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A Study of Relationships Between Family Members Using Familial Correlations

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A STUDY OF RELATIONSHIPS BETWEEN FAMILY MEMBERS USING FAMILIAL CORRELATIONS

by

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ABSTRACT

A STUDY OF RELATIONSHIPS BETWEEN FAMILY MEMBERS USING FAMILIAL CORRELATIONS

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Familial correlations measure the resemblance between family members and are used in many fields of study including epidemiology, genetics, heredity, and psychology. Here, an analysis of familial correlations where male and female children of the same family can have different correlations in the unequal family size case is presented. First, three likelihood based tests, namely the likelihood ratio test, Rao score test, and Wald test, and two more asymptotic tests which use Srivastava's estimator of the intraclass correlation coefficient are considered to test the null hypothesis of equality of the intraclass correlation coefficients when families have unequal numbers of children. These methods are implemented on Galton's data set on human stature and a simulation study is conducted to compare the different tests. The simulations show the alternative tests to be better or comparable to the likelihood based tests in certain situations. Additionally, testing the equality of interclass correlations from g independent populations is considered where male and female children of the same family can have different correlations and the family sizes within populations are unequal. For this problem, the likelihood ratio test is compared with two asymptotic alternative tests using Srivastava's estimator of the interclass correlation coefficient that are easier to compute. Simulations are used to study the size and power of these tests. Based on the simulation study, the alternative tests perform well when compared to the likelihood ratio test. Finally, the likelihood ratio test is compared with an asymptotic alternative test of interclass correlation for testing the equality of two parent interclass correlations coefficients, namely, parent-son and parent-daughter interclass correlation coefficients, within families from a single population with unequal family sizes. Both tests are illustrated on Galton's data set on human stature and the results of a simulation study are shown. The results show the alternative test to perform better for certain cases.

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CHAPTER I

INTRODUCTION

I.1 FAMILIAL CORRELATIONS

Familial data is observed in many different fields of research including epidemiology, genetics, heredity, and psychology. A common assumption of familial data is a likeness or dependency between family members, as relatives tend to have similar attributes. There is an extended history of research on estimating this dependency using familial correlations. For example, in genetics, famialial correlations have been used by Bouzigon et. al. (2004) to study the role of genetic traits in asthma development and by Atramentova and Belyaeva (2003) to study the development of lung cancer and large-intestine cancer. In epidemiology and psychology, Provencher et. al. (2005) studied the familial similarities of eating behavior traits, and in heredity and psychology, Knuiman et. al. (1996) studied the familial resemblance of cardiovascular risk factors. Familial correlations also have application in other fields, such as sports medicine and business. Bouchard et. al. (1998) used familial correlations when studying the genetic influences on maximal oxygen uptake, and Hackett and Parmanto (2009) used familial correlations when studying usability of the homepage of a website.

Formally, familial correlations measure the degree of resemblance between family members with respect to some specified quantitative characteristic such as height, weight, or blood pressure. The *intraclass* correlation (ρ) measures the degree of resemblance between members of the same group. In familial correlations, it might refer to the measure of resemblance between the children of a family (ρ_c), the sons of a family (ρ_1), or the daughters of a family (ρ_2). The *interclass* correlation measures the degree of resemblance between members of different groups. In familial correlations, it can refer to the measure of resemblance between the parents and children of a family (ρ_{pc}), the parents and sons of a family (ρ_{p1}), the parents and daughters of a family (ρ_{p2}), or the sons and daughters of a family (ρ_{12}). As mentioned, all these types of familial correlations have applications in several areas of study. Estimation of these correlations and testing for relationships between these correlations is of interest here.

I.2 FAMILIAL CORRELATION LITERATURE REVIEW

One of the first methods of estimating the intraclass correlation, ρ, is the pairwise Pearson correlation, which is computed from all possible pairs of measurements within families. Another commonly adopted estimator of ρ is the analysis of variance estimator, r_A, proposed by Fisher (1925), which estimates ρ as a ratio of variance components in a one-way random effects model. Under the assumption of normality, Donner and Koval (1980) derived the maximum likelihood estimator of ρ, which can only be obtained in closed form when family sizes are balanced, but can be obtained numerically in the unbalanced case. Donner (1986) gives a more detailed summary and review of the research done on these more common estimators of ρ, including significance testing and interval estimation. In practice, other problems arise for which these standard methods of estimating ρ may not apply.

Consider the case when two independent populations or samples of familial data are available for testing equality of the two intraclass coefficients. If family sizes in each population are fixed and equal, the distribution of the ratio of F-statistics from the one-way analysis of variance from each sample can be used to test the equality of the intraclass correlations, as was worked out by Schumann and Bradley (1957), Bross (1959), and Zerbe and Goldgar (1980). Donner and Bull (1983) derived the likelihood ratio test for the equality of two independent intraclass correlations for fixed family sizes. Their methods can accomodate the unequal family size case, but require an iterative solution to maximize the likelihood function. Other methods for testing the two independent populations case, in both the equal and unequal family size situations, have been developed and are discussed in Donner (1986) and Young and Bhandary (1998). Bhandary and Alam (2000) considered testing equality of three intraclass correlation coefficients from independent populations when family sizes are unequal. The more general case of testing the null hypothesis of equality of g intraclass correlations was considered by Naik and Helu (2007). However, in their work each family is grouped together as an entity and different levels of dependency between family members are not considered.

Many authors have worked with the parent-children correlation structure using the parent-child, ρ_{pc}, and child-child, ρ_c, correlations. In this set up, a parent score and children scores are available for each family. One such author is Srivastava (1984), who worked with one population of families and derived the iterative maximum likelihood estimators of ρ_{pc} and ρ_c using a canonical reduction of the data. He also

proposed two sets of alternative estimators based on the canonical reduction that do not require an iterative procedure and have better distributional properties. All three sets of estimators allow families to have different numbers of children. Srivastava and Katapa (1986) compared the asymptotic distributions of the maximum likelihood estimators and alternative estimators proposed in Srivastava (1984).

Srivastava's estimators of ρ_c have been extended by several authors to other familial correlation situations. Young and Bhandary (1998), Bhandary and Alam (2000), and Naik and Helu (2007) all used Srivastava's estimator of intraclass correlation, ρ_c. Naik and Helu (2007) developed tests to compare several intraclass correlations from independent populations. They compared three maximum likelihood asymptotic tests, namely the likelihood ratio test, Wald test, and Rao score test, and two other tests based on Srivastava's estimators. An illustration of their procedures tested to see if the correlation between the daughters of one group of families equaled the correlation between the sons of another group of families. While only an illustration of methods, this approach assumed no dependency between the sons and daughters.

Consider the familial model where data is available for the sons and daughters of each family in one population. Shoukri, Mian, and Tracy (1991) took a linear regression model approach to finding the maximum likelihood estimates of the brother-brother (ρ_1), sister-sister (ρ_2), and sister-brother (ρ_{12}) correlations. The maximum likelihood estimates for this familial model require a numerical solution. Donner and Zou (2002) presented several procedures for testing the equality of two dependent intraclass correlations, $H_0 : \rho_1 = \rho_2$, when family structures are identical, i.e. each family has the same number of sons and the same number of daughters. Bross (1959) noted that an exact test of this hypothesis, $H_0 : \rho_1 = \rho_2$, is available only when the number of sons equals the number of daughters for each family and this number is the same across all families; additionally, no dependency between sons and daughters can exist, that is $\rho_{12} = 0$. Another problem of interest would be to test the equality of the two dependent intraclass correlations, $H_0 : \rho_1 = \rho_2$, when families are allowed different numbers of both boys and girls. This problem will be considered in Chapter II.

The interclass correlations ρ_{pc} and ρ_{12} are very important in familial studies as they account for the dependency between two groups within families. As noted above, Srivastava (1984) and Srivastava and Katapa (1986) developed alternative estimates of the interclass correlation ρ_{pc} for familial data from one population with unequal

family sizes. Srivastava and Keen (1988) developed other noniterative techniques for estimating ρ_{pc} as alternatives to the iterative maximum likelihood estimator. Donner, Eliasziw, and Shoukri (1998) reviewed different procedures for estimating and testing ρ_{pc} for families from one population with both equal and unequal family sizes. The interclass correlation coefficient ρ_{12} was estimated by Shoukri, Mian, and Tracy (1991) using the maximum likelihood approach as already noted. Paul (1996) considered a score test for testing the significance of ρ_{12} for families from one population with unequal family sizes. Of further interest, is to test the equality of several interclass correlations from independent populations. This problem will be considered in Chapter III.

Family data can include parent data as well as data from the sons and daughters as suggested above. Shoukri, Mian, and Tracy (1991) also incorporated a parent score into their familial model and found maximum likelihood estimators for the parent-brother correlation (ρ_{p1}) and the parent-sister correlation (ρ_{p2}). Another test of interest is to determine if the correlation between the parent and sons of a family, ρ_{p1}, equals that of the correlation between the parent and daughters of a family, ρ_{p2}, when families come from one population with unequal family sizes. This problem will be considered in Chapter IV.

I.3 OVERVIEW OF THESIS

As noted above, Chapter II will focus on testing the equality of two dependent intraclass correlations, namely ρ_1 and ρ_2, when families from one population are allowed to have unequal family sizes. Specifically following the methods of Naik and Helu (2007), the likelihood ratio test (LRT), Rao Score test, Wald's test, and two other asymptotic tests based on Srivastava's estimator of intraclass correlation are developed. These five tests are illustrated on Galton's data set on human stature. Simulation studies are presented to compare the performance of the proposed tests. In Chapter III, testing the equality of several son-daughter interclass coefficients from independent populations of familial data when families have unequal family sizes is considered. Here the LRT, and two other asymptotic tests based on Srivastava's estimator of interclass correlation are developed and compared in simulation. In Chapter IV, the problem of testing the equality of two parent interclass coefficients within families, namely ρ_{p1} and ρ_{p2}, from one population with unequal family sizes is considered. For this problem, the LRT is again compared with two other asymptotic

tests based on Srivastava's estimator of interclass correlation. Galton's data set on human stature is again used as an illustration and further simulation studies are presented. Finally, a summary of the methods and findings presented is given in Chapter V, along with future areas of research.

CHAPTER II

FAMILIAL CORRELATIONS: ONE POPULATION

II.1 INTRODUCTION

In this chapter, we consider the situation where familial data is available for the sons and daughters of families from one population. The problem of interest is in testing the equality of brother-brother and sister-sister intraclass correlations, namely ρ_1 and ρ_2, assuming that the brother-sister interclass correlation is not zero. This problem has been considered in Donner and Zou (2002) under the assumption of equal family sizes. Shoukri, Mian, and Tracy (1991) have considered this problem under the unequal family sizes case and have taken a linear regression model approach to finding the maximum likelihood estimates of the brother-brother (ρ_1), sister-sister (ρ_2), and sister-brother (ρ_{12}) correlations. The maximum likelihood estimates for this familial model require a numerical solution. As noted by Bross (1959), an exact test of this hypothesis, $H_0 : \rho_1 = \rho_2$, is not available when the number of sons and the number of daughters are different and dependency between sons and daughters exists, that is, $\rho_{12} \neq 0$. In this chapter, we provide three maximum likelihood based asymptotic tests, namely, the likelihood ratio test, Wald's test, and Rao's score test, along with certain alternative tests based on estimators similar to the ones proposed by Srivastava (1984). Explicit expressions for both the score functions and the elements of Fisher information matrix are provided as well. When compared with the maximum likelihood based asymptotic tests, the alternative tests are easy to compute and perform quite well.

Suppose data on the children of n randomly selected families are available from a population. The number of boys and girls in each family is allowed to be different. Denote the number of boys and girls in the i^{th} family as m_{1i} and m_{2i}, respectively, for $i = 1, \ldots, n$. Suppose x_{1ij}, $j = 1, \ldots, m_{1i}$; $i = 1, \ldots, n$ is the observation on the j^{th} boy of the i^{th} family. Similarly, x_{2ij}, $j = 1, \ldots, m_{2i}$; $i = 1, \ldots, n$ is the observation on the j^{th} girl of the i^{th} family.

Assume that the mean of the son scores is $E(x_{1ij}) = \mu_1$, the mean of the daughter scores is $E(x_{2ij}) = \mu_2$, the variance of the son scores is $Var(x_{1ij}) = \sigma_1^2$, and the variance of the daughter scores is $Var(x_{2ij}) = \sigma_2^2$. Denote the son-son *intraclass*

correlation as ρ_1, the daughter-daughter *intraclass* correlation as ρ_2, and the son-daughter *interclass* correlation as ρ_{12}. Assume for each family of fixed i $(1 \leq i \leq n)$, $Corr(x_{1ij}, x_{1ij'}) = \rho_1$ for $j \neq j'$; $1 \leq j, j' \leq m_{1i}$, $Corr(x_{2ij}, x_{2ij'}) = \rho_2$ for $j \neq j'$; $1 \leq j, j' \leq m_{2i}$, and $Corr(x_{1ij}, x_{2ij'}) = \rho_{12}$ for all j, j'; $1 \leq j \leq m_{1i}$ and $1 \leq j' \leq m_{2i}$.

Let the vector of observations on the i^{th} family be

$$\mathbf{x}_i = \begin{pmatrix} \mathbf{x}_{1i} \\ \mathbf{x}_{2i} \end{pmatrix} = \begin{pmatrix} x_{1i1} \\ \vdots \\ x_{1im_{1i}} \\ x_{2i1} \\ \vdots \\ x_{2im_{2i}} \end{pmatrix}$$

with

$$E(\mathbf{x}_i) = \boldsymbol{\mu}_i = \begin{pmatrix} \mu_1 \mathbf{1}_{m_{1i}} \\ \mu_2 \mathbf{1}_{m_{2i}} \end{pmatrix}$$

and

$$Var(\mathbf{x}_i) = \mathbf{\Sigma}_i = \begin{pmatrix} \sigma_1^2\{(1-\rho_1)\mathbf{I}_{m_{1i}} + \rho_1 \mathbf{J}_{m_{1i}}\} & \rho_{12}\sigma_1\sigma_2 \mathbf{J}_{m_{1i},m_{2i}} \\ \rho_{12}\sigma_1\sigma_2 \mathbf{J}_{m_{2i},m_{1i}} & \sigma_2^2\{(1-\rho_2)\mathbf{I}_{m_{2i}} + \rho_2 \mathbf{J}_{m_{2i}}\} \end{pmatrix},$$

where $\mathbf{1}_m$ is a unit vector of length m, $\mathbf{I}_m$ is an identity matrix of order m, $\mathbf{J}_m$ is the $m \times m$ matrix of all ones, and $\mathbf{J}_{m,n}$ is the $m \times n$ matrix of all ones. Note that $-\infty < \mu_1 < \infty$ and $-\infty < \mu_2 < \infty$.

If there are both sons and daughters in a family, $m_{1i} > 0$ and $m_{2i} > 0$, then the determinant of $\mathbf{\Sigma}_i$ is

$$\begin{aligned} |\mathbf{\Sigma}_i| &= \sigma_1^{2m_{1i}}\sigma_2^{2m_{2i}}(1-\rho_1)^{m_{1i}-1}(1-\rho_2)^{m_{2i}-1} \\ &\times \left((1+(m_{1i}-1)\rho_1)(1+(m_{2i}-1)\rho_2) - m_{1i}m_{2i}\rho_{12}^2\right). \end{aligned}$$

Restrictions on the parameters so that $\mathbf{\Sigma}_i$ positive definite are $\sigma_1^2 > 0$, $\sigma_2^2 > 0$, $\rho_1 < 1$, $\rho_2 < 1$, and

$$(1+(m_{1i}-1)\rho_1)(1+(m_{2i}-1)\rho_2) > m_{1i}m_{2i}\rho_{12}^2. \tag{1}$$

If $m_{1i} > 0$ and $m_{2i} > 0$, then the inverse of $\mathbf{\Sigma}_i$ is

$$\mathbf{\Sigma}_i^{-1} = \begin{pmatrix} \frac{1}{\sigma_1^2}\mathbf{A}_i & \frac{1}{\sigma_1\sigma_2}\mathbf{B}_i \\ \frac{1}{\sigma_1\sigma_2}\mathbf{B}_i' & \frac{1}{\sigma_2^2}\mathbf{C}_i \end{pmatrix},$$

where

$$\begin{aligned}
\mathbf{A}_i &= \frac{1}{1-\rho_1}\left[\mathbf{I}_{m_{1i}} - \frac{\rho_1(1+(m_{2i}-1)\rho_2) - m_{2i}\rho_{12}^2}{(1+(m_{1i}-1)\rho_1)(1+(m_{2i}-1)\rho_2) - m_{1i}m_{2i}\rho_{12}^2}\mathbf{J}_{m_{1i}}\right], \\
\mathbf{B}_i &= \frac{-\rho_{12}}{(1+(m_{1i}-1)\rho_1)(1+(m_{2i}-1)\rho_2) - m_{1i}m_{2i}\rho_{12}^2}\mathbf{J}_{m_{1i},m_{2i}}, \\
\mathbf{C}_i &= \frac{1}{1-\rho_2}\left[\mathbf{I}_{m_{2i}} - \frac{\rho_2(1+(m_{1i}-1)\rho_1) - m_{1i}\rho_{12}^2}{(1+(m_{1i}-1)\rho_1)(1+(m_{2i}-1)\rho_2) - m_{1i}m_{2i}\rho_{12}^2}\mathbf{J}_{m_{2i}}\right].
\end{aligned}$$

If there are no sons in the family, $m_{1i} = 0$, then the determinant of $\mathbf{\Sigma}_i$ is

$$|\mathbf{\Sigma}_i| = \sigma_2^{2m_{2i}}(1-\rho_2)^{m_{2i}-1}(1+(m_{2i}-1)\rho_2)$$

and the inverse of $\mathbf{\Sigma}_i$ is

$$\mathbf{\Sigma}_i^{-1} = \tfrac{1}{\sigma_2^2(1-\rho_2)}\left[I_{m_{2i}} - \tfrac{\rho_2}{(1+(m_{2i}-1)\rho_2)}J_{m_{2i}}\right].$$

Similarly, if there are no daughters in the family, $m_{2i} = 0$, then the determinant of $\mathbf{\Sigma}_i$ is

$$|\mathbf{\Sigma}_i| = \sigma_1^{2m_{1i}}(1-\rho_1)^{m_{1i}-1}(1+(m_{1i}-1)\rho_1)$$

and the inverse of $\mathbf{\Sigma}_i$ is

$$\mathbf{\Sigma}_i^{-1} = \tfrac{1}{\sigma_1^2(1-\rho_1)}\left[I_{m_{1i}} - \tfrac{\rho_1}{(1+(m_{1i}-1)\rho_1)}J_{m_{1i}}\right].$$

II.2 LIKELIHOOD FUNCTION

Assume that $\mathbf{x}_i \sim N_{m_{1i}+m_{2i}}(\boldsymbol{\mu}_i, \mathbf{\Sigma}_i)$, $i = 1, \ldots, n$. Let

$$\boldsymbol{\theta} = (\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho_1, \rho_2, \rho_{12})'$$

then

$$L(\boldsymbol{\theta}) = \textstyle\prod_{i=1}^n L_i(\boldsymbol{\theta}) = \prod_{i=1}^n \frac{1}{(2\pi)^{(m_{1i}+m_{2i})/2}|\mathbf{\Sigma}_i|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}_i-\boldsymbol{\mu}_i)'\mathbf{\Sigma}_i^{-1}(\mathbf{x}_i-\boldsymbol{\mu}_i)}$$

and

$$log(L(\boldsymbol{\theta})) = \sum_{i=1}^n log(L_i(\boldsymbol{\theta})).$$

If $m_{1i} > 0$ and $m_{2i} > 0$, then

$$\begin{aligned}
log(L_i(\boldsymbol{\theta})) &= -\frac{m_{1i}}{2}log(2\pi\sigma_1^2) - \frac{m_{2i}}{2}log(2\pi\sigma_2^2) \\
&- \frac{1}{2}(m_{1i}-1)log(1-\rho_1) - \frac{1}{2}(m_{2i}-1)log(1-\rho_2) \\
&- \frac{1}{2}log\left[(1+(m_{1i}-1)\rho_1)(1+(m_{2i}-1)\rho_2) - m_{1i}m_{2i}\rho_{12}^2\right] \\
&- \frac{1}{2}(\mathbf{x}_i - \boldsymbol{\mu}_i)'\boldsymbol{\Sigma}_i^{-1}(\mathbf{x}_i - \boldsymbol{\mu}_i).
\end{aligned}$$

Note $\mathbf{x}_i = \begin{pmatrix} \mathbf{x}_{1i} \\ \mathbf{x}_{2i} \end{pmatrix}$, where $\mathbf{x}_{1i} = (x_{1i1}, \ldots, x_{1im_{1i}})'$ and $\mathbf{x}_{2i} = (x_{2i1}, \ldots, x_{2im_{2i}})'$, therefore

$$\begin{aligned}
& (\mathbf{x}_i - \boldsymbol{\mu}_i)'\boldsymbol{\Sigma}_i^{-1}(\mathbf{x}_i - \boldsymbol{\mu}_i) \\
= & \left[(\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}})'(\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}})'\right] \begin{bmatrix} \frac{1}{\sigma_1^2}\mathbf{A}_i & \frac{1}{\sigma_1\sigma_2}\mathbf{B}_i \\ \frac{1}{\sigma_1\sigma_2}\mathbf{B}_i' & \frac{1}{\sigma_2^2}\mathbf{C}_i \end{bmatrix} \begin{bmatrix} (\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}}) \\ (\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}}) \end{bmatrix} \\
= & (\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}})'\frac{1}{\sigma_1^2}\mathbf{A}_i(\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}}) + (\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}})'\frac{1}{\sigma_1\sigma_2}\mathbf{B}_i'(\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}}) \\
+ & (\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}})'\frac{1}{\sigma_1\sigma_2}\mathbf{B}_i(\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}}) + (\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}})'\frac{1}{\sigma_2^2}\mathbf{C}_i(\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}}).
\end{aligned}$$

