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Duality of the weak parallelogram laws on Banach spaces

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ABSTRACT

This paper explores a family of weak parallelogram laws for Banach spaces. Some basic properties of such spaces are obtained. The main result is that a Banach space satisfies a lower weak parallelogram law if and only if its dual satisfies an upper weak parallelogram law, and vice versa. Connections are established between the weak parallelogram laws and the following: subspaces, quotient spaces, Cartesian products, and the Rademacher type and co-type properties.

1. Introduction

The parallelogram law states that for any vectors x and y in a Hilbert space $\mathcal{H}$, we have

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2. \tag{1}$$

Any normed space that satisfies the parallelogram law must in fact be an inner product space. However, by modifying condition (1) somewhat, we arrive at a family of conditions that may be applied to a much broader class of spaces. Let $1 < p < \infty$ and $C > 0$. Following [8], we say that a Banach space $\mathcal{X}$ satisfies the p-lower weak parallelogram law with constant C if

$$\|x+y\|^p + C\|x-y\|^p \leq 2^{p-1}(\|x\|^p + \|y\|^p) \tag{2}$$

for all x and y in $\mathcal{X}$. Let us abbreviate this condition as p-LWP(C). By analogy there is also an upper weak parallelogram law. We say that $\mathcal{X}$ is p-UWP(C) if condition (2) holds with the inequality reversed.

Note that with the substitutions $X = x + y$ and $Y = x - y$, we can express (2) equivalently as

$$\|X+Y\|^p + \|X-Y\|^p \geq 2(\|X\|^p + C\|Y\|^p). \tag{3}$$

Furthermore, it is obvious that the constant C could just as well appear in the other term, i.e.,

$$C\|x+y\|^p + \|x-y\|^p \leq 2^{p-1}(\|x\|^p + \|y\|^p)$$

in the defining condition for p-LWP(C). Actually, the definitions make sense even on a linear space $\mathcal{X}$ that is not complete, or if $\|\cdot\|$ is merely a semi-norm, or even a quasi-norm. From time to time it is convenient to suppress the parameters p and C, and to speak of "weak parallelogram" laws and spaces with the obvious meaning. No assertion is made, at this point, that the constants p or C are "optimal" in any way, and they are certainly not unique. These matters will be addressed more precisely in due course.

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The weak parallelogram laws with $p = 2$ were introduced by Bynum and Drew [3,4], who were interested in the geometry of Banach spaces. They obtained convexity, smoothness and duality conditions associated with this case.

Cheng and Ross [8] extended this investigation to the parameter range $1 < p < \infty$, with the goal of applying the results to the prediction of p-stationary processes. Indeed, they obtained: a sort of Pythagorean theorem for weak parallelogram spaces; smoothness and convexity properties; growth estimates for moving average coefficients; a Baxter-type inequality; and a characterization of "regularity" (in the sense of linear non-determinism) for a broad class of processes. This built upon the works [6,5,7,9], concerning the linear prediction of such classes of processes. It was also shown in [8] that for $1 < p < \infty$, the L^p spaces satisfy a range of weak parallelogram laws, and examples were given of weak parallelogram spaces that are not L^p spaces.

Against this background, a number of reasonable conjectures immediately come to mind. For instance, a duality relationship among weak parallelogram spaces suggests itself. This state of affairs would seem to justify further exploration of the weak parallelogram laws on a more fundamental level, and on a broader class of spaces. Our present purpose, then, is to present a general duality theorem for weak parallelogram spaces, and establish some consequences and related results.

2. Basic properties

Let us proceed by exploring some basic properties of weak parallelogram spaces. This will include the behavior of the parameters p and C.

First, the following assertion about subspaces is obvious but worth recording. It will enable us to build additional examples of weak parallelogram spaces.

