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Nonperturbative evolution of parton quasi-distributions



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ABSTRACT

Using the formalism of parton virtuality distribution functions (VDFs) we establish a connection between the transverse momentum dependent distributions (TMDs) $\mathcal{F}(x, k_\perp^2)$ and quasi-distributions (PQDs) $Q(y, p_3)$ introduced recently by X. Ji for lattice QCD extraction of parton distributions $f(x)$. We build models for PQDs from the VDF-based models for soft TMDs, and analyze the p_3 dependence of the resulting PQDs. We observe a strong nonperturbative evolution of PQDs for small and moderately large values of p_3 reflecting the transverse momentum dependence of TMDs. Thus, the study of PQDs on the lattice in the domain of strong nonperturbative effects opens a new perspective for investigation of the 3-dimensional hadron structure.

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1. Introduction

The parton distribution functions (PDFs) $f(x)$, being related to matrix elements of nonlocal operators near the light cone $z^2 = 0$ are notoriously difficult objects for a calculation using the lattice gauge theory. The latter is formulated in the Euclidean space where light-like separations do not exist. Recently, X. Ji [1] proposed to use purely space-like separations $z = (0, 0, 0, z_3)$ to overcome this problem.

The parton *quasi-distributions* $Q(y, p_3)$ introduced by X. Ji, differ from PDFs $f(x)$, but tend to them in the $p_3 \rightarrow \infty$ limit, displaying a usual perturbative evolution [2–5] with respect to p_3 for large p_3 . Refs. [1,6–17] discuss the properties of PQDs in the large p_3 limit and their matching with scale-dependent PDFs $f(x, \mu)$. The results of lattice calculations of PQDs were reported in Refs. [18–24].

These results show a significant variation of PQDs with p_3 . However, since the values of p_3 used in these calculations are not very large, the observed p_3 evolution does not have a perturbative form. The nonperturbative aspects of the p_3 -evolution were studied in diquark spectator models [25–27] for parton distributions. The evolution patterns observed in these papers are in a qualitative agreement with the lattice results. The authors also discuss the $p_3 \rightarrow \infty$ extrapolation of results obtained for moderately large p_3 values.

Our goal in the present paper is to study nonperturbative evolution of parton quasi-distributions using the formalism of *virtuality distribution functions* proposed and developed in our recent papers [28,29], where it was applied to the transverse momentum dependent pion distribution amplitude and the exclusive $\gamma^* \gamma \rightarrow \pi^0$ process.

To this end, in Section 2 we extend the VDF formalism onto the parton distribution functions, and show how the basic VDF $\Phi(x, \sigma)$ is related to PDFs, to TMDs and to PQDs. In particular, we show that PQDs are completely determined by TMDs through a rather simple transformation. Since the basic relations between the parton distributions are rather insensitive to complications brought by spin, in Section 2 we refer to a simple scalar model. In Section 3 we discuss modifications related to quark spin and gauge nature of gluons in quantum chromodynamics (QCD). In Section 4 we discuss VDF-based models for soft TMDs, and in Section 5 we present our results for nonperturbative evolution of PQDs obtained in these models. The transition to perturbative evolution for large p_3 is discussed in Section 6. Our conclusions are given in Section 7.

2. Parton distributions

2.1. Virtuality distribution functions

Historically, parton distributions [30] were introduced to describe inclusive deep inelastic scattering involving spin-1/2 quarks. Since complications related to spin do not affect the very concept of parton distributions, we start with a simple example of a scalar theory. Then information about the target is accumulated in the

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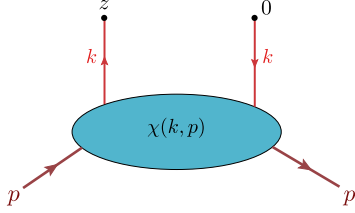


Fig. 1. Structure of parton-hadron matrix element.

generic matrix element $\langle p|\phi(0)\phi(z)|p\rangle$. Transforming to the momentum space

$$\langle p|\phi(0)\phi(z)|p\rangle = \frac{1}{\pi^2} \int d^4k e^{-ikz} \chi(k, p) \quad (2.1)$$

we switch to the description in terms of $\chi(k, p)$ which is an analog of the Bethe–Salpeter amplitude [31].

A crucial observation is that the contribution of any (uncut) diagram to $\chi(k, p)$ may be written as

$$i\chi(k, p) = \int_0^\infty d\lambda \int_{-1}^1 dx e^{i\lambda[k^2 - 2x(kp) + i\epsilon]} F(x, \lambda; M^2) \quad (2.2)$$