If $m_{1i} = 0$ and $m_{2i} > 1$, then

$$\begin{aligned}
log(L_i(\boldsymbol{\theta})) &= -\frac{m_{2i}}{2}log(2\pi\sigma_2^2) - \frac{1}{2}(m_{2i}-1)log(1-\rho_2) \\
&- \frac{1}{2}log(1+(m_{2i}-1)\rho_2) - \frac{1}{2}(\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}})'\boldsymbol{\Sigma}_i^{-1}(\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}}).
\end{aligned}$$

And, if $m_{1i} = 0$ and $m_{2i} = 1$, then

$$log(L_i(\boldsymbol{\theta})) = -\tfrac{1}{2}log(2\pi\sigma_2^2) - \tfrac{1}{2\sigma_2^2}(x_{2i} - \mu_2)^2.$$

Similarly, if $m_{1i} > 1$ and $m_{2i} = 0$, then

$$\begin{aligned}
log(L_i(\boldsymbol{\theta})) &= -\frac{m_{1i}}{2}log(2\pi\sigma_1^2) - \frac{1}{2}(m_{1i}-1)log(1-\rho_1) \\
&- \frac{1}{2}log(1+(m_{1i}-1)\rho_1) - \frac{1}{2}(\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}})'\boldsymbol{\Sigma}_i^{-1}(x_{1i} - \mu_1\mathbf{1}_{m_{1i}}).
\end{aligned}$$

And, if $m_{1i} = 1$ and $m_{2i} = 0$, then

$$log(L_i(\boldsymbol{\theta})) = -\tfrac{1}{2}log(2\pi\sigma_1^2) - \tfrac{1}{2\sigma_1^2}(x_{1i} - \mu_1)^2.$$

The likelihood function $L(\boldsymbol{\theta})$ or $log(L(\boldsymbol{\theta}))$ can be maximized to obtain $\hat{\boldsymbol{\theta}}$, the maximum likelihood estimator (MLE) of $\boldsymbol{\theta}$. Let $\hat{\boldsymbol{\theta}} = (\hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}_1^2, \hat{\sigma}_2^2, \hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_{12})'$. From the theory of maximum likelihood estimation, the asymptotic distribution of $\hat{\boldsymbol{\theta}}$ is multivariate normal with mean vector $\boldsymbol{\theta}$ and variance matrix $V_{\hat{\boldsymbol{\theta}}} = \mathcal{I}(\hat{\boldsymbol{\theta}})^{-1}$, where $\mathcal{I}(\hat{\boldsymbol{\theta}})$ is the Fisher information matrix, details of which will be given a little later below.

Suppose we are interested in testing the hypothesis that the two intraclass correlation coefficients are equal, that is, $H_0 : \rho_1 = \rho_2 = \rho\ (say)$. Under H_0, $\boldsymbol{\theta} = (\mu_1, \mu_2, \sigma_1, \sigma_2, \rho, \rho, \rho_{12})'$. The likelihood function, $L(\boldsymbol{\theta})$ or $log(L(\boldsymbol{\theta}))$ can also be maximized under the null hypothesis $H_0 : \rho_1 = \rho_2$ to obtain $\hat{\boldsymbol{\theta}}_0$.

The maximization procedure used will need to be provided with initial estimates of the parameters. The initial values could be selected from the alternative estimates given in Sections II.9 and II.10.

II.3 LIKELIHOOD RATIO TEST

The likelihood ratio test (LRT) for testing H_0 is to reject H_0 for large values of

$$LRT = 2logL(\hat{\boldsymbol{\theta}}) - 2logL(\hat{\boldsymbol{\theta}}_0) \tag{2}$$

This test statistic has a χ^2 asymptotic distribution with 1 degree of freedom.

The other two asymptotic tests for testing H_0 are Wald's test and Rao's score test.

II.4 MODIFIED WALD'S TEST

The null hypothesis $H_0 : \rho_1 = \rho_2$ can be written as $H_0 : \mathbf{C}\boldsymbol{\theta} = 0$, where $\mathbf{C} = (0,\ 0,\ 0,\ 0,\ 1,\ -1,\ 0)$. The Wald's test then rejects H_0 for large values of

$$Wald = (\mathbf{C}\hat{\boldsymbol{\theta}})'[\mathbf{C}\mathbf{V}_{\hat{\boldsymbol{\theta}}_0}\mathbf{C}']^{-1}(\mathbf{C}\hat{\boldsymbol{\theta}}). \tag{3}$$

This test statistic has a χ^2 asymptotic distribution with 1 degree of freedom.

The standard Wald's test uses $V_{\hat{\boldsymbol{\theta}}}$ instead of $V_{\hat{\boldsymbol{\theta}}_0}$, but during analysis of this problem the slightly modified Wald's test performed uniformly better than the usual Wald's test.

II.5 RAO'S SCORE TEST

The score test rejects H_0 for large values of

$$Score = \mathbf{S}(\hat{\boldsymbol{\theta}}_0)'\mathcal{I}(\hat{\boldsymbol{\theta}}_0)^{-1}\mathbf{S}(\hat{\boldsymbol{\theta}}_0), \tag{4}$$

where $\mathbf{S}(\boldsymbol{\theta})$ is the 7 x 1 score function vector,

$$\mathbf{S}(\boldsymbol{\theta}) = \sum_{i=1}^{n} \mathbf{S}_i(\boldsymbol{\theta}) = \sum_{i=1}^{n} \frac{\delta log L_i(\boldsymbol{\theta})}{\delta \boldsymbol{\theta}},$$

where $\mathbf{S}_i(\boldsymbol{\theta}) = \frac{\delta log L_i(\boldsymbol{\theta})}{\delta \boldsymbol{\theta}}$ and $\mathcal{I}(\boldsymbol{\theta})$ is the 7 x 7 Fisher information matrix,

$$\mathcal{I}(\boldsymbol{\theta}) = \sum_{i=1}^{n} E\left[\left(\frac{\delta log L_i(\boldsymbol{\theta})}{\delta \boldsymbol{\theta}}\right)\left(\frac{\delta log L_i(\boldsymbol{\theta})}{\delta \boldsymbol{\theta}}\right)'\right].$$

This test statistic also has a χ^2 asymptotic distribution with 1 degree of freedom.

In the next section, we provide the elements of the score function vector $\mathbf{S}(\boldsymbol{\theta})$ and those of Fisher information matrix $\mathcal{I}(\boldsymbol{\theta})$.

II.6 SCORE FUNCTION

The score function is

$$\mathbf{S}(\boldsymbol{\theta}) = \sum_{i=1}^{n} \mathbf{S}_i(\boldsymbol{\theta}) = \sum_{i=1}^{n} \frac{\delta log L_i(\boldsymbol{\theta})}{\delta \boldsymbol{\theta}},$$

where $\mathbf{S}_i(\boldsymbol{\theta}) = \frac{\delta log L_i(\boldsymbol{\theta})}{\delta \boldsymbol{\theta}}$.

Here we provide the elements of $\mathbf{S}_i(\boldsymbol{\theta})$ denoted $\mathbf{S}_i[k], 1 \le k \le 7$. If $m_{1i} > 1$ and $m_{2i} > 1$, then let

$$\begin{aligned}
a_i &= m_{2i}\rho_{12}^2 - \rho_1(1 + (m_{2i} - 1)\rho_2), \\
b_i &= m_{1i}\rho_{12}^2 - \rho_2(1 + (m_{1i} - 1)\rho_1), \\
c_i &= (1 + (m_{1i} - 1)\rho_1)(1 + (m_{2i} - 1)\rho - 2) - m_{1i}m_{2i}\rho_{12}^2, \\
d_i &= (m_{1i} - 1)(1 + (m_{2i} - 1)\rho_2), \\
e_i &= (m_{2i} - 1)(1 + (m_{1i} - 1)\rho_1),
\end{aligned}$$

and

$$
\begin{aligned}
\mathbf{S}_i[1] &= \frac{1}{\sigma_1^2(1-\rho_1)}\mathbf{1}'_{m_{1i}}\left[\mathbf{I}_{m_{1i}} + \frac{a_i}{c_i}\mathbf{J}_{m_{1i}}\right](\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}}) \\
&- \frac{\rho_{12}}{2\sigma_1\sigma_2 c_i}(\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}})'\mathbf{J}_{m_{2i},m_{1i}}\mathbf{1}_{m_{1i}} - \frac{\rho_{12}}{2\sigma_1\sigma_2 c_i}\mathbf{1}'_{m_{1i}}\mathbf{J}_{m_{1i},m_{2i}}(\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}}), \\
\mathbf{S}_i[2] &= \frac{1}{\sigma_2^2(1-\rho_2)}\mathbf{1}'_{m_{2i}}\left[\mathbf{I}_{m_{2i}} + \frac{b_i}{c_i}\mathbf{J}_{m_{2i}}\right](\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}}) \\
&- \frac{\rho_{12}}{2\sigma_1\sigma_2 c_i}\mathbf{1}'_{m_{2i}}\mathbf{J}_{m_{2i},m_{1i}}(\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}}) - \frac{\rho_{12}}{2\sigma_1\sigma_2 c_i}(\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{2i}})'\mathbf{J}_{m_{1i},m_{2i}}\mathbf{1}'_{m_{2i}}, \\
\mathbf{S}_i[3] &= -\frac{m_{1i}}{2\sigma_1^2} + \frac{1}{2\sigma_1^4(1-\rho_1)}(\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}})'\left[\mathbf{I}_{m_{1i}} + \frac{a_i}{c_i}\mathbf{J}_{m_{1i}}\right](\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}}) \\
&- \frac{\rho_{12}}{4\sigma_1^3\sigma_2 c_i}(\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}})'\mathbf{J}_{m_{2i},m_{1i}}(\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}}) \\
&- \frac{\rho_{12}}{4\sigma_1^3\sigma_2 c_i}(\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}})'\mathbf{J}_{m_{1i},m_{2i}}(\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}}), \\
\mathbf{S}_i[4] &= -\frac{m_{2i}}{2\sigma_2^2} + \frac{1}{2\sigma_2^4(1-\rho_2)}(\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}})'\left[\mathbf{I}_{m_{2i}} + \frac{b_i}{c_i}\mathbf{J}_{m_{2i}}\right](\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}}) \\
&- \frac{\rho_{12}}{4\sigma_1\sigma_2^3 c_i}(\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}})'\mathbf{J}_{m_{2i},m_{1i}}(\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}}) \\
&- \frac{\rho_{12}}{4\sigma_1\sigma_2^3 c_i}(\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}})'\mathbf{J}_{m_{1i},m_{2i}}(\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}}), \\
\mathbf{S}_i[5] &= \frac{m_{1i}-1}{2(1-\rho_1)} - \frac{d_i}{2c_i} \\
&- \frac{1}{2\sigma_1^2(1-\rho_1)^2}(\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}})'\left[\mathbf{I}_{m_{1i}} + \frac{a_i c_i - d_i(1-\rho_1)\left(\frac{c_i}{m_{1i}-1} + a_i\right)}{c_i^2}\mathbf{J}_{m_{1i}}\right] \\
&\times (\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}}) \\
&- \frac{\rho_{12} d_i}{2\sigma_1\sigma_2 c_i^2}(\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}})'\mathbf{J}_{m_{2i},m_{1i}}(\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}}) \\
&- \frac{\rho_{12} d_i}{2\sigma_1\sigma_2 c_i^2}(\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}})'\mathbf{J}_{m_{1i},m_{2i}}(\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}}) \\
&+ \frac{(m_{1i}-1)\rho_2 c_i + b_i d_i}{2\sigma_2^2(1-\rho_2)c_i^2}(\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}})'\mathbf{J}_{m_{2i}}(\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}}),
\end{aligned}
$$

$$\begin{aligned}
\mathbf{S}_i[6] &= \frac{m_{2i}-1}{2(1-\rho_2)} - \frac{e_i}{2c_i} + \frac{(m_{2i}-1)\rho_1 c_i + a_i e_i}{2\sigma_1^2(1-\rho_1)c_i^2}(\mathbf{x}_{1i}-\mu_1\mathbf{1}_{m_{1i}})'\mathbf{J}_{m_{1i}}(\mathbf{x}_{1i}-\mu_1\mathbf{1}_{m_{1i}}) \\
&- \frac{\rho_{12}e_i}{2\sigma_1\sigma_2 c_i^2}(\mathbf{x}_{2i}-\mu_2\mathbf{1}_{m_{2i}})'\mathbf{J}_{m_{2i},m_{1i}}(\mathbf{x}_{1i}-\mu_1\mathbf{1}_{m_{1i}}) \\
&- \frac{\rho_{12}e_i}{2\sigma_1\sigma_2 c_i^2}(\mathbf{x}_{1i}-\mu_1\mathbf{1}_{m_{1i}})'\mathbf{J}_{m_{1i},m_{2i}}(\mathbf{x}_{2i}-\mu_2\mathbf{1}_{m_{2i}}) \\
&- \frac{1}{2\sigma_2^2(1-\rho_2)^2}(\mathbf{x}_{2i}-\mu_2\mathbf{1}_{m_{2i}})'\left[\mathbf{I}_{m_{2i}} + \frac{b_i c_i - e_i(1-\rho_2)\left(\frac{c_i}{m_{2i}-1}-b_i\right)}{c^2}\mathbf{J}_{m_{2i}}\right] \\
&\times (\mathbf{x}_{2i}-\mu_2\mathbf{1}_{m_{2i}}), \\
\mathbf{S}_i[7] &= \frac{m_{1i}m_{2i}\rho_{12}}{c_i} \\
&- \frac{1}{2\sigma_1^2(1-\rho_1)}\left(\frac{2m_{2i}\rho_{12}}{c_i} + \frac{2m_{1i}m_{2i}\rho_{12}a_i}{c_i^2}\right)(\mathbf{x}_{1i}-\mu_1\mathbf{1}_{m_{1i}})'\mathbf{J}_{m_{1i}}(\mathbf{x}_{1i}-\mu_1\mathbf{1}_{m_{1i}}) \\
&+ \frac{1}{2\sigma_1\sigma_2}\left(\frac{1}{c_i} + \frac{2m_{1i}m_{2i}\rho_{12}^2}{c_i^2}\right)(\mathbf{x}_{2i}-\mu_2\mathbf{1}_{m_{2i}})'\mathbf{J}_{m_{2i},m_{1i}}(\mathbf{x}_{1i}-\mu_1\mathbf{1}_{m_{1i}}) \\
&+ \frac{1}{2\sigma_1\sigma_2}\left(\frac{1}{c_i} + \frac{2m_{1i}m_{2i}\rho_{12}^2}{c_i^2}\right)(\mathbf{x}_{1i}-\mu_1\mathbf{1}_{m_{1i}})'\mathbf{J}_{m_{1i},m_{2i}}(\mathbf{x}_{2i}-\mu_2\mathbf{1}_{m_{2i}}) \\
&- \frac{1}{2\sigma_2^2(1-\rho_2)}\left(\frac{2m_{1i}\rho_{12}}{c_i} + \frac{2m_{1i}m_{2i}\rho_{12}b_i}{c_i^2}\right)(\mathbf{x}_{2i}-\mu_2\mathbf{1}_{m_{2i}})'\mathbf{J}_{m_{2i}}(\mathbf{x}_{2i}-\mu_2\mathbf{1}_{m_{2i}}).
\end{aligned}$$

If $m_{1i}=0$ and $m_{2i}=1$, then

$$\begin{aligned}
\mathbf{S}_i[2] &= \frac{1}{\sigma_2^2}(x_{2i}-\mu_2), \\
\mathbf{S}_i[4] &= \frac{1}{2\sigma_2^4}(x_{2i}-\mu_2)^2 - \frac{1}{2\sigma_2^2}.
\end{aligned}$$

Other entries of $\mathbf{S}_i(\boldsymbol{\theta})$ are zero for $m_{1i}=0$ and $m_{2i}=1$.

If $m_{1i}=1$ and $m_{2i}=0$, then

$$\begin{aligned}
\mathbf{S}_i[1] &= \frac{1}{\sigma_1^2}(x_{1i}-\mu_1), \\
\mathbf{S}_i[3] &= \frac{1}{2\sigma_1^4}(x_{1i}-\mu_1)^2 - \frac{1}{2\sigma_1^2}.
\end{aligned}$$

Other entries of $\mathbf{S}_i(\boldsymbol{\theta})$ are zero for $m_{1i}=1$ and $m_{2i}=0$.

If $m_{1i} = 0$ and $m_{2i} > 1$, then

$$
\begin{aligned}
\mathbf{S}_i[2] &= \frac{1}{\sigma_2^2(1-\rho_2)}\mathbf{1}'_{m_{2i}}\left[\mathbf{I}_{m_{2i}} - \frac{\rho_2}{1+(m_{2i}-1)\rho_2}\mathbf{J}_{m_{2i}}\right](\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}}),\\
\mathbf{S}_i[4] &= \frac{1}{2\sigma_2^4(1-\rho_2)}(\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}})'\left[\mathbf{I}_{m_{2i}} - \frac{\rho_2}{1+(m_{2i}-1)\rho_2}\mathbf{J}_{m_{2i}}\right](\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}})\\
&\quad - \frac{m_{2i}}{2\sigma_2^2},\\
\mathbf{S}_i[6] &= \frac{m_{2i}-1}{2(1-\rho_2)} - \frac{m_{2i}-1}{2(1+(m_{2i}-1)\rho_2)}\\
&\quad - \frac{1}{2\sigma_2^2(1-\rho_2)^2}(\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}})'\,[\mathbf{I}_{m_{2i}}\\
&\quad - \left.\frac{(1-\rho_2)(1+(m_{2i}-1)\rho_2) - \rho_2[(m_{2i}-1)(1-2\rho_2)-1]}{(1+(m_{2i}-1)\rho_2)^2}\mathbf{J}_{m_{2i}}\right](\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}}).
\end{aligned}
$$

Other entries of $\mathbf{S}_i(\boldsymbol{\theta})$ are zero for $m_{1i} = 0$ and $m_{2i} > 1$.

If $m_{1i} > 1$ and $m_{2i} = 0$, then

$$
\begin{aligned}
\mathbf{S}_i[1] &= \frac{1}{\sigma_1^2(1-\rho_1)}\mathbf{1}'_{m_{1i}}\left[\mathbf{I}_{m_{1i}} - \frac{\rho_1}{1+(m_{1i}-1)\rho_1}\mathbf{J}_{m_{1i}}\right](\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}}),\\
\mathbf{S}_i[3] &= \frac{1}{2\sigma_1^4(1-\rho_1)}(\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}})'\left[\mathbf{I}_{m_{1i}} - \frac{\rho_1}{1+(m_{1i}-1)\rho_1}\mathbf{J}_{m_{1i}}\right](\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}})\\
&\quad - \frac{m_{1i}}{2\sigma_1^2},\\
\mathbf{S}_i[5] &= \frac{m_{1i}-1}{2(1-\rho_1)} - \frac{m_{1i}-1}{2(1+(m_{1i}-1)\rho_1)}\\
&\quad - \frac{1}{2\sigma_1^2(1-\rho_1)^2}(\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}})'\,[\mathbf{I}_{m_{1i}}\\
&\quad - \left.\frac{(1-\rho_1)(1+(m_{1i}-1)\rho_1) - \rho_1[(m_{1i}-1)(1-2\rho_1)-1]}{(1+(m_{1i}-1)\rho_1)^2}\mathbf{J}_{m_{1i}}\right](\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}}).
\end{aligned}
$$

Other entries of $\mathbf{S}_i(\boldsymbol{\theta})$ are zero for $m_{1i} > 1$ and $m_{2i} = 0$.