Proposition 2.1. *Let $\mathcal{X}$ be a Banach space, and let $\mathcal{M}$ be a subspace of $\mathcal{X}$. If $\mathcal{X}$ satisfies p-LWP(C), then $\mathcal{M}$ satisfies p-LWP(C); if $\mathcal{X}$ satisfies p-UWP(C), then $\mathcal{M}$ satisfies p-UWP(C).*

The L^p example motivates the following result, which tells us that a general weak parallelogram space always satisfies weak parallelogram laws for a wide range of parameter values. It will later help us handle the case $p = q = 2$ in the proof of the main duality theorem.

Proposition 2.2. *Suppose that $\mathcal{X}$ is a Banach space, C is a positive constant, and $1 < p < \infty$. If $\mathcal{X}$ is p-LWP(C), then $0 < C \le 1$, $2 \le p < \infty$, and $\mathcal{X}$ is r-LWP(B) whenever $r \ge p$ and $B \le C^{r/p}$. If $\mathcal{X}$ is p-UWP(C), then $1 \le C < \infty$, $1 < p \le 2$, and $\mathcal{X}$ is r-UWP(B) whenever $r \le p$ and $B \ge C^{r/p}$.*

Proof. Assume that $\mathcal{X}$ is p-LWP(C). Then when $y = -x$, we have

$$\begin{aligned} 2^pC\|x\|^p &= \|x-x\|^p + C\|x+x\|^p \\ &\le 2^{p-1}(\|x\|^p + \|x\|^p) \\ &= 2^p\|x\|^p \end{aligned}$$

which forces $C \le 1$.

Next, suppose that $r \ge p$, and apply the inequality for $l^p(\{1,2\})$ norms, the p-LWP(C) condition, and Hölder's inequality to get

$$\begin{aligned} 2\Big(\|x\|^r + C^{r/p}\|y\|^r\Big) &\le 2\Big(\|x\|^p + C\|y\|^p\Big)^{r/p} \\ &\le 2\Big([1/2]\cdot[\|x+y\|^p + \|x-y\|^p]\Big)^{r/p} \\ &= 2\cdot 2^{-r/p}\Big(\|x+y\|^p + \|x-y\|^p\Big)^{r/p} \\ &\le 2^{1-r/p}\Big(\|x+y\|^{p\cdot r/p} + \|x-y\|^{p\cdot r/p}\Big)^{(r/p)(p/r)}\Big(1^{(r/p)'} + 1^{(r/p)'}\Big)^{(r/p)/(r/p)'} \\ &= \|x+y\|^r + \|x-y\|^r. \end{aligned}$$

Here $(r/p)' = 1 - (r/p)$, and we used

$$\begin{aligned} 1-(r/p)+(r/p)/(r/p)' &= 1-(r/p)+(r/p)\Big[1-1/(r/p)\Big] \\ &= 0. \end{aligned}$$

This affirms that $\mathcal{X}$ is r-LWP$(C^{r/p})$.

Now observe that if $\mathcal{X}$ is p-LWP(C), then we can obviously replace C with a smaller positive constant B in the defining inequality (2), and it remains true for all vectors x and y in $\mathcal{X}$.

Finally, consider the inequality

$$\|x+y\|^r + C\|x-y\|^r \le 2^{r-1}(\|x\|^r + \|y\|^r)$$

where x is a unit vector in $\mathcal{X}$, and $y = bx$ for some constant b, $0 < b < 1$. Evidently,

$$C \leq \frac{2^{r-1}(1+b^r) - (1+b)^r}{(1-b)^r}.$$

Take the limit of the right hand side as b increases to 1. This requires two applications of L'Hôpital's rule, with the result

$$C \leq \lim_{b\to 1-} \frac{2^{r-1}r(r-1)b^{r-2} - r(r-1)(1+b)^{r-2}}{r(r-1)(1-b)^{r-2}}.$$

If $r < 2$, then the right hand side is zero, which means that $\mathcal{X}$ cannot be r-LWP. Therefore, the condition p-LWP(C) requires that $p \geq 2$.

This proves half of the assertion, and the other half is similarly handled. □

We now turn to the issue of optimal constants.

