \]  

(6.25)

The selection of \( a, \theta_0, \dot{\theta}_0 \) provides a good match to the projected initial halo orbit state but likely will incur a larger error at other locations around the orbit. The base solution modulus value, computed from Equation (5.10), with the above initial conditions is \( k = k_b = 0.198 \). The \( d_x \) selection at the collinear equilibrium point \( L_1 \) is based on insights from Section 5.4 and provides averaged correct \( x \) axis motion as the supposition plane is located in the mean of the motion variation. Figure 6.1 shows overlay plots of the base and exact solutions. The temporally unsynchronized maximum \( x,y,z \) positional errors are \( 0.0274 r_{12}, 0.0435 r_{12}, 0.0045 r_{12} \). The orbital period error is \( 1.347 / \omega \). The base solution roughly captures the halo orbit \( xy \) plane geometry, but further refinement is required.
Computation of first corrections in the $x$ and $y$ axes is considered next. Recall no first correction for the $z$ axis is available ($z_{cl0} = 0, \dot{z}_{cl0} = 0$). Rather than using the general formulation given in Section 6.2, a truncated version will be considered to show that the full procedure may not be required for every application. The main departure from the general formulation is truncation of the $2\omega \dot{y}_b(t)$ Coriolis forcing term in Equations (6.5a)-(6.5b). A two term nome expansion for the forcing is
\[
\frac{2 \alpha \dot{y}_b(t)}{\left( \frac{2 \omega^2 a}{k} \right)} = \frac{\pi^2 \sqrt{q}}{k K^2(k)(1-q)} \left[ \sin\{\nu(t)\} + \frac{q}{1+q+q^2} \sin\{3\nu(t)\} + \frac{4q}{1+q^2} \sin\{\nu(t)\} \cos\{2\nu(t)\} + \frac{4q^2}{(1+q+q^2)(1+q^2)} \sin\{3\nu(t)\} \cos\{2\nu(t)\} \right]
\]

where \( \nu(t) = \pi \tau(t) / 2K(k) \). Using the following trigonometric identities

\[
\sin\{\nu(t)\} \cos\{2\nu(t)\} = \frac{1}{2} \sin\{3\nu(t)\} - \frac{1}{2} \sin\{\nu(t)\} \tag{6.27a}
\]
\[
\sin\{3\nu(t)\} \cos\{2\nu(t)\} = \frac{1}{2} \sin\{5\nu(t)\} + \frac{1}{2} \sin\{\nu(t)\} \tag{6.27b}
\]

Equation (6.26) is reformulated to

\[
\frac{2 \alpha \dot{y}_b(t)}{\left( \frac{2 \omega^2 a}{k} \right)} = \frac{\pi^2 \sqrt{q}}{k K^2(k)(1-q)} \left[ \frac{1}{1+q+q^2} + \frac{2q^2}{(1+q+q^2)(1+q^2)} \right] \sin\{\nu(t)\} + \left( \frac{q}{1+q+q^2} + \frac{2q}{1+q^2} \right) \sin\{3\nu(t)\} + \frac{2q^2}{(1+q+q^2)(1+q^2)} \sin\{5\nu(t)\} \right]
\]

Conversion of the forcing signal to a pure sinusoidal nature will simplify calculations. As \( k \) decreases the nome \( q \) also decreases, and the nome dependent harmonic coefficients in Equation (6.28) decrease, leaving a simple sine wave which the full expansion would also collapse to. In contrast, when \( k \rightarrow 1 \) the nome dependent harmonic coefficients increase in significance, and differences can be expected between the two term and full expansion signals. Figure 6.2 shows the approximate and exact forcing signal versus dimensionless time for a group of \( k \) values. The two term nome formulation should be accurate for \( k < 0.85 \).
Another smaller departure from the general formulation is retention of the nonzero $a^2$ term in Equation (5.40). This retention leads to an additional constant forcing term in Equation (6.4a) which improves correction solution accuracy. In this case the solution for the correction in the $x$ axis is found to be

$$x_{c_1}(t) = A_x e^{k_1t} + B_x e^{-k_1t} + C_x + D_x \sin \{v(t)\} + E_x \sin \{3v(t)\} + F_x \sin \{5v(t)\}$$

(6.29)

where $A_x$ and $B_x$ are constants to be determined from initial conditions, and
\[ C_x = -\frac{C_{x_{cl}}}{\lambda_{cl}^2} \]

\[ D_x = -\frac{D_{x_{cl}}}{\left(\frac{\pi \omega}{2kK(k)}\right)^2 + \lambda_{cl}^2} \]

\[ E_x = -\frac{E_{x_{cl}}}{\left(\frac{3\pi \omega}{2kK(k)}\right)^2 + \lambda_{cl}^2} \]

\[ F_x = -\frac{F_{x_{cl}}}{\left(\frac{5\pi \omega}{2kK(k)}\right)^2 + \lambda_{cl}^2} \]

\[ C_{x_{cl}} = \omega^2 x_b - \frac{Gm_2(x_b - x_1)}{\rho_1^5} - \frac{Gm_2(x_b - x_3)}{\rho_2^5} \]

\[ D_{x_{cl}} = \frac{2\omega^2 a}{k} \frac{\pi^2 \sqrt{q}}{kK^2(k)(1 - q)} \left\{ 1 - \frac{2q}{1 + q^2} + \frac{2q^2}{(1 + q + q^2)(1 + q^2)} \right\} \]

\[ E_{x_{cl}} = \frac{2\omega^2 a}{k} \frac{\pi^2 \sqrt{q}}{kK^2(k)(1 - q)} \left\{ \frac{q}{1 + q + q^2} + \frac{2q}{1 + q^2} \right\} \]

\[ F_{x_{cl}} = \frac{2\omega^2 a}{k} \frac{\pi^2 \sqrt{q}}{kK^2(k)(1 - q)} \left\{ \frac{2q^2}{(1 + q + q^2)(1 + q^2)} \right\} \]

(6.30)

Applying the initial conditions \( x_{cl_0}(t_0) = x_{cl_0} \), \( \dot{x}_{cl}(t_0) = 0 \) to \( t_0 = 0 \) yields

\[ x_{cl_0} = A_x + B_x + C_x + D_x - E_x + F_x \]  \hspace{1cm} (6.31a)

\[ 0 = A_x - B_x \]  \hspace{1cm} (6.31b)

To remove the nonperiodic terms (\( A_x = B_x = 0 \)) in Equation (6.29) the required condition is

\[ x_{cl_0} = C_x + D_x - E_x + F_x \]  \hspace{1cm} (6.31c)

Equation (6.31c) is equivalent to Equation (6.9a) for the general formulation.

The required condition in Equation (6.31c) facilitates a correction to the modulus value. The initial position \( x_{cl_0} \) is first computed from the halo orbit data as

\[ x_{cl_0} = x_{h_0} - d_x = 0.01764 \, r_{12} \]

Observing that coefficients \( C_x, D_x, E_x, F_x \) are functions of \( k \) (see Equation (6.30)), Equation (6.31c) is reformulated as

\[ f(k) - x_{cl_0} = 0 \]  \hspace{1cm} (6.32)
Equation (6.32) is to be solved for $k$ numerically. Since not all the harmonics are included in the solution, one cannot expect Equation (6.32) to be fully satisfied for any value of $k$. An appropriate value for $k$ which corresponds to the minimum of the left hand side magnitude of Equation (6.32) will be considered here. Figure 6.3 shows the absolute value of the left hand side of Equation (6.32) versus $k$. The $x$ axis first correction modulus value corresponding to the minimum Equation (6.32) error is $k = k_{x_1} = 0.279$. Finally, the solution for the first correction in the $x$ axis is written as follows.

$$x_{c_1}(t) = C_x + D_x \sin\{v(t)\} + E_x \sin\{3v(t)\} + F_x \sin\{5v(t)\}$$

(6.33)

Since the value of $k$ does not completely satisfy Equation (6.32), the initial value $x_{c_10}$ is automatically adjusted by Equation (6.33) when the coefficients are computed from $k$ obtained in Equation (6.32).

![Figure 6.3 Equation (6.32) Error vs. Elliptic Modulus](image)
Using the result from the $x$ axis to correct $y(t)$, the solution for the $y$ axis first correction is found to be

$$y_{c1}(t) = A_y \cos \{\omega_{c1} t\} + B_y \sin \{\omega_{c1} t\} + D_y \cos \{\nu(t)\} + E_y \cos \{3\nu(t)\} + F_y \cos \{5\nu(t)\} \quad (6.34a)$$

where $A_y$ and $B_y$ are constants to be determined from initial conditions, and

$$D_y = \frac{D_{y_{c1}}}{\omega_{c1}^2 - \left(\frac{\pi \omega}{2kK(k)}\right)^2} \quad D_x = \frac{\pi \omega^2}{kK(k)} D_x$$

$$E_y = \frac{E_{y_{c1}}}{\omega_{c1}^2 - \left(\frac{3\pi \omega}{2kK(k)}\right)^2} \quad E_x = \frac{3\pi \omega^2}{kK(k)} E_x$$

$$F_y = \frac{F_{y_{c1}}}{\omega_{c1}^2 - \left(\frac{5\pi \omega}{2kK(k)}\right)^2} \quad F_x = \frac{5\pi \omega^2}{kK(k)} F_x \quad (6.34b)$$

The initial conditions $y_{c1}(t_0) = 0$, $\dot{y}_{c1}(t_0) = \ddot{y}_{c10}$, $t_0 = 0$, when applied to Equation (6.34a) and its derivative, result in

$$0 = A_y \quad (6.35a)$$

$$\dot{y}_{c10} = \omega_{c1} B_y + \frac{\pi \omega}{2kK(k)} \left(D_y - 3E_y + 5F_y\right) \quad (6.35b)$$

To eliminate the homogeneous part of the solution ($B_y = 0$), the following condition should be satisfied.

$$\ddot{y}_{c10} = \frac{\pi \omega}{2kK(k)} \left(D_y - 3E_y + 5F_y\right) \quad (6.36)$$

In the general formulation, Equation (6.36) corresponds to Equation (6.21).