If $m_{1i} = 1$ and $m_{2i} = 1$, then

$$
\begin{aligned}
\mathbf{S}_i[1] &= \frac{1}{\sigma_1^2(1-\rho_{12}^2)}(x_{1i} - \mu_1) - \frac{\rho_{12}}{\sigma_1\sigma_2(1-\rho_{12}^2)}(x_{2i} - \mu_2),\\
\mathbf{S}_i[2] &= \frac{1}{\sigma_2^2(1-\rho_{12}^2)}(x_{2i} - \mu_2) - \frac{\rho_{12}}{\sigma_1\sigma_2(1-\rho_{12}^2)}(x_{1i} - \mu_1),\\
\mathbf{S}_i[3] &= \frac{1}{2\sigma_1^4(1-\rho_{12}^2)}(x_{1i} - \mu_1)^2 - \frac{1}{2\sigma_1^2} - \frac{\rho_{12}}{2\sigma_1^3\sigma_2(1-\rho_{12}^2)}(x_{1i} - \mu_1)(x_{2i} - \mu_2),
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{S}_i[4] &= \frac{1}{2\sigma_2^4(1-\rho_{12}^2)}(x_{2i}-\mu_2)^2 - \frac{1}{2\sigma_2^2} - \frac{\rho_{12}}{2\sigma_1\sigma_2^3(1-\rho_{12}^2)}(x_{1i}-\mu_1)(x_{2i}-\mu_2),\\
\mathbf{S}_i[7] &= \frac{\rho_{12}}{1-\rho_{12}^2} - \frac{\rho_{12}}{\sigma_1^2(1-\rho_{12}^2)^2}(x_{1i}-\mu_1)^2 - \frac{\rho_{12}}{\sigma_2^2(1-\rho_{12}^2)^2}(x_{2i}-\mu_2^2)\\
&+ \frac{1+\rho_{12}^2}{\sigma_1\sigma_2(1-\rho_{12}^2)^2}(x_{1i}-\mu_1)(x_{2i}-\mu_2).
\end{aligned}
$$

Other entries of $\mathbf{S}_i(\boldsymbol{\theta})$ are zero for $m_{1i}=1$ and $m_{2i}=1$.

If $m_{1i}=1$ and $m_{2i}>1$, then

$$
c_i = 1 + (m_{2i}-1)\rho_2 - m_{2i}\rho_{12}^2,
$$

and

$$
\begin{aligned}
\mathbf{S}_i[1] &= \frac{1+(m_{2i}-1)\rho_2}{\sigma_1^2 c_i}(x_{1i}-\mu_1)\\
&- \frac{\rho_{12}}{2\sigma_1\sigma_2 c_i}\mathbf{1}'_{m_{2i}}(\mathbf{x}_{2i}-\mu_2\mathbf{1}_{m_{2i}})\\
&- \frac{\rho_{12}}{2\sigma_1\sigma_2 c_i}(\mathbf{x}_{2i}-\mu_2\mathbf{1}_{m_{2i}})'\mathbf{1}_{m_{2i}},\\
\mathbf{S}_i[2] &= \frac{1}{\sigma_2^2(1-\rho_2)}\mathbf{1}'_{m_{2i}}\left[\mathbf{I}_{m_{2i}} - \frac{\rho_2-\rho_{12}^2}{c_i}\mathbf{J}_{m_{2i}}\right](\mathbf{x}_{2i}-\mu_2\mathbf{1}_{m_{2i}})\\
&- \frac{m_{2i}\rho_{12}}{\sigma_1\sigma_2 c_i}(x_{1i}-\mu_1),\\
\mathbf{S}_i[3] &= -\frac{1}{2\sigma_1^2} + \frac{1+(m_{2i}-1)\rho_2}{2\sigma_1^4 c_i}(x_{1i}-\mu_1)^2\\
&- \frac{1}{4\sigma_1^3\sigma_2 c_i}(x_{1i}-\mu_1)\mathbf{1}'_{m_{2i}}(\mathbf{x}_{2i}-\mu_2\mathbf{1}_{m_{2i}})\\
&- \frac{1}{4\sigma_1^3\sigma_2 c_i}(x_{1i}-\mu_1)(\mathbf{x}_{2i}-\mu_2\mathbf{1}_{m_{2i}})'\mathbf{1}_{m_{2i}},\\
\mathbf{S}_i[4] &= -\frac{m_{2i}}{2\sigma_2^2}\\
&+ \frac{1}{2\sigma_2^4(1-\rho_2)}(\mathbf{x}_{2i}-\mu_2\mathbf{1}_{m_{2i}})'\left[\mathbf{I}_{m_{2i}} - \frac{\rho_2-\rho_{12}^2}{c_i}\mathbf{J}_{m_{2i}}\right](\mathbf{x}_{2i}-\mu_2\mathbf{1}_{m_{2i}})\\
&- \frac{\rho_{12}}{4\sigma_1\sigma_2^3 c_i}(x_{1i}-\mu_1)(\mathbf{x}_{2i}-\mu_2\mathbf{1}_{m_{2i}})'\mathbf{1}_{m_{2i}}\\
&- \frac{\rho_{12}}{4\sigma_1\sigma_2^3 c_i}(x_{1i}-\mu_1)\mathbf{1}'_{m_{2i}}(\mathbf{x}_{2i}-\mu_2\mathbf{1}_{m_{2i}}),
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{S}_i[6] &= \frac{m_{2i}-1}{2(1-\rho_2)} - \frac{m_{2i}-1}{2c_i} \\
&+ \frac{m_{2i}(m_{2i}-1)\rho_{12}^2}{2\sigma_1^2 c_i^2}(x_{1i}-\mu_1)^2 \\
&- \frac{(m_{2i}-1)\rho_{12}}{2\sigma_1\sigma_2 c_i^2}(x_{1i}-\mu_1)\mathbf{1}'_{m_{2i}}(\mathbf{x}_{2i}-\mu_2\mathbf{1}_{m_{2i}}) \\
&- \frac{(m_{2i}-1)\rho_{12}}{2\sigma_1\sigma_2 c_i^2}(x_{1i}-\mu_1)(\mathbf{x}_{2i}-\mu_2\mathbf{1}_{m_{2i}})'\mathbf{1}_{m_{2i}} \\
&- \frac{1}{2\sigma_2^2(1-\rho_2)^2}(\mathbf{x}_{2i}-\mu_2\mathbf{1}_{m_{2i}})'\left[\mathbf{I}_{m_{2i}}\right. \\
&- \left.\frac{(1-\rho_{12}^2)c_i-(m_{2i}-1)(1-\rho_2)(\rho_2-\rho_{12}^2)}{c_i^2}\mathbf{J}_{m_{2i}}\right](\mathbf{x}_{2i}-\mu_2\mathbf{1}_{m_{2i}}), \\
\mathbf{S}_i[7] &= \frac{m_{2i}\rho_{12}}{c_i} - \frac{m_{2i}\rho_{12}(1+(m_{2i}-1)\rho_2)}{\sigma_1^2 c_i^2}(x_{1i}-\mu_1)^2 \\
&+ \frac{1+(m_{2i}-1)\rho_2+m_{2i}\rho_{12}^2}{2\sigma_1\sigma_2 c_i}(x_{1i}-\mu_1)(\mathbf{x}_{2i}-\mu_2\mathbf{1}_{m_{2i}})'\mathbf{1}_{m_{1i}} \\
&+ \frac{1+(m_{2i}-1)\rho_2+m_{2i}\rho_{12}^2}{2\sigma_1\sigma_2 c_i}(x_{1i}-\mu_1)\mathbf{1}'_{m_{2i}}(\mathbf{x}_{2i}-\mu_2\mathbf{1}_{m_{2i}}) \\
&+ \frac{m_{2i}\rho_{12}(\rho_2-\rho 12^2)-\rho_{12}c_i}{\sigma_2^2(1-\rho_2)c_i^2}(\mathbf{x}_{2i}-\mu_2\mathbf{1}_{m_{2i}})'\mathbf{J}_{m_{2i}}(\mathbf{x}_{2i}-\mu_2\mathbf{1}_{m_{2i}}).
\end{aligned}
$$

Other entries of $\mathbf{S}_i(\boldsymbol{\theta})$ are zero for $m_{1i}=1$ and $m_{2i}>1$.

If $m_{1i}>1$ and $m_{2i}=1$, then

$$
c_i = 1+(m_{1i}-1)\rho_1 - m_{1i}\rho_{12}^2,
$$

and

$$
\begin{aligned}
\mathbf{S}_i[1] &= \frac{1}{\sigma_1^2(1-\rho_1)}\mathbf{1}'_{m_{1i}}\left[\mathbf{I}_{m_{1i}} - \frac{\rho_1-\rho_{12}^2}{c_i}\mathbf{J}_{m_{1i}}\right](\mathbf{x}_{1i}-\mu_1\mathbf{1}_{m_{1i}}) \\
&- \frac{m_{1i}\rho_{12}}{\sigma_1\sigma_2 c_i}(x_{2i}-\mu_2), \\
\mathbf{S}_i[2] &= \frac{1+(m_{1i}-1)\rho_1}{\sigma_2^2 c_i}(x_{2i}-\mu_2) \\
&- \frac{\rho_{12}}{2\sigma_1\sigma_2 c_i}\mathbf{1}'_{m_{1i}}(\mathbf{x}_{1i}-\mu_1\mathbf{1}_{m_{1i}}) \\
&- \frac{\rho_{12}}{2\sigma_1\sigma_2 c_i}(\mathbf{x}_{1i}-\mu_2\mathbf{1}_{m_{1i}})'\mathbf{1}_{m_{1i}},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{S}_i[3] = & -\frac{m_{1i}}{2\sigma_1^2} \\
& + \frac{1}{2\sigma_1^4(1-\rho_1)}(\mathbf{x}_{1i} - \mu_1 \mathbf{1}_{m_{1i}})' \left[\mathbf{I}_{m_{1i}} - \frac{\rho_1 - \rho_{12}^2}{c_i} \mathbf{J}_{m_{1i}} \right] (\mathbf{x}_{1i} - \mu_1 \mathbf{1}_{m_{1i}}) \\
& - \frac{\rho_{12}}{4\sigma_1^3 \sigma_2 c_i}(x_{2i} - \mu_2)(\mathbf{x}_{1i} - \mu_1 \mathbf{1}_{m_{1i}})' \mathbf{1}_{m_{1i}} \\
& - \frac{\rho_{12}}{4\sigma_1^3 \sigma_2 c_i}(x_{2i} - \mu_2) \mathbf{1}'_{m_{1i}} (\mathbf{x}_{1i} - \mu_1 \mathbf{1}_{m_{1i}}), \\
\mathbf{S}_i[4] = & -\frac{1}{2\sigma_2^2} + \frac{1 + (m_{1i} - 1)\rho_2}{2\sigma_2^4 c_i}(x_{2i} - \mu_2)^2 \\
& - \frac{1}{4\sigma_1 \sigma_2^3 c_i}(x_{2i} - \mu_2) \mathbf{1}'_{m_{1i}} (\mathbf{x}_{1i} - \mu_1 \mathbf{1}_{m_{1i}}) \\
& - \frac{1}{4\sigma_1 \sigma_2^3 c_i}(x_{2i} - \mu_2)(\mathbf{x}_{1i} - \mu_1 \mathbf{1}_{m_{1i}})' \mathbf{1}_{m_{1i}}, \\
\mathbf{S}_i[5] = & \frac{m_{1i} - 1}{2(1-\rho_1)} - \frac{m_{1i} - 1}{2c_i} \\
& + \frac{m_{1i}(m_{1i} - 1)\rho_{12}^2}{2\sigma_2^2 c_i^2}(x_{2i} - \mu_2)^2 \\
& - \frac{(m_{1i} - 1)\rho_{12}}{2\sigma_1 \sigma_2 c_i^2}(x_{2i} - \mu_2) \mathbf{1}'_{m_{1i}} (\mathbf{x}_{1i} - \mu_1 \mathbf{1}_{m_{1i}}) \\
& - \frac{(m_{1i} - 1)\rho_{12}}{2\sigma_1 \sigma_2 c_i^2}(x_{2i} - \mu_2)(\mathbf{x}_{1i} - \mu_1 \mathbf{1}_{m_{1i}})' \mathbf{1}_{m_{1i}} \\
& - \frac{1}{2\sigma_1^2(1-\rho_1)^2}(\mathbf{x}_{1i} - \mu_1 \mathbf{1}_{m_{1i}})' \left[\mathbf{I}_{m_{1i}} \right. \\
& \left. - \frac{(1-\rho_{12}^2)c_i - (m_{1i} - 1)(1-\rho_1)(\rho_1 - \rho_{12}^2)}{c_i^2} \mathbf{J}_{m_{1i}} \right] (\mathbf{x}_{1i} - \mu_1 \mathbf{1}_{m_{1i}}), \\
\mathbf{S}_i[7] = & \frac{m_{1i}\rho_{12}}{c_i} - \frac{m_{1i}\rho_{12}(1 + (m_{1i} - 1)\rho_1)}{\sigma_2^2 c_i^2}(x_{2i} - \mu_2)^2 \\
& + \frac{1 + (m_{1i} - 1)\rho_1 + m_{1i}\rho_{12}^2}{2\sigma_1 \sigma_2 c_i}(x_{2i} - \mu_2)(\mathbf{x}_{1i} - \mu_1 \mathbf{1}_{m_{1i}})' \mathbf{1}_{m_{2i}} \\
& + \frac{1 + (m_{1i} - 1)\rho_1 + m_{1i}\rho_{12}^2}{2\sigma_1 \sigma_2 c_i}(x_{2i} - \mu_2) \mathbf{1}'_{m_{1i}} (\mathbf{x}_{1i} - \mu_1 \mathbf{1}_{m_{1i}}) \\
& + \frac{m_{1i}\rho_{12}(\rho_1 - \rho 12^2) - \rho_{12} c_i}{\sigma_1^2(1-\rho_1)c_i^2}(\mathbf{x}_{1i} - \mu_1 \mathbf{1}_{m_{1i}})' \mathbf{J}_{m_{1i}} (\mathbf{x}_{1i} - \mu_1 \mathbf{1}_{m_{1i}}).
\end{aligned}
$$

Other entries of $\mathbf{S}_i(\boldsymbol{\theta})$ are zero for $m_{1i} > 1$ and $m_{2i} = 1$.

II.7 INFORMATION MATRIX

The information matrix is the 7×7 matrix

$$\mathcal{I}(\boldsymbol{\theta}) = \sum_{i=1}^{n} \mathcal{I}_i(\boldsymbol{\theta}) = \sum_{i=1}^{n} E\left[\left(\frac{\delta log L_i(\boldsymbol{\theta})}{\delta \boldsymbol{\theta}}\right)\left(\frac{\delta log L_i(\boldsymbol{\theta})}{\delta \boldsymbol{\theta}}\right)'\right].$$

If $m_{1i} > 1$ and $m_{2i} > 1$, then recall

$$\begin{aligned}
a_i &= m_{2i}\rho_{12}^2 - \rho_1(1 + (m_{2i} - 1)\rho_2), \\
b_i &= m_{1i}\rho_{12}^2 - \rho_2(1 + (m_{1i} - 1)\rho_1), \\
c_i &= (1 + (m_{1i} - 1)\rho_1)(1 + (m_{2i} - 1)\rho - 2) - m_{1i}m_{2i}\rho_{12}^2, \\
d_i &= (m_{1i} - 1)(1 + (m_{2i} - 1)\rho_2), \\
e_i &= (m_{2i} - 1)(1 + (m_{1i} - 1)\rho_1).
\end{aligned}$$

In this case, the entries of the information matrix are

$$\begin{aligned}
&\mathcal{I}_i[1,1] = \frac{1}{\sigma_1^2(1-\rho_1)}\left(m_{1i} + \frac{m_{1i}^2 a_i}{c_i}\right), \\
&\mathcal{I}_i[1,2] = \mathcal{I}_i[2,1] = \frac{-m_{1i}m_{2i}\rho_{12}}{\sigma_1\sigma_2 c_i}, \\
&\mathcal{I}_i[1,3] = \mathcal{I}_i[3,1] = 0, \\
&\mathcal{I}_i[1,4] = \mathcal{I}_i[4,1] = 0, \\
&\mathcal{I}_i[1,5] = \mathcal{I}_i[5,1] = 0, \\
&\mathcal{I}_i[1,6] = \mathcal{I}_i[6,1] = 0, \\
&\mathcal{I}_i[1,7] = \mathcal{I}_i[7,1] = 0, \\
&\mathcal{I}_i[2,2] = \frac{1}{\sigma_2^2(1-\rho_2)}\left(m_{2i} + \frac{m_{2i}^2 b_i}{c_i}\right), \\
&\mathcal{I}_i[2,3] = \mathcal{I}_i[3,2] = 0, \\
&\mathcal{I}_i[2,4] = \mathcal{I}_i[4,2] = 0, \\
&\mathcal{I}_i[2,5] = \mathcal{I}_i[5,2] = 0, \\
&\mathcal{I}_i[2,6] = \mathcal{I}_i[6,2] = 0, \\
&\mathcal{I}_i[2,7] = \mathcal{I}_i[7,2] = 0, \\
&\mathcal{I}_i[3,3] = \frac{m_{1i}}{\sigma_1^4(1-\rho_1)}\left(1 + \frac{a_i(1 + (m_{1i} - 1)\rho_1)}{c_i}\right) - \frac{m_{1i}}{2\sigma_1^4} - \frac{3\rho_{12}^2 m_{1i}m_{2i}}{4\sigma_1^4 c_i}, \\
&\mathcal{I}_i[3,4] = \mathcal{I}_i[4,3] = \frac{-m_{1i}m_{2i}\rho_{12}^2}{4\sigma_1^2\sigma_2^2 c_i},
\end{aligned}$$

$$\mathcal{I}_i[3,5] = \mathcal{I}_i[5,3] = -\frac{m_{1i}m_{2i}\rho_{12}^2 d_i}{2\sigma_1^2 c_i^2}$$
$$- \frac{m_{1i}}{2\sigma_1^2(1-\rho_1)^2}\left[1 + \frac{(1+(m_{1i}-1)\rho_1)\left[a_i c_i - d_i(1-\rho_1)(\frac{c_i}{m_{1i}-1}+a_i)\right]}{c_i^2}\right],$$

$$\mathcal{I}_i[3,6] = \mathcal{I}_i[6,3] = \frac{m_{1i}(1+(m_{1i}-1)\rho_1)(\rho_1(m_{2i}-1)c_i + a_i e_i)}{2\sigma_1^2(1-\rho_1)c_i^2} - \frac{\rho_{12}^2 m_{1i}m_{2i}e_i}{2\sigma_1^2 c_i^2},$$

$$\mathcal{I}_i[3,7] = \mathcal{I}_i[7,3] = \frac{m_{1i}m_{2i}\rho_{12}(c_i + 2m_{1i}m_{2i}\rho_{12}^2)}{2\sigma_1^2 c_i^2}$$
$$- \frac{m_{1i}m_{2i}\rho_{12}(1+(m_{1i}-1\rho_1))(c_i + m_{1i}a_i)}{\sigma_1^2(1-\rho_1)c_i^2},$$

$$\mathcal{I}_i[4,4] = \frac{m_{2i}}{\sigma_2^4(1-\rho_2)}\left(1 + \frac{b_i(1+(m_{2i}-1)\rho_2)}{c_i}\right) - \frac{m_{2i}}{2\sigma_2^4} - \frac{3\rho_{12}^2 m_{1i}m_{2i}}{4\sigma_2^4 c_i},$$

$$\mathcal{I}_i[4,5] = \mathcal{I}_i[5,4] = \frac{m_{2i}(1+(m_{2i}-1)\rho_2)((m_{1i}-1)\rho_2 c_i + b_i d_i)}{2\sigma_2^2(1-\rho_2)c_i^2} - \frac{m_{1i}m_{2i}\rho_{12}^2 d_i}{2\sigma_2^2 c_i^2},$$

$$\mathcal{I}_i[4,6] = \mathcal{I}_i[6,4] = -\frac{m_{1i}m_{2i}\rho_{12}^2 e_i}{2\sigma_2^2 c_i^2}$$
$$- \frac{m_{2i}}{2\sigma_2^2(1-\rho_2)^2}\left(1 + \frac{(1+(m_{2i}-1)\rho_2)\left[b_i c_i - e_i(1-\rho_2)\left(\frac{c_i}{m_{2i}-1}+b_i\right)\right]}{c_i^2}\right),$$

$$\mathcal{I}_i[4,7] = \mathcal{I}_i[7,4] = \frac{m_{1i}m_{2i}\rho_{12}(c_i + 2m_{1i}m_{2i}\rho_{12}^2)}{2\sigma_2^2 c_i^2}$$
$$- \frac{m_{1i}m_{2i}\rho_{12}(1+(m_{2i}-1)\rho_2)(c_i + m_{2i}b_i)}{\sigma_2^2(1-\rho_2)c_i^2},$$

$$\mathcal{I}_i[5,5] = \frac{m_{1i}}{(1-\rho_1)^3}\left[1 + \frac{(1+(m_{1i}-1)\rho_1)(1-\rho_1)^2 Star_1}{2c}\right] - \frac{m_{1i}-1}{2(1-\rho_1)^2} - \frac{d_i^2}{2c_i^2}$$
$$- \frac{2m_{1i}m_{2i}\rho_{12}^2 d_i^2}{c_i^3} - \frac{m_{2i}(1+(m_{2i}-1)\rho_2)}{(1-\rho_2)c_i^2}\left[(m_{1i}-1)\rho_2 d_i + \frac{b_i d_i^2}{c}\right],$$

where

$$Star_1 = \frac{2a_i}{(1-\rho_1)^2} - \frac{1+(m_{2i}-1)\rho_2}{1-\rho_1} - \frac{2a_i d_i}{(1-\rho_1)c_i}$$
$$+ \frac{2a_i d_i^2}{c_i^2} + \frac{d_i(1+(m_{2i}-1)\rho_1)}{c_i} + \frac{d_i^2(1-\rho_1) - c_i d_i}{(m_{1i}-1)(1-\rho_1)c_i},$$