Proposition 2.3. *Suppose that the Banach space $\mathcal{X}$ is p-LWP(B) for some B. There exists a unique constant $C \leq 1$ such that $\mathcal{X}$ is p-LWP(C), and $B \leq C$ whenever $\mathcal{X}$ is p-LWP(B). If the Banach space $\mathcal{X}$ is p-UWP(B) for some B, then there exists a unique constant $C \geq 1$ such that $\mathcal{X}$ is p-UWP(C), and $B \geq C$ whenever $\mathcal{X}$ is p-UWP(B).*

Proof. The collection of constants B for which $\mathcal{X}$ is p-LWP(B) is nonempty and bounded above by 1. Therefore, it has a supremum C which is bounded above by 1. If, for the sake of argument, it should happen that

$$\|x+y\|^p + C\|x-y\|^p > 2^{p-1}(\|x\|^p + \|y\|^p) \tag{4}$$

for some x and y in $\mathcal{X}$, then (4) also holds with C replaced by a value of B sufficiently close to C. This contradicts the assumption that p-LWP(B) holds, and thus $\mathcal{X}$ must also be p-LWP(C).

Once again, the UWP case is analogous. □

Continuing with the matter of optimal constants, suppose that a Banach space $\mathcal{X}$ is given, and it satisfies r-LWP(C_r), where C_r is the optimal constant associated with r. We already know that $\mathcal{X}$ must satisfy s-LWP($C^{s/r}$) for all $s > r$. Let us write C_s for the optimal constant in that case. It follows that $C_s \geq C_r^{s/r}$, or $C_s^{1/s} \geq C_r^{1/r}$. Consequently the expression $C_r^{1/r}$, viewed as a function of r, is non-increasing as r decreases. Let p be the infimum of the collection of parameters r for which $\mathcal{X}$ is r-LWP(C) for some constant C, and define $C_{(p)} = (\lim_{r\to p+} C_r^{1/r})^p$. It may happen that $C_{(p)} = 0$. Otherwise, suppose for the sake of argument that there exist vectors x and y with

$$\|x+y\|^p + C_{(p)}\|x-y\|^p > 2^{p-1}(\|x\|^p + \|y\|^p). \tag{5}$$

Then, by continuity, (5) also holds with p replaced by $p+\epsilon$, and $C_{(p)}$ replaced by $C_{(p)}^{(p+\epsilon)/p}$, for ϵ sufficiently small and positive. But we can further replace $C_{(p)}^{(p+\epsilon)/p}$ by the equal or larger constant $C_{p+\epsilon}$. This contradicts the condition $(p+\epsilon)$-LWP($C_{p+\epsilon}$), and we may then conclude that $\mathcal{X}$ is p-LWP($C_{(p)}$). Naturally, a similar thing holds for the upper weak parallelogram case. Let us summarize this as follows.

Lemma 2.4. *Suppose that the Banach space $\mathcal{X}$ satisfies a lower weak parallelogram law. Let p be the infimum of the set of parameters r for which $\mathcal{X}$ satisfies an r-lower weak parallelogram law, and let C_r be the optimal constant for each r-LWP condition. If $C = (\lim_{r\to p+} C_r^{1/r})^p$ is positive, then $\mathcal{X}$ is p-LWP(C).*

Suppose that the Banach space $\mathcal{X}$ satisfies an upper weak parallelogram law. Let p be the supremum of the set of parameters r for which $\mathcal{X}$ satisfies an r-upper weak parallelogram law, and let C_r be the optimal constant for each r-UWP condition. If $C = (\lim_{r\to p-} C_r^{1/r})^p$ is finite, then $\mathcal{X}$ is p-UWP(C).

We rely on this result in the next section.

3. Duality

It was shown by Cheng and Ross [8, Propositions 2.1 and 2.2] that the L^p spaces, for $1 < p < \infty$, are weak parallelogram spaces, each for a wide range of parameter values. More precisely:

If $1 < p \leq 2$, and q is the conjugate index, then L^p is:

r-UWP(1) when $1 < r \leq p$;
r-LWP($(p-1)^{r/2}$) when $2 \leq r \leq q$; and
r-LWP(1) when $r \geq q$.