There appears to be at least three plausible interpretations to the meaning and utilization of Equation (6.36) in conjunction with Equations (6.31c) and (5.10). The solution theory in this paper is analytically consistent implying existence of a single
unique $k$ for the exact solution. Since approximations have been invoked and only a single iteration has been considered, a unique $k$ satisfying Equations (5.10), (6.31c), and (6.36) simultaneously does not exist. The first interpretation is to compute three separate moduli ($k = k_b, k = k_{x_1}, k = k_y$) and use them in the appropriate solution components ($k_b \rightarrow y_b(t)$ and $z_b(t), k_{x_1} \rightarrow x_{c1}(t), k_y \rightarrow y_{c1}(t)$). Moduli $k_b$ and $k_{x_1}$ have already been considered. Modulus $k_y$ would be computed from Equation (6.36) with $\dot{y}_{c10} = 0$ since the full halo orbit initial velocity $\dot{y}_{h0}$ has been previously applied to the base solution (see Equation (6.25)). This interpretation was investigated. Although rigorous solutions to Equation (6.36) with $\dot{y}_{c10} = 0$ exist for the limiting cases $k : k \rightarrow 0$ and $k \rightarrow 1$, no other values of $k$ precisely satisfy the required condition. However, broad regions were found where the right hand side of Equation (6.36) was nearly zero and independent of the value of $k$, implying the possibility of vastly different values of the three moduli or the insignificance of the Equation (6.36) condition. Consequently, this interpretation was abandoned. The second interpretation is to compute a single optimum $k$ ($k = k_b = k_{x_1} = k_y$) which best satisfies all three equations. The problem formulation would involve unknowns $k, x_{c10}$, and the percentage distribution of $\dot{y}_{h0}$ to initial conditions $\dot{\theta}_0$ and $\dot{y}_{c10}$. This interpretation appears overly complicated and was not considered.

The third interpretation, lying somewhere in between the first and second interpretations, is to compute a single optimum $k$ which best satisfies a single equation (Equation (6.31c), $k = k_{x_1}$). This single $k$ would then be used consistently throughout all
solution components \((k_{x_{c1}} \rightarrow y_b(t))\) and \(z_b(t), \ k_{x_{c1}} \rightarrow x_{c1}(t), \ k_{x_{c1}} \rightarrow y_{c1}(t)\).

Implementation here is simpler and is congruent with numerical differential correction logic in the following sense. With the appropriate value of the modulus determined from Equation (6.31c), Equation (6.36) is used to compute the change in the initial \(y\) axis velocity. Thus, at each step of the iteration process there should be an increment in the initial \(y\) axis velocity. The mechanism of the iterative solution procedure is acting like a differential correction technique. However, this increment in the initial \(y\) axis velocity is not used in the analytical orbit construction process because numerical integration is not employed.

The third interpretation was adopted for this work, and the computation effort was already presented in Figure 6.3. Using \(k = k_{x_{c1}} = 0.279\) in Equation (6.34b) to compute coefficients \(D_y, E_y, F_y\), the solution for the first correction in \(y\) is

\[
y_{c1}(t) = D_y \cos \{v(t)\} + E_y \cos \{3v(t)\} + F_y \cos \{5v(t)\} (6.37)
\]

Care should be taken to avoid resonance forcing singularity in the solution, which occurs when

\[
\frac{\omega_{c1, y}}{\omega} = \frac{n\pi}{2kK(k)}, \ n = 1,3,5 \tag{6.38}
\]

Figure 6.4 shows the dimensionless natural frequency to be avoided as a function of \(k\) for the three possible forcing signals parameterized by \(n\).
Figure 6.4 Natural Frequency at Singularity Conditions

Figure 6.5 shows the analytic orbit construction after the first correction iteration using the two term nome expansion with the true orbit. The temporally unsynchronized maximum x, y, z positional errors are 0.0131 \( r_{12} \), 0.0165 \( r_{12} \), 0.0045 \( r_{12} \). Using the true halo orbit dimensions as the reference, the first correction has reduced the maximum positional error in the x axis from 60% to 28% and in the y axis from 26% to 9.7%. No change has occurred in the z axis. Also, the period error has reduced to 34% from 52%. The orbital period error is \( 0.89/\omega \). The corrected orbit has the sloped \( xz \) plane track and the flattened \( yz \) plane closed path signatures commonly exhibited by halo-class orbits. The \( xz \) plane track does not show any significant curvature at this iteration, but the rectilinear track reasonably captures the true motion behavior. Note how coefficient \( C_x \) has pushed the track away from the \( L_1 \) point. Also note how the \( D_y \) coefficient has
amplified the distance the third body travels from the $x$ axis when it traverses from the upper quadrants to the lower quadrants. Significant improvement in the halo orbit prediction is noted in Figure 6.5 after one correction iteration. Depending on the application and required accuracy, a second iteration for the analytic solution can be considered. Determination of sufficiency of the required number of corrections can be judged by the respective contributions to the total solution and the intended application.

![Figure 6.5 True and Two Term Once Corrected Orbit](image)

Geometry of halo orbits for mission design can be reduced, in preliminary studies, to the average slope of the $xz$ plane track, $M$, and the approximate vertical-to-horizontal aspect ratio of the $yz$ plane path, $A$. Suitable definitions for these parameters for the analytic solution displayed in Figure 6.5 are
\[ M = \frac{z(t_0) - z}{x(t_0) - x} \left( t_0 + \frac{k}{\omega} K(k) \right) \]  
with \( \theta_0 = 0, t_0 = 0 \)  \hspace{1cm} (6.39a)

\[ A = \frac{y(t_0)}{y} \left( t_0 + \frac{k}{\omega} K(k) \right) \]  
with \( \theta_0 = 0, t_0 = 0 \)  \hspace{1cm} (6.39b)

Note, \( t = 0 \) corresponds to \( \theta = 0 \) while \( t = (k/\omega)K(k) \) corresponds to \( \theta = \pi / 2 \) rad.

Using the analytic results and simplifying by neglecting \( E_x, F_x, E_y, F_y \) \( (O(E_x, F_x, E_y, F_y) \ll O(D_x, D_y) \) holds for the numerical example), the slope and aspect ratio can be expressed symbolically as

\[
M(k) = \frac{a}{D_x} = -\frac{\left( \frac{\pi \omega}{2kK(k)} \right)^2 + \lambda_{c_1}^2}{8 \left( \frac{\pi \omega}{2kK(k)} \right)^2 f(q(k))} \\
= \frac{1 + \frac{4}{\pi^2} k^2 K^2(k) \left[ 1 - \frac{G}{\omega^2} \left\{ m_1 \left( \frac{1}{\rho_1^3} - \frac{3(d_z - x_z)^2}{\rho_1^5} \right) + m_2 \left( \frac{1}{\rho_2^3} - \frac{3(d_z - x_z)^2}{\rho_2^5} \right) \right]\right]}{8f(q(k))} \\
\] \hspace{1cm} (6.40a)

\[
A(k) = \left( 1 + \frac{D_x}{a} \right)^{-1} = \left[ 1 + \frac{8\pi \omega^2}{kK(k)} \left( \frac{\pi \omega}{2kK(k)} \right)^2 f(k) \left[ \left( \frac{\pi \omega}{2kK(k)} \right)^2 - \omega_{c_1}^2 \right] \left( \frac{\pi \omega}{2kK(k)} \right)^2 + \lambda_{c_1}^2 \right]^{-1} \\
= \left[ 1 + \frac{32}{\pi kK(k)} f(k) \right]^{-1} \\
\hspace{1cm} = \left[ 1 + \frac{4k^2 K^2(k)}{\pi^2} \left\{ 1 - \frac{G}{\omega^2} \left( \frac{m_1}{\rho_1^3} + \frac{m_2}{\rho_2^3} \right) \right\} \right]^{-1} \hspace{1cm} (6.40b)
\]

where
These expressions can be used in various ways. For example, mission sensing or communication objectives may impose certain requirements on a satellite to periodically transit various regions in the three-body spatial system. Given a preliminary set of values for $d_x$, $a$, $x_{c_0}$, the modulus $k$ can be computed, and from this, the halo orbit slope and aspect ratio can be computed. If the orbit does not meet the requirements satisfactorily, sensitivities of $M$ and $A$ with respect to variations in $k$ can be analyzed with Equations (6.40a)-(6.40b) for modification purposes. The analytical expressions could also be useful with inverse problems where parameters $M, A$ are specified along with $d_x$ to satisfy mission objectives and Equations (6.40a)-(6.40b) are used to compute compatible $a, k$ values. Additionally, Equations (6.40a)-(6.40b) could be differentiated with respect to $k$ to identify extremal conditions for slope or aspect ratio. Assuming all assumptions taken in the derivation of the expressions are maintained, Equations (6.40a)-(6.40b) could provide physical insight and avoid costly numerical propagation within iterative searches, i.e., when the slope and aspect ratio are preselected and the appropriate orbital parameters are calculated. Even in other numerical differential correction techniques, such as numerically constructing a periodic orbit by finding appropriate initial position-velocity pairs, the supposed orbit may provide a better initial guess than the linearized equation solution, and this directly affects the required computational convergence time. By combining Equations (6.40a)-(6.40b), a direct relation between $M$ and $A$ for design purposes can also be derived. Finally, if mission requirements involve time, the orbital period through Equation (5.12) can be coupled to the process.
A hypothetical elliptic integral solution to Jacobi's integral equation in the circular restricted three-body problem, under planar circular assumptions with nonuniform speed, has been offered. The solution, which satisfies one motion equation precisely and the other two approximately, provides closed-form analytical results for the orbital period and path in terms of several parameters including the orbit radius, the plane location, and the elliptic modulus (initial conditions). These suppositional results are found to be mathematically rich and insightful. However, to bridge the gap from hypothetical to factual, an iterative analytical approximate procedure that computes successive corrections to the hypothetical solution is also offered. An initial test case using a small $L_1$ periodic halo orbit showed, after a single correction step using simplifying assumptions, the corrections bring the hypothetical solution closer to the true orbit.