The reason is that for a general scalar handbag diagram d_i (see Fig. 1) one can write (see, e.g., [32])

$$\begin{aligned} i\chi_{d_i}(k, p) &= i^l \frac{P(\text{c.c.})}{(4\pi i)^{2L}} \int \prod_{j=1}^l d\alpha_j [D(\alpha)]^{-2} \\ &\times \exp \left\{ ik^2 \frac{A(\alpha)}{D(\alpha)} + i \frac{(p-k)^2 B_s(\alpha) + (p+k)^2 B_u(\alpha)}{D(\alpha)} \right\} \\ &\times \exp \left\{ iM^2 \frac{C(\alpha)}{D(\alpha)} - i \sum_j \alpha_j (m_j^2 - i\epsilon) \right\}, \end{aligned} \quad (2.3)$$

where $M^2 = p^2$, $P(\text{c.c.})$ is the relevant product of coupling constants, L is the number of loops of the diagram, and l is the number of its lines. For our purposes, the most important property of this representation is that $A(\alpha)$, $B_s(\alpha)$, $B_u(\alpha)$, $C(\alpha)$, $D(\alpha)$ are positive (or better, non-negative) functions (sums of products) of the non-negative α_j -parameters of a diagram. Using it, we get the representation (2.2) with

$$\lambda = \frac{A(\alpha) + B_s(\alpha) + B_u(\alpha)}{D(\alpha)}, \quad (2.4)$$

$$x = \frac{B_s(\alpha) - B_u(\alpha)}{A(\alpha) + B_s(\alpha) + B_u(\alpha)}, \quad (2.5)$$

and a function $F(x, \lambda; M^2)$ specific for each diagram. Evidently, $0 \leq \lambda \leq \infty$. The limits for x in general case are $-1 \leq x \leq 1$, the negative x appearing when $B_u(\alpha) \neq 0$, which happens for some nonplanar diagrams.

Integrating over λ in Eq. (2.2) gives a Nakanishi-type representation (see, e.g. [33]) for this amplitude. We prefer, however, to use the representation involving both x and λ as integration variables.

Note that no restrictions (like being lightlike, etc.) are imposed on k and p in Eq. (2.2). In particular, p is the actual external momentum with $p^2 = M^2$. Basically, Eq. (2.2) expresses an obvious fact that, due to the Lorentz invariance, the function $\chi(k, p)$ depends on k through (kp) and k^2 . It may be treated as a double Fourier representation of $\chi(k, p)$ in both (kp) and k^2 .

Transforming Eq. (2.2) to the coordinate representation and changing $\lambda = 1/\sigma$ gives

$$\begin{aligned} \langle p|\phi(0)\phi(z)|p\rangle &= \int_0^\infty d\sigma \int_{-1}^1 dx e^{-ix(pz) - i\sigma(z^2 - i\epsilon)/4} \\ &\times e^{-ix^2 M^2/\sigma} F(x, 1/\sigma; M^2). \end{aligned} \quad (2.6)$$

Defining the *Virtuality Distribution Function*

$$\Phi(x, \sigma; M^2) = \exp[-ix^2 M^2/\sigma] F(x, 1/\sigma; M^2) \quad (2.7)$$

we arrive at the *VDF representation*

$$\begin{aligned} \langle p|\phi(0)\phi(z)|p\rangle &= \int_0^\infty d\sigma \int_{-1}^1 dx \Phi(x, \sigma; M^2) \\ &\times e^{-ix(pz) - i\sigma(z^2 - i\epsilon)/4} \end{aligned} \quad (2.8)$$

that reflects the fact that the matrix element $\langle p|\phi(0)\phi(z)|p\rangle$ depends on z through (pz) and z^2 , and may be treated as a double Fourier representation with respect to these variables. On general grounds, one would expect that such a Fourier representation should be valid for a very wide class of functions. The main non-trivial feature of the representations (2.2), (2.8) is in their specific limits of integration over x and λ (or σ). For an arbitrary function, one cannot insist on such limits.

However, our matrix element is not an arbitrary function. It is given by a sum of handbag Feynman diagrams, and the limits on x and λ (or σ) are dictated by the properties of these diagrams, in particular, by positivity of the functions A, B, D determining x and λ . It should be emphasized that these functions are determined purely by denominators of propagators, and are not affected by their numerators present in non-scalar theories.

Thus, the VDF representation (2.8) is valid for any diagram and reflects very general features of quantum field theory. On these grounds, we will assume that it holds nonperturbatively. An important point is that Eq. (2.8) gives a covariant definition of x as a variable that is Fourier-conjugate to (pz) . There is no need to assume that $p^2 = 0$ or $z^2 = 0$ to define x . The parameter σ , being conjugate to z^2 , may be interpreted as some measure of parton virtuality, hence the name of the function. In particular, VDF contains higher-twist contributions describing transverse momentum effects.

2.2. Collinear PDFs and TMDs

While the VDF representation holds for any z and p , nothing prevents us from considering some special cases, like a projection on the light cone $z^2 = 0$. This may be implemented, e.g., by choosing z that has the minus component only. Then one can parameterize the matrix element in terms of the twist-2 parton distribution $f(x)$

$$\langle p|\phi(0)\phi(z_-)|p\rangle = \int_{-1}^1 dx f(x) e^{-ixp_+ z_-} \quad (2.9)$$

that depends on the fraction x of the target momentum component p_+ carried by the parton. The relation between VDF $\Phi(x, \sigma)$ and the collinear twist-2 PDF $f(x)$ is formally given by

$$\int_0^\infty \Phi(x, \sigma) d\sigma = f(x). \quad (2.10)$$

Of course, this construction of $f(x)$ works only if the $z^2 \rightarrow 0$ limit is finite, e.g. in the super-renormalizable φ^3 theory. In the renormalizable φ^4 theory, the function $\Phi(x, \sigma)$ has a $\sim 1/\sigma$ hard

part, and the integral (2.10) is logarithmically divergent, reflecting the perturbative evolution of parton densities in such a theory.