If $2 \le p < \infty$, and q is the conjugate index, then L^p is:

r-LWP(1) when $p \le r < \infty$;
r-UWP($(p-1)^{r/2}$) when $q \le r \le 2$; and
r-UWP(1) when $r \le q$.

These are extensions of the well known Clarkson's Inequalities. Evidently, an L^p space is LWP if and only if its dual L^q is UWP. Our main result is a duality theorem for general weak parallelogram spaces, suggested by this behavior of the L^p spaces.

Theorem 3.1. *Let $1 < p < \infty$, let q be its conjugate index, and let $C > 0$;* (a) *a Banach space $\mathcal{X}$ is p-LWP(C) if and only if its dual space $\mathcal{X}^*$ is q-UWP($C^{-q/p}$);* (b) *$\mathcal{X}$ is p-UWP(C) if and only if $\mathcal{X}^*$ is q-LWP($C^{-q/p}$).*

The next lemma takes the proof of this theorem a long way. Oddly, the method of the lemma fails when $p = q = 2$, however; additional steps are therefore needed.

Lemma 3.2. *Let $1 < p < \infty$, $p \ne 2$, let q be its conjugate index, and let $C > 0$;* (a) *if the Banach space $\mathcal{X}$ is p-LWP(C), then its dual space $\mathcal{X}^*$ is q-UWP($C^{-q/p}$);* (b) *if $\mathcal{X}$ is p-UWP(C), then $\mathcal{X}^*$ is q-LWP($C^{-q/p}$).*

Proof. Let u and v be positive numbers, and define the function f_1 on the two points $\{1, 2\}$ by $f_1(1) = u$ and $f_1(2) = v$. Similarly define $f_2(1) = u$ and $f_2(2) = -v$. Let the measure μ be the mass v^2 at the point $\{1\}$, and the mass u^2 at the point $\{2\}$. It is readily seen that $\int_{\{1,2\}} f_1 f_2 \, d\mu = 0$.

For any x_1 and x_2 in $\mathcal{X}$; x_1^* and x_2^* in $\mathcal{X}^*$; and scalars a_1 and a_2; we have

$$\int [f_1(t)x_1^* + f_2(t)x_2^*][f_1(t)a_1x_1 + f_2(t)a_2x_2] \, d\mu(t) = (2u^2v^2)[a_1x_1^*(x_1) + a_2x_2^*(x_2)]. \tag{6}$$

Furthermore, the usual estimates provide that

$$\begin{aligned}
&\left| \int [f_1(t)x_1^* + f_2(t)x_2^*][f_1(t)a_1x_1 + f_2(t)a_2x_2] \, d\mu(t) \right| \\
&\le \int |[f_1(t)x_1^* + f_2(t)x_2^*][f_1(t)a_1x_1 + f_2(t)a_2x_2]| \, d\mu(t) \\
&\le \int \|f_1(t)x_1^* + f_2(t)x_2^*\| \, \|f_1(t)a_1x_1 + f_2(t)a_2x_2\| \, d\mu(t) \\
&\le \left[\int \|f_1(t)x_1^* + f_2(t)x_2^*\|^q \, d\mu(t) \right]^{1/q} \left[\int \|f_1(t)a_1x_1 + f_2(t)a_2x_2\|^p \, d\mu(t) \right]^{1/p} \\
&\le \left[u^q v^2 \|x_1^* + x_2^*\|^q + u^2 v^q \|x_1^* - x_2^*\|^q \right]^{1/q} \cdot \left[u^p v^2 \|a_1x_1 + a_2x_2\|^p + u^2 v^p \|a_1x_1 - a_2x_2\|^p \right]^{1/p}.
\end{aligned} \tag{7}$$