Results presented in Chapters 5 and 6 capture the essence of, and are dynamically relevant to, highly inclined orbits located near the collinear equilibrium points in the circular restricted three-body problem. The findings provide a window for deeper physical understanding of detail characteristics associated with this class of orbits. Results may provide practical utility for mission design and celestial analysis.
CHAPTER 7
APPLICATIONS

7.1 Introduction

In this chapter the solution developed in Chapter 5, which is a circular orbit in a plane perpendicular to the line joining the two primaries, is used as a nominal orbit around a collinear equilibrium point. Since the equation of motion is not generally satisfied when substituting the proposed nominal motion from Equation (6.1), to have the third body traverse an exact circular orbit in the $y'z'$ plane, external thrust forces must be applied. These thrust forces are applied to keep the spacecraft traversing the nominal orbit, and characteristics and parameters of the nominal orbit are chosen so that motion starts with the minimum initial thrust. If the third body is an artificial satellite with an actively controlled propulsion subsystem, such capability can be exploited to render the suppositional motion described in Section 5.2 exact. Such an orbit could be useful for communications, in-situ space measurements, observation platforms, loitering, etc., particularly in regions where natural halo orbits do not exist or are expensive to maintain due to instabilities. The proposed exact circular vertical orbit maintained by thrust would then add a mission design freedom or a potential design alternative. In Section 7.2 thrust control inputs are included in the differential equation of motion, and thrust components are obtained in a cylindrical coordinate system. In Section 7.3 an approach for minimizing initial thrust is discussed. In Section 7.4 analytical relations for thrust supplied velocity increments over one period of motion are obtained in both normal and bi-normal directions. Parameters for minimum velocity increment are analyzed. In
Section 7.5 an example of a vertical circular orbit in a three-body system with zero mass parameter is introduced.

7.2 Thrust Control Inputs

In this section, the required thrust will be calculated and presented in a non-dimensional way such that any user can convert the data to specific fuel/thrust requirements for a specific engine-vehicle system. Rewriting the equations of motion including the thrust control inputs in the right hand side yields

\[ \ddot{x} - 2\omega \dot{y} = J_x + u_x \]  
\[ \ddot{y} + 2\omega \dot{x} = J_y + u_y \]  
\[ \ddot{z} = J_z + u_z \]

where \( u_x, u_y, u_z \) are the thrust accelerations in the \( x, y, z \) directions respectively.

When substituting the coordinates of the proposed motion from Equation (6.1) into Equation (7.1), the thrust accelerations are assumed to balance or negate the residue in each direction.

\[ u_x = -2\omega a \dot{\theta} \cos \{ \theta(t) \} - J_x(d_x, a) \]  
\[ u_y = -a \dot{\theta}^2 \sin \{ \theta(t) \} + a \dot{\theta} \cos \{ \theta(t) \} - J_y(d_y, a, \theta(t)) \]  
\[ u_z = -a \dot{\theta}^2 \cos \{ \theta(t) \} - a \dot{\theta} \sin \{ \theta(t) \} - J_z(d_z, a, \theta(t)) \]

Before exploring the detail properties of each term in the right hand side of Equation (7.2), reconfiguring Equation (7.2) to another coordinate system is considered. In such a system the thrust acceleration components are applied in the bi-normal, tangential, and normal directions, or
\[
\begin{bmatrix}
u_b \\
u_t \\
u_n
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
u_x \\
u_y \\
u_z
\end{bmatrix}
\]

\[u_b = -a\left(2\omega \dot{\theta} \cos \{\theta(t)\} + C_x\right)\] \hspace{1cm} (7.3a)

\[u_t = 0\] \hspace{1cm} (7.3b)

\[u_n = -a\left(\ddot{\theta}^2 + \omega^2 \sin^2 \{\theta(t)\} + C_z\right)\] \hspace{1cm} (7.3d)

where the subscripts \(b, t, n\) in Equation (7.3) denote the bi-normal, tangential, and normal directions respectively. \(C_x, C_z\) are constants defined as follows.

\[C_x = \frac{1}{a}\left(\omega^2 d_x - \frac{Gm_1}{\rho_1^3} (d_x - x_1) - \frac{Gm_2}{\rho_2^3} (d_x - x_2)\right)\] \hspace{1cm} (7.4a)

\[C_z = -\frac{Gm_1}{\rho_1^3} - \frac{Gm_2}{\rho_2^3}\] \hspace{1cm} (7.4b)

Equation (7.4) shows that both \(C_x\) and \(C_z\) depend on the three-body system under question: the location of the plane motion, i.e., the location of the orbit center on the line joining the two primaries, and the orbit radius. Equation (7.3) indicates that there is no thrust in the tangential direction required to keep the third body on the nominal orbit, which means that the nominal orbit solution satisfies the equation of motion in the tangential direction.

Unfortunately, this is not the case in either the normal or the bi-normal directions. However, for certain regions within the \(xyz\) space, the centripetal and gravitational accelerations appearing in the right hand side of the governing radial expression were shown to be approximately balanced, in a banded sense. At any specified time, to exactly balance these terms and preserve the suppositional motion, the radial thrust acceleration
must equal the difference between these centripetal and gravitational terms. By substituting for $\dot{\theta}$ from Jacobi’s integral equation under the supposition motion into Equation (7.3) the radial accelerative thrust can be expressed as

$$u_n = -2a\omega^2 \sin^2 \{\theta\} + c_T$$  \hspace{1cm} (7.5a)

where $c_T$ is a constant which depends on the characteristics of the three-body system and the suppositional motion. This constant can be either positive, negative, or zero.

$$c_T = G \left( \frac{m_1}{\rho_1} + \frac{m_2}{\rho_2} \right) \left( a - \left( a\dot{\theta}^2 - \omega^2 a \sin^2 \{\theta_0\} \right) \right)$$  \hspace{1cm} (7.5b)

Equation (7.2) gives the required propulsive $x$ axis acceleration to maintain planar circular motion for a general $y'z'$ plane. By selecting $d_x$ to coincide with libration points $L_1, L_2, \text{or} L_3$, the thrust requirement simplifies (see Equation (5.40)) to

$$u_b = -2\omega a\dot{\theta} \cos \{\theta\} - \frac{1}{2} \left. \frac{\partial^2 J_x (d_x, a)}{\partial a^2} \right|_{a=0} a^2 + ...$$  \hspace{1cm} (7.6)

The residual gravitational terms appearing in the right hand side of Equation (7.6) were shown to be small, in a bounded averaged sense. Thus, the required cylindrical thrust component is dominated by the Coriolis acceleration. Figure 7.1 shows the various thrust acceleration components in the suppositional plane $y'z'$ axes and the radial-tangential $rt$ axes.
Figures 7.2-7.3 show the non-dimensional accelerative thrust demands for this special case against normalized time for half an orbit, across a family of elliptic moduli, for the Earth-Moon system using an orbit radius of $a = 0.05r_{12}$. In Figure 7.2, all accelerative thrust curves in the radial direction have similar behavior (constant plus elliptic $cn(\tau(t),k)$ square nature) with time except the limiting case $k : k \to 1$. In this case, results indicate a sharp reduction in thrust demand at $\theta = n\pi / 2$, $n = 1, 3, 5, \ldots$, and between these values thrust can be approximated to be constant. Also note all curves have minimums at $\theta = n\pi / 2$, $n = 1, 3, 5, \ldots$ and maximums at $\theta = n\pi$, $n = 0, 1, 2, 3, \ldots$. Further, the radial thrust acceleration is nearly always positive. For the considered orbit radius and elliptic moduli values, the gravitational acceleration is larger than the centripetal acceleration, and outward (positive) radial thrust is required to maintain the circular path.
Figure 7.3 shows the thrust acceleration component in the cylindrical direction (x axis). Note all curves intersect at common points $\theta = n\pi / 2$, $n=1,3,5,...$ at which they all have the same value equaling the constant gravitational part in Equation (7.6). With the value of $k$ getting smaller, the thrust in the $x$ direction approaches a cosine wave that is shifted above the zero value. Generally, the required thrust profiles take the shape of a constant plus an elliptic $\text{sn}(\tau(t),k)$ and $\text{dn}(\tau(t),k)$ product. For the considered orbit radius, the second order gravitational attraction that exists in the $L_1$ libration plane is towards the second primary, thus requiring a positive (toward the first primary) bias in the cylindrical thrust. Overall, the rate of change of $u_n, u_b$ is nonuniform and strongly depends on $k$. The peak values of $u_n, u_b$ are also a strong function of $k$. For the limiting case $k : k \rightarrow 1$ in Figure 7.3, the smallest peak propulsive acceleration and smallest integrated propulsive acceleration (impulse) are experienced. For smaller values of $k$, required peak and integrated thrust accelerations to maintain circular motion increase. The case $k : k \rightarrow 0$ requires infinite propulsive capability. Opposite but similar trends are noted in Figure 7.2 The largest (positive) propulsive demand occurs in the $k : k \rightarrow 1$ case. For $k : k \rightarrow 0$ the required thrust approaches an infinite negative bias.
Figure 7.2 Radial Thrust Component vs. Time

Figure 7.3 Cylindrical Thrust Component vs. Time
Transverse: $(\theta - \theta_0)/2\pi$, Radial: $u_n/(\omega^2 r_{12})$