Treating the target momentum p as purely longitudinal, $p = (E, \mathbf{0}_\perp, P)$, one can introduce the parton's transverse momentum. In the light-front variables [we use the convention $(ab) = a_+b_- + a_-b_+ - a_\perp b_\perp$], we write $p = (p_+, p_- = p^2/2p_+, p_\perp = 0)$. Taking z that has z_- and z_\perp components only, i.e., projecting on the light front $z_+ = 0$, we define the *transverse momentum dependent distribution* in the usual way as a Fourier transform with respect to remaining coordinates z_- and z_\perp :

$$\mathcal{F}(x, k_\perp^2) = \frac{p_+}{(2\pi)^3} \int_{-\infty}^{\infty} dz_- \int d^2z_\perp e^{i(k_\perp z_\perp)} e^{ixz_- - p_+} \times \langle p | \phi(0) \phi(z_-, z_\perp) | p \rangle_{|p_\perp=0}. \quad (2.11)$$

Because of the rotational invariance in z_\perp plane, this TMD depends on k_\perp^2 only, the fact already reflected in the notation. The TMD may be written in terms of VDF as

$$\mathcal{F}(x, k_\perp^2) = \frac{i}{\pi} \int_0^\infty \frac{d\sigma}{\sigma} \Phi(x, \sigma) e^{-i(k_\perp^2 - i\epsilon)/\sigma}. \quad (2.12)$$

Note that having a covariantly defined VDF $\Phi(x, \sigma)$, one can use this representation to analytically continue $\mathcal{F}(x, k_\perp^2)$ into a region of negative and even complex values of k_\perp^2 .

The integrated TMD

$$f(x, \mu^2) \equiv \pi \int_0^{\mu^2} dk_\perp^2 \mathcal{F}(x, k_\perp^2) \quad (2.13)$$

may be interpreted as a scale-dependent parton distribution. Indeed, when the $\mu^2 \rightarrow \infty$ limit exists, we have $f(x, \infty) = f(x)$. One can write $f(x, \mu^2)$ in terms of VDF,

$$f(x, \mu^2) = \int_0^\infty d\sigma \left[1 - e^{-i(\mu^2 - i\epsilon)/\sigma} \right] \Phi(x, \sigma). \quad (2.14)$$

Note that $f(x, \mu^2)$ has μ^2 -dependence, i.e. evolves with μ^2 even if the limit $\mu^2 \rightarrow \infty$ is finite, e.g. in a super-renormalizable theory. The evolution equation

$$\mu^2 \frac{d}{d\mu^2} f(x, \mu^2) = \pi \mu^2 \mathcal{F}(x, \mu^2) \quad (2.15)$$

follows from the definition (2.13). When the TMD $\mathcal{F}(x, k_\perp^2)$ vanishes faster than $1/k_\perp^2$ (such a TMD will be called “soft”), the evolution essentially stops at large μ^2 .

In a renormalizable theory, it makes sense to represent $\Phi(x, \sigma)$ as a sum of a soft part $\Phi^{\text{soft}}(x, \sigma)$, generating a nonperturbative evolution of $f(x, \mu^2)$, and a $\sim 1/\sigma$ hard tail. Namely, the lowest-order hard-tail term

$$\Phi^{\text{hard}}(x, \sigma) = \Delta(x)/\sigma, \quad (2.16)$$

with $\Delta(x)$ given by

$$\Delta(x) = a \int_x^1 \frac{dz}{z} P(x/z) f^{\text{soft}}(z) \quad (2.17)$$

(where $P(x/z)$ is the evolution kernel and a is the appropriate coupling constant) generates the perturbative evolution

$$\mu^2 \frac{d}{d\mu^2} f^{\text{hard}}(x, \mu^2) = a \int_x^1 \frac{dy}{z} P(x/z) f^{\text{soft}}(z). \quad (2.18)$$

The theory of perturbative evolution (which includes also the subtleties of using the running coupling constant, higher-order corrections, scheme-dependence, etc.) is well developed, and it is not of much interest for us in this paper. Our main subject in what follows is the nonperturbative evolution generated by the soft part of VDF $\Phi(x, \sigma)$ [or TMD $\mathcal{F}(x, k_\perp^2)$], in application to parton quasi-distributions, introduced recently by X. Ji [1].