Now, suppose that $\mathcal{X}$ satisfies p-LWP(C), and assume that $p > q$. Apply the above bound, taking $v = 1$ and $u = C^{-1/(p-2)}$. The calculation then yields

$$\begin{aligned}
(2u^2)|a_1x_1^*(x_1) + a_2x_2^*(x_2)| &\le \left[u^q \|x_1^* + x_2^*\|^q + u^2 \|x_1^* - x_2^*\|^q \right]^{1/q} \cdot \left[u^p \|a_1x_1 + a_2x_2\|^p + u^2 \|a_1x_1 - a_2x_2\|^p \right]^{1/p} \\
&\le u^2 \cdot \left[\|x_1^* + x_2^*\|^q + u^{2-q} \|x_1^* - x_2^*\|^q \right]^{1/q} \cdot \left[\|a_1x_1 + a_2x_2\|^p + C \|a_1x_1 - a_2x_2\|^p \right]^{1/p} \\
&\le u^2 \cdot [\|x_1^* + x_2^*\|^q + u^{2-q} \|x_1^* - x_2^*\|^q]^{1/q} \cdot \left[2^{p-1} (\|a_1x_1\|^p + \|a_2x_2\|^p) \right]^{1/p} \\
&= 2^{1/q} u^2 \cdot [\|x_1^* + x_2^*\|^q + u^{2-q} \|x_1^* - x_2^*\|^q]^{1/q} \cdot \left[(\|a_1x_1\|^p + \|a_2x_2\|^p) \right]^{1/p}.
\end{aligned}$$

There is a common factor of u^2 which can be canceled. Also, it is easy to check that $u^{2-q} = C^{-q/p}$. Thus we may deduce that

$$2^{1-1/q} |a_1x_1^*(x_1) + a_2x_2^*(x_2)| \le \left(\|x_1^* + x_2^*\|^q + C^{-q/p} \|x_1^* - x_2^*\|^q \right)^{1/q} \left((\|a_1x_1\|^p + \|a_2x_2\|^p) \right)^{1/p}.$$

Choose x_1 and x_2 to be unit vectors that are approximately norming for x_1^* and x_2^*, respectively. Then, take the supremum of the left hand side over the condition $(|a_1|^p + |a_2|^p)^{1/p} = 1$. The conclusion is

$$2^{1-1/q} (\|x_1^*\|^q + \|x_2^*\|^q)^{1/q} \le [\|x_1^* + x_2^*\|^q + C^{-q/p} \|x_1^* - x_2^*\|^q]^{1/q}.$$

In other words, $\mathcal{X}^*$ is q-UWP($C^{-q/p}$).

If, on the other hand, $\mathcal{X}$ satisfies p-LWP(C) and $p < q$, then choose $u = 1$ and $v = C^{-1/(2-p)}$. The argument proceeds as before, with the result that $\mathcal{X}^*$ is q-UWP($C^{-q/p}$). (Note that the case $p = q = 2$ cannot be handled this way, as then u or v ends up being zero, and the associated calculation tells us nothing.) This proves assertion (a).

To get (b), assume that $\mathcal{X}$ is p-UWP(C), and take $u = v = 1$ in (6) and (7). We invoke the weak parallelogram laws in the form (3) to get

$$\begin{aligned} 2|a_1x_1^*(x_1) + a_2x_2^*(x_2)| &\le \left[\|x_1^* + x_2^*\|^q + \|x_1^* - x_2^*\|^q\right]^{1/q} \cdot \left[\|a_1x_1 + a_2x_2\|^p + \|a_1x_1 - a_2x_2\|^p\right]^{1/p} \\ &\le \left[\|x_1^* + x_2^*\|^q + \|x_1^* - x_2^*\|^q\right]^{1/q} \cdot \left[2(\|a_1x_1\|^p + C\|a_2x_2\|^p)\right]^{1/p}. \end{aligned}$$