**Figure 7.4 Radial Thrust Component vs. Angle**

Transverse: $(\theta - \theta_0)/2\pi$, Radial: $u_b/(\omega^2 r_{12})$

**Figure 7.5 Cylindrical Thrust Component vs. Angle**
Figures 7.4-7.5 show the non-dimensional thrust acceleration against the normalized angle. These figures help with visualizing the thrust demands at different locations in the physical coordinates. Figure 7.4 indicates that the amount of radial thrust in the upper half \((z>0)\) of the orbit is the same as in the lower half of the orbit. Acceleration demand curves are symmetrical with respect to both the \(y\) axis and \(z\) axis. Figure 7.5 shows that the amount of cylindrical thrust required in the lower half of the orbit is larger than that in the upper half of the orbit. Thrust demand curves are symmetrical with respect to the \(z\) axis but unsymmetrical with respect to the \(y\) axis. In the lower half, the Coriolis acceleration is aligned with the gravitational bias, thus requiring more thrust to maintain planar motion. Figures 7.2-7.3 illustrate when thrust components are positive or negative, which is critical for determining the instantaneous direction of the total thrust vector in space. In contrast, Figures 7.4-7.5 show only absolute values of thrust at different locations along the orbit. Together the two sets of figures may be helpful in determining suboptimal thrust logic to maintain the required orbit. The radial and cylindrical thrust vector components vary with the change in modulus \(k\) for the same value of circular orbit radius \(a\). Since the only way to minimize the total amount of thrust is to minimize these components, care should be taken when determining a proper modulus \(k\).
7.3 Minimum Initial Thrust

Equations (7.3b) and (7.3d) indicate the general view to minimize the thrust accelerations. The right hand sides of these two equations consist of two parts: a periodic part with period \( T \) and a constant part. Once the three-body system is determined and the location of the orbit center is chosen, the only freedom to minimize the thrust accelerations is the choice of the initial conditions that determine the effect of the periodic part. When the third body starts the motion on the \( z \) axis at time \( t = t_0 \) and an angle \( \theta = \theta_0 = 0 \), the magnitude \( u \) of the total thrust vector at this moment is

\[
u_0 = a \left( 2\omega \theta_0 + C_x \right)^2 + (\dot{\theta}_0^2 + C_z^2)^{1/2}\]

(7.7)

Substituting the initial angular velocity from Equation (5.16) with the use of Jacobi elliptic function identities, the initial thrust vector magnitude is rewritten as a function of the modulus of the elliptic integral and the two constants \( C_x \) and \( C_z \). These two constants are functions of the radius of the circular orbit, and the location of the orbit center, given the three-body system, is determined.

\[
u_0(k, a, d_x) = a \left( 2\omega^2 \frac{\sqrt{1-k^2}}{k} + C_x \right)^2 + \left( \omega^2 \frac{1-k^2}{k^2} + C_z \right)^{1/2}\]

(7.8)

Figures 7.6 to 7.8 show three dimensional graphs representing the initial thrust vector magnitude as expressed in Equation (7.8) for three different orbital plane locations corresponding to the collinear equilibrium points of the Earth-Moon system \( L_1, L_2, L_3 \). The Earth-Moon system is chosen merely as an example.

Equation (7.8) indicates that singularity occurs when the elliptic integral modulus approaches its lower limit \( k : k \to 0 \). In this case velocity of the third body approaches
infinity, and the amount of required thrust is unbounded. When the elliptic modulus approaches its upper limit \( k : k \to 1 \), velocity of the third body approaches zero, and the amount of required thrust also approaches its minimum.

\[
\begin{align*}
    k &\to 0, \: \dot{\theta}_0 \to \infty, \: u_0 \to \infty \quad (7.9a) \\
    k &\to 1, \: \dot{\theta}_0 \to 0, \: u_0 \to a(C_x^2 + C_z^2)^{1/2} \quad (7.9b)
\end{align*}
\]

Equation (7.8) and Figures 7.6, 7.7, and 7.8 indicate that the initial thrust vector magnitude is monotonically increasing with the increase of the orbit radius. This behavior can be explained as follows: as the spatial size of the orbit increases, at constant values of initial angular velocity (or elliptic integral modulus magnitude), the effect of gravitational nonlinearity increases, and the required initial thrust increases.

The value of \( u_0 \) in Equation (7.8) depends on two squared quantities, and for \( u_0 \) to be zero, the two squared quantities should vanish. It is noteworthy to investigate values of the two constants \( C_x \) and \( C_z \) over the domain of orbit radius and location of the orbit center at the three Lagrange collinear points. The locations in the \( a-d_x \) space, at which either one or both of the constants \( C_x \) and \( C_z \) are negative, represent possible minima for the squared quantities in the right hand side of Equation (7.8).
Figure 7.6 Initial Thrust in $L_1$ Plane of the Earth-Moon System

Figure 7.7 Initial Thrust in $L_2$ Plane of the Earth-Moon System
Figure 7.8 Initial Thrust in $L_3$ Plane of the Earth-Moon System

Figure 7.9 Constant $C_x$ vs. Parameters $a$ and $d_x$
In Figure 7.9 it is noticed that the sign of $C_z$ varies between positive and negative when the orbit center is located at the Lagrange point $L_1$, while this is not the case at the two other Lagrange points. At point $L_3$ the value of $C_z$ is always positive while at point $L_2$ the value of $C_z$ is always negative. Figure 7.10 shows that the value of $C_z$ is always negative at the three Lagrange points.

Since the elliptic modulus represents a selectable initial condition parameter, a rational process is to choose the elliptic modulus to give minimum initial thrust in one or both directions. When the derivative of the initial thrust vector magnitude with respect to elliptic modulus is set equal to zero, one obtains

$$0 = C_x \frac{k^3}{\sqrt{1-k^2}} + (\omega^2 + C_z)k^2 + \omega^2$$  \hspace{1cm} (7.10)
since $C_x$ and $C_z$ only depend on $a$ and $d_x$. Equation (7.10) may be expanded into a sixth order polynomial of $k$. However, for small values of the elliptic modulus ($k \ll 1$), Equation (7.10) is simplified to a third order polynomial of $k$ as follows.

$$0 = C_x k^3 + (\omega^2 + C_z) k^2 + \omega^2$$  

(7.11)

Existence and number of positive real roots for $k$ in Equation (7.11) depends on the number of sign change of the coefficients of the first and second term. Three situations for sign change exist:

1. Two sign changes ($C_x/\omega^2 > 0$ and $C_z/\omega^2 < -1$), which is possible only at $L_1$ for the range of orbit radius $a < 0.52106 r_{12}$,

2. One sign change ($C_x/\omega^2 < 0$ and $C_z/\omega^2 > -1$), which is possible at $L_1$ for the range of orbit radius $a > 0.56061 r_{12}$ and at $L_2$ for the range of orbit radius $a > 0.25650 r_{12}$,

3. One sign change ($C_x/\omega^2 < 0$ and $C_z/\omega^2 < -1$), which is possible at $L_1$ for the range of orbit radius $0.52106 r_{12} < a < 0.56061 r_{12}$ and at $L_2$ for the range of orbit radius $a < 0.25650 r_{12}$.

Figures 7.9 and 7.10 indicate that at $L_3$, constants $C_x/\omega^2 > 0$ and $C_z/\omega^2 > -1$; thus no sign change occurs in Equation (7.11), and no real positive value of the modulus can give a minimum initial thrust.

### 7.4 Velocity Increment

The velocity increment to be supplied by thrusters after one complete period of the third body motion is analyzed next. The magnitude of the accumulated velocity increment is obtained in the bi-normal and normal directions when integrating with respect to time Equations (7.3b) and (7.3d) respectively.
\[
\Delta v_b = \int_{t_0}^{t_f+\tau_0} |u_b(t)| \, dt = -\frac{k}{\omega} \int_{t_0}^{t_f} |u_b(\tau, k)| \, d\tau
\]

(7.12a)

\[
\Delta v_n = \int_{t_0}^{t_f+\tau_0} |u_n(t)| \, dt = -\frac{k}{\omega} \int_{t_0}^{t_f} |u_n(\tau, k)| \, d\tau
\]

(7.12b)

Variable \( \tau \) is the argument of an elliptic function and

\[
\tau_0 = F\left(\frac{\pi}{2} - \theta_0, k\right)
\]

(7.13a)

\[
\tau_f = -\frac{\omega}{k} (T) + F\left(\frac{\pi}{2} - \theta_0, k\right)
\]

(7.13b)

Properties of the periodic terms in the integrands of Equation (7.12) allow setting the limits of integration as given.

Figures 7.11a to 7.11c show the time history of the periodic terms in the binormal and normal thrust accelerations, respectively. Periods of these thrust acceleration components are 4K in Figure 7.11a, 2K in Figure 7.11b, and 2K in Figure 7.11c. When the initial conditions are chosen such that \( t_0 = 0, \theta_0 = 0 \) one obtains \( \tau_0 = K, \tau_f = -3K \), and integration is thus taken from \( \tau = K \) to \( \tau = 0 \), and the results are quadrupled.
Figure 7.11a Time History of Periodic Term in the Bi-Normal Thrust Acceleration \((\text{sn}(\tau)\text{dn}(\tau))\)

Figure 7.11b Time History of Periodic First Term in the Normal Thrust Acceleration \((\text{dn}^2(\tau))\)
The results of integration in Equation (7.12) are

$$\Delta v_b = 4a \frac{k}{\omega} KC_x + 8a\omega \quad (7.14a)$$

$$\Delta v_c = 4a \frac{\omega}{k} \left\{ 2E - \left[ 1 - k^2 - \left( \frac{k}{\alpha} \right)^2 C_x \right] K \right\} \quad (7.14b)$$

where $K$ and $E$ are the complete elliptic integrals of the first and second kind respectively. Knowing that $aC_x = J_x$ Equation (7.14) is rewritten as follows

$$\Delta v_b = 4a \frac{k}{\omega} KJ_x + 8a\omega \quad (7.14c)$$

Both velocity increments vanish when the third body orbit collapses to a zero radius orbit at one of the Lagrange points, i.e., $a = 0 = J_x(d_{x_i})$, $i=1,2,3$. However, if the third body is located at any general point on the x axis, i.e., $a = 0$, $J_x(d_{x_i}) \neq 0$, $d_x \neq d_{x_i}$, $i=1,2,3$, there
is an increment still needed in the bi-normal direction, and no velocity increment is needed in the normal direction to keep the third body in its location.