2.3. Quasi-distributions

The basic idea of Ref. [1] is to consider equal-time bilocal operator corresponding to $z = (0, 0, 0, z_3)$ [or, for brevity, $z = z_3$]. Then

$$\langle p | \phi(0) \phi(z_3) | p \rangle = \int_0^\infty d\sigma \int_{-1}^1 dx \Phi(x, \sigma) e^{ixp_3 z_3 + i\sigma z_3^2/4}. \quad (2.19)$$

Using again the frame in which $p = (E, \mathbf{0}_\perp, P)$, and introducing quasi-distributions [1] through

$$\langle p | \phi(0) \phi(z_3) | p \rangle = \int_{-\infty}^\infty dy Q(y, P) e^{iyPz_3} \quad (2.20)$$

we get a relation between PQDs and VDFs,

$$Q(y, P) = \int_0^\infty d\sigma \sqrt{\frac{iP^2}{\pi\sigma}} \int_{-1}^1 dx \Phi(x, \sigma) e^{-i(x-y)^2 P^2/\sigma}. \quad (2.21)$$

For large P , we have

$$\sqrt{\frac{iP^2}{\pi\sigma}} e^{-i(x-y)^2 P^2/\sigma} = \delta(x-y) + \frac{\sigma}{4P^2} \delta''(x-y) + \dots \quad (2.22)$$

and $Q(y, P \rightarrow \infty)$ tends to the integral (2.10) producing $f(y)$. This observation suggests that one may be able to extract the “light-cone” parton distribution $f(y)$ from the studies of the purely “space-like” function $Q(y, P)$ for large P , which can be done on the lattice [1].

The nonperturbative evolution of $Q^{\text{soft}}(y, P)$ with respect to P has the area-preserving property. Namely, since

$$\int_{-\infty}^\infty dy e^{-i(x-y)^2 P^2/\sigma} = \sqrt{\frac{\pi\sigma}{iP^2}}, \quad (2.23)$$

we formally have

$$\int_{-\infty}^\infty dy Q(y, P) = \int_0^\infty d\sigma \int_{-1}^1 dx \Phi(x, \sigma). \quad (2.24)$$

For the soft part, the integral over σ converges, and we may write

$$\int_{-\infty}^\infty dy Q^{\text{soft}}(y, P) = \int_{-1}^1 dx f^{\text{soft}}(x), \quad (2.25)$$

which means that $Q^{\text{soft}}(y, P)$ for any P has the same area normalization as $f^{\text{soft}}(x)$. Note also that the result of Eq. (2.25) may be obtained by formally taking $z_3 = 0$ in the definition (2.20) of $Q(y, P)$.

Similarly, since

$$\int_{-\infty}^\infty dy y e^{-i(x-y)^2 P^2/\sigma} = x \sqrt{\frac{\pi\sigma}{iP^2}}, \quad (2.26)$$

we formally have

$$\int_{-\infty}^{\infty} dy y Q(y, P) = \int_0^{\infty} d\sigma \int_{-1}^1 dx x \Phi(x, \sigma). \quad (2.27)$$

Again, the integral over σ converges for the soft part, and we have the momentum sum rule

$$\int_{-\infty}^{\infty} dy y Q^{\text{soft}}(y, P) = \int_{-1}^1 dx x f^{\text{soft}}(x). \quad (2.28)$$

Finally, comparing the VDF representation (2.21) for $Q(y, P)$ with that for the TMD $\mathcal{F}(x, k_{\perp}^2)$ [see (2.12)] we conclude that

$$Q(y, P) = \int_{-\infty}^{\infty} dk_1 \int_{-1}^1 dx P \mathcal{F}(x, k_1^2 + (x - y)^2 P^2). \quad (2.29)$$

Thus, the quasi-distribution $Q(y, P)$ [both its soft and hard parts] is completely determined by the form of the TMD $\mathcal{F}(x, k_{\perp}^2)$.

3. QCD

3.1. Spinor quarks

In spinor case, one deals with the matrix element of a

$$B^{\alpha}(z, p) \equiv \langle p | \bar{\psi}(0) \gamma^{\alpha} \psi(z) | p \rangle \quad (3.1)$$

type. It may be decomposed into p^{α} and z^{α} parts: $B^{\alpha}(z, p) = p^{\alpha} B_p(z, p) + z^{\alpha} B_z(z, p)$, or in the VDF representation

$$B^{\alpha}(z, p) = \int_0^{\infty} d\sigma \int_{-1}^1 dx \times [2p^{\alpha} \Phi(x, \sigma) + z^{\alpha} Z(x, \sigma)] e^{-ix(pz) - i\sigma(z^2 - i\epsilon)/4}. \quad (3.2)$$

If we take $z = (z_{-}, z_{\perp})$ in the $\alpha = +$ component of \mathcal{O}^{α} , the purely higher-twist z^{α} -part drops out and we can introduce the TMD $\mathcal{F}(x, k_{\perp}^2)$ that is related to the VDF $\Phi(x, \sigma)$ by the scalar formula (2.12).

The easiest way to avoid the effects of the z^{α} contamination in the quasi-distributions is to take the time component of $B^{\alpha}(z = z_3, p)$ and define

$$B^0(z_3, p) = 2p^0 \int_{-1}^1 dx Q(y, P) e^{iyPz_3} \quad (3.3)$$

(here we differ from the original definition of PQDs by X. Ji [1] who uses $\alpha = 3$). The connection between $Q(y, P)$ and $\Phi(x, \sigma)$ is given then by the same formula (2.21) as in the scalar case. As a result, we have the sum rules (2.25) and (2.28) corresponding to charge and momentum conservation. Furthermore, the quasi-distributions $Q_i(y, P)$ are related to TMDs $\mathcal{F}_i(x, k_{\perp}^2)$ by the scalar conversion formula (2.29).