Proceeding as before, we take x_1 and x_2 to be unit vectors that are approximately norming for x_1^* and x_2^*, respectively. Then take the supremum of the left hand side over the condition $(|a_1|^p + C|a_2|^p)^{1/p} = 1$. The left hand side can be expressed as

$$2|a_1x_1^*(x_1) + (C^{1/p}a_2)C^{-1/p}x_2^*(x_2)|.$$

We deduce that

$$2^{1/q}\left(\|x_1^*\|^q + C^{-q/p}\|x_2^*\|^q\right)^{1/q} \le \left[\|x_1^* + x_2^*\|^q + \|x_1^* - x_2^*\|^q\right]^{1/q}$$

which is to say that $\mathcal{X}^*$ is q-LWP($C^{-q/p}$). □

We are now equipped to complete the proof of Theorem 3.1. First let us handle the $p = q = 2$ case omitted from Lemma 3.2. Suppose that $\mathcal{X}$ is 2-LWP(C), with C being the optimal constant. Then $\mathcal{X}$ is r-LWP($C^{r/2}$) for every $r > 2$. It follows from Lemma 3.2 that the dual space $\mathcal{X}^*$ is r'-UWP($C^{-r'/2}$), where r' is the conjugate index to r. Clearly the supremum of all the values of r' that arise in this way is 2. Furthermore,

$$\left(\lim_{r\to 2+} C_{r'}^{1/r'}\right)^2 \le \left(\lim_{r\to 2+} C^{(-r'/2)/r'}\right)^2 = 1/C.$$

Therefore, by Lemma 2.4, $\mathcal{X}^*$ satisfies 2-UWP(C_2) for some constant $C_2 \le 1/C$. A second application of Lemma 3.2 now provides that the second dual $\mathcal{X}^{**}$ satisfies 2-LWP($1/C_2$). But $\mathcal{X}$ is isometrically isomorphic to a subspace of $\mathcal{X}^{**}$, and so $\mathcal{X}$ is 2-LWP($1/C_2$). Since C was chosen to be the optimal constant for $\mathcal{X}$ satisfying the condition 2-LWP($\cdot$), and $C \le 1/C_2$, it must be the case that $C_2 = 1/C$. (Actually, a LWP space is already uniformly convex, and hence reflexive; however, this does not hold for UWP spaces, and thus it does not help the other half of the proof.)

This verifies that if $\mathcal{X}$ is 2-LWP(C), then $\mathcal{X}^*$ is 2-UWP($1/C$). As ever, a similar argument shows that if $\mathcal{X}$ is 2-UWP(C), then $\mathcal{X}^*$ is 2-LWP($1/C$). (This improves on the constants in [3, Theorem 9], a duality statement for the 2-weak parallelogram laws.)

We have, as another corollary to Lemma 2.4 and Proposition 2.1, the following statements. If $\mathcal{X}^*$ is p-LWP(C), then $\mathcal{X}$ is q-UWP($C^{-q/p}$); if $\mathcal{X}^*$ is p-UWP(C), then $\mathcal{X}$ is q-LWP($C^{-q/p}$). As before, this is because $\mathcal{X}$ is isometrically isomorphic to a subspace of $\mathcal{X}^{**}$.

At last, the proof of Theorem 3.1 is complete.

4. Further developments

With the duality theorem in hand, we are in a position to relate weak parallelogram laws to quotient spaces. Let us also connect the weak parallelogram laws to Cartesian products. This enables us to identify additional examples of weak parallelogram spaces. Finally, we establish a link to the notions of Rademacher type and co-type.

As a quick consequence of duality, we have this statement about quotient spaces.

Proposition 4.1. *Let $\mathcal{X}$ be a Banach space, and let $\mathcal{M}$ be a subspace of $\mathcal{X}$;* (a) *if $\mathcal{X}$ satisfies p-LWP(C), then $\mathcal{X}/\mathcal{M}$ satisfies p-LWP(C);* (b) *if $\mathcal{X}$ satisfies p-UWP(C), then $\mathcal{X}/\mathcal{M}$ satisfies p-UWP(C).*

Proof. Let $\mathcal{X}$ be p-LWP(C), and $\mathcal{M}$ a subspace of $\mathcal{X}$. Then $(\mathcal{X}/\mathcal{M})^*$ is isometrically isomorphic to $\mathcal{M}^\perp$, a subspace of $\mathcal{X}^*$. Thus $(\mathcal{X}/\mathcal{M})^*$ is q-UWP($C^{-q/p}$), where q is the conjugate index of p. But $(\mathcal{X}/\mathcal{M})^{**}$ must then be p-LWP(C), as must $\mathcal{X}/\mathcal{M}$ itself. That proves (a), and (b) is proved analogously. □

Let us add that the weak parallelogram laws are preserved under Cartesian products, endowed with an associated norm.