![Figure 7.12a Total Velocity Increment at Lagrange Point L₁](image1)

**Figure 7.12a Total Velocity Increment at Lagrange Point L₁**

![Figure 7.12b Contour of Total Velocity Increment at Lagrange Point L₁](image2)

**Figure 7.12b Contour of Total Velocity Increment at Lagrange Point L₁**
Figure 7.12c Total Velocity Increment at Lagrange Point $L_2$

Figure 7.12d Contour of Total Velocity Increment at Lagrange Point $L_2$
Figures 7.12a to 7.12f show total velocity increments versus orbit radius and modulus of elliptic integral at three different locations of the orbit center on the x axis representing the three Lagrange collinear points for the Earth-Moon system. In Figures
7.12a to 7.12f $\Delta v_i$ is the summation of the absolute values of both $\Delta v_b$ and $\Delta v_n$ which are velocity increments in the bi-normal and normal directions respectively. Generally, at a certain value of the elliptic modulus the total velocity increment decreases as the orbit radius decreases. However, this is not the rule for the relation between the total velocity increment and the modulus of elliptic integral. Let the orbit radius and elliptic modulus corresponding to certain minimum total velocity increment along the line $(\Delta v_i / \omega r_{12})_{\text{min}}$ be denoted by $a^*, k^*$, respectively. At a certain value of the orbit radius $a = a^*$, the total velocity increment increases for any $k \neq k^*$.

To investigate the dimensional values of the velocity increment at Lagrange point $L_1$, Figure 7.12b shows that the level curve of $\Delta v_i / \omega r_{12} = 0.2$, i.e., $\Delta v_i < 204.96 \text{ m/s}$ is attainable for values of orbit radius $a / r_{12} < 0.02$, i.e., $a < 7680 \text{ km}$ for all values of elliptic modulus. However, for values of orbit radius $a > 7680 \text{ km}$ this level curve is not attainable at any value of the elliptic modulus. In this case the next minimum level curve in Figure 7.12b, is that corresponding to $\Delta v_i / \omega r_{12} = 0.4$, i.e., $\Delta v_i < 409.92 \text{ m/s}$ applicable for values of elliptic modulus in the range $0.3 < k < 0.6$. Thus, as the value of the orbit radius increases the level curve corresponding to minimum velocity increment is attainable for a smaller range of values of the elliptic modulus, and this level curve is not attainable for any value of the elliptic modulus when the orbit radius exceeds a certain limit.

A problem of great importance is the determination of initial velocity corresponding to a certain orbit radius. Existence of the modulus in the proposed
minimization solution gives an implicit relation between orbit radius and initial velocity. Rewriting Equation (5.10), the initial angular velocity is

\[ \dot{\theta}_0 = \frac{\omega}{k} \sqrt{1 - k^2 \cos^2 \{\theta_0\}} \]  

(7.15)

Assuming the third body starts the motion on the z axis, i.e., \( \theta_0 = 0 \), the initial angular velocity \( \dot{\theta}_0^* \) that gives minimum total velocity increment is determined from Equation (7.15) when values of the elliptic modulus and the orbit radius are chosen to give \( (\Delta v_t / \omega r_{12})_{\text{min}} \) as shown in Figures 7.12a to 7.12f. Figure 7.13 shows that the relation between \( a \) and \( \dot{\theta}_0^* \) is nearly the same when the orbit is located at the \( L_1, L_2 \) Lagrange points. At these two points the value of \( \dot{\theta}_0^* \) decreases with \( a \), while at the point \( L_3 \) the value of \( \dot{\theta}_0^* \) is nearly constant with \( a \).

![Figure 7.13 Initial Angular Velocity vs. Orbit Radius for Minimum Total Velocity Increment](image)

**Figure 7.13 Initial Angular Velocity vs. Orbit Radius for Minimum Total Velocity Increment**
7.5 Example

In Chapter 6 the nominal orbit is used as a base solution for an analytic correction process which brings it closer to one of the halo type orbits at the collinear point \( L_1 \) in a three-body system with mass parameter \( \mu = m_2 / m_1 + m_2 = 0.04 \). However, in the current research with the nominal solution maintained by control inputs, potential applications lie in utilizing the fact that the nominal solution is kept in a vertical plane maintaining constant distances from the two primaries all the time. Using a spacecraft for communication purposes between the two primaries is a feasible application. The libration point \( L_1 \) is a candidate location for such an orbit. Studying natural properties or phenomena that happen in vertical planes at locations \( d_x \) other than libration points is also considered. This freedom is actually one of the properties of the nominal orbit that allow applications at locations other than libration points. The difference between these locations is the cost in terms of thrust fuel demands required to maintain the nominal orbit at that location. Another advantage of the nominal solution is that it can be applied to any three-body system with mass parameter in the range \( 0 \leq \mu \leq 1/2 \).

One of the important applications is vertical orbits located at the primaries which correspond to polar orbits in the two-body problem. When the vertical plane is located at the first primary, i.e., \( d_x = x_1 \), the constants \( C_x, C_z \) take the following forms

\[
C_x = \omega^2 \left\{ \frac{\mu}{a/r_2} \left( 1 - \frac{1}{(\rho_2/r_2)^3} \right) \right\} \\
C_z = \omega^2 \left\{ -\frac{1}{(\rho_1/r_1)^3} + \frac{\mu}{(\rho_1/r_1)^3} \left( 1 - \left( \frac{\rho_1}{\rho_2} \right)^3 \right) \right\}
\]  

(7.16a)  

(7.16b)
where \( \mu = m_2 / (m_1 + m_2) \) is known as the mass parameter. Substituting \( d_x = x_1 \) in Equation (5.2) and knowing that \( x_1 - x_2 = r_{12} \), one obtains

\[
\rho_1 = a \\
\rho_2 = \left( r_{12}^2 + a^2 \right)^{1/2}
\] (7.17a, 7.17b)

When the second primary mass is very small compared to the first primary the mass parameter approaches its lower bound. From Equation (2.58b) the angular velocity of the system is rewritten in case of zero mass parameter as follows

\[
\omega = \sqrt{\frac{Gm_1}{r_{12}^3}} = \sqrt{\frac{Gm_1}{a^3 \left( \frac{a}{r_{12}} \right)^3}} = n \left( \frac{a}{r_{12}} \right)^{3/2}
\] (7.18)

where \( n \) is the mean motion of a satellite in a two-body orbit. In that case the velocity increments in both bi-normal and radial directions are

\[
\Delta \nu_b = 8a \omega \\
\Delta \nu_r = 4a \frac{\omega}{k} \left\{ 2E - \left[ 1 - k^2 + k^2 \left( \frac{n}{\omega} \right)^2 \right] K \right\}
\] (7.19a, 7.19b)

The modulus of the elliptic integral \( k \) can be substituted from Equation (5.10) when the initial angular position \( \theta_0 = 0 \), or

\[
k = \left( 1 + \left( \frac{\dot{\theta}_0}{\omega} \right)^2 \right)^{-1/2}
\] (7.20)

In this case the center of mass is considered originated at the first primary, and a position vector \( R \) can be transformed from the inertial to rotating frame using the following relation

\[
r = AR
\] (7.21a)
where \( A \) is the transformation matrix:

\[
A = \begin{pmatrix}
\cos \omega t & \sin \omega t & 0 \\
-\sin \omega t & \cos \omega t & 0 \\
0 & 0 & 1
\end{pmatrix}
\] (7.21b)

Differentiating Equation (7.21a) with respect to time, the relation between velocity vectors \( \mathbf{V}, \mathbf{v} \) in inertial and rotating frames respectively is

\[
\mathbf{v} = \dot{\mathbf{A}} \mathbf{R} + \mathbf{A} \mathbf{V}
\] (7.21c)

At time \( t = 0 \), Equation (7.21c) is used to transform the initial velocity from the inertial to rotating frame. The value of the initial angular velocity is found to be \( \dot{\theta}_0 = \omega \).

Substituting this result into Equation (7.20) and using Equation (7.18), the modulus \( k \) takes the following form

\[
k = \left( 1 + \left( \frac{n}{\omega} \right)^2 \right)^{-1/2} = \left( 1 + \left( \frac{r_{12}}{a} \right)^3 \right)^{-1/2}
\] (7.22a)

\[
k^2 \left( \frac{n}{\omega} \right)^2 = 1 - k^2
\] (7.22b)

Substituting Equation (7.22b) into Equation (7.19b) yields

\[
\Delta v_b = 8a\omega
\] (7.19a)

\[
\Delta v_n = 8a\omega \left\{ kK - \frac{K - E}{k} \right\}
\] (7.19b)

For values of \( a \ll r_{12} \) the modulus \( k \) is very small, and the complete elliptic integrals of the first and second kinds can be expanded using hypergeometric functions \( \mathcal{R}, \mathcal{I} \) respectively.\(^{112}\)

\[
K(k) = \frac{\pi}{2} \mathcal{R}\left\{ \frac{1}{2}, \frac{1}{2}; 1; k^2 \right\} = \frac{\pi}{2} \left( 1 + \frac{1}{4} k^2 + \frac{9}{64} k^4 + ... \right)
\] (7.24a)
Substituting Equation (7.24) into Equation (7.23), one obtains

\[ \Delta v_b = 8a\omega \quad (7.25a) \]

\[ \Delta v_n = (2\pi k)a\omega \quad (7.25b) \]

Equation (7.25) shows that if \( \omega = 0 \) and the rotating frame coincide with the inertial frame, the control inputs vanish in both bi-normal and normal directions. This result means that the nominal orbit exists without correction in the inertial frame, but it cannot be maintained naturally without correction in the rotating frame. From Equation (7.22a) the modulus \( k \) can be approximated to

\[ k \approx \left( \frac{a}{r_{12}} \right)^{3/2} \]

and noting that the velocity of an Earth satellite on a circular orbit is \( V = \frac{a}{n} \), the velocity increments in Equation (7.25) can be normalized using \( V \) as follows.

\[ \frac{\Delta v_b}{V} = 8\left( \frac{a}{r_{12}} \right)^{3/2} \quad (7.25a) \]

\[ \frac{\Delta v_n}{V} = 2\pi \left( \frac{a}{r_{12}} \right)^3 \quad (7.25b) \]

Equation (7.26) shows that the control input in the normal direction is very small compared to that in the bi-normal direction.