3.2. Gauge fields

In QCD, one should take the operator

$$\mathcal{O}_q^{\alpha}(0, z; A) \equiv \bar{\psi}(0) \gamma^{\alpha} \hat{E}(0, z; A) \psi(z) \quad (3.4)$$

involving a straight-line path-ordered exponential

$$\hat{E}(0, z; A) \equiv P \exp \left[ig z_{\nu} \int_0^1 dt A^{\nu}(tz) \right] \quad (3.5)$$

in the quark (adjoint) representation. As is well-known, its Taylor expansion has the same structure as that for the original $\bar{\psi}(0) \gamma^{\alpha} \psi(z)$ operator, with the only change that one should use covariant derivatives $D^{\nu} = \partial^{\nu} - ig A^{\nu}$ instead of the ordinary ∂^{ν} ones:

$$\hat{E}(0, z; A) \psi(z) = \sum_{n=0}^{\infty} \frac{1}{n!} (zD)^n \psi(0). \quad (3.6)$$

Again, the z^{α} contamination is avoided if the quasi-distributions are defined through the time component of \mathcal{O}^{α} . Then we have the same relations between the VDFs and PQDs as in the scalar case.

3.3. Sum rules

Converting Eq. (2.24) into the sum rule (2.25) we noted that in general it holds for the soft part only, because the hard part $\Phi^{\text{hard}}(x, \sigma)$ (2.17) is proportional to $1/\sigma$ and its σ -integral logarithmically diverges. However, the x -integral of $\Phi^{\text{hard}}(x, \sigma)$ vanishes (the zeroth x -moment of the evolution kernel $P_{qq}(x/z)$ is proportional to the anomalous dimension of the vector current, which is zero due to the vector current conservation). As a result, we have the valence quark sum rules

$$\int_{-\infty}^{\infty} dy [Q_q(y, P) - Q_{\bar{q}}(y, P)] = \int_{-1}^1 dx [f_q(x, \mu^2) - f_{\bar{q}}(x, \mu^2)] \quad (3.7)$$

involving full PQDs and PDFs.

Since the first x -moment of $P_{qq}(x/z)$ is non-zero, Eq. (2.28) may be only used to derive the momentum sum rule involving the soft parts of quark distributions

$$\int_{-\infty}^{\infty} dy y [Q_q^{\text{soft}}(y, P) + Q_{\bar{q}}^{\text{soft}}(y, P)] = \int_{-1}^1 dx x [f_q^{\text{soft}}(x) + f_{\bar{q}}^{\text{soft}}(x)]. \quad (3.8)$$

To include gluons, one should consider the operator

$$\mathcal{O}_g^{\alpha\beta}(0, z; A) \equiv G^{\alpha\nu}(0) \tilde{E}(0, z; A) G_{\nu}^{\beta}(z). \quad (3.9)$$

Here \tilde{E} is the straight-line path-ordered exponential in the gluon (fundamental) representation. The matrix element of $\mathcal{O}_g^{\alpha\beta}(0, z; A)$ contains the basic $p^{\alpha} p^{\beta}$ structure that produces the twist-2 PDF, but it also has the contaminating structures containing z^{α} , z^{β} or $g^{\alpha\beta}$. When one takes, as usual, $\alpha = \beta = +$ and $z = (z_{-}, z_{+})$, the z -structures and $g^{\alpha\beta}$ do not contribute to the matrix element of the operator \mathcal{O}_g^{++} defining the gluon PDF. In case of the quasi-distribution, the contaminating structures containing z_3 are avoided when we take $\alpha = 0$, $\beta = 0$ (again, another definition of the gluon PQD corresponding to $\alpha = 3$, $\beta = 3$ was chosen in Ref. [1]). Still, there remains contamination from the $g^{\alpha\beta}$ structure and the momentum sum rule for gluons

$$\int_{-\infty}^{\infty} dy y Q_g^{\text{soft}}(y, P) = \int_{-1}^1 dx x f_g^{\text{soft}}(x) + \mathcal{O}(\Lambda^2/P^2) \quad (3.10)$$

is spoiled by the $\mathcal{O}(\Lambda^2/P^2)$ term brought in by the $g^{\alpha\beta}$ admixture.

3.4. Primordial TMDs

One may notice that the $\mathcal{O}^\alpha(0, z; A)$ operator involves a straight-line link from 0 to z rather than a stapled link usually used in the definitions of TMDs appearing in the description of Drell–Yan and semi-inclusive DIS processes. As is well-known, the stapled links reflect initial or final state interactions inherent in these processes. The “straight-link” TMDs, in this sense, describe the structure of a hadron when it is in its non-disturbed or “primordial” state. While it is unlikely that such a TMD can be measured in a scattering experiment, it is a well-defined QFT object, and one may hope that it can be measured on the lattice through its connection (2.29) to the quasi-distributions.