$$
\begin{aligned}
\mathcal{I}_i[5,6] = \mathcal{I}_i[6,5] &= \frac{(m_{1i}-1)(m_{2i}-1)c_i - d_i e_i}{2c_i^2} \\
&+ \frac{m_{1i}m_{2i}\rho_{12}^2[(m_{1i}-1)(m_{2i}-1)c_i - 2d_i e_i]}{c_i^3} \\
&- \frac{m_{1i}(1+(m_{1i}-1)\rho_1)}{2(1-\rho_1)^2}\left[\frac{\rho_1(m_{2i}-1)c_i + a_i e_i}{c_i^2} + \frac{(1-\rho_1)(m_{2i}-1)}{c_i}\right. \\
&- \left.\frac{(1-\rho_1)d_i e_i}{(m_{1i}-1)c_i^2} + \frac{(1-\rho_1)[(m_{1i}-1)(m_{2i}-1)a_i c_i - 2a_i d_i e_i - \rho_1(m_{2i}-1)d_i c_i]}{c_i^3}\right] \\
&- \frac{m_{2i}(1+(m_{2i}-1)\rho_2)}{2(1-\rho_2)c_i}\left[(m_{1i}-1)\left(1+\frac{\rho_2}{1-\rho_2} - \frac{\rho_2 e_i}{c_i}\right)\right. \\
&- \left.\left(\frac{(1+(m_{1i}-1)\rho_1)d_i + (m_{1i}-1)(m_{2i}-1)b_i}{c_i} - \frac{2b_i d_i e_i}{c_i^2} + \frac{b_i d_i}{(1-\rho_2)c_i}\right)\right].
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}_i[5,7] = \mathcal{I}_i[7,5] &= \frac{m_{1i}m_{2i}\rho_{12}d_i}{c_i^2} + m_{1i}m_{2i}\rho_{12}d_i\left[\frac{1}{c_i^2} + \frac{4m_{1i}m_{2i}\rho_{12}^2}{c_i^3}\right] \\
&+ \frac{m_{1i}m_{2i}(1+(m_{1i}-1)\rho_1)\rho_{12}}{(1-\rho_1)^2 c_i^2}\left[c_i + m_{1i}a_i - \frac{m_{1i}(1-\rho_1)d_i}{m_{1i}-1}\right. \\
&- \left.\frac{(1-\rho_1)(c_i+2m_{1i}a_i)d_i}{c_i}\right] \\
&- \frac{m_{1i}m_{2i}(1+(m_{2i}-1)\rho_2)\rho_{12}}{(1-\rho_2)c_i^2}\left[d_i + m_{2i}(m_{1i}-1)\rho_2 + \frac{2m_{2i}b_i d_i}{c_i}\right],
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}_i[6,6] &= \frac{m_{1i}(1+(m_{1i}-1)\rho_1)}{(1-\rho_1)c_i^2}\left[(m_{2i}-1)\rho_1 e_i + \frac{a_i e_i^2}{c_i}\right] - \frac{m_{2i}-1}{2(1-\rho_2)^2} - \frac{e_i^2}{2c_i^2} \\
&- \frac{2m_{1i}m_{2i}\rho_{12}^2 e_i^2}{c_i^3} + \frac{m_{2i}}{(1-\rho_2)^3}\left[1 + \frac{(1+(m_{2i}-1)\rho_2)(1-\rho_2)^2 Star_2}{2c_i}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
Star_2 &= \frac{2b_i}{(1-\rho_2)^2} - \frac{1+(m_{1i}-1)\rho_1}{1-\rho_2} + \frac{e_i(1+(m_{1i}-1)\rho_1)}{c_i} \\
&- \frac{2b_i e_i}{(1-\rho_2)c_i} + \frac{2b_i e_i^2}{c_i^2} + \frac{e_i^2(1-\rho_2) - c_i e_i}{(m_{2i}-1)(1-\rho_2)c_i},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}_i[6,7] = \mathcal{I}_i[7,6] &= \frac{m_{1i}m_{2i}\rho_{12}e_i}{c_i^2} + m_{1i}m_{2i}\rho_{12}e_i\left[\frac{1}{c^2} + \frac{4m_{1i}m_{2i}\rho_{12}^2}{c^3}\right] \\
&- \frac{m_{1i}m_{2i}\rho_{12}(1+(m_{1i}-1)\rho_1)}{(1-\rho_1)c_i^2}\left[d_i + m_{1i}(m_{2i}-1)\rho_1 + \frac{2m_{1i}a_i e_i}{c_i}\right] \\
&- \frac{m_{1i}m_{2i}\rho_{12}(1+(m_{2i}-1)\rho_2)}{2(1-\rho_2)^2 c_i^2}\left[c + m_{2i}d_i - \frac{m_{2i}(1-\rho_2)e_i}{m_{2i}-1}\right. \\
&- \left.\frac{(1-\rho_2)(c_i+2m_{2i}b_i)e_i}{c_i}\right],
\end{aligned}
$$

$$\mathcal{I}_i[7,7] = \frac{m_{1i}m_{2i}}{c_i} + \frac{2m_{1i}^2 m_{2i}^2 \rho_{12}^2}{c_i^2}$$
$$- \frac{m_{1i}m_{2i}(1+(m_{1i}-1)\rho_1)}{1-\rho_1}\left[\frac{1}{c_i} + \frac{4m_{1i}m_{2i}\rho_{12}^2}{c_i^2} + \frac{m_{1i}a_i}{c_i^2} + \frac{4m_{1i}^2 m_{2i}\rho_{12}^2 a_i}{c_i^3}\right]$$
$$+ \frac{2m_{1i}^2 m_{2i}^2 \rho_{12}^2}{c_i^2}\left[3 + \frac{4m_{1i}m_{2i}\rho_{12}^2}{c_i}\right]$$
$$- \frac{m_{1i}m_{2i}(1+(m_{2i}-1)\rho_2)}{1-\rho_2}\left[\frac{1}{c_i} + \frac{4m_{1i}m_{2i}\rho_{12}^2}{c_i^2} + \frac{m_{2i}b_i}{c_i^2} - \frac{4m_{1i}m_{2i}^2\rho_{12}^2 b_i}{c_i^3}\right].$$

If $m_{1i} = 0$ and $m_{2i} = 1$, then

$$\mathcal{I}_i[2,2] = \frac{1}{\sigma_2^2},$$
$$\mathcal{I}_i[2,4] = \mathcal{I}_i[4,2] = 0,$$
$$\mathcal{I}_i[4,4] = \frac{1}{2\sigma_2^4}.$$

Other entries of $\mathcal{I}_i(\boldsymbol{\theta})$ are zero for $m_{1i} = 0$ and $m_{2i} = 1$.

If $m_{1i} = 1$ and $m_{2i} = 0$, then

$$\mathcal{I}_i[1,1] = \frac{1}{\sigma_1^2},$$
$$\mathcal{I}_i[1,3] = \mathcal{I}_i[3,1] = 0,$$
$$\mathcal{I}_i[3,3] = \frac{1}{2\sigma_1^4}.$$

Other entries of $\mathcal{I}_i(\boldsymbol{\theta})$ are zero for $m_{1i} = 1$ and $m_{2i} = 0$.

If $m_{1i} = 0$ and $m_{2i} > 1$, then

$$\mathcal{I}_i[2,2] = \frac{1}{\sigma_2^2(1-\rho_2)}\left[m_{2i} - \frac{m_{2i}^2\rho_2}{1+(m_{2i}-1)\rho_2}\right],$$
$$\mathcal{I}_i[2,4] = \mathcal{I}_i[4,2] = 0,$$
$$\mathcal{I}_i[2,6] = \mathcal{I}_i[6,2] = 0,$$
$$\mathcal{I}_i[4,4] = \frac{m_{2i}}{2\sigma_2^4},$$
$$\mathcal{I}_i[4,6] = \mathcal{I}_i[6,4] = \frac{-m_{2i}}{2\sigma_2^2(1-\rho_2)^2}\left[1 - \frac{1+\rho_2^2(m_{2i}-1)}{(1+(m_{2i}-1)\rho_2)^2}\right],$$
$$\mathcal{I}_i[6,6] = \frac{m_{2i}}{(1-\rho_2)^2}\left[\frac{(m_{2i}-1)(1+\rho_2^2(m_{2i}-1))}{(1+(m_{2i}-1)\rho_2)^2}\right] - \frac{m_{2i}-1}{2(1-\rho_2)^2}$$
$$- \frac{(m_{2i}-1)^2}{2(1+(m_{2i}-1)\rho_2)^2}.$$

Other entries of $\mathcal{I}_i(\boldsymbol{\theta})$ are zero for $m_{1i} = 0$ and $m_{2i} > 1$.

If $m_{1i} > 1$ and $m_{2i} = 0$, then

$$\begin{aligned}
\mathcal{I}_i[1,1] &= \frac{1}{\sigma_1^2(1-\rho_1)}\left[m_{1i} - \frac{m_{1i}^2\rho_1}{1+(m_{1i}-1)\rho_1}\right],\\
\mathcal{I}_i[1,3] &= \mathcal{I}_i[3,1] = 0,\\
\mathcal{I}_i[1,5] &= \mathcal{I}_i[5,1] = 0,\\
\mathcal{I}_i[3,3] &= \frac{m_{1i}}{2\sigma_1^4},\\
\mathcal{I}_i[3,5] &= \mathcal{I}_i[5,3] = \frac{-m_{1i}}{2\sigma_1^2(1-\rho_1)^2}\left[1 - \frac{1+\rho_1^2(m_{1i}-1)}{(1+(m_{1i}-1)\rho_1)^2}\right],\\
\mathcal{I}_i[5,5] &= \frac{m_{1i}}{(1-\rho_1)^2}\left[\frac{(m_{1i}-1)(1+\rho_1^2(m_{1i}-1))}{(1+(m_{1i}-1)\rho_1)^2}\right] - \frac{m_{1i}-1}{2(1-\rho_1)^2}\\
&\quad - \frac{(m_{1i}-1)^2}{2(1+(m_{1i}-1)\rho_1)^2}.
\end{aligned}$$

Other entries of $\mathcal{I}_i(\boldsymbol{\theta})$ are zero for $m_{1i} > 1$ and $m_{2i} = 0$.

If $m_{1i} = 1$ and $m_{2i} = 1$, then

$$\begin{aligned}
\mathcal{I}_i[1,1] &= \frac{1}{\sigma_1^2(1-\rho_{12}^2)},\\
\mathcal{I}_i[1,2] &= \mathcal{I}_i[2,1] = \frac{-\rho_{12}}{\sigma_1\sigma_2(1-\rho_{12}^2)},\\
\mathcal{I}_i[1,3] &= \mathcal{I}_i[3,1] = 0,\\
\mathcal{I}_i[1,4] &= \mathcal{I}_i[4,1] = 0,\\
\mathcal{I}_i[1,7] &= \mathcal{I}_i[7,1] = 0,\\
\mathcal{I}_i[2,2] &= \frac{1}{\sigma_2^2(1-\rho_{12}^2)},\\
\mathcal{I}_i[2,3] &= \mathcal{I}_i[3,2] = 0,\\
\mathcal{I}_i[2,4] &= \mathcal{I}_i[4,2] = 0,\\
\mathcal{I}_i[2,7] &= \mathcal{I}_i[7,2] = 0,\\
\mathcal{I}_i[3,3] &= \frac{1}{\sigma_1^4(1-\rho_{12}^2)} - \frac{1}{2\sigma_1^4} - \frac{3\rho_{12}^2}{4\sigma_1^4(1-\rho_{12}^2)},\\
\mathcal{I}_i[3,4] &= \mathcal{I}_i[4,3] = \frac{-\rho_{12}^2}{4\sigma_1^2\sigma_2^2(1-\rho_{12}^2)},\\
\mathcal{I}_i[3,7] &= \mathcal{I}_i[7,3] = \frac{\rho_{12}(1+\rho_{12}^2)}{2\sigma_1^2(1-\rho_{12}^2)^2} - \frac{\rho_{12}}{\sigma_1^2(1-\rho_{12}^2)^2},\\
\mathcal{I}_i[4,4] &= \frac{1}{\sigma_2^4(1-\rho_{12}^2)} - \frac{1}{2\sigma_2^4} - \frac{3\rho_{12}^2}{4\sigma_2^4(1-\rho_{12}^2)},
\end{aligned}$$

$$\mathcal{I}_i[4,7] = \mathcal{I}_i[7,4] = \frac{\rho_{12}(1+\rho_{12}^2)}{2\sigma_2^2(1-\rho_{12}^2)^2} - \frac{\rho_{12}}{\sigma_2^2(1-\rho_{12}^2)^2},$$
$$\mathcal{I}_i[7,7] = \frac{2(1+3\rho_{12}^2)}{(1-\rho_{12}^2)^3} - \frac{1+\rho_{12}^2}{(1-\rho_{12}^2)^2} - \frac{2\rho_{12}^2(3+\rho_{12}^2)}{(1-\rho_{12}^2)^3}.$$

Other entries of $\mathcal{I}_i(\boldsymbol{\theta})$ are zero for $m_{1i}=1$ and $m_{2i}=1$.

If $m_{1i}=1$ and $m_{2i}>1$, then

$$c_i = 1+(m_{2i}-1)\rho_2 - m_{2i}\rho_{12}^2,$$

and

$$\mathcal{I}_i[1,1] = \frac{1+(m_{2i}-1)\rho_2}{\sigma_1^2 c_i},$$
$$\mathcal{I}_i[1,2] = \mathcal{I}_i[2,1] = -\frac{m_{2i}\rho_{12}}{\sigma_1\sigma_2 c_i},$$
$$\mathcal{I}_i[1,3] = \mathcal{I}_i[3,1] = 0,$$
$$\mathcal{I}_i[1,4] = \mathcal{I}_i[4,1] = 0,$$
$$\mathcal{I}_i[1,6] = \mathcal{I}_i[6,1] = 0,$$
$$\mathcal{I}_i[1,7] = \mathcal{I}_i[7,1] = 0,$$
$$\mathcal{I}_i[2,2] = \frac{1}{\sigma_2^2(1-\rho_2)}\left[m_{2i} - \frac{m_{2i}^2(\rho_2-\rho_{12}^2)}{c_i}\right],$$
$$\mathcal{I}_i[2,3] = \mathcal{I}_i[3,2] = 0,$$
$$\mathcal{I}_i[2,4] = \mathcal{I}_i[4,2] = 0,$$
$$\mathcal{I}_i[2,6] = \mathcal{I}_i[6,2] = 0,$$
$$\mathcal{I}_i[2,7] = \mathcal{I}_i[7,2] = 0,$$
$$\mathcal{I}_i[3,3] = \frac{1+(m_{2i}-1)\rho_2}{\sigma_1^4 c_i} - \frac{1}{2\sigma_1^4} - \frac{3m_{2i}\rho_{12}^2}{4\sigma_1^4 c_i},$$
$$\mathcal{I}_i[3,4] = \mathcal{I}_i[4,3] = -\frac{m_{2i}\rho_{12}^2}{4\sigma_1^2\sigma_2^2(1+(m_{2i}-1)\rho_2 - m_{2i}\rho_{12}^2)},$$
$$\mathcal{I}_i[3,6] = \mathcal{I}_i[6,3] = 0,$$
$$\mathcal{I}_i[3,7] = \mathcal{I}_i[7,3] = \frac{m_{2i}\rho_{12}(1+(m_{2i}-1)\rho_2 + m_{2i}\rho_{12}^2)}{2\sigma_1^2 c_i)} - \frac{m_{2i}(1+(m_{2i}-1)\rho_2)\rho_{12}}{\sigma_1^2 c_i^2},$$

$$\mathcal{I}_i[4,4] = \frac{m_{2i}}{\sigma_2^4(1-\rho_2)}\left[1 - \frac{(1+(m_{2i}-1)\rho_2)(\rho_2-\rho_{12}^2)}{c_i}\right] - \frac{3m_{2i}\rho_{12}^2}{4\sigma_2^4 c_i} - \frac{m_{2i}}{2\sigma_2^4},$$

$$\mathcal{I}_i[4,6] = \mathcal{I}_i[6,4] = -\frac{m_{2i}(m_{2i}-1)\rho_{12}^2}{2\sigma_2^2 c_i} - \frac{m_{2i}}{2\sigma_2^2(1-\rho_2)^2}\left[1 - \frac{(1-\rho_{12}^2)(1+(m_{2i}-1)\rho_2)}{c_i} + \frac{(m_{2i}-1)(1+(m_{2i}-1)\rho_2)(1-\rho_2)(\rho_2-\rho_{12}^2)}{c_i^2}\right],$$

$$\mathcal{I}_i[4,7] = \mathcal{I}_i[7,4] = \frac{m_{2i}\rho_{12}(1+(m_{2i}-1)\rho_2+m_{2i}\rho_{12}^2)}{2\sigma_2^2 c_i^2} + \frac{m_{2i}\rho_{12}(1+(m_{2i}-1)\rho_2)}{\sigma_2^2(1-\rho_2)}\left[\frac{m_{2i}(\rho_2-\rho_{12}^2)-c_i}{c_i^2}\right],$$

$$\mathcal{I}_i[6,6] = \frac{m_{2i}(m_{2i}-1)^2\rho_{12}^2}{c_i^3} - \frac{(m_{2i}-1)^2+4m_{2i}(m_{2i}-1)^2\rho_{12}^2}{2c_i^2} - \frac{m_{2i}-1}{2(1-\rho_2)^2} + \frac{m_{2i}(1-(1+(m_{2i}-1)\rho_2)Star_3)}{c_i^3},$$

where

$$Star_3 = \frac{1-\rho_{12}^2}{c_i} - \frac{(m_{2i}-1)(1-\rho_2)(1-\rho_{12}^2)}{c_i^2} + \frac{(m_{2i}-1)^2(1-\rho_2)^2(\rho_2-\rho_{12}^2)}{c_i^3},$$

$$\mathcal{I}_i[6,7] = \mathcal{I}_i[7,6] = \frac{m_{2i}(m_{2i}-1)\rho_{12}}{c_i^2} + \frac{m_{2i}(m_{2i}-1)\left[\rho_2\rho_{12}+(m_{2i}-1)\rho_{12}^2\right]+m_{2i}\rho_{12}}{c_i^3},$$

$$\mathcal{I}_i[7,7] = \frac{m_{2i}(1+(m_{2i}-1)\rho_2)(1+(m_{2i}-1)\rho_2+3m_{2i}\rho_{12}^2)}{c_i^3} - \frac{m_{2i}(1+(m_{2i}-1)\rho_2+m_{2i}\rho_{12}^2)}{c_i^2} - \frac{2m_{2i}^2\rho_{12}^2(3+3(m_{2i}-1)\rho_2+m_{2i}\rho_{12}^2)}{c_i^3} - \frac{m_{2i}(1+(m_{2i}-1)\rho_2)}{1-\rho_2}\left[\frac{4m_{2i}^2\rho_{12}^2(\rho_2-\rho_{12}^2)-(1-\rho_2+4m_{2i}\rho_{12}^2)c_i}{c_i^3}\right].$$

Other entries of $\mathcal{I}_i(\boldsymbol{\theta})$ are zero for $m_{1i}=1$ and $m_{2i}>1$.