An exact vertical circular orbit at a three-body Lagrange collinear point is achievable using thrust control inputs. Coordinates of a spacecraft moving along such an orbit can be predicted at any time, and the period of motion can be calculated once the initial conditions are given. The cost for maintaining this orbit in terms of velocity
increment per period can be estimated and minimized according to a given minimization criteria, as outlined in this chapter.
CHAPTER 8

CONCLUSIONS

An approximate analytical circular orbit can be established in the plane of motion of the primary masses. Conditions of small mass parameter and motion in the neighborhood of the first primary should be satisfied. The advantage of such an orbit is that results can be compared to the case of zero mass parameter and still maintain the characteristics of the three-body problem. Control input in the radial direction is needed to keep the orbit circular in the strict sense; however, motion is bounded without applying any control, and overall the approximate solution is stable. When the mass parameter goes to zero the solution gives the traditional two-body solution in a rotating coordinate system. This solution covers the gap between motion in the theory of the three-body problem and the two-body problem.

A nominal circular orbit in a plane perpendicular to the line joining the two primaries can also be established. The nominal solution is nothing but the projection of the three dimensional motion in the vertical plane. The solution results can be applied to any plane parallel to the vertical plane. Motion needs to be stabilized in the normal and bi-normal directions. Control inputs depend on the location of the vertical plane relative to the two primary masses. Overall, motion is unstable, and correcting the nominal orbit is necessary. Since the nominal orbit needs to be strictly maintained, thrusters should provide the required velocity increment during each period. The velocity increment is found to be in the applicable range of thrusters, and further minimization processes can be performed by choosing suitable parameters and initial conditions. This orbit will enhance onboard trajectory planning and maintenance in formation flying near collinear
libration points. The results also can be applied to exploring polar like orbits near any of the primaries. The fact that it can be applied to any three-body system gives it a potential advantage in many applications.
REFERENCES


APPENDIX A

Geometrical Significance and Convergence of Legendre Polynomials

From Figure A.1 the distance from the first primary to the third primary is calculated as follows

\[ \rho_i^2 = a^2 + x_i^2 - 2x_i a \cos \theta \quad (A.1) \]

\[ \frac{a}{\rho_i} = \left( 1 - \left( \frac{x_i}{a} \right)^2 + 2 \left( \frac{x_i}{a} \right) \cos \theta \right)^{-\frac{1}{2}} \quad (A.2) \]

The right hand side of Equation (A.2) is nothing but the generating function \( G(v; \chi) \) of Legendre polynomials, where \( v = \cos \theta \) and \( \chi = x_i / a \).

\[ \frac{a}{\rho_i} = G(v; \chi) = \sum_{m=0}^{\infty} \chi^m P_m(\cos \theta) \quad (A.3) \]

The minimum of the expansion in Equation (A.3) at a certain constant orbit radius is obtained at \( \rho_{i\text{max}} = x_i + a \) when the third body crosses the line of syzygy between the two
primaries. In contrast, the maximum of the expansion is obtained when \( \rho_{\text{min}} = a - x_1 \) occurs when the third body crosses the line of syzygy from the side far from the second primary.

\[
\left( \frac{a}{\rho_1} \right)_{\max} = 1 + \chi + \chi^2 + \\
\left( \frac{a}{\rho_1} \right)_{\min} = 1 - \chi + \chi^2 -
\]

(A.4a)

(A.4b)

From Equations (A.3), (A.4) the function \( G(v; \chi) \) can be considered analytic about unity. Generally, as indicated in Chapter 3, because of the property of a Legendre polynomial, \( |P_m(\cos \theta)| < 1 \) as long as the condition \( x_1 / a < 1 \) in the expansion of the generating function holds. However, defining two real variables, \( \delta_m, \delta_{m-1} \), convergence can usually be tested as follows

\[
\delta_m = \frac{\chi^m P_m(v)}{\chi^{m-1} P_{m-1}(v)} = \chi \frac{P_m(v)}{P_{m-1}(v)}
\]

(A.5a)

\[
\delta_{m-1} = \frac{\chi^{m-1} P_{m-1}(v)}{\chi^{m-2} P_{m-2}(v)} = \chi \frac{P_{m-1}(v)}{P_{m-2}(v)}
\]

(A.5b)

and from the recurrence formula of a Legendre polynomial,

\[
mP_m(v) - (2m - 1)vP_{m-1}(v) + (m - 1)P_{m-2}(v) = 0
\]

(A.6a)

the following formula is obtained

\[
m \frac{P_m(v)}{P_{m-1}(v)} = (2m - 1)v - (m - 1) \frac{P_{m-2}(v)}{P_{m-1}(v)}
\]

(A.6b)

\[
\chi \frac{P_m(v)}{P_{m-1}(v)} = \left( \frac{2m - 1}{m} \right) \chi v - \left( \frac{m - 1}{m} \right) \chi \frac{P_{m-2}(v)}{P_{m-1}(v)}
\]

(A.6c)

Substituting Equation (A.5) into Equation (A.6), one obtains a new recurrence formula:
\[
\delta_m = \frac{(2m-1)}{m} \chi^v - \frac{(m-1)}{m} \chi \frac{\delta}{\delta_{m-1}} \quad (A.7a)
\]

\[
\delta_m = \left(2 - \frac{1}{m}\right) \chi^v - \left(1 - \frac{1}{m}\right) \chi \frac{\chi}{\delta_{m-1}} \quad (A.7b)
\]

and for higher values of \( m = M, M >> 1 \)

\[
\delta_M \approx \chi \left(2v - \frac{\chi}{\delta_{M-1}}\right) \quad (A.7c)
\]

Knowing that \( P_0 = 1 \), then \( \delta_1 = \chi v \) and after substituting into Equation (A.7b) for \( M = 2 \), one finds

\[
\delta_2 = \frac{3}{2} \chi v - \frac{1}{2} \frac{\chi^2}{\delta_1} \quad (A.8a)
\]

\[
\delta_2 = \frac{3}{2} \chi v - \frac{1}{2} \frac{\chi}{v} \quad (A.8b)
\]

\[
\delta_2 = \chi v - \frac{1}{2} \chi \left(\frac{1}{v} - v\right) \quad (A.8c)
\]

For a value of \( v = 1 - \varepsilon \) where \( \varepsilon \) is a small real number, one has

\[
\delta_2 = \chi v - \varepsilon \chi \quad (A.9)
\]

In this case \( \delta_2 < \delta_1 \) and the expansion of the generating function converges.

This analysis also is valid for negative values of \( v \); however, Equation (A.8c) has a singularity when \( v = 0 \). This is because the expansion of the generating function has two sets of polynomials one in odd powers of \( v \) and one in even powers of \( v \). At \( v = 0 \) the set of odd polynomials vanish and convergence testing should be applied on even polynomials only. This test can be obtained from the recurrence formula in Equation (A.6a) when neglecting, for small values of \( v \), the odd polynomial which is multiplied by \( v \).
\[
\frac{P_m(v)}{P_{m-2}(v)} = -\frac{(m-1)}{m} \tag{A.10a}
\]

\[
\frac{\chi^mP_m(v)}{\chi^{m-2}P_{m-2}(v)} = -\frac{(m-1)}{m} \chi^2 \tag{A.10b}
\]

\[
|s'_m| = \left| \frac{\chi^mP_m(v)}{\chi^{m-2}P_{m-2}(v)} \right| = \frac{(m-1)}{m} \chi^2 , \quad m > 1 \tag{A.10c}
\]

For higher values of \( m = M \), \( M \gg 1 \), Equation (A.10b) can be reformulated as follows.

\[
|s'_m| \approx \chi^2 \tag{A.10d}
\]

Together, Equations (A.7b) and (A.10c) show the convergence of the generating function of Legendre polynomials.
APPENDIX B

Allowable Motion Regions in the PCRTBP with Zero Mass Parameter

The Jacobi integral equation is

\[ v^2 = 2J - C = 2\left( \frac{1}{2} \omega^2 (x^2 + y^2) + \frac{Gm_1}{\rho_1} + \frac{Gm_2}{\rho_2} \right) - C \] (B.1)

For simplicity, a normalized version of this equation will be used when dividing by \( \omega^2 r_{12}^2 \). Define the normalized radial distance from the first primary in case of zero mass parameter as \( r^2 = \rho_1^2 / r_{12}^2 = (x^2 + y^2) / r_{12}^2 \), and the distance from the second primary is arbitrary. Equation (B.1) then becomes

\[ rv^2 = r^3 - \frac{C}{\omega^2 r_{12}^2} r + 2 \] (B.2)

Assume that \( R_1 < 1, R_2 > 1 \) are two real positive roots satisfying Equation (B.2) when velocity is zero. Eq. (B.1) then becomes

\[ rv^2 = R_1^3 - \frac{C}{\omega^2 r_{12}^2} R_1 + 2 = 0 \] (B.3a)

\[ rv^2 = R_2^3 - \frac{C}{\omega^2 r_{12}^2} R_2 + 2 = 0 \] (B.3b)

Case I Motion About the First Primary \( a = R_1 < 1 \)

Substituting by \( r = R_1 + \Delta R_1 < 1 \) with \( \Delta R_1 > 0 \) in Equation (B.2) where \( C' \) denotes the normalized \( C \) yields

\[ rv^2 = (R_1^3 - C'R_1 + 2) + (3R_1^2 + 3R_1\Delta R_1 + \Delta R_1^2 - C')\Delta R_1 \] (B.4)

Substituting Equation (B.3a) into Equation (B.4) one obtains
\[ rv^2 = (3R_i^2 + 3R_iAR_i + \Delta R_i^2 - C')\Delta R_i \]  
(B.5a)

The constant \( C' \) can be substituted from the zero velocity boundary in Equation (B.3a), and Equation (B.5a) is rewritten as follows.

\[ rv^2 = \left( 2R_i^2 + 3R_iAR_i + \Delta R_i^2 - \frac{2}{R_i} \right)\Delta R_i \]  
(B.5b)