4. Models for soft part

Let us now discuss some explicit models of the k_\perp dependence of soft TMDs $\mathcal{F}(x, k_\perp^2)$. In general, they are functions of two independent variables x and k_\perp^2 . For simplicity, we will consider here the case of factorized models

$$\mathcal{F}(x, k_\perp^2) = f(x) \psi(k_\perp^2), \quad (4.1)$$

in which x -dependence and k_\perp -dependence appear in separate factors. Since, with our definitions, the relations between VDFs and TMDs are the same in scalar and spinor cases, we will refer for brevity to scalar operators.

4.1. Gaussian model

It is popular to assume a Gaussian dependence on k_\perp ,

$$\mathcal{F}_G(x, k_\perp^2) = \frac{f(x)}{\pi \Lambda^2} e^{-k_\perp^2/\Lambda^2}. \quad (4.2)$$

Writing

$$\mathcal{F}_G(x, k_\perp^2) = \frac{f(x)}{2\pi^2 i \Lambda^2} \int_{-\infty}^{\infty} \frac{d\sigma}{\sigma - i\Lambda^2} e^{-ik_\perp^2/\sigma}, \quad (4.3)$$

we see that the integral here involves both positive and negative σ , i.e. formally $\mathcal{F}_G(x, k_\perp^2)$ cannot be written in the VDF representation (2.12). This is a consequence of the fact that the analytic continuation of $\mathcal{F}_G(x, k_\perp^2)$ into the region of negative k_\perp^2 has an exponential increase.

However, since we are interested in positive k_\perp^2 only, in our modeling we will just use the conversion formula (2.29) for all k_\perp^2 profiles for which it gives convergent results. For the Gaussian model we have then

$$Q_G(y, P) = \frac{P}{\Lambda \sqrt{\pi}} \int_{-1}^1 dx f(x) e^{-(x-y)^2 P^2/\Lambda^2}. \quad (4.4)$$

4.2. Simple non-Gaussian models

In the space of impact parameters z_\perp , the Gaussian model gives a $e^{-z_\perp^2 \Lambda^2/4}$ fall-off, and one may argue that the decrease is too fast for large z_\perp . In particular, propagators $D^c(z, m)$ of massive particles have an exponential $e^{-m|z|}$ fall-off for spacelike intervals z^2 .

To build models for TMDs that resemble more closely the perturbative propagators in the deep spacelike region, we recall that the propagator of a scalar particle with mass m may be written as

$$D^c(z, m) = \frac{1}{(4\pi)^2} \int_0^\infty e^{-i\sigma z^2/4 - i(m^2 - i\epsilon)/\sigma} d\sigma. \quad (4.5)$$

It is the mass term that assures that the propagator falls off exponentially $\sim e^{-|z|m}$ for large spacelike distances. At small intervals z^2 , however, the free particle propagator has a $1/z^2$ singularity while we want the soft part of $\langle p|\phi(0)\phi(z)|0\rangle$ to be finite at $z=0$. The simplest way is to add a constant term ($-4/\Lambda^2$) to z^2 in the VDF representation (2.8). So, we take

$$\Phi(x, \sigma) = \frac{f(x)}{2im\Lambda K_1(2m/\Lambda)} e^{i\sigma/\Lambda^2 - im^2/\sigma - \epsilon\sigma} \quad (4.6)$$

as a model for the VDF, where K_1 is the modified Bessel function. The sign of the Λ^2 term is fixed from the requirement that $(4/\Lambda^2 - z^2)^{-1}$ should not have singularities for space-like z^2 . This model corresponds to the following TMD

$$\mathcal{F}_m(x, k_\perp^2) = f(x) \frac{K_0\left(2\sqrt{k_\perp^2 + m^2/\Lambda}\right)}{\pi m \Lambda K_1(2m/\Lambda)}. \quad (4.7)$$

It is finite for $k_\perp = 0$ reflecting the exponential $\sim e^{-m|z_\perp|}$ fall-off for large z_\perp . To avoid a two-parameter modeling, one may take $m=0$, i.e.

$$\Phi_{m=0}(x, \sigma) = f(x) \frac{e^{i\sigma/\Lambda^2 - \epsilon\sigma}}{i\Lambda^2}, \quad (4.8)$$

which corresponds to

$$\mathcal{F}_{m=0}(x, k_\perp^2) = 2f(x) \frac{K_0(2|k_\perp|/\Lambda)}{\pi \Lambda^2}. \quad (4.9)$$

It has a logarithmic singularity for small k_\perp that reflects a too slow $\sim 1/(1 + z_\perp^2 \Lambda^2/4)$ fall-off for large z_\perp . For the quasi-distribution, we have

$$Q_{m=0}(y, P) = \frac{P}{\Lambda} \int_{-1}^1 dx f(x) e^{-2|x-y|P/\Lambda}. \quad (4.10)$$

Note that the Gaussian model and the $m=0$ models have the same $\sim (1 - z_\perp^2 \Lambda^2/4)$ behavior for small z_\perp , i.e. they correspond to the same value of the $\langle p|\varphi(0)\partial^2\varphi(0)|p\rangle$ matrix element, provided that one takes the same value of Λ in both models. For large z_\perp , however, the fall-off of the Gaussian model is too fast, while that of the $m=0$ model is too slow. Thus, they look like two extreme cases of one-parameter models, and we will use them for illustration of the nonperturbative evolution of quasi-distributions, expecting that other models (e.g. $m \neq 0$ model) will produce results somewhere in between of these two cases.