Proposition 4.2. *Let $(\mathcal{X}_1, \|\cdot\|_{(1)})$ and $(\mathcal{X}_2, \|\cdot\|_{(2)})$ be Banach spaces.*

If $\mathcal{X}_1$ is p_1-LWP(C_1), and $\mathcal{X}_2$ is p_2-LWP(C_2), then the Cartesian product $\mathcal{X}_1 \times \mathcal{X}_2$, endowed with the norm

$$\|\langle x_1, x_2\rangle\| = \left(\|x_1\|_{(1)}^p + \|x_2\|_{(2)}^p\right)^{1/p},$$

is p-LWP(C), where $p = \max\{p_1, p_2\}$ and $C = \min\{C_1^{p/p_1}, C_2^{p/p_2}\}$.

If $\mathcal{X}_1$ is p_1-UWP(C_1), and $\mathcal{X}_2$ is p_2-UWP(C_2), then the Cartesian product $(\mathcal{X}_1 \times \mathcal{X}_2, \|\cdot\|)$ is p-UWP(C), where $p = \min\{p_1, p_2\}$ and $C = \max\{C_1^{p/p_1}, C_2^{p/p_2}\}$.

Proof. Suppose that $\mathcal{X}_1$ is p_1-LWP(C_1), and $\mathcal{X}_2$ is p_2-LWP(C_2). Then by Proposition 2.2, both $\mathcal{X}_1$ and $\mathcal{X}_2$ are p-LWP(C). Now for x_1 and y_1 in $\mathcal{X}_1$, and x_2 and y_2 in $\mathcal{X}_2$,

$$\begin{aligned}\|\langle x_1, x_2\rangle + \langle y_1, y_2\rangle\|^p + C\|\langle x_1, x_2\rangle - \langle y_1, y_2\rangle\|^p &= \|x_1 + y_1\|_{(1)}^p + \|x_2 + y_2\|_{(2)}^p + C\|x_1 - y_1\|_{(1)}^p + C\|x_2 - y_2\|_{(2)}^p \\ &\le 2^{p-1}\Big(\|x_1\|_{(1)}^p + \|y_1\|_{(1)}^p\Big) + 2^{p-1}\Big(\|x_2\|_{(2)}^p + \|y_2\|_{(2)}^p\Big) \\ &= 2^{p-1}\Big(\|\langle x_1, x_2\rangle\|^p + \|\langle y_1, y_2\rangle\|^p\Big).\end{aligned}$$

The rest is similar. $\square$

The above assertion directly extends to Cartesian products of any finite or countably infinite length. This, in conjunction with the previous results, helps us to identify many more examples of weak parallelogram spaces. The Sobolev space $W^{1,p}(\Omega)$ of functions on a domain Ω, for instance, can be identified with that subspace of $L^p(\Omega) \times L^p(\Omega)$ consisting of pairs of the form $\langle f, Df\rangle$, where D is the appropriate weak derivative.

The Besov space $B_{p,r}^{\alpha}$ can similarly be expressed as a subspace of an infinite Cartesian product of copies of $L^p([0, 2\pi))$. The norm in this case is

$$\|f\|_{p,r}^{\alpha} = \left(\sum_{k=0}^{\infty} 2^{\alpha k}\|W_k * f\|_p^r\right)^{1/r}$$

where $\{W_k\}_{k=0}^{\infty}$ is a standard sequence of convolution kernels [2]. Thus for $\alpha > 0$, the space $B_{p,r}^{\alpha}$ is r-LWP when $2 \le p \le r < \infty$, and it is r-UWP when $1 < r \le p \le 2$.

The Rademacher type and co-type properties (see [1, p. 145ff]) constitute another way to generalize the parallelogram law. Here is the precise connection between them and the weak parallelogram laws.