Now \( R_i < 1 \); thus, \( 1/R_i > 1 \), and from the fact that \( R_i + \Delta R_i < 1 \), the following inequality can be introduced

\[ 2 \left( (R_i + \Delta R_i)^2 - \frac{1}{R_i} \right)\Delta R_i < 0 \]  
(B.6a)

which can be reformulated as follows

\[ \left( 2R_i^2 + 4R_i\Delta R_i + 2\Delta R_i^2 - \frac{2}{R_i} \right)\Delta R_i < 0 \]  
(B.6b)

\[ \left( 2R_i^2 + 3R_i\Delta R_i + \Delta R_i^2 - \frac{2}{R_i} \right)\Delta R_i + \left( R_i\Delta R_i + \Delta R_i^2 \right)\Delta R_i < 0 \]  
(B.6c)

Substituting Equation (B.5a) into the inequality in Equation (B.6c) one obtains

\[ rv^2 + \left( R_i\Delta R_i + \Delta R_i^2 \right)\Delta R_i < 0 \]  
(B.7a)

\[ rv^2 < -\left( R_i + \Delta R_i \right)\Delta R_i^2 \]  
(B.7b)

From the fact that \( r = R_i + \Delta R_i \), one obtains the following concluding inequality

\[ \nu^2 < -\Delta R_i^2 \]  
(B.7c)

\[ \nu < i\Delta R_i \]  
(B.7d)

On the other hand, if \( r = R_i - \Delta R_i < 1 \) is a solution of Equation (B.2) and following the same procedures as from Equations (B.4)–(B.5), one obtains
rv^2 = \left( -2R_1^2 + 3R_1 \Delta R_1 - \Delta R_1^2 + \frac{2}{R_1} \right) \Delta R_1 \tag{B.8}

With \( R_1 < 1 \), \( 1/R_1 > 1 \), and from the fact that \( R_1 - \Delta R_1 < 1 \) the following inequality can be introduced

\[ 2 \left( \frac{1}{R_1} - R_1(1 - \Delta R_1) \right) \Delta R_1 > 0 \tag{B.9a} \]

which can be rewritten as follows.

\[ \left( \frac{2}{R_1} - 2R_1^2 + 3R_1 \Delta R_1 - \Delta R_1^2 \right) \Delta R_1 - (R_1 - \Delta R_1) \Delta R_1 > 0 \tag{B.9b} \]

Substituting Equation (B.8) into Equation (B.9b) and using the fact that \( r = R_1 - \Delta R_1 \), one obtains

\[ v > \Delta R_1 \tag{B.10} \]

Equations (B.7d) and (B.10) indicate that motion around the first primary is allowable inside the zero velocity curve with radius \( a = R_1 < 1 \).

**Case II Motion About Both Primaries** \( a = R_2 > 1 \)

The analysis procedures in this case are typically the same as in Case I. When substituting \( r = R_2 + \Delta R_2 > 1 \) in Equation (B.2) and eliminating the Jacobi constant using Equation (B.3b), one obtains

\[ rv^2 = \left( 2R_2^2 + 3R_2 \Delta R_2 + \Delta R_2^2 - \frac{2}{R_2} \right) \Delta R_2 \tag{B.11} \]

From the fact that \( R_2 > 1 \) and \( 1/R_2 < 1 \), the following inequality is introduced

\[ 2 \left( R_2 (R_2 + \Delta R_2) - \frac{1}{R_2} \right) \Delta R_2 > 0 \tag{B.12a} \]

which can be rewritten as follows.
Substituting Equation (B.11) into Equation (B.12b) one obtains

\[ v > \Delta R_2 \]  \hspace{1cm} \text{(B.12c)}

When substituting \( r = R_2 - \Delta R_2 > 1 \) in Equation (B.2) and eliminating the Jacobi constant using Equation (B.3b), the result is

\[ rv^2 = \left( -2R_2^2 + 3R_2\Delta R_2 - \Delta R_2^2 + \frac{2}{R_2} \right) \Delta R_2 \]  \hspace{1cm} \text{(B.13)}

From the fact that \( R_2 > 1 \) and \( 1 / R_2 < 1 \), the following inequality is introduced

\[ 2 \left( \frac{1}{R_2} - (R_2 - \Delta R_2)^2 \right) \Delta R_2 < 0 \]  \hspace{1cm} \text{(B.14a)}

which can be rewritten as follows.

\[ \left( -2R_2^2 + 3R_2\Delta R_2 - \Delta R_2^2 + \frac{2}{R_2} \right) \Delta R_2 + (R_2 - \Delta R_2)\Delta R_2^2 < 0 \]  \hspace{1cm} \text{(B.14b)}

Substituting Equation (B.13) into Equation (B.14b) one obtains

\[ v < i\Delta R_2 \]  \hspace{1cm} \text{(B.14c)}

Equations (B.12c) and (B.14c) indicate that motion around the two primaries is allowable outside the zero velocity curve with radius \( a = R_2 > 1 \).
APPENDIX C

Circular Orbits Periodicity Conditions for Zero Mass Parameter

The angle $\phi$ of the total acceleration vector with the $x$ axis is

$$\tan \phi = \frac{\dot{y}}{\dot{x}}$$  \hspace{1cm} (C.1)

The angle $\theta$ of the radial distance with the $x$ axis is calculated from Equation (4.67) as follows.

$$\tan \theta = \frac{\ddot{y} + 2\omega \dot{x} - \frac{y}{r} J_r}{\ddot{x} - 2\omega \dot{y} - \frac{x}{r} J_r}$$  \hspace{1cm} (C.2)

When the two angles $\phi, \theta$ have the relation $\phi = \theta + n\pi$ where $n = 0, 1, 2\ldots$, the right hand sides of Equations (C.1) and (C.2) are equal. From this property one obtains the following equation

$$2\omega (\dddot{x} + \dddot{y}) = \frac{1}{r} J_r (\dddot{x} - \dddot{y})$$  \hspace{1cm} (C.3)

which can be rewritten as follows:

$$\omega \frac{d}{dt} (\dot{x}^2 + \dot{y}^2) = \frac{1}{r} J_r \frac{d}{dt} (\dot{x} \dot{y} - x \dot{y})$$  \hspace{1cm} (C.4)

From the Jacobi integral equation

$$\frac{d}{dt} (\dot{x}^2 + \dot{y}^2) = \frac{d}{dt} (2J - C) = 2 \left( \frac{\delta J}{\delta r} \frac{dr}{dt} + \frac{\partial J}{\partial t} \right) = 2J \frac{dr}{dt}$$  \hspace{1cm} (C.5)

Substituting Equation (C.5) into (C.4) and separating variables one obtains

$$\omega \frac{dr^2}{dt} = \frac{d}{dt} (\dot{x} \dot{y} - x \dot{y})$$  \hspace{1cm} (C.6)

By integrating Equation (C.6) one obtains
\[ \omega r^2 + (x\dot{y} - \dot{x}y) = \omega r_0^2 + (x_0\dot{y}_0 - \dot{x}_0y_0) \]  

(C.7)

The angular momentum vector \( H \) is expressed in the inertial coordinate system as follows

\[ H = R \times V \]  

(C.8)

where \( R \), \( V \) are the position vector and velocity vector of the third body in the inertial coordinate system. These vectors are related to the position and velocity vectors in the rotating coordinate system as follows

\[ R = T r \]  

(C.9)

\[ V = \dot{r} = \dot{T} r + T \dot{r} = T^T \omega T r + \dot{r} = T (T^T \omega T r + \dot{r}) \]  

(C.10)

where \( T \), \( \omega \) are a transformation and skew-symmetric matrix respectively.

\[
T = \begin{pmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \bar{\omega} = \begin{pmatrix}
0 & -\omega & 0 \\
\omega & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]  

(C.11)

Substituting Equations (C.9) and (C.10) into Equation (C.8) one obtains

\[ H = T \times \left(T^T \bar{\omega} T r + \dot{r}\right) \]  

(C.12a)

Utilizing the property of the transformation matrix, Equation (C.12a) can be written as

\[ H = T \left[r \times (\bar{\omega}' r + \dot{r})\right] \]  

(C.12b)

where \( \bar{\omega}' = T^T \bar{\omega} T \). Equation (C.12b) is still expressed in inertial coordinates. Multiplying Equation (C.12b) from the left by \( T^T \) the angular momentum vector \( h \) in the rotating coordinates is

\[ h = \tilde{r} \bar{\omega}' r + \tilde{r} \dot{r} \]  

(C.13)

where \( \tilde{r} \) is a skew-symmetric matrix defined as
The right hand side of Equation (C.13) is expanded in a scalar form as follows

\[ h = \omega r^2 + (x\dot{y} - \dot{x}y) \]  

which is the left hand side of Equation (C.7).
APPENDIX D

Solution Boundness Of A Differential Equations System With Periodic Coefficients

A fundamental matrix $\Phi(t)$ is known to satisfy the homogeneous equation

$$\dot{\Phi}(t) = A(t)\Phi(t) \tag{D.1a}$$

$$A(t) = \dot{\Phi}(t)\Phi^{-1}(t) \tag{D.1b}$$

where $A(t)$ is a $T$ periodic matrix $A(t + T) = A(t)$. From the property $\text{tr}A(t) = 0$ the determinant of the fundamental matrix, $\det \Phi(t)$, is a nonzero constant.

$$\det \Phi(t) = \det \Phi(t_0)e^{\int_{t_0}^{t} \text{tr}A(t')dt} = \text{constant} \tag{D.2}$$

Thus the fundamental matrix is invertible at any time in addition to being continuous, and Equation (D.1b) can be integrated over one complete period. Define the norm of matrix $A$, $\|A\|$, where for a $4 \times 4$ matrix

$$\|A(t)\| = \int_0^T |A(t)|dt = \sqrt{\sum_{i,j=1}^{4} |a_{ij}|^2} \tag{D.3}$$

where $a_{ij}$ is the element in the $i^{th}$ row and $j^{th}$ column. For any other matrices $\Phi_1(t)$, $A_1(t)$ satisfying Equation (D.1), a metric $\rho(\Phi_1(t), \Phi(t))$ is defined as follows.

$$\rho(\Phi_1(t), \Phi(t)) = \sup_{0 \leq s \leq T} |\Phi_1(t) - \Phi(t)| + \int_0^T |\dot{\Phi}_1(t) - \dot{\Phi}(t)|dt \tag{D.4}$$

Whether small or large, the value of the metric $\rho$ determines if a matrix solution $\Phi_1(t)$ converges or diverges from a nominal matrix solution $\Phi(t)$.