5. Numerical results

The full $-1 \leq x \leq 1$ PDF-support segment is usually split into the positive- x “quark” region and negative- x “antiquark” region. As we will see below, the PQDs $Q(y, P)$ live on the whole $-\infty < y < \infty$ axis, even when they are generated from a TMD model that is non-zero for positive x only. Thus, to avoid confusion of what generates PQD for negative y , it makes sense to separate the parts of PQDs coming from positive- x and negative- x parts of TMDs.

To illustrate the pattern of the nonperturbative evolution of quasi-distributions, we apply Eqs. (4.4) and (4.10) to a simple PDF $f(x) = (1-x)^3 \theta(0 \leq x \leq 1)$ resembling nucleon valence distributions (an enthusiastic reader can easily obtain curves for more realistic $(1-x)^3/\sqrt{x}$ valence models, for sea distribution models, etc.).

As one can see from Figs. 2, 4, the evolution patterns in our two models are very close to each other. They also resemble the pattern observed in actual lattice calculations [18–24] and in the diquark spectator model [25–27]. The quasi-distributions are wider

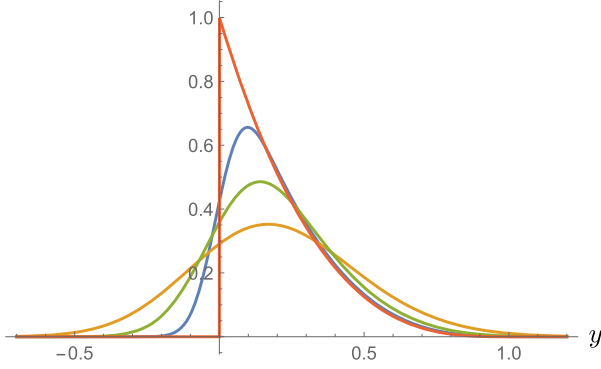


Fig. 2. Evolution of $Q(y, P)$ in the Gaussian model for $P/\Lambda = 3, 5, 10$ (from bottom to top at $y = 0.2$) compared to the limiting PDF $f(y) = (1 - y)^3 \theta(y)$.

for small P , with their support visibly extending beyond the $0 \leq y \leq 1$ segment, becoming narrower (and higher in their maxima) with increasing P .

The approach to the limiting $(1 - y)^3$ shape is not uniform, as illustrated in Figs. 3, 5. For large $y = 0.7$, the ratio $Q(y, P)/f(y)$ considerably exceeds 1 for small P tending to the limiting value from above. For smaller $y = 0.1$ and $y = 0.3$, the ratio curves tend to 1 from below. One can see that $P/\Lambda \gtrsim 10$ is needed (or P of the order of several GeV) to get $Q(y, P)/f(y)$ close to 1 for these y values.

6. Leading-order hard tail

The nonperturbative evolution of $Q(y, P)$ essentially stops for $P/\Lambda \gtrsim 20$, and for larger values of P the dominant role is played by the perturbative evolution generated by the hard part.

The simplest $\Phi \sim 1/\sigma$ hard tail model (2.16) corresponds to a $\sim 1/k_{\perp}^2$ TMD. It is singular for $k_{\perp} = 0$ while we want TMDs be finite in this limit. The simplest regularization $1/k_{\perp}^2 \rightarrow 1/(k_{\perp}^2 + m^2)$ corresponds to the change $1/\sigma \rightarrow e^{-im^2/\sigma}/\sigma$ in the hard part of VDF,

$$\Phi^{\text{hard}}(x, \sigma) \rightarrow \frac{\Delta(x)}{\sigma} e^{-im^2/\sigma}. \quad (6.1)$$

To proceed with the conversion formula, one needs the integral over σ

$$\begin{aligned} I(x, y, P) &= \int_0^{\infty} \frac{d\sigma}{\sqrt{\pi\sigma}} \frac{P}{\sigma} e^{-(x-y)^2 P^2/\sigma - m^2/\sigma} \\ &= \frac{1}{\sqrt{(x-y)^2 + m^2/P^2}}. \end{aligned} \quad (6.2)$$

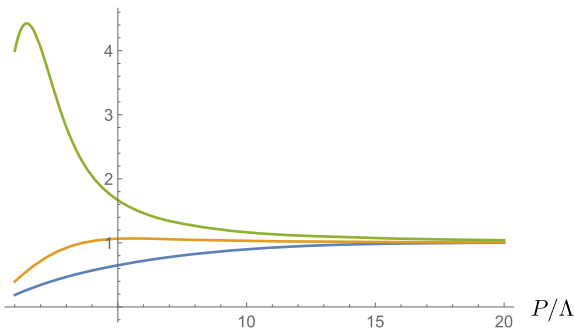


Fig. 3. Ratio $Q(y, P)/f(y)$ in the Gaussian model for $y = 0.1, 0.3, 0.7$ (from bottom to top) and $f(y) = (1 - y)^3$.