If $m_{1i}>1$ and $m_{2i}=1$, then

$$c_i = 1+(m_{1i}-1)\rho_1 - m_{1i}\rho_{12}^2,$$

and

$$
\begin{aligned}
\mathcal{I}_i[1,1] &= \frac{1}{\sigma_1^2(1-\rho_1)}\left[m_{1i} - \frac{m_{1i}^2(\rho_1-\rho_{12}^2)}{c_i}\right],\\
\mathcal{I}_i[1,2] &= \mathcal{I}_i[2,1] = -\frac{m_{1i}\rho_{12}}{\sigma_1\sigma_2 c_i},\\
\mathcal{I}_i[1,3] &= \mathcal{I}_i[3,1] = 0,\\
\mathcal{I}_i[1,4] &= \mathcal{I}_i[4,1] = 0,\\
\mathcal{I}_i[1,5] &= \mathcal{I}_i[5,1] = 0,\\
\mathcal{I}_i[1,7] &= \mathcal{I}_i[7,1] = 0,\\
\mathcal{I}_i[2,2] &= \frac{1+(m_{1i}-1)\rho_1}{\sigma_2^2 c_i},\\
\mathcal{I}_i[2,3] &= \mathcal{I}_i[3,2] = 0,\\
\mathcal{I}_i[2,4] &= \mathcal{I}_i[4,2] = 0,\\
\mathcal{I}_i[2,5] &= \mathcal{I}_i[5,2] = 0,\\
\mathcal{I}_i[2,7] &= \mathcal{I}_i[7,2] = 0,\\
\mathcal{I}_i[3,3] &= \frac{m_{1i}}{\sigma_1^4(1-\rho_1)}\left[1 - \frac{(1+(m_{1i}-1)\rho_1)(\rho_1-\rho_{12}^2)}{c_i}\right]\\
&\quad - \frac{3m_{1i}\rho_{12}^2}{4\sigma_1^4 c_i} - \frac{m_{1i}}{2\sigma_1^4},\\
\mathcal{I}_i[3,4] &= \mathcal{I}_i[4,3] = -\frac{m_{1i}\rho_{12}^2}{4\sigma_1^2\sigma_2^2 c_i},\\
\mathcal{I}_i[3,5] &= \mathcal{I}_i[5,3] = -\frac{m_{1i}(m_{1i}-1)\rho_{12}^2}{2\sigma_1^2 c_i}\\
&\quad - \frac{m_{1i}}{2\sigma_1^2(1-\rho_1)^2}\left[1 - \frac{(1-\rho_{12}^2)(1+(m_{1i}-1)\rho_1)}{c_i}\right.\\
&\quad \left. + \frac{(m_{1i}-1)(1+(m_{1i}-1)\rho_1)(1-\rho_1)(\rho_1-\rho_{12}^2)}{c_i^2}\right],\\
\mathcal{I}_i[3,7] &= \mathcal{I}_i[7,3] = \frac{m_{1i}\rho_{12}(1+(m_{1i}-1)\rho_1+m_{1i}\rho_{12}^2)}{2\sigma_1^2 c_i^2}\\
&\quad + \frac{m_{1i}\rho_{12}(1+(m_{1i}-1)\rho_1)}{\sigma_1^2(1-\rho_1)}\left[\frac{m_{1i}(\rho_1-\rho_{12}^2)-c_i}{c_i^2}\right],\\
\mathcal{I}_i[4,4] &= \frac{1+(m_{1i}-1)\rho_1}{\sigma_2^4 c_i} - \frac{1}{2\sigma_2^4} - \frac{3m_{1i}\rho_{12}^2}{4\sigma_2^4 c_i},\\
\mathcal{I}_i[4,5] &= \mathcal{I}_i[5,4] = 0,\\
\mathcal{I}_i[4,7] &= \mathcal{I}_i[7,4] = \frac{m_{1i}\rho_{12}(1+(m_{1i}-1)\rho_1+m_{1i}\rho_{12}^2)}{2\sigma_2^2 c_i} - \frac{m_{1i}(1+(m_{1i}-1)\rho_1)\rho_{12}}{\sigma_2^2 c_i^2},
\end{aligned}
$$

$$\mathcal{I}_i[5,5] = \frac{m_{1i}(m_{1i}-1)^2\rho_{12}^2}{c_i^3} - \frac{(m_{1i}-1)^2 + 4m_{1i}(m_{1i}-1)^2\rho_{12}^2}{2c_i^2}$$
$$- \frac{m_{1i}-1}{2(1-\rho_1)^2} + \frac{m_{1i}(1-(1+(m_{1i}-1)\rho_1)Star_4)}{c_i^3},$$

where

$$Star_4 = \frac{1-\rho_{12}^2}{c_i} - \frac{(m_{1i}-1)(1-\rho_1)(1-\rho_{12}^2)}{c_i^2}$$
$$+ \frac{(m_{1i}-1)^2(1-\rho_1)^2(\rho_1-\rho_{12}^2)}{c_i^3},$$

$$\mathcal{I}_i[5,7] = \mathcal{I}_i[7,6] = \frac{m_{1i}(m_{1i}-1)\rho_{12}}{c_i^2}$$
$$+ \frac{m_{2i}(m_{2i}-1)\left[\rho_2\rho_{12} + (m_{2i}-1)\rho_{12}^2\right] + m_{2i}\rho_{12}}{c_i^3},$$

$$\mathcal{I}_i[7,7] = \frac{m_{1i}(1+(m_{1i}-1)\rho_1)(1+(m_{1i}-1)\rho_1+3m_{1i}\rho_{12}^2)}{c_i^3}$$
$$- \frac{m_{1i}(1+(m_{1i}-1)\rho_1+m_{1i}\rho_{12}^2)}{c_i^2} - \frac{2m_{1i}^2\rho_{12}^2(3+3(m_{1i}-1)\rho_1+m_{1i}\rho_{12}^2)}{c_i^3}$$
$$- \frac{m_{1i}(1+(m_{1i}-1)\rho_1)}{1-\rho_1}\left[\frac{4m_{1i}^2\rho_{12}^2(\rho_1-\rho_{12}^2)-(1-\rho_1+4m_{1i}\rho_{12}^2)c_i}{c_i^3}\right].$$

Other entries of $\mathcal{I}_i(\boldsymbol{\theta})$ are zero for $m_{1i} > 1$ and $m_{2i} = 1$.

The three asymptotic tests that we just discussed are based on the asymptotic chi-square distribution. The success of these tests is contingent upon the fact that the sample size is large. Further, the computation of the maximum likelihood estimates have to be obtained by numerically maximizing the likelihood functions. This process many times leads to non-convergence and in these cases it is hard to obtain the estimates.

Next, we will provide certain transformations which enables us to provide some simple procedures for testing the intended null hypothesis.

II.8 CANONICAL TRANSFORMATION

In this section, we apply a canonical transformation to the familial data simplifying the variance-covariance structure of the model. Doing this, will provide transformed data that can easily be used to estimate the model parameters as will be done in the following sections.

Recall, $\mathbf{x}_i = \begin{pmatrix} \mathbf{x}_{1i} \\ \mathbf{x}_{2i} \end{pmatrix}$ is distributed with mean $\boldsymbol{\mu}_i = \begin{pmatrix} \mu_1 \mathbf{1}_{m_{1i}} \\ \mu_2 \mathbf{1}_{m_{2i}} \end{pmatrix}$ and covariance matrix

$$\boldsymbol{\Sigma}_i = \begin{pmatrix} \sigma_1^2\{(1-\rho_1)\mathbf{I}_{m_{1i}} + \rho_1 \mathbf{J}_{m_{1i}}\} & \rho_{12}\sigma_1\sigma_2 \mathbf{J}_{m_{1i},m_{2i}} \\ \rho_{12}\sigma_1\sigma_2 \mathbf{J}_{m_{2i},m_{1i}} & \sigma_2^2\{(1-\rho_2)\mathbf{I}_{m_{2i}} + \rho_2 \mathbf{J}_{m_{2i}}\} \end{pmatrix}.$$

Let

$$\boldsymbol{\Gamma}_{i,(m_{1i}+m_{2i} \times m_{1i}+m_{2i})} = \begin{pmatrix} \boldsymbol{\Gamma}_{1i} & \mathbf{0}_{m_{1i},m_{2i}} \\ \mathbf{0}_{m_{2i},m_{1i}} & \boldsymbol{\Gamma}_{2i} \end{pmatrix}$$

where

$$\boldsymbol{\Gamma}_{1i} = \begin{pmatrix} \frac{1}{m_{1i}} & \frac{1}{m_{1i}} & \frac{1}{m_{1i}} & \cdots & \frac{1}{m_{1i}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{m_{1i}(m_{1i}-1)}} & \frac{1}{\sqrt{m_{1i}(m_{1i}-1)}} & \frac{1}{\sqrt{m_{1i}(m_{1i}-1)}} & \cdots & \frac{-(m_{1i}-1)}{\sqrt{m_{1i}(m_{1i}-1)}} \end{pmatrix},$$

$$\boldsymbol{\Gamma}_{2i} = \begin{pmatrix} \frac{1}{m_{2i}} & \frac{1}{m_{2i}} & \frac{1}{m_{2i}} & \cdots & \frac{1}{m_{2i}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{m_{2i}(m_{2i}-1)}} & \frac{1}{\sqrt{m_{2i}(m_{2i}-1)}} & \frac{1}{\sqrt{m_{2i}(m_{2i}-1)}} & \cdots & \frac{-(m_{2i}-1)}{\sqrt{m_{2i}(m_{2i}-1)}} \end{pmatrix},$$

and $\mathbf{0}_{m,n}$ is the $m \times n$ matrix of all zeros.

Transform the family scores by making a Srivastava type transformation (Srivastava, 1984) to create $\mathbf{y}_i$, the transformed vector of family scores,

$$\mathbf{y}_i = \begin{pmatrix} \mathbf{y}_{1i} \\ \mathbf{y}_{2i} \end{pmatrix} = \boldsymbol{\Gamma}_i \begin{pmatrix} \mathbf{x}_{1i} \\ \mathbf{x}_{2i} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Gamma}_{1i} & \mathbf{0}_{m_{1i},m_{2i}} \\ \mathbf{0}_{m_{2i},m_{1i}} & \boldsymbol{\Gamma}_{2i} \end{pmatrix} \begin{pmatrix} \mathbf{x}_{1i} \\ \mathbf{x}_{2i} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Gamma}_{1i}\mathbf{x}_{1i} \\ \boldsymbol{\Gamma}_{2i}\mathbf{x}_{2i} \end{pmatrix}.$$

Now, the expected value of the transformed son scores is

$$E(\mathbf{y}_{1i}) = \begin{pmatrix} \mu_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

and the variance matrix of the vector of transformed son scores is

$$\begin{aligned} Var(\mathbf{y}_{1i}) &= \sigma_1^2 \mathbf{\Gamma}_{1i}((1-\rho_1)\mathbf{I}_{m_{1i}} + \rho_1 \mathbf{J}_{m_{1i}})\mathbf{\Gamma}'_{1i} \\ &= \begin{pmatrix} \frac{1}{2}\sigma_1^2(1+(m_{1i}-1)\rho_1) & 0 & \cdots & 0 \\ 0 & \sigma_1^2(1-\rho_1) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma_1^2(1-\rho_1) \end{pmatrix}. \end{aligned}$$

Similarly, the expected value of the transformed daughter scores is

$$E(\mathbf{y}_{2i}) = \begin{pmatrix} \mu_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

and the variance matrix of the vector of transformed daughter scores is

$$\begin{aligned} Var(\mathbf{y}_{2i}) &= \sigma_2^2 \mathbf{\Gamma}_{2i}((1-\rho_2)\mathbf{I}_{m_{2i}} + \rho_2 \mathbf{J}_{m_{2i}})\mathbf{\Gamma}'_{2i} \\ &= \begin{pmatrix} \frac{1}{2}\sigma_2^2(1+(m_{2i}-1)\rho_2) & 0 & \cdots & 0 \\ 0 & \sigma_2^2(1-\rho_2) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma_2^2(1-\rho_2) \end{pmatrix}. \end{aligned}$$

The covariance matrix between the vector of transformed son scores and the vector of transformed daughter scores is

$$\begin{aligned} Cov(\mathbf{y}_{1i}, \mathbf{y}_{2i}) &= \sigma_1\sigma_2\rho_{12}\mathbf{\Gamma}_{1i}\mathbf{J}_{m_{1i},m_{2i}}\mathbf{\Gamma}'_{2i} \\ &= \sigma_1\sigma_2\rho_{12} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{m_{1i},m_{2i}}. \end{aligned}$$

Note that only the first transformed son score and the first transformed daughter score, namely y_{1i1} and y_{2i1}, are correlated. Also, the vector $\begin{pmatrix} y_{1i1} \\ y_{2i1} \end{pmatrix}$ is bivariate

normal with mean $\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$ and variance covariance matrix

$$\begin{pmatrix} \frac{1}{m_{1i}}\sigma_1^2(1+(m_{1i}-1)\rho_1) & \sigma_1\sigma_2\rho_{12} \\ \sigma_1\sigma_2\rho_{12} & \frac{1}{m_{2i}}\sigma_2^2(1+(m_{2i}-1)\rho_2) \end{pmatrix}. \tag{5}$$

Additionally, y_{1i1} and y_{2i1} are independent of $y_{1i2}, \ldots, y_{1im_{1i}} \sim N(0, \sigma_1^2(1-\rho_1))$ and $y_{2i2}, \ldots, y_{2im_{2i}} \sim N(0, \sigma_2^2(1-\rho_2))$.

In terms of $\mathbf{x}_{1i}$ and $\mathbf{x}_{2i}$, the first transformed son score, y_{1i1}, is the average of all the boy scores of the family. As well, the first transformed daughter score, y_{2i1}, is the average of all the girl scores of the family. That is,

$$y_{1i1} = \tfrac{1}{m_{1i}} \textstyle\sum_{j=1}^{m_{1i}} x_{1ij}, \quad y_{2i1} = \tfrac{1}{m_{2i}} \textstyle\sum_{j=1}^{m_{2i}} x_{2ij}.$$

Hence, the average of the first transformed son scores is an average of the mean score of sons for each family. Similarly, the average of the first transformed daughter scores is an average of the mean score of daughters for each family. Thus,

$$\bar{y}_{1i1} = \tfrac{1}{n} \textstyle\sum_{i=1}^{n} y_{1i1} = \tfrac{1}{n} \sum_{i=1}^{n} \tfrac{1}{m_{1i}} \sum_{j=1}^{m_{1i}} x_{1ij},$$
$$\bar{y}_{2i1} = \tfrac{1}{n} \textstyle\sum_{i=1}^{n} y_{2i1} = \tfrac{1}{n} \sum_{i=1}^{n} \tfrac{1}{m_{2i}} \sum_{j=1}^{m_{2i}} x_{2ij}.$$

Also, the sum of squares of the "left-over" transformed son scores, $y_{1i2}, \ldots, y_{1im_{1i}}$, for a family can be written in terms of the second through last son of the family. Similarly, the sum of squares of the "left-over" transformed daughter scores, $y_{2i2}, \ldots, y_{2im_{2i}}$, for a family can be written in terms of the second through last daughter of the family. Specifically,

$$\begin{aligned} \sum_{j=2}^{m_{1i}} y_{1ij}^2 &= (x_{1i2}, \ldots, x_{1im_{1i}})' \, \mathbf{\Gamma}_i' \mathbf{\Gamma}_i \, (x_{1i2}, \ldots, x_{1im_{1i}}) \\ &= (x_{1i2}, \ldots, x_{1im_{1i}})' \, (I_{m_{1i}} - \frac{1}{m_{1i}} J_{m_{1i}}) \, (x_{1i2}, \ldots, x_{1im_{1i}}) \\ &= \sum_{j=2}^{m_{1i}} x_{1ij}^2 - \frac{1}{m_{1i}} (\sum_{j=2}^{m_{1i}} x_{1ij})^2, \end{aligned}$$

$$\begin{aligned} \sum_{j=2}^{m_{2i}} y_{2ij}^2 &= (x_{2i2}, \ldots, x_{2im_{2i}})' \, \mathbf{\Gamma}_i' \mathbf{\Gamma}_i \, (x_{2i2}, \ldots, x_{2im_{2i}}) \\ &= (x_{2i2}, \ldots, x_{2im_{2i}})' \, (I_{m_{2i}} - \frac{1}{m_{2i}} J_{m_{2i}}) \, (x_{2i2}, \ldots, x_{2im_{2i}}) \\ &= \sum_{j=2}^{m_{2i}} x_{2ij}^2 - \frac{1}{m_{2i}} (\sum_{j=2}^{m_{2i}} x_{2ij})^2. \end{aligned}$$

In order to simplify the transformed model, let $\mathbf{\Gamma}_i'\mathbf{\Sigma}_i\mathbf{\Gamma}_i =$

$$\begin{bmatrix} \eta_{1i}^2 & \mathbf{0}'_{m_{1i}-1} & \sigma_{12} & \mathbf{0}'_{m_{2i}-1} \\ \mathbf{0}_{m_{1i}-1} & \gamma_1^2\mathbf{I}_{m_{1i}-1} & \mathbf{0}_{m_{1i}-1} & \mathbf{0}_{m_{1i}-1,m_{2i}-1} \\ \sigma_{12} & \mathbf{0}'_{m_{1i}-1} & \eta_{2i}^2 & \mathbf{0}'_{m_{2i}-1} \\ \mathbf{0}_{m_{2i}-1} & \mathbf{0}_{m_{1i}-1,m_{2i}-1} & \mathbf{0}_{m_{2i}-1} & \gamma_2^2\mathbf{I}_{m_{2i}-1} \end{bmatrix},$$

where

$$\begin{aligned} \eta_{1i}^2 &= \sigma_1^2(1+(m_{1i}-1)\rho_1)/m_{1i}, \\ \eta_{2i}^2 &= \sigma_2^2(1+(m_{2i}-1)\rho_2)/m_{2i}, \\ \gamma_1^2 &= \sigma_1^2(1-\rho_1), \\ \gamma_2^2 &= \sigma_2^2(1-\rho_2), \\ \sigma_{12} &= \sigma_1\sigma_2\rho_{12}. \end{aligned}$$

Note $\eta_{1i}^2 = \sigma_1^2 - a_{1i}\gamma_1^2$ and $\eta_{2i}^2 = \sigma_2^2 - a_{2i}\gamma_2^2$, where $a_{1i} = 1 - m_{1i}^{-1}$ and $a_{2i} = 1 - m_{2i}^{-1}$. Also, there is a 1-1 transformation from the old parameters to a new set of parameters, namely,

$$\begin{aligned} \xi_1 &= \frac{\sigma_1^2}{\gamma_1^2}, \\ \xi_2 &= \frac{\sigma_2^2}{\gamma_2^2}, \\ \xi_{12} &= \frac{\sigma_{12}^2}{\gamma_1^2\gamma_2^2}. \end{aligned}$$

II.9 FIRST SET OF ALTERNATIVE ESTIMATORS

From the distribution of the transformed familial data, alternative estimates can be developed that do not require maximization of a non-linear constraint, as is the case for finding the MLEs used in the LRT, Wald, and Score tests. Let n_1 be the number of families with $m_{1i} > 0$, n_2 be the number of families with $m_{2i} > 0$, and n_{12} be the number of families with $m_{1i} > 0$ and $m_{2i} > 0$. Since $y_{1i2}, \ldots, y_{1im_{1i}} \sim N(0, \gamma_1^2)$ then $\sum_{i=1}^{n_1}\sum_{j=2}^{m_{1i}} y_{1ij}^2$ is a complete, sufficient statistic for γ_1^2. Also, by the Weak Law of Large Numbers $\sum_{i=1}^{n_1}\sum_{j=2}^{m_{1i}} y_{1ij}^2$ is a consistent estimator of γ_1^2 because $E[\sum_{i=1}^{n_1}\sum_{j=2}^{m_{1i}} y_{1ij}^2] = \sum_{i=1}^{n_1}\sum_{j=2}^{m_{1i}} \gamma_1^2 = \gamma_1^2\sum_{i=1}^{n_1}(m_{1i}-1)$. Hence an unbiased and consistent estimator of γ_1^2 is proposed as

$$\tilde{\gamma}_1^2 = \frac{\sum_{i=1}^{n_1}\sum_{j=2}^{m_{1i}} y_{1ij}^2}{\sum_{i=1}^{n_1}(m_{1i}-1)}. \tag{6}$$

Similarly, γ_2^2 can be estimated by

$$\tilde{\gamma}_2^2 = \frac{\sum_{i=1}^{n_2}\sum_{j=2}^{m_{2i}} y_{2ij}^2}{\sum_{i=1}^{n_2}(m_{2i}-1)}. \tag{7}$$

Next take

$$\tilde{\sigma}_{12} = \sum_{i=1}^{n_{12}} \frac{(y_{1i1}-\bar{y}_{11}^*)(y_{2i1}-\bar{y}_{21}^*)}{n_{12}-1},$$

where $\bar{y}_{11}^* = \frac{1}{n_{12}}\sum_{i=1}^{n_{12}} y_{1i1}$ and $\bar{y}_{21}^* = \frac{1}{n_{12}}\sum_{i=1}^{n_{12}} y_{2i1}$. Since

$$\begin{pmatrix} y_{1i1} \\ y_{2i1} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \eta_{1i}^2 & \sigma_{12} \\ \sigma_{12} & \eta_{2i}^2 \end{pmatrix}\right),$$

$\tilde{\sigma}_{12}$ is an unbiased estimator of σ_{12}.