Actually, the value of this metric depends on the structure of the state matrix $A(t)$ as follows.
\[ \| \mathbf{A}_t(t) - \mathbf{A}(t) \| = \int_0^T \| \mathbf{A}_t(t) - \mathbf{A}(t) \| dt = \int_0^T | \mathbf{A}_t(t) \mathbf{A}_t^{-1}(t) - \mathbf{A}(t) \mathbf{A}_t^{-1}(t) | dt \]
\[ = \int_0^T | \mathbf{A}_t(t) \mathbf{A}_t^{-1}(t) + \mathbf{A}(t) \mathbf{A}_t^{-1}(t) - \mathbf{A}(t) \mathbf{A}_t^{-1}(t) - \mathbf{A}(t) \mathbf{A}_t^{-1}(t) | dt \]
\[ = \int_0^T | -\mathbf{A}(t) (\mathbf{A}_t^{-1}(t) - \mathbf{A}_t^{-1}(t)) - (\mathbf{A}(t) - \mathbf{A}(t)) \mathbf{A}_t^{-1}(t) | dt \] (D.5)

From the identity of a norm of two matrices \( \mathbf{P}, \mathbf{Q} \), the following relation holds.
\[ \| \mathbf{P} + \mathbf{Q} \| \leq \| \mathbf{P} \| + \| \mathbf{Q} \| \] (D.6)

Equation (D.5) is then written as follows.
\[ \| \mathbf{A}_t(t) - \mathbf{A}(t) \| = \int_0^T | \mathbf{A}(t) (\mathbf{A}_t^{-1}(t) - \mathbf{A}_t^{-1}(t)) | dt + \int_0^T | (\mathbf{A}(t) - \mathbf{A}(t)) \mathbf{A}_t^{-1}(t) | dt \] (D.7)

Also note the two equalities for \( |\mathbf{P}| |\mathbf{Q}| \) and \( |\mathbf{PQ}| \).
\[ |\mathbf{P}| \mathbf{Q} = \sqrt{\sum_{i,j} |p_{ij}|^2 \sum_{i,j} |q_{ij}|^2} = \sqrt{\sum_{i,j} |p_{ij}|^2 |q_{ij}|^2} = |\mathbf{Q}| \mathbf{P} \] (D.8)
\[ |\mathbf{PQ}| = \sqrt{\sum_{i,k} |p_{ij} q_{jk}|^2} \] (D.9)

From the inequality \( |p_{ij} q_{jk}| \leq |p_{ij}| |q_{jk}| \), it can be concluded from Equations (D.8) and (D.9) that
\[ |\mathbf{PQ}| \leq |\mathbf{P}| \mathbf{Q} | \mathbf{P} | \] (D.10)

Thus, Equation (D.7) can be written as follows.
\[ \| \mathbf{A}_t(t) - \mathbf{A}(t) \| \leq \int_0^T | \mathbf{A}(t) | |\mathbf{A}_t^{-1}(t) - \mathbf{A}_t^{-1}(t) | dt + \int_0^T | (\mathbf{A}(t) - \mathbf{A}(t)) \mathbf{A}_t^{-1}(t) | dt \] (D.11a)
\[ \| \mathbf{A}_t(t) - \mathbf{A}(t) \| \leq \int_0^T \Phi(t) || \Phi^{-1}(t) - \Phi^{-1}(t) | dt + \int_0^T | (\Phi(t) - \Phi(t)) \Phi^{-1}(t) | dt \] (D.11b)
For any two functions \( f_1(t) \), \( f_2(t) \) the following inequality is always true.

\[
\int_0^T f_1(t)f_2(t)dt \leq \int_0^T \left( \sup_{0 \leq s \leq T} f_1(t) \right) f_2(t)dt
\]

(D.12a)

\[
\int_0^T f_1(t)f_2(t)dt \leq \left( \sup_{0 \leq s \leq T} f_1(t) \right) \int_0^T f_2(t)dt
\]

(D.12b)

Using Equation (D.12b), Equation (D.11c) can now be written as follows.

\[
\|A_1(t) - A(t)\| \leq \left( \sup_{0 \leq s \leq T} \Phi^{-1}(t) - \Phi^{-1}(t) \right) \int_0^T \Phi(t)dt + \left( \sup_{0 \leq s \leq T} \Phi^{-1}(t) \right) \int_0^T |\Phi(t) - \Phi_1(t)|dt
\]

(D.13)

Equation (D.13) indicates that when a solution \( \Phi_1(t) \) is close to a nominal solution \( \Phi(t) \) the metric \( \rho(\Phi_1(t), \Phi(t)) \) is small and the variation is bounded. The question now is how does the matrix \( A(t) \) affect the value of the quantity \( \sup_{0 \leq s \leq T} \Phi_1(t) - \Phi(t) \). Both matrices \( \Phi_1(t) \), \( \Phi(t) \) satisfy the matrix equation; it then follows that \( \Phi_1(t) - \Phi(t) \) is also a solution to the matrix equation. Assuming that this variation in the solution space is a function of both \( \Phi(t) \) and \( A(t) \) one obtains

\[
\frac{d}{dt} \left( \Phi_1(t) - \Phi(t) \right) = \Delta A(t)\Phi(t) + A(t)\Delta \Phi(t)
\]

\[
= A(t)\left( \Phi_1(t) - \Phi(t) \right) + (A_1(t) - A(t))\Phi(t)
\]

(D.14)

Equation (D.14) is a nonhomogeneous matrix equation. Assuming that \( \Phi_1(t)D \) is the solution of the associated homogeneous equation, where \( D \) is a constant matrix, and then using the method of variation of parameters the total solution of Equation (D.14) can be written as follows.
\[ \Phi_1(t) - \Phi(t) = \Phi_1(t)D + \Phi_1(t) \int_{t_0}^{t} \Phi_1^{-1}(\tau) (A_1(\tau) - A(\tau)) \Phi(\tau) d\tau \]  

(D.15)

If \( \Phi_1(t_0) = \Phi(t_0) \neq 0 \) the first part of the solution vanishes and only the particular solution survives. Equation (D.15) is then reduced to

\[ \Phi_1(t) - \Phi(t) = \Phi_1(t) \int_{t_0}^{t} \Phi_1^{-1}(\tau) (A_1(\tau) - A(\tau)) \Phi(\tau) d\tau \]  

(D.16)

Using the property indicated by Equation (D.10), then

\[ |\Phi_1(t) - \Phi(t)| \leq |\Phi_1(t)| \int_{t_0}^{t} |\Phi_1^{-1}(\tau)| |(A_1(\tau) - A(\tau))| |\Phi(\tau)| d\tau \]  

(D.17a)

\[ |\Phi_1(t) - \Phi(t)| \leq |\Phi_1(t)| \int_{t_0}^{t} |\Phi_1^{-1}(\tau)| |\Phi(\tau)| |(A_1(\tau) - A(\tau))| d\tau \]  

(D.17b)

Since \( \Phi_1(t) \) satisfies the matrix homogeneous Equation (D.1a) with a general solution,

\[ \Phi_1(t) = \Phi_1(t_0) e^{A_1(t)} \]  

(D.18)

Two additional inequalities result:

\[ |\Phi_1(t)| = |\Phi_1(t_0)| e^{\int_{t_0}^{t} |A_1(\tau)| d\tau} \]  

(D.19a)

\[ |\Phi_1(t)| \leq |\Phi_1(t_0)| e^{\int A_1 d\tau} \]  

(D.19b)

From the identity \( \Phi_1^{-1}(t) \Phi_1^{-1}(t) = I_{4 \times 4} \), one obtains

\[ \dot{\Phi}_1^{-1}(t) \Phi_1(t) + \Phi_1^{-1}(t) \dot{\Phi}_1(t) = 0 \]  

(D.20a)

\[ \dot{\Phi}_1^{-1}(t) = -\Phi_1^{-1}(t) \dot{\Phi}_1(t) \Phi_1^{-1}(t) \]  

(D.20b)

From Equation (D.20b) in a similar form to Equation (D.19), one can write
Substituting Equations (D.19) and (D.20) into Equation (D.17b), one finds

\[ |\Phi^{-1}(t)| \leq |\Phi^{-1}(t_0)| e^{|A||t - t_0|} \]  \hspace{1cm} (D.21)

Substituting Equations (D.19) and (D.20) into Equation (D.17b), one finds

\[ |\Phi_1(t) - \Phi(t)| \leq |\Phi(t_0)| e^{|A||t - t_0|} |\Phi^{-1}(t_0)| e^{\|A\|t} \int_{t_0}^{t} (A_1(\tau) - A(\tau)) d\tau \]  \hspace{1cm} (D.22a)

\[ \sup_{0 \leq t \leq T} |\Phi_1(t) - \Phi(t)| \leq |\Phi(t_0)| e^{2|A||t - t_0|} \int_{t_0}^{t} |(A_1(\tau) - A(\tau))| d\tau \]  \hspace{1cm} (D.22b)

When \( \Phi_1(t_0) = \Phi(t_0) = I_{4 \times 4} \), Equation (D.22b) can be written as follows

\[ \sup_{0 \leq t \leq T} |\Phi_1(t) - \Phi(t)| \leq n \sqrt{n} e^{2|A||t - t_0|} \left\| (A_1(\tau) - A(\tau)) \right\| \]  \hspace{1cm} (D.22c)

where \( \sqrt{n} = |\Phi(t_0)| = |I_{4 \times 4}| \). Equation (D.22c) indicates that the ultimate variation of \( \sup_{0 \leq t \leq T} (\Phi_1(t) - \Phi(t)) \) in the solution space is governed by the possible variation in the state matrix \( A(t) \).
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