Proposition 4.3. *Let $\mathcal{X}$ be a Banach space, and let $1 < p \le 2$, and $2 \le q < \infty$. The space $\mathcal{X}$ is Rademacher type p with constant $T_p(\mathcal{X}) = 1$ if and only if $\mathcal{X}$ is p-UWP(1); and $\mathcal{X}$ is Rademacher co-type q with constant $C_q(\mathcal{X}) = 1$ if and only if $\mathcal{X}$ is q-LWP(1).*

Proof. Let $\{r_n(t)\}_{n=1}^{\infty}$ be the Rademacher functions on the interval $[0, 1]$. If $\mathcal{X}$ is Rademacher type p with constant $T_p(\mathcal{X}) = 1$, then for any x and y in $\mathcal{X}$,

$$\begin{aligned}\|x + y\|^p + \|x - y\|^p &= 2\int_0^1 \|r_1(t)x + r_2(t)y\|^p\,dt \\ &\le 2\,T_p(\mathcal{X})^p\Big(\|x\|^p + \|y\|^p\Big) \\ &= 2\Big(\|x\|^p + 1 \cdot \|y\|^p\Big)\end{aligned}$$

which shows that $\mathcal{X}$ is p-UWP(1).

Conversely, if $\mathcal{X}$ is p-UWP(1), and $\{x_n\}_{n=1}^{N}$ are vectors in $\mathcal{X}$, then

$$\begin{aligned}&\|x_1 + x_2 + x_3\|^p + \|x_1 - x_2 - x_3\|^p \le 2\Big(\|x_1\|^p + \|x_2 + x_3\|^p\Big) \\ &\|x_1 + x_2 - x_3\|^p + \|x_1 - x_2 + x_3\|^p \le 2\Big(\|x_1\|^p + \|x_2 - x_3\|^p\Big).\end{aligned}$$

Add the corresponding sides, and apply the p-UWP(1) property to the right side, to get

$$\begin{aligned}&\|x_1 + x_2 + x_3\|^p + \|x_1 - x_2 + x_3\|^p + \|x_1 + x_2 - x_3\|^p + \|x_1 - x_2 - x_3\|^p \\ &\quad\le 2\Big(\|x_1\|^p + \|x_1\|^p + \|x_2 + x_3\|^p + \|x_2 - x_3\|^p\Big) \\ &\quad\le 4\Big(\|x_1\|^p + \|x_2\|^p + \|x_3\|^p\Big).\end{aligned}$$

Note that the set of points t on which $r_1(t)$, $r_2(t)$ and $r_3(t)$ have the same sign has measure $\frac{1}{4}$, and likewise with other permutations of signs. It follows that

$$\int_0^1 \|r_1(t)x_1 + r_2(t)x_2 + r_3(t)x_3\|^p\,dt \le \|x_1\|^p + \|x_2\|^p + \|x_3\|^p.$$

Repeating this argument with $x_4, x_5, \ldots, x_N$ leads to

$$\int_0^1 \left\| \sum_{n=1}^{N} r_n(t)x_n \right\|^p dt \le \sum_{n=1}^{N} \|x_n\|^p.$$

That is, $\mathcal{X}$ is type p with constant 1. The other case is treated in the same way. □

Thus many of the present results agree with the Rademacher type theory within their common scope, e.g., the L^p spaces with $1 < p < \infty$. However, an important difference is that the full duality theorem for Rademacher type only holds in one direction [1, p. 139], whereas the duality theorem for weak parallelogram laws holds in both directions.

5. Conclusions

The main result of this paper asserts that a Banach space satisfies a lower weak parallelogram law if and only if its dual satisfies an upper weak parallelogram law, and vice versa. It was also shown that the weak parallelogram laws are well behaved with respect to subspaces, quotient spaces and Cartesian products; and that these properties overlap with the notions of being Rademacher type and co-type. Examples of weak parallelogram spaces were identified: they include not only the L^p spaces, but also certain Sobolev and Besov spaces. The present results, in conjunction with the applications of the weak parallelogram laws established in [8], assure us that the weak parallelogram laws are a worthwhile subject for study and merit further investigation.

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