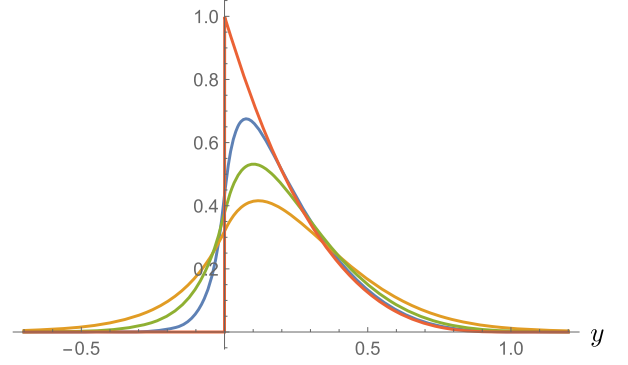


Fig. 4. Evolution of $Q(y, P)$ in the $m = 0$ model for $P/\Lambda = 3, 5, 10$ (from bottom to top at $y = 0.2$) compared to the limiting PDF $f(y) = (1 - y)^3 \theta(y)$.

This gives the hard part of a quasi-distribution

$$Q^{\text{hard}}(y, P) = \int_0^1 dx \frac{\Delta(x)}{\sqrt{(x-y)^2 + m^2/P^2}} \quad (6.3)$$

($x > 0$ is taken for definiteness) generating evolution with respect to P^2 in the form

$$P^2 \frac{d}{dP^2} Q^{\text{hard}}(y, P) = \frac{m^2}{2P^2} \int_0^1 dx \frac{\Delta(x)}{[(x-y)^2 + m^2/P^2]^{3/2}}. \quad (6.4)$$

In the $m/P \rightarrow 0$ limit we have

$$\frac{m^2}{2P^2} \int_0^1 dx \frac{P(x/z)}{[(x-y)^2 + m^2/P^2]^{3/2}} = P(y/z) + \mathcal{O}(m^2/P^2), \quad (6.5)$$

i.e. for large P^2 the quasi-distributions evolve according to the perturbative evolution equation with respect to P^2 .

The pattern of the sub-asymptotic m^2/P^2 dependence for the hard part may be illustrated by taking $P(x/z) \rightarrow 1$. Then

$$\begin{aligned} &\frac{m^2}{2P^2} \int_0^1 dx \frac{1}{[(x-y)^2 + m^2/P^2]^{3/2}} \\ &= \frac{1}{2} \left(\frac{y}{\sqrt{y^2 + m^2/P^2}} + \frac{1-y}{\sqrt{(1-y)^2 + m^2/P^2}} \right) \\ &= \theta(0 \leq y \leq 1) + \mathcal{O}(m^2/P^2). \end{aligned} \quad (6.6)$$

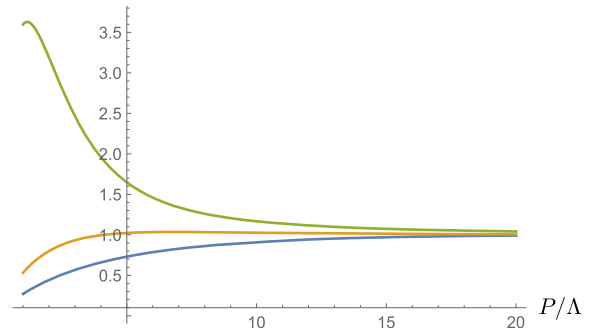


Fig. 5. Ratio $Q(y, P)/f(y)$ in the $m = 0$ model for $y = 0.1, 0.3, 0.7$ (from bottom to top) and $f(y) = (1 - y)^3$.

7. Conclusions

In this paper, we applied the formalism of parton virtuality distributions to study the p_3 -dependence of quasi-distributions $Q(y, p_3)$. We established a simple relation between PQDs and TMDs that allows to derive models for PQDs from the models for TMDs. Our model results show a pronounced nonperturbative evolution of PQDs for small and moderately large values of p_3 reflecting the transverse momentum dependence of TMDs, i.e. the spatial structure of the hadrons. Using two rather different models for the k_\perp dependence of TMDs, we obtained very similar patterns of the p_3 dependence of PQDs $Q(y, p_3)$ for each particular y . This observation may be used for a guided extrapolation of the moderate- p_3 lattice results to the $p_3 \rightarrow \infty$ limit. The basic idea is to find analytic models for soft TMDs that would successfully fit lattice PQDs for several values of p_3 , and then take the $p_3 \rightarrow \infty$ limit. A practical implementation of this program should be a subject of future studies.

Summarizing, the study of PQDs on the lattice in the domain of strong nonperturbative effects opens a new perspective in investigations of the three-dimensional structure of the hadrons.

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