Since, $\sigma_1^2 = \eta_{1i}^2 + a_{1i}\gamma_1^2 = \frac{1}{n_1}(\sum_{i=1}^{n_1}\eta_{1i}^2) + \frac{1}{n_1}\gamma_1^2(\sum_{i=1}^{n_1} a_{1i})$ and since $y_{1i2}, \ldots, y_{1im_{1i}} (i = 1, \ldots, n)$ were used for estimating γ_1^2, Srivastava (1984) proposed using $y_{1i1} (i = 1, \ldots, n)$ to estimate σ_1^2. Consider $E\left[\sum_{i=1}^{n_1}(y_{1i1} - \bar{y}_{11})^2\right]$, where $\bar{y}_{11} = \frac{1}{n_1}\sum_{i=1}^{n_1} y_{1i1}$. That is,

$$\begin{aligned} E\left(\sum_{i=1}^{n_1} y_{1i1}^2 - n_1\bar{y}_{11}^2\right) &= \sum_{i=1}^{n_1}(\eta_{1i}^2+\mu^2) - n_1\left(\frac{1}{n_1^2}\sum_{i=1}^{n_1}\eta_{1i}^2 + \mu_1^2\right) \\ &= \left(1-\frac{1}{n_1}\right)\sum_{i=1}^{n_1}\eta_{1i}^2 \\ &= \left(1-\frac{1}{n_1}\right)n_1\sigma_1^2 - \left(1-\frac{1}{n_1}\right)\gamma_1^2\left(\sum_{i=1}^{n_1} a_{1i}\right) \\ &= (n_1-1)\sigma_1^2 - \left(1-\frac{1}{n_1}\right)\gamma_1^2\left(\sum_{i=1}^{n_1} a_{1i}\right). \end{aligned}$$

Then estimate σ_1^2 by

$$\tilde{\sigma}_1^2 = \frac{1}{n_1-1}\sum_{i=1}^{n_1}(y_{1i1} - \bar{y_{11}})^2 + \frac{1}{n_1}\tilde{\gamma}_1^2(\sum_{i=1}^{n_1} a_{1i}).$$

Similarly, one can estimate σ_2^2 by

$$\tilde{\sigma}_2^2 = \frac{1}{n_2-1}\sum_{i=1}^{n_2}(y_{2i1} - \bar{y_{21}})^2 + \frac{1}{n_2}\tilde{\gamma}_2^2(\sum_{i=1}^{n_2} a_{2i}).$$

From these, estimates of the other parameters are

$$\begin{aligned}\tilde{\rho}_1 &= 1-(\tilde{\gamma}_1^2/\tilde{\sigma}_1^2),\\ \tilde{\rho}_2 &= 1-(\tilde{\gamma}_2^2/\tilde{\sigma}_2^2),\\ \tilde{\rho}_{12} &= \frac{\tilde{\sigma}_{12}}{\tilde{\sigma}_1\tilde{\sigma}_2}.\end{aligned}$$

Lastly, μ_1 and μ_2 can be estimated by the maximum likelihood method from the distribution of y_{1i1} and y_{2i1}. These are not simple averages since the variance of y_{1i1} and the variance of y_{2i1} are dependent on the family size.

$$\tilde{\mu}_1 = \frac{Sub_4\left[Sub_1 - 2\tilde{\sigma}_{12}\left(\frac{Sub_1}{Sub_2}\right)\sum_{i=1}^{n_{12}}\frac{1}{\tilde{\eta}_{1i}^2\tilde{\eta}_{2i}^2-\tilde{\sigma}_{12}^2}\right]}{Sub_2 Sub_4 - 4\tilde{\sigma}_{12}^2\left[\sum_{i=1}^{n_{12}}\frac{1}{\tilde{\eta}_{1i}^2\tilde{\eta}_{2i}^2-\tilde{\sigma}_{12}^2}\right]^2},$$

$$\tilde{\mu}_2 = \frac{Sub_2\left[Sub_3 - 2\tilde{\sigma}_{12}\left(\frac{Sub_3}{Sub_4}\right)\sum_{i=1}^{n_{12}}\frac{1}{\tilde{\eta}_{1i}^2\tilde{\eta}_{2i}^2-\tilde{\sigma}_{12}^2}\right]}{Sub_2 Sub_4 - 4\tilde{\sigma}_{12}^2\left[\sum_{i=1}^{n_{12}}\frac{1}{\tilde{\eta}_{1i}^2\tilde{\eta}_{2i}^2-\tilde{\sigma}_{12}^2}\right]^2},$$

$$Sub_1 = \sum_{i=1}^{n_1}\frac{y_{1i1}}{\tilde{\eta}_{1i}^2} + \sum_{i=1}^{n_{12}}\frac{\tilde{\eta}_{2i}^2 y_{1i1}}{\tilde{\eta}_{1i}^2\tilde{\eta}_{2i}^2-\tilde{\sigma}_{12}^2} + 2\tilde{\sigma}_{12}\sum_{i=1}^{n_{12}}\frac{y_{2i1}}{\tilde{\eta}_{1i}^2\tilde{\eta}_{2i}^2-\tilde{\sigma}_{12}^2},$$

$$Sub_2 = \sum_{i=1}^{n_{12}}\frac{1}{\tilde{\eta}_{1i}^2} + \sum_{i=1}^{n_{12}}\frac{\tilde{\eta}_{2i}^2}{\tilde{\eta}_{1i}^2\tilde{\eta}_{2i}^2-\tilde{\sigma}_{12}^2},$$

$$Sub_3 = \sum_{i=1}^{n_1}\frac{y_{2i1}}{\tilde{\eta}_{2i}^2} + \sum_{i=1}^{n_{12}}\frac{\tilde{\eta}_{1i}^2 y_{2i1}}{\tilde{\eta}_{1i}^2\tilde{\eta}_{2i}^2-\tilde{\sigma}_{12}^2} + 2\tilde{\sigma}_{12}\sum_{i=1}^{n_{12}}\frac{y_{1i1}}{\tilde{\eta}_{1i}^2\tilde{\eta}_{2i}^2-\tilde{\sigma}_{12}^2},$$

$$Sub_4 = \sum_{i=1}^{n_{12}}\frac{1}{\tilde{\eta}_{2i}^2} + \sum_{i=1}^{n_{12}}\frac{\tilde{\eta}_{1i}^2}{\tilde{\eta}_{1i}^2\tilde{\eta}_{2i}^2-\tilde{\sigma}_{12}^2}.$$

II.10 SECOND SET OF ALTERNATIVE ESTIMATORS

A second transformation can be considered that has better distributional properties providing simpler estimates of μ_1, μ_2, σ_1^2, and σ_2^2.

Consider

$$\tilde{y}_{1i1} = y_{1i1} - \frac{1}{\sqrt{m_{1i}}}\sum_{j=2}^{m_{1i}} y_{1ij} \quad (i = 1, \ldots, n)$$

and

$$\tilde{y}_{2i1} = y_{2i1} - \frac{1}{\sqrt{m_{2i}}}\sum_{j=2}^{m_{2i}} y_{2ij} \quad (i = 1, \ldots, n).$$

Then

$$E(\tilde{y}_{1i1}) = \mu_1, \quad Var(\tilde{y}_{1i1}) = \eta_{1i}^2 + a_{1i}\gamma_1^2 = \sigma_1^2,$$

$$E(\tilde{y}_{2i1}) = \mu_2, \quad Var(\tilde{y}_{2i1}) = \eta_{2i}^2 + a_{2i}\gamma_2^2 = \sigma_2^2,$$

and

$$Cov(\tilde{y}_{1i1}, \tilde{y}_{2i1}) = \sigma_{12}.$$

Hence, there are n_{12} independent bivariate pairs of observations $(\tilde{y}_{1i1}, \tilde{y}_{2i1})$ for $i = 1, \ldots, n_{12}$ with

$$\begin{pmatrix} \tilde{y}_{1i1} \\ \tilde{y}_{2i1} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}\right). \tag{8}$$

Note the variances of $\tilde{y}_{1i1}$ and $\tilde{y}_{2i1}$ are not dependent on the family size as was the case in the first transformation (5).

Therefore, natural estimates of $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$, and σ_{12} are

$$\begin{aligned}
\tilde{\mu}_{1a} &= \frac{1}{n_1}\sum_{i=1}^{n_1} \tilde{y}_{1i1} = \tilde{y}_{1\cdot 1}, &(9)\\
\tilde{\mu}_{2a} &= \frac{1}{n_2}\sum_{i=1}^{n_2} \tilde{y}_{2i1} = \tilde{y}_{2\cdot 1}, &(10)\\
\tilde{\sigma}_{1a}^2 &= \frac{1}{n_1 - 1}\sum_{i=1}^{n_1} (\tilde{y}_{1i1} - \tilde{y}_{1\cdot 1})^2, \\
\tilde{\sigma}_{2a}^2 &= \frac{1}{n_2 - 1}\sum_{i=1}^{n_2} (\tilde{y}_{2i1} - \tilde{y}_{2\cdot 1})^2, \\
\tilde{\sigma}_{12a} &= \frac{1}{n_{12} - 1}\sum_{i=1}^{n_{12}} (\tilde{y}_{1i1} - \tilde{y}_{1\cdot 1})(\tilde{y}_{2i1} - \tilde{y}_{2\cdot 1}),
\end{aligned}$$

where $\tilde{y}_{1\cdot 1}^* = \frac{1}{n_{12}}\sum_{i=1}^{n_{12}} \tilde{y}_{1i1}$ and $\tilde{y}_{2\cdot 1}^* = \frac{1}{n_{12}}\sum_{i=1}^{n_{12}} \tilde{y}_{2i1}$. Hence, the estimate of ρ_{12} is

$$\tilde{\rho}_{12a} = \frac{\tilde{\sigma}_{12a}}{\tilde{\sigma}_{1a}\tilde{\sigma}_{2a}}.$$

While γ_1^2 and γ_2^2 can still be estimated as before by (6) and (7), a change could be

$$\begin{aligned}
\tilde{\gamma}_{1a}^2 &= \frac{\sum_{i=1}^{n_1^*}\sum_{j=2}^{m_{1i}} (y_{1ij} - \bar{y}_{1i\cdot})^2}{\sum_{i=1}^{n_1^*} (m_{1i} - 1)}, \\
\tilde{\gamma}_{2a}^2 &= \frac{\sum_{i=1}^{n_2^*}\sum_{j=2}^{m_{2i}} (y_{2ij} - \bar{y}_{2i\cdot})^2}{\sum_{i=1}^{n_2^*} (m_{2i} - 1)},
\end{aligned}$$

where

$$\bar{y}_{1i\cdot} = \frac{1}{m_{1i}-1}\sum_{j=2}^{m_{1i}} y_{1ij},$$
$$\bar{y}_{2i\cdot} = \frac{1}{m_{2i}-1}\sum_{j=2}^{m_{2i}} y_{2ij},$$

$n_1^* =$ number of families with $m_{1i} > 1$, and $n_2^* =$ number of families with $m_{2i} > 1$.

Note, $\tilde{\sigma}_{1a}^2$ and $\tilde{\gamma}_{1a}^2$ are independently distributed as are $\tilde{\sigma}_{2a}^2$ and $\tilde{\gamma}_{2a}^2$. Hence,

$$\tilde{\rho}_{1a} = 1 - \frac{\tilde{\gamma}_{1a}^2}{\tilde{\sigma}_{1a}^2},$$
$$\tilde{\rho}_{2a} = 1 - \frac{\tilde{\gamma}_{2a}^2}{\tilde{\sigma}_{2a}^2}.$$

Both sets of alternative estimates are easier to compute than the MLEs as they do not require an iterative procedure that does not always converge. However, the second set of alternative estimators, $\tilde{\rho_{1a}}$ and $\tilde{\rho_{2a}}$, requires the average number of boys in the familial data set, $\bar{m}_1$, and the average number of girls in the familial data set, $\bar{m}_2$, to be greater than 2 so that the variances given below in equations (11) and (12) are positive. The first set of alternative estimates, $\tilde{\rho}_1$ and $\tilde{\rho}_2$, do not have this restriction. However, it is possible for both sets of alternative estimates to violate the model constraint (1). This will be discussed further after the simulation results are presented.

II.11 VARIANCE OF ALTERNATIVE ESTIMATORS

This section gives the asymptotic variance of both sets of alternative estimates. The distributions of these estimates will be used to construct alternative tests to test the null hypothesis later in the chapter. In order to determine the variance of the first set of alternative estimators, consider the following asymptotic distributions

$$n_1^{1/2}\begin{bmatrix} \tilde{\gamma}_1^2 - \gamma_1^2 \\ \tilde{\sigma}_1^2 - \sigma_1^2 \end{bmatrix} \rightarrow N(0, \Sigma_1),$$

where

$$\Sigma_1 = 2\begin{bmatrix} \gamma_1^4(\bar{m}_1 - 1)^{-1} & \gamma_1^4(\bar{m}_1 - 1)^{-1}n_1^{-1}\sum_{i=1}^{n_1} a_{1i} \\ \gamma_1^4(\bar{m}_1 - 1)^{-1}n_1^{-1}\sum_{i=1}^{n_1} a_{1i} & c_1^2\sigma_1^4 \end{bmatrix},$$

with $c_1^2 = 1 - 2(1-\rho_1)n_1^{-1}\sum_{i=1}^{n_1} a_{1i} + (1-\rho_1)^2\left[n_1^{-1}\sum_{i=1}^{n_1} a_{1i}^2 + (\bar{m}_1 - 1)^{-1}(n_1^{-1}\sum_{i=1}^{n_1} a_{1i})^2\right]$,

and $\bar{m}_1 = n_1^{-1}\sum_{i=1}^{n_1} m_{1i}$.

Using the delta method, the asymptotic variance of $\tilde{\rho}_1$ is

$$AV_1 = 2(1-\rho_1)^2\frac{1}{n_1}\left[(\bar{m}_1 - 1)^{-1} + c_1^2 - 2(1-\rho_1)(\bar{m}_1 - 1)^{-1}n_1^{-1}\sum_{i=1}^{n_1} a_{1i}\right].$$

Similarly, for the first alternative estimators based only on daughter scores

$$n_2^{1/2}\begin{bmatrix} \tilde{\gamma}_2^2 - \gamma_2^2 \\ \tilde{\sigma}_2^2 - \sigma_2^2 \end{bmatrix} \to N(0, \Sigma_2),$$

where

$$\Sigma_2 = 2\begin{bmatrix} \gamma_2^4(\bar{m}_2 - 1)^{-1} & \gamma_2^4(\bar{m}_2 - 1)^{-1}n_2^{-1}\sum_{i=1}^{n_2} a_{2i}, \\ \gamma_2^4(\bar{m}_2 - 1)^{-1}n_2^{-1}\sum_{i=1}^{n_2} a_{2i} & c_2^2\sigma_2^4 \end{bmatrix},$$

with $c_2^2 = 1 - 2(1-\rho_2)n_2^{-1}\sum_{i=1}^{n_2} a_{2i} + (1-\rho_2)^2\left[n_2^{-1}\sum_{i=1}^{n_2} a_{2i}^2 + (\bar{m}_2 - 1)^{-1}(n_2^{-1}\sum_{i=1}^{n_2} a_{2i})^2\right]$

and $\bar{m}_2 = n_2^{-1}\sum_{i=1}^{n_2} m_{2i}$.

Hence, the asymptotic variance of $\tilde{\rho}_2$ is

$$AV_2 = 2(1-\rho_2)^2\frac{1}{n_2}\left[(\bar{m}_2 - 1)^{-1} + c_2^2 - 2(1-\rho_2)(\bar{m}_2 - 1)^{-1}n_2^{-1}\sum_{i=1}^{n_2} a_{2i}\right].$$

In order to find the covariance of $\tilde{\rho}_1$ and $\tilde{\rho}_2$, consider

$$\sqrt{n}\begin{bmatrix} \tilde{\gamma}_1^2 - \gamma_1^2 \\ \tilde{\sigma}_1^2 - \sigma_1^2 \\ \tilde{\gamma}_2^2 - \gamma_2^2 \\ \tilde{\sigma}_2^2 - \sigma_2^2 \end{bmatrix} \to N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \delta_{11} & \delta_{12} & 0 & 0 \\ \delta_{12} & \delta_{22} & 0 & \delta_{24} \\ 0 & 0 & \delta_{33} & \delta_{34} \\ 0 & \delta_{24} & \delta_{34} & \delta_{44} \end{bmatrix}\right).$$

Expressions for the terms $\delta_{11}, \delta_{12}, \delta_{22}, \delta_{33}, \delta_{34}$, and δ_{44} are clearly specified in Σ_1 and Σ_2. We only need to specify δ_{24}. Using the distribution of $\begin{pmatrix} y_{1i} \\ y_{2i} \end{pmatrix}$ given in (5) and the distribution of the sample covariance matrix as given in Appendix A.2,

$$\begin{aligned} \delta_{24} &= Cov(\tilde{\sigma}_1^2, \tilde{\sigma}_2^2) \\ &= 2(n_{12}-1)^{-1}\sigma_{12}^2. \end{aligned}$$

Using the delta method, we have the following:

$$Cov(\tilde{\rho}_1, \tilde{\rho}_2) = 2\frac{1}{n_{12}-1}\rho_{12}^2(1-\rho_1)(1-\rho_2).$$

Let $\tilde{AV}(\tilde{\rho}_1)$ be the estimated $AV_1(\tilde{\rho}_1)$ obtained by substituting the alternative estimator $\tilde{\rho}_1$ for the unknown parameter, and let $\tilde{AV}(\tilde{\rho}_2)$ be the estimated $AV_2(\tilde{\rho}_2)$ obtained by substituting the alternative estimator $\tilde{\rho}_2$ for the unknown parameter. Also, let $\tilde{Cov}(\tilde{\rho}_1, \tilde{\rho}_2)$ be the estimated $Cov(\tilde{\rho}_1, \tilde{\rho}_2)$ obtained by substituting the alternative estimators, $\tilde{\rho}_1, \tilde{\rho}_2$, and $\tilde{\rho}_{12}$ for the unknown parameters.

In order to determine the variance of the second set of alternative estimators, consider the following asymptotic distributions

$$n_1^{1/2}\begin{bmatrix} \tilde{\gamma}_{1a}^2 - \gamma_1^2 \\ \tilde{\sigma}_{1a}^2 - \sigma_1^2 \end{bmatrix} \to N(0, \Sigma_{1a}),$$

where

$$\Sigma_{1a} = 2\begin{bmatrix} \gamma_1^4(\bar{m}_1 - 2)^{-1} & 0 \\ 0 & \sigma_1^4 \end{bmatrix},$$

$$\bar{m}_1 = n_1^{-1}\sum_{i=1}^{n_1} m_{1i}.$$

Using the delta method, the asymptotic variance of $\tilde{\rho}_{1a}$ is

$$AV_{1a} = 2(1-\rho_1)^2\frac{1}{n_1}\left[(\bar{m}_1 - 2)^{-1} + 1\right]. \tag{11}$$

Similarly, for the second alternative estimates based only on daughter scores

$$n_2^{1/2}\begin{bmatrix} \tilde{\gamma}_{2a}^2 - \gamma_2^2 \\ \tilde{\sigma}_{2a}^2 - \sigma_2^2 \end{bmatrix} \to N(0, \Sigma_{2a}),$$

where

$$\Sigma_{2a} = 2\begin{bmatrix} \gamma_2^4(\bar{m}_2-2)^{-1} & 0 \\ 0 & \sigma_2^4 \end{bmatrix},$$

$$\bar{m}_2 = n_2^{-1}\sum_{i=1}^{n_2} m_{2i}.$$

Therefore using the delta method, the asymptotic variance of $\tilde{\rho}_{2a}$ is

$$AV_{2a} = 2(1-\rho_2)^2\frac{1}{n_2}\left[(\bar{m}_2-2)^{-1}+1\right]. \tag{12}$$

In order to find the covariance of $\tilde{\rho}_{1a}$ and $\tilde{\rho}_{2a}$, consider

$$\sqrt{n}\begin{bmatrix} \tilde{\gamma}_{1a}^2-\gamma_1^2 \\ \tilde{\sigma}_{1a}^2-\sigma_1^2 \\ \tilde{\gamma}_{2a}^2-\gamma_2^2 \\ \tilde{\sigma}_{2a}^2-\sigma_2^2 \end{bmatrix} \to N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \delta_{11a} & 0 & 0 & 0 \\ 0 & \delta_{22a} & 0 & \delta_{24a} \\ 0 & 0 & \delta_{33a} & 0 \\ 0 & \delta_{24a} & 0 & \delta_{44a} \end{bmatrix}\right),$$

where $\delta_{11a}, \delta_{22a}, \delta_{33a}$, and δ_{44a} are already specified in Σ_{1a} and Σ_{2a}. We only need δ_{24a}. Using the distribution of $\begin{pmatrix} \bar{y}_{1i} \\ \bar{y}_{2i} \end{pmatrix}$ given in (8) and the distribution of the sample covariance matrix as given in Appendix A.2,

$$\begin{aligned}\delta_{24} &= Cov(\tilde{\sigma}_{1a}^2, \tilde{\sigma}_{2a}^2) \\ &= 2(n_{12}-1)^{-1}\sigma_{12}^2.\end{aligned}$$

Using the delta method, we have the following:

$$Cov(\tilde{\rho}_{1a}, \tilde{\rho}_{2a}) = 2\frac{1}{n_{12}-1}\rho_{12}^2(1-\rho_1)(1-\rho_2).$$

Let $\tilde{AV}(\tilde{\rho}_{1a})$ be the estimated $AV_{1a}(\tilde{\rho}_{1a})$ obtained by substituting the alternative estimator $\tilde{\rho}_{1a}$ for the unknown parameter, and let $\tilde{AV}(\tilde{\rho}_{2a})$ be the estimated $AV_{2a}(\tilde{\rho}_{2a})$ obtained by substituting the alternative estimator $\tilde{\rho}_{2a}$ for the unknown parameter. Also, let $\tilde{Cov}(\tilde{\rho}_{1a}, \tilde{\rho}_{2a})$ be the estimated $Cov(\tilde{\rho}_{1a}, \tilde{\rho}_{2a})$ obtained by substituting the alternative estimators, $\tilde{\rho}_{1a}, \tilde{\rho}_{2a}$, and $\tilde{\rho}_{12a}$ for the unknown parameters.

II.12 ALTERNATIVE TESTS

The two tests we propose are

$$TS_1 = \left(\frac{\tilde{\rho}_1-\tilde{\rho}_2}{S.E.(\tilde{\rho}_1-\tilde{\rho}_2)}\right)^2 \sim \chi_1^2, \tag{13}$$

where

$$S.E.(\tilde{\rho}_1 - \tilde{\rho}_2) = \tilde{AV}(\tilde{\rho}_1) + \tilde{AV}(\tilde{\rho}_2) - 2\tilde{Cov}(\tilde{\rho}_1, \tilde{\rho}_2),$$

and

$$TS_2 = \left(\frac{\tilde{\rho}_{1a} - \tilde{\rho}_{2a}}{S.E.(\tilde{\rho}_{1a} - \tilde{\rho}_{2a})}\right)^2 \sim \chi^2_1, \tag{14}$$

where

$$S.E.(\tilde{\rho}_{1a} - \tilde{\rho}_{2a}) = \tilde{AV}(\tilde{\rho}_{1a}) + \tilde{AV}(\tilde{\rho}_{2a}) - 2\tilde{Cov}(\tilde{\rho}_{1a}, \tilde{\rho}_{2a}).$$

The two alternative tests, TS_1 and TS_2, are simpler to implement than the LRT, Wald, and Score tests.

II.13 ANALYSIS OF GALTON'S DATA

An example of a familial data set on which these procedures can be implemented is Galton's data set on human stature. Galton collected family heights from family records and published his analysis on hereditary stature during the 1880s (Galton, 1886, 1889). Hanley (2004) worked directly with Galton's notebooks to make the raw familial data publicly available. Naik and Helu (2007) used Galton's data set as an illustration of their techniques to test the equality of independent intraclass correlation coefficients. To do this, Galton's data set was split into 2 groups from which they tested if the son intraclass correlation from one group of families equaled the daughter intraclass correlation from the other group of families. The tests proposed here allow one to test the null hypothesis of equal son and daughter intraclass correlation coefficients, $H_0 : \rho_1 = \rho_2$, while accounting for any dependency between the boys and girls of a family.

Galton's data set consists of heights from 205 families with children. Of these 205 families, 197 had numerical heights for all their children. The other 8 families had at least one child height recorded verbally, for example "tallish" (see Hanley 2004). Family sizes range from 1 to 15 with the number of sons ranging from 0 to 10 and the number of daughters ranging from 0 to 9. The distribution of family sizes is given in

TABLE 1. For Galton's data set, the maximum likelihood estimates $\boldsymbol{\theta}$ and $\boldsymbol{\theta}_0$ are

$$\hat{\boldsymbol{\theta}} = \begin{bmatrix} \hat{\mu}_1 \\ \hat{\mu}_2 \\ \hat{\sigma}_1^2 \\ \hat{\sigma}_2^2 \\ \hat{\rho}_1 \\ \hat{\rho}_2 \\ \hat{\rho}_{12} \end{bmatrix} = \begin{bmatrix} 69.305345 \\ 64.155985 \\ 7.1050891 \\ 5.3345446 \\ 0.3954526 \\ 0.4066034 \\ 0.3860179 \end{bmatrix}, \quad \hat{\boldsymbol{\theta}}_0 = \begin{bmatrix} \hat{\mu}_1 \\ \hat{\mu}_2 \\ \hat{\sigma}_1^2 \\ \hat{\sigma}_2^2 \\ \hat{\rho} \\ \hat{\rho} \\ \hat{\rho}_{12} \end{bmatrix} = \begin{bmatrix} 69.306173 \\ 64.155481 \\ 7.1242155 \\ 5.3212658 \\ 0.4008412 \\ 0.4008412 \\ 0.3863638 \end{bmatrix}.$$

The alternative estimates of $\boldsymbol{\theta}$ for Galton's data set are

$$\tilde{\boldsymbol{\theta}} = \begin{bmatrix} \tilde{\mu}_1 \\ \tilde{\mu}_2 \\ \tilde{\sigma}_1^2 \\ \tilde{\sigma}_2^2 \\ \tilde{\rho}_1 \\ \tilde{\rho}_2 \\ \tilde{\rho}_{12} \end{bmatrix} = \begin{bmatrix} 69.060144 \\ 63.824326 \\ 7.7280211 \\ 6.4325128 \\ 0.3754032 \\ 0.4427573 \\ 0.4087418 \end{bmatrix}, \quad \tilde{\boldsymbol{\theta}}_a = \begin{bmatrix} \tilde{\mu}_{1a} \\ \tilde{\mu}_{2a} \\ \tilde{\sigma}_1^{2a} \\ \tilde{\sigma}_2^{2a} \\ \tilde{\rho}_{1a} \\ \tilde{\rho}_{2a} \\ \tilde{\rho}_{12a} \end{bmatrix} = \begin{bmatrix} 67.785214 \\ 63.063426 \\ 6.6747275 \\ 5.9345296 \\ 0.3474392 \\ 0.4743360 \\ 0.4478932 \end{bmatrix}.$$

The results of the five proposed tests for the null hypothesis that the correlation between the boy heights equals the correlation between the girls heights , $H_0 : \rho_1 = \rho_2$, in Galton's data set are given in Table 2. All 5 tests fail to reject H_0 at the $\alpha = 0.05$ significance level.

TABLE 1: Frequency Table of Galton's Family Sizes.

	# of Daughters									
# of Sons	0	1	2	3	4	5	6	7	8	9
0	0	15	4	3	0	1	0	0	1	0
1	17	10	6	4	1	1	1	0	0	0
2	6	10	13	9	6	3	1	0	0	1
3	3	10	8	6	7	2	2	2	0	0
4	2	8	6	3	4	2	1	1	0	0
5	0	0	1	6	1	0	0	0	0	0
6	1	1	1	2	1	0	0	0	0	0
7	0	1	0	0	0	0	0	1	0	0
10	0	1	0	0	0	0	0	0	0	0

TABLE 2: Galton's Data, $H_0 : \rho_1 = \rho_2$.

Test	LRT	Score	Wald	TS_1	TS_2
Statistic	0.030	0.036	0.026	1.240	0.845
Pvalue	0.861	0.850	0.873	0.265	0.358

II.14 SIMULATION EXPERIMENTS AND RESULTS

All five tests are expected to behave similarly for large sample sizes, since they all have an asymptotic chi-square distribution with 1 degree of freedom. A good comparison of the tests is to assess their performance when applied to small samples. As previously noted, in order for the variance of $\tilde{\rho}_{1a}$ (11) and the variance of $\tilde{\rho}_{2a}$ (12) to be positive the average number of boys per family, $\bar{m}_1$, and the average number of girls per family, $\bar{m}_2$, need to be greater than 2. Therefore, two simulation experiments were designed to examine the small sample performance of the tests. The first simulation experiment has smaller family sizes and compares the LRT (2), Score (4), Wald (3), and the first proposed test, TS_1 (13). The second experiment compares all five tests: LRT (2), Wald (3), Score (4), TS_1 (13), and the second proposed test, TS_2 (14). In both simulation studies, only positive values of the familial correlations, ρ_1, ρ_2, and ρ_{12}, are considered because the model constraint (1) restricts the negative values the familial correlations can attain based on a family's size.

For the first experiment, 50 family scores are simulated as multivariate normal random vectors. The family size for each vector is simulated from a truncated negative binomial distribution with the number of children ranging from 1 to 15. The mean of the negative binomial distribution is taken as 2.84 and the success probability as 0.483. This distribution was suggested by Brass (1958) as the distribution of U.S. births and has been used in several other previous simulation experiments including Rosner, Donner, and Hennekens (1977), Srivastava and Keen (1988), Young and Bhandary (1998), and Naik and Helu (2007). Gender was then assigned to each child in the family using a discrete uniform distribution. The choices of parameters are $\mu_1 = 0$, $\mu_2 = 0$, $\sigma_1^2 = 1$, $\sigma_2^2 = 2$, and ρ_1 and ρ_2 range from 0.1 to 0.9 by increments of 0.1. The interclass correlation ρ_{12} is set as the midpoint between 0 and the upper bound for ρ_{12}:

$$\rho_{12} = \frac{\sqrt{(\rho_1 + (1-\rho_1)/m_1)(\rho_2 + (1-\rho_2)/m_2)}}{2},$$

where m_1 is the maximum number of boys in the simulation and m_2 is the maximum number of girls in the simulation. For each choice of parameters, 10,000 simulations were run. For an arbitrary case of $\rho_1 = \rho_2 = 0.5$, Table 3 shows the family size distribution, Table 4 shows the distribution of the number of sons per family, and Table 5 shows the distribution of the number of daughters per family. For this case, the simulations produced 866,284 boys total and 866,584 girls total with an average of 1.73 boys and 1.73 girls per family.

TABLE 3: Distribution of Family Sizes, First Simulation Experiment.

# Children	# of Families
1	106089
2	105862
3	88523
4	66841
5	47089
6	31678
7	20764
8	13048
9	8194
10	5036
11	3076
12	1792
13	1036
14	617
15	355

For each choice of ρ_1 and ρ_2, estimated size and power values are computed for testing, $H_0 : \rho_1 = \rho_2 = \rho$. Tables 6-8 give the estimated sizes for $\alpha = 0.01, 0.05,$ and 0.10, respectively. Table 6 also gives the percentage of simulations for which the maximum likelihood procedure did not converge and the percentage of simulations for which the alternative estimates, $\tilde{\rho}_1, \tilde{\rho}_2$, and $\tilde{\rho}_{12}$, violated the model constraint (1). These percentages apply for all 3 size tables, since the different sizes are estimated from the same run. Similar percentages are also given in the rejection proportion tables.

TABLE 4: Distribution of Male Family Sizes, First Simulation Experiment.

# Sons	# of Families
0	96819
1	168088
2	112797
3	63089
4	32165
5	15423
6	6672
7	3045
8	1218
9	457
10	159
11	52
12	14
13	2
14	0
15	0

TABLE 5: Distribution of Female Family Sizes, First Simulation Experiment.

# Daughters	# of Families
0	96960
1	168224
2	112229
3	63274
4	32034
5	15468
6	6939
7	2966
8	1246
9	437
10	156
11	57
12	10
13	0
14	0
15	0

TABLE 6: Sizes, $\alpha = 0.01$, $H_0 : \rho_1 = \rho_2 = \rho$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)
0.1	0.0103 (5.41%)	0.0114	0.0126	0.0114 (35.90%)
0.2	0.0146 (2.98%)	0.0123	0.0154	0.0097 (21.33%)
0.3	0.0182 (2.24%)	0.0141	0.0164	0.0112 (12.13%)
0.4	0.0161 (1.27%)	0.0140	0.0183	0.0093 (5.94%)
0.5	0.0158 (0.61%)	0.0123	0.0317	0.0120 (2.05%)
0.6	0.0131 (0.32%)	0.0104	0.0499	0.0108 (0.86%)
0.7	0.0115 (0.16%)	0.0123	0.0746	0.0114 (0.26%)
0.8	0.0118 (0.11%)	0.0329	0.0601	0.0107 (0.09%)
0.9	0.0121 (0.40%)	0.0842	0.0159	0.0111 (0.11%)

TABLE 7: Sizes, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$.

ρ_2	LRT	Score	Wald	TS_1
0.1	0.0548	0.0391	0.0642	0.0520
0.2	0.0644	0.0534	0.0605	0.0521
0.3	0.0678	0.0642	0.0625	0.0542
0.4	0.0667	0.0561	0.0691	0.0552
0.5	0.0650	0.0501	0.0900	0.0553
0.6	0.0590	0.0374	0.1123	0.0536
0.7	0.0546	0.0321	0.1365	0.0557
0.8	0.0548	0.0508	0.0954	0.0552
0.9	0.0587	0.1153	0.0261	0.0618

The following observations can be made. The Score test tends to be larger than the assumed level except for $\rho_1 = \rho_2 = 0.9$ for which the Score test is notably lower for the $\alpha = 0.05$ and 0.10 levels. Wald's test is erratic as estimated sizes are sometimes larger than the assumed level and other times lower than the assumed level. The LRT performs well although it tends to be slightly larger than the assumed level. The alternative test TS_1 also performs well and tends to be slightly larger than the

TABLE 8: Sizes, $\alpha = 0.10$, $H_0 : \rho_1 = \rho_2 = \rho$.

ρ_2	LRT	Score	Wald	TS_1
0.1	0.1020	0.0811	0.1255	0.1098
0.2	0.1226	0.1100	0.1170	0.1056
0.3	0.1284	0.1218	0.1204	0.1072
0.4	0.1229	0.1102	0.1271	0.1106
0.5	0.1188	0.0928	0.1496	0.1128
0.6	0.1052	0.0688	0.1738	0.1041
0.7	0.1045	0.0559	0.1921	0.1086
0.8	0.1072	0.0674	0.1260	0.1171
0.9	0.1113	0.1325	0.0337	0.1205

assumed level, but generally closer to the assumed level than the LRT. Specifically, TS_1 is closest to the assumed level in 15 of the 27 cases. When comparing TS_1 to only the LRT, TS_1 is closer to the assumed level than the LRT in 18 of the 27 cases.

Tables 9-17 give estimated power values adjusted to the level each test attained in the size calculations. For each table, the rejection proportions are based on the 95^{th} percentiles of the test statistics from the size simulation for the value of ρ_1. For example, Table 9 shows the proportion of simulations with test statistics greater than 3.98822 for the LRT, 4.26378 for the Score test, 3.48331 for Wald's test, and 3.91926 for TS_1 which were the 95^{th} percentiles from the simulation of $H_0 : \rho_1 = \rho_2 = 0.1$.

TABLE 9: Adjusted Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.1$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)
0.1	0.0548 (5.41%)	0.0391	0.0642	0.0520 (35.90%)
0.2	0.0812 (3.57%)	0.0803	0.0739	0.0625 (29.03%)
0.3	0.1899 (2.51%)	0.1949	0.1641	0.1232 (24.73%)
0.4	0.3635 (1.63%)	0.3821	0.3220	0.2365 (22.33%)
0.5	0.5902 (1.28%)	0.6104	0.5417	0.4216 (20.83%)
0.6	0.8019 (1.02%)	0.8139	0.7698	0.6547 (19.38%)
0.7	0.9410 (0.57%)	0.9355	0.9217	0.8666 (18.86%)
0.8	0.9937 (0.68%)	0.9532	0.9642	0.9783 (18.37%)
0.9	0.9996 (0.96%)	0.8023	0.8196	0.9999 (18.47%)

TABLE 10: Adjusted Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.2$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_2	**LRT (*)**	**Score**	**Wald**	TS_1 **(**)**
0.1	0.0655 (3.35%)	0.0624	0.0756	0.0626 (28.21%)
0.2	0.0644 (2.98%)	0.0534	0.0605	0.0521 (21.33%)
0.3	0.0816 (2.43%)	0.0811	0.0784	0.0685 (17.08%)
0.4	0.1775 (2.02%)	0.1755	0.1732	0.1372 (14.05%)
0.5	0.3622 (1.45%)	0.3623	0.3672	0.2863 (12.57%)
0.6	0.6019 (1.06%)	0.5906	0.6208	0.5145 (10.95%)
0.7	0.8438 (0.89%)	0.8065	0.8562	0.7793 (10.23%)
0.8	0.9777 (0.63%)	0.8936	0.9360	0.9590 (9.36%)
0.9	0.9986 (1.04%)	0.7493	0.7797	0.9993 (9.35%)

TABLE 11: Adjusted Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.3$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_2	**LRT (*)**	**Score**	**Wald**	TS_1 **(**)**
0.1	0.1515 (2.27%)	0.1313	0.1628	0.1190 (25.30%)
0.2	0.0782 (2.40%)	0.0711	0.0780	0.0680 (17.49%)
0.3	0.0678 (2.24%)	0.0642	0.0625	0.0542 (12.13%)
0.4	0.0813 (1.52%)	0.0745	0.0901	0.0774 (9.35%)
0.5	0.1810 (1.17%)	0.1621	0.2097	0.1634 (7.18%)
0.6	0.3959 (0.76%)	0.3465	0.4476	0.3622 (6.14%)
0.7	0.6914 (1.03%)	0.5853	0.7392	0.6537 (5.48%)
0.8	0.9303 (0.54%)	0.7598	0.8838	0.9122 (4.80%)
0.9	0.9981 (0.78%)	0.6589	0.7132	0.9974 (4.38%)

TABLE 12: Adjusted Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.4$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)
0.1	0.3287 (1.56%)	0.3104	0.3049	0.2325 (21.99%)
0.2	0.1851 (1.79%)	0.1785	0.1721	0.1459 (14.13%)
0.3	0.0887 (1.83%)	0.0844	0.0790	0.0767 (9.19%)
0.4	0.0667 (1.27%)	0.0561	0.0691	0.0552 (5.94%)
0.5	0.0832 (0.76%)	0.0765	0.0966	0.0778 (3.64%)
0.6	0.2209 (0.78%)	0.1834	0.2603	0.2099 (3.27%)
0.7	0.5103 (0.40%)	0.3899	0.5652	0.4841 (2.27%)
0.8	0.8487 (0.29%)	0.6024	0.7935	0.8296 (1.85%)
0.9	0.9955 (0.66%)	0.5812	0.6474	0.9940 (1.76%)

TABLE 13: Adjusted Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.5$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)
0.1	0.5632 (0.99%)	0.5797	0.4528	0.4166 (20.63%)
0.2	0.3551 (1.60%)	0.3707	0.2757	0.2792 (12.67%)
0.3	0.1865 (1.40%)	0.1991	0.1424	0.1558 (6.70%)
0.4	0.0834 (0.89%)	0.0870	0.0686	0.0826 (3.90%)
0.5	0.0650 (0.61%)	0.0501	0.0900	0.0553 (2.05%)
0.6	0.0907 (0.47%)	0.0788	0.1029	0.0937 (1.72%)
0.7	0.2801 (0.25%)	0.2057	0.3103	0.2765 (0.91%)
0.8	0.6858 (0.25%)	0.4136	0.6146	0.6719 (0.76%)
0.9	0.9810 (0.52%)	0.4785	0.5535	0.9778 (0.74%)

TABLE 14: Adjusted Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.6$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_2	**LRT** (*)	**Score**	**Wald**	$\boldsymbol{TS_1}$ (**)
0.1	0.7961 (0.91%)	0.8344	0.6008	0.6503 (20.22%)
0.2	0.6184 (0.97%)	0.6520	0.4289	0.5007 (11.14%)
0.3	0.4269 (0.99%)	0.4489	0.2763	0.3519 (5.90%)
0.4	0.2355 (0.61%)	0.2412	0.1516	0.2116 (2.95%)
0.5	0.1010 (0.21%)	0.1016	0.0729	0.0978 (1.30%)
0.6	0.0590 (0.32%)	0.0374	0.1123	0.0536 (0.86%)
0.7	0.1246 (0.19%)	0.0935	0.1220	0.1288 (0.16%)
0.8	0.4497 (0.07%)	0.2554	0.3535	0.4505 (0.33%)
0.9	0.9370 (0.31%)	0.3961	0.4255	0.9294 (0.28%)

TABLE 15: Adjusted Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.7$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_2	**LRT** (*)	**Score**	**Wald**	$\boldsymbol{TS_1}$ (**)
0.1	0.9406 (0.78%)	0.9525	0.7256	0.8599 (19.61%)
0.2	0.8632 (0.84%)	0.8708	0.5837	0.7708 (10.68%)
0.3	0.7253 (0.75%)	0.7388	0.4330	0.6440 (5.26%)
0.4	0.5396 (0.55%)	0.5226	0.2804	0.4725 (2.56%)
0.5	0.3092 (0.31%)	0.2811	0.1550	0.2731 (1.18%)
0.6	0.1254 (0.22%)	0.1141	0.0792	0.1150 (0.48%)
0.7	0.0546 (0.16%)	0.0321	0.1365	0.0557 (0.26%)
0.8	0.1940 (0.08%)	0.1234	0.1273	0.1892 (0.13%)
0.9	0.8000 (0.31%)	0.3058	0.2410	0.7878 (0.06%)

TABLE 16: Adjusted Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.8$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_2	**LRT** (*)	**Score**	**Wald**	$\boldsymbol{TS_1}$ (**)
0.1	0.9928 (0.61%)	0.9435	0.9260	0.9802 (18.68%)
0.2	0.9778 (0.75%)	0.8844	0.8684	0.9551 (9.83%)
0.3	0.9483 (0.59%)	0.7884	0.7942	0.9179 (4.92%)
0.4	0.8662 (0.31%)	0.6278	0.6567	0.8291 (1.74%)
0.5	0.6969 (0.20%)	0.4039	0.4760	0.6585 (0.72%)
0.6	0.4574 (0.09%)	0.2222	0.2987	0.4328 (0.32%)
0.7	0.1918 (0.06%)	0.0928	0.1275	0.1874 (0.17%)
0.8	0.0548 (0.11%)	0.0508	0.0954	0.0552 (0.09%)
0.9	0.4082 (0.46%)	0.1810	0.0953	0.4031 (0.05%)

TABLE 17: Adjusted Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.9$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)
0.1	0.9997 (0.72%)	0.6630	0.8268	0.9993 (17.79%)
0.2	0.9995 (0.87%)	0.5557	0.7791	0.9987 (9.70%)
0.3	0.9989 (0.65%)	0.4316	0.7276	0.9976 (4.46%)
0.4	0.9959 (0.32%)	0.3273	0.6621	0.9929 (1.92%)
0.5	0.9856 (0.18%)	0.2458	0.5814	0.9763 (0.67%)
0.6	0.9464 (0.15%)	0.1797	0.4996	0.9313 (0.33%)
0.7	0.7947 (0.04%)	0.1329	0.3832	0.7745 (0.08%)
0.8	0.4089 (0.06%)	0.1002	0.2049	0.3923 (0.09%)
0.9	0.0587 (0.40%)	0.1153	0.0261	0.0618 (0.11%)

Since only the LRT and TS_1 performed consistently well in the size calculations, it is reasonable to only compare power calculations of the LRT and TS_1. From the tables, one can see that the LRT achieves higher power levels than TS_1, but TS_1 is not far behind. In 32% of the simulations, the power of TS_1 is greater than the LRT or within 0.01. In 76% of the simulations, the power of TS_1 is greater than the LRT or within 0.05.

Tables 18-26 give the power values for a nominal level $\alpha = 0.05$.

TABLE 18: Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.1$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)
0.1	0.0548 (5.41%)	0.0391	0.0642	0.0520 (35.90%)
0.2	0.0871 (3.57%)	0.0635	0.0886	0.0656 (29.03%)
0.3	0.2008 (2.51%)	0.1673	0.1903	1.1237 (24.73%)
0.4	0.3799 (1.63%)	0.3374	0.3610	0.2439 (22.33%)
0.5	0.6028 (1.28%)	0.5687	0.5839	0.4295 (20.83%)
0.6	0.8108 (1.02%)	0.7919	0.7956	0.6631 (19.38%)
0.7	0.9442 (0.57%)	0.9201	0.9337	0.8702 (18.86%)
0.8	0.9944 (0.68%)	0.9494	0.9654	0.9790 (18.37%)
0.9	0.9996 (0.96%)	0.8002	0.8205	0.9999 (18.47%)

TABLE 19: Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.2$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)
0.1	0.0856 (3.35%)	0.0663	0.0919	0.0648 (28.21%)
0.2	0.0644 (2.98%)	0.0534	0.0605	0.0521 (21.33%)
0.3	0.1020 (2.43%)	0.0863	0.0932	0.0711 (17.08%)
0.4	0.2066 (2.02%)	0.1847	0.1997	0.1412 (14.05%)
0.5	0.4043 (1.45%)	0.3736	0.4004	0.2922 (12.57%)
0.6	0.6486 (1.06%)	0.6007	0.6539	0.5201 (10.95%)
0.7	0.8681 (0.89%)	0.8124	0.8729	0.7837 (10.23%)
0.8	0.9830 (0.63%)	0.8964	0.9394	0.9603 (9.36%)
0.9	0.9986 (1.04%)	0.7508	0.7808	0.9995 (9.35%)

TABLE 20: Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.3$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)
0.1	0.1977 (2.27%)	0.1645	0.1887	0.1246 (25.30%)
0.2	0.1041 (2.40%)	0.0924	0.0948	0.0724 (17.49%)
0.3	0.0678 (2.24%)	0.0642	0.0625	0.0542 (12.13%)
0.4	0.1099 (1.52%)	0.0974	0.1055	0.0820 (9.35%)
0.5	0.2232 (1.17%)	0.1955	0.2383	0.1725 (7.18%)
0.6	0.5431 (0.76%)	0.3925	0.4850	0.3732 (6.14%)
0.7	0.7415 (1.03%)	0.6306	0.7647	0.6644 (5.48%)
0.8	0.9487 (0.54%)	0.7881	0.8920	0.9174 (4.80%)
0.9	0.9991 (0.78%)	0.6684	0.7155	0.9980 (4.38%)

TABLE 21: Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.4$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)
0.1	0.3743 (1.56%)	0.3348	0.3543	0.2399 (21.99%)
0.2	0.2193 (1.79%)	0.1950	0.2093	0.1523 (14.13%)
0.3	0.1086 (1.83%)	0.0965	0.1032	0.0803 (9.19%)
0.4	0.0667 (1.27%)	0.0561	0.0691	0.0552 (5.94%)
0.5	0.1020 (0.76%)	0.0843	0.1220	0.0815 (3.64%)
0.6	0.2577 (0.78%)	0.1984	0.3062	0.2170 (3.27%)
0.7	0.5540 (0.40%)	0.4103	0.6094	0.4935 (2.27%)
0.8	0.8711 (0.29%)	0.6202	0.8126	0.8365 (1.85%)
0.9	0.9967 (0.66%)	0.5872	0.6507	0.9944 (1.76%)

TABLE 22: Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.5$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)
0.1	0.6077 (0.99%)	0.5799	0.5883	0.4332 (20.63%)
0.2	0.3999 (1.60%)	0.3709	0.3967	0.2963 (12.67%)
0.3	0.2247 (1.40%)	0.1993	0.2367	0.1670 (6.70%)
0.4	0.1040 (0.89%)	0.0870	0.1229	0.0900 (3.90%)
0.5	0.0650 (0.61%)	0.0501	0.0900	0.0553 (2.05%)
0.6	0.1122 (0.47%)	0.0788	0.1672	0.1024 (1.72%)
0.7	0.3226 (0.25%)	0.2059	0.4068	0.2930 (0.91%)
0.8	0.7250 (0.25%)	0.4141	0.6805	0.6888 (0.76%)
0.9	0.9863 (0.52%)	0.4788	0.5647	0.9810 (0.74%)

TABLE 23: Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.6$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)
0.1	0.8146 (0.91%)	0.7941	0.7998	0.6630 (20.22%)
0.2	0.6420 (0.97%)	0.5961	0.6433	0.5134 (11.14%)
0.3	0.4575 (0.99%)	0.3923	0.4843	0.3637 (5.90%)
0.4	0.2576 (0.61%)	0.1998	0.3047	0.2184 (2.95%)
0.5	0.1145 (0.21%)	0.0794	0.1673	0.1030 (1.30%)
0.6	0.0590 (0.32%)	0.0374	0.1123	0.0536 (0.86%)
0.7	0.1386 (0.19%)	0.0747	0.2220	0.1343 (0.61%)
0.8	0.4783 (0.07%)	0.2225	0.4800	0.4613 (0.33%)
0.9	0.9455 (0.31%)	0.3756	0.4622	0.9335 (0.28%)

TABLE 24: Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.7$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)
0.1	0.9452 (0.78%)	0.9221	0.9347	0.8681 (19.61%)
0.2	0.8714 (0.84%)	0.8116	0.8755	0.7842 (10.68%)
0.3	0.7374 (0.75%)	0.6361	0.7654	0.6635 (5.26%)
0.4	0.5540 (0.55%)	0.4124	0.6049	0.4893 (2.56%)
0.5	0.3255 (0.31%)	0.2036	0.4003	0.2870 (1.18%)
0.6	0.1356 (0.22%)	0.0729	0.2155	0.1262 (0.48%)
0.7	0.0546 (0.16%)	0.0321	0.1365	0.0557 (0.26%)
0.8	0.2058 (0.08%)	0.0924	0.2472	0.2013 (0.13%)
0.9	0.8105 (0.31%)	0.2741	0.3278	0.7988 (0.06%)

TABLE 25: Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.8$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)
0.1	0.9940 (0.61%)	0.9443	0.9639	0.9823 (18.68%)
0.2	0.9797 (0.75%)	0.8866	0.9297	0.9587 (9.83%)
0.3	0.9525 (0.59%)	0.7922	0.8973	0.9241 (4.92%)
0.4	0.8765 (0.31%)	0.6320	0.8091	0.8403 (1.74%)
0.5	0.7125 (0.20%)	0.4097	0.6662	0.6763 (0.72%)
0.6	0.4773 (0.09%)	0.2261	0.4743	0.4481 (0.32%)
0.7	0.2055 (0.06%)	0.0947	0.2337	0.2013 (0.17%)
0.8	0.0548 (0.11%)	0.0508	0.0954	0.0552 (0.09%)
0.9	0.4248 (0.46%)	0.1810	0.1467	0.4031 (0.05%)

TABLE 26: Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.9$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)
0.1	0.9997 (0.72%)	0.7991	0.8205	0.9993 (17.79%)
0.2	0.9996 (0.87%)	0.7429	0.7713	0.9990 (9.70%)
0.3	0.9990 (0.65%)	0.6718	0.7160	0.9983 (4.46%)
0.4	0.9964 (0.32%)	0.5907	0.6453	0.9940 (1.92%)
0.5	0.9874 (0.18%)	0.4842	0.5551	0.9815 (0.67%)
0.6	0.9543 (0.15%)	0.3893	0.4593	0.9429 (0.33%)
0.7	0.8133 (0.04%)	0.2759	0.3234	0.8001 (0.08%)
0.8	0.4367 (0.06%)	0.1736	0.1454	0.4297 (0.09%)
0.9	0.0587 (0.40%)	0.1153	0.0261	0.0618 (0.11%)

It can be seen that similar to the adjusted power levels, the LRT achieves higher power levels than TS_1 but TS_1 is not too far behind. Since the estimated sizes for the LRT are generally larger than the estimated sizes for TS_1 one can expect the LRT powers to have an advantage when not adjusted for the size levels the tests achieved.

For the second experiment, 50 family scores are simulated as multivariate normal random vectors. The family size for each vector is simulated from a truncated negative binomial distribution with the number of children ranging from 1 to 15. The mean of the negative binomial distribution is taken as 6.72 and the success probability as 0.302. This is the estimated distribution of Australian births as proposed by Brass (1958). This distribution has larger family sizes which satisfies the requirements for TS_2 that both the average number of boys and the average number of girls are greater than 2. The discrete uniform distribution was used to assign gender to each child. Again, the choices of the parameters are $\mu_1 = 0$, $\mu_2 = 0$, $\sigma_1^2 = 1$, $\sigma_2^2 = 0$, and ρ_1 and ρ_2 range from 0.1 to 0.9 by increments of 0.1. The interclass correlation ρ_{12} is set as the midpoint between 0 and the upper bound for ρ_{12}:

$$\rho_{12} = \frac{\sqrt{(\rho_1 + (1-\rho_1)/m_1)(\rho_2 + (1-\rho_2)/m_2)}}{2},$$

where m_1 is the maximum number of boys in the simulation and m_2 is the maximum number of girls in the simulation. For each choice of parameters, 10,000 simulations were run. For an arbitrary case of $\rho_1 = \rho_2 = 0.5$, Table 27 shows the family size distribution, Table 28 shows the distribution of the number of sons per family, and Table 29 shows the distribution of the number of daughters per family. For this case, the simulations produced 2,600,132 boys and 2,602,861 girls with an average of 5.2 boys and 5.2 girls per family.

TABLE 27: Distribution of Family Sizes, Second Simulation Experiment.

# of Children	# of Families
1	1413
2	3659
3	7388
4	12570
5	19013
6	25736
7	32589
8	39160
9	44655
10	49212
11	52213
12	54140
13	53956
14	52972
15	51324

For each choice of ρ_1 and ρ_2, estimated size and power values are computed for testing, $H_0 : \rho_1 = \rho_2 = \rho$. Tables 30-32 give the estimated sizes for $\alpha = 0.01, 0.05$, and 0.10, respectively. Table 30 also gives the percentage of simulations for which the MLE procedure did not converge, the percentage of simulations for which the alternative estimates, $\tilde{\rho}_1, \tilde{\rho}_2$, and $\tilde{\rho}_{12}$, violate the model constraint (1), and the percentage of simulations for which the alternative estimates, $\tilde{\rho}_{1a}, \tilde{\rho}_{2a}$, and $\tilde{\rho}_{12a}$, violate the model constraint (1). These percentages apply for all 3 size tables, since the different sizes are estimated from the same run. Similar percentages are also given in the rejection proportion tables to come.

TABLE 28: Distribution of Male Family Sizes, Second Simulation Experiment.

# of Sons	# of Families
0	4968
1	18776
2	38704
3	59271
4	75789
5	81882
6	76767
7	61476
8	42035
9	23918
10	11064
11	4022
12	1085
13	212
14	30
15	1

TABLE 29: Distribution of Female Family Sizes, Second Simulation Experiment.

# of Daughters	# of Families
0	4881
1	18612
2	38616
3	59561
4	75468
5	81750
6	76767
7	61669
8	42171
9	24066
10	11022
11	4032
12	1146
13	205
14	34
15	0

TABLE 30: Sizes, $\alpha = 0.01$, $H_0 : \rho_1 = \rho_2 = \rho$.

*Percent that did not converge; **Percent that violated the model constraints; ***Percent that violated model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)	TS_2 (***)
0.1	0.0109 (0.17%)	0.0073	0.0095	0.0097 (4.15%)	0.0074 (14.03%)
0.2	0.0131 (0.04%)	0.0116	0.0107	0.0101 (0.45%)	0.0092 (7.55%)
0.3	0.0131 (0.03%)	0.0124	0.0109	0.0117 (0.03%)	0.0070 (2.83%)
0.4	0.0117 (0.09%)	0.0113	0.0093	0.0112 (0%)	0.0099 (1.06%)
0.5	0.0117 (0.08%)	0.0120	0.0099	0.0114 (0%)	0.0087 (0.24%)
0.6	0.0107 (0.07%)	0.0112	0.0100	0.0098 (0%)	0.0079 (0.06%)
0.7	0.0111 (0.10%)	0.0115	0.0139	0.0108 (0%)	0.0095 (0%)
0.8	0.0123 (0.15%)	0.0152	0.0274	0.0108 (0%)	0.0113 (0%)
0.9	0.0130 (0.22%)	0.0387	0.0372	0.0119 (0%)	0.0092 (0%)

TABLE 31: Sizes, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$.

ρ_2	LRT	Score	Wald	TS_1	TS_2
0.1	0.0589	0.0510	0.0551	0.0494	0.0500
0.2	0.0585	0.0563	0.0506	0.0507	0.0498
0.3	0.0588	0.0580	0.0519	0.0544	0.0530
0.4	0.0545	0.0549	0.0502	0.0501	0.0473
0.5	0.0559	0.0562	0.0525	0.0537	0.0508
0.6	0.0542	0.0528	0.0536	0.0523	0.0521
0.7	0.0558	0.0524	0.0611	0.0548	0.0490
0.8	0.0587	0.0521	0.0832	0.0567	0.0507
0.9	0.0592	0.0798	0.0804	0.0559	0.0512

TABLE 32: Sizes, $\alpha = 0.10$, $H_0 : \rho_1 = \rho_2 = \rho$.

ρ_2	LRT	Score	Wald	TS_1	TS_2
0.1	0.1105	0.1056	0.1066	0.0978	0.1027
0.2	0.1096	0.1092	0.1030	0.0991	0.1059
0.3	0.1130	0.1126	0.1044	0.1087	0.1069
0.4	0.1041	0.1057	0.0978	0.1000	0.1037
0.5	0.1054	0.1062	0.1023	0.1030	0.1085
0.6	0.1039	0.1039	0.1034	0.1017	0.1028
0.7	0.1097	0.1018	0.1187	0.1082	0.1042
0.8	0.1121	0.0997	0.1381	0.1092	0.1068
0.9	0.1124	0.1217	0.1220	0.1089	0.1062

The following observations can be made. The Score and Wald tests perform well except for large values ρ for which the tests are larger than the assumed level. The LRT performs well, although it is uniformly larger than the assumed level. Both the alternative tests, TS_1 and TS_2 perform well. Specifically, TS_1 is closest to the assumed level in 7 of the 27 cases and TS_2 is closest to the assumed level in 11 of the 27 cases. When comparing the alternative tests, TS_1 and TS_2, to only the LRT, TS_1 is closest to the assumed level 11 of the 27 cases and TS_2 is closest to the assumed level in the other 16 of the 27 cases. The LRT was not closer to the assumed level than the alternative estimates in any of the simulations.

Tables 33-41 give estimated power values adjusted to the level each test attained in the size calculations. For each table, the rejection proportions are based on the 95^{th} percentiles of the test statistics from the size simulations for the value of ρ_1. For example, Table 33 shows the proportion of simulations with test statistics greater than 4.17172 for the LRT, 4.00311 for the Score test, 3.86216 for the Wald test, 3.80180 for TS_1, and 3.84007 for TS_2 which were the 95^{th} percentiles from the simulation of $H_0 : \rho_1 = \rho_2 = 0.1$.

TABLE 33: Adjusted Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.1$.

*Percent that did not converge; **Percent that violated the model constraints; ***Percent that violated model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)	TS_2 (***)
0.1	0.0589 (0.17%)	0.0510	0.0551	0.0494 (4.15%)	0.0500 (14.03%)
0.2	0.2234 (0.04%)	0.2401	0.2217	0.1683 (1.74%)	0.0614 (11.02%)
0.3	0.6259 (0.02%)	0.6526	0.6227	0.5044 (1.43%)	0.1173 (8.35%)
0.4	0.9114 (0%)	0.9214	0.9099	0.8357 (1.19%)	0.2301 (6.95%)
0.5	0.9906 (0%)	0.9925	0.9904	0.9751 (1.19%)	0.4326 (6.94%)
0.6	0.9998 (0%)	0.9998	0.9998	0.9991 (1.04%)	0.6943 (6.45%)
0.7	1.0000 (0.01%)	1.0000	1.0000	1.0000 (0.97%)	0.9177 (0.97%)
0.8	1.0000 (0%)	0.9994	0.9994	1.0000 (0.99%)	0.9962 (6.09%)
0.9	1.0000 (0%)	0.9653	0.9651	1.0000 (0.83%)	1.0000 (5.81%)

TABLE 34: Adjusted Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.2$.

*Percent that did not converge; **Percent that violated the model constraints; ***Percent that violated model constraints.

ρ_2	**LRT** (*)	**Score**	**Wald**	TS_1 (**)	TS_2 (***)
0.1	0.2376 (0.04%)	0.2358	0.2349	0.1674 (1.82%)	0.0644 (10.88%)
0.2	0.0585 (0.04%)	0.0563	0.0506	0.0507 (0.45%)	0.0498 (7.55%)
0.3	0.1882 (0.04%)	0.1901	0.1880	0.1622 (0.10%)	0.0670 (0.10%)
0.4	0.5852 (0%)	0.5852	0.5841	0.5111 (0.05%)	0.1377 (4.21%)
0.5	0.8992 (0%)	0.8999	0.8994	0.8484 (0.06%)	0.2954 (3.47%)
0.6	0.9910 (0%)	0.9918	0.9909	0.9812 (0.07%)	0.5522 (2.87%)
0.7	0.9998 (0%)	0.9998	0.9998	0.9996 (0.05%)	0.8515 (3.25%)
0.8	1.0000 (0%)	0.9999	0.9999	1.0000 (0.05%)	0.9873 (2.58%)
0.9	1.0000 (0%)	0.9627	0.9631	1.0000 (0.03%)	1.0000 (2.39%)

TABLE 35: Adjusted Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.3$.

*Percent that did not converge; **Percent that violated the model constraints; ***Percent that violated model constraints.

ρ_2	**LRT** (*)	**Score**	**Wald**	TS_1 (**)	TS_2 (***)
0.1	0.6534 (0%)	0.6466	0.6510	0.5057 (1.45%)	0.1099 (8.76%)
0.2	0.1951 (0.03%)	0.1907	0.1946	0.1581 (0.14%)	0.0641 (5.37%)
0.3	0.0588 (0.03%)	0.0580	0.0519	0.0544 (0.03%)	0.0530 (2.83%)
0.4	0.1797 (0.02%)	0.1776	0.1812	0.1690 (0%)	0.0692 (2.04%)
0.5	0.5630 (0%)	0.5573	0.5635	0.5199 (0%)	0.1705 (1.52%)
0.6	0.9039 (0%)	0.9021	0.9062	0.8777 (0.01%)	0.4076 (1.42%)
0.7	0.9952 (0%)	0.9952	0.9957	0.9915 (0.01%)	0.7340 (1.25%)
0.8	1.0000 (0%)	0.9991	0.9994	0.9999 (0%)	0.9667 (0.96%)
0.9	1.0000 (0%)	0.9562	0.9568	1.0000 (0%)	0.9999 (0.70%)

TABLE 36: Adjusted Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.4$.

*Percent that did not converge; **Percent that violated the model constraints; ***Percent that violated model constraints.

ρ_2	**LRT** (*)	**Score**	**Wald**	TS_1 (**)	TS_2 (***)
0.1	0.9235 (0.01%)	0.9225	0.9220	0.8414 (1.37%)	0.2331 (7.31%)
0.2	0.5796 (0.01%)	0.5786	0.5768	0.5095 (0.12%)	0.1428 (4.22%)
0.3	0.1919 (0.05%)	0.1905	0.1931	0.1801 (0%)	0.0800 (2.24%)
0.4	0.0545 (0.09%)	0.0549	0.0502	0.0501 (0%)	0.0473 (1.06%)
0.5	0.1934 (0.02%)	0.1928	0.1965	0.1900 (0%)	0.0847 (0.73%)
0.6	0.6101 (0.01%)	0.6043	0.6159	0.5877 (0%)	0.2403 (0.53%)
0.7	0.9464 (0%)	0.9441	0.9505	0.9406 (0%)	0.5846 (0.26%)
0.8	0.9993 (0%)	0.9971	0.9978	0.9991 (0%)	0.9351 (0.18%)
0.9	1.0000 (0%)	0.9469	0.9475	1.0000 (0%)	0.9999 (0.35%)

TABLE 37: Adjusted Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.5$.

*Percent that did not converge; **Percent that violated the model constraints; ***Percent that violated model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)	TS_2 (***)
0.1	0.9938 (0%)	0.9936	0.9931	0.9741 (1.06%)	0.4182 (6.82%)
0.2	0.8958 (0%)	0.8934	0.8903	0.8407 (0.04%)	0.2998 (3.78%)
0.3	0.5730 (0%)	0.5688	0.5673	0.5226 (0.01%)	0.1753 (1.83%)
0.4	0.1829 (0.02%)	0.1793	0.1790	0.1742 (0%)	0.0803 (0.59%)
0.5	0.0559 (0.08%)	0.0562	0.0525	0.0537 (0%)	0.0508 (0.24%)
0.6	0.2111 (0.04%)	0.2073	0.2120	0.2107 (0%)	0.1051 (0.12%)
0.7	0.7044 (0.01%)	0.6951	0.7128	0.6919 (0%)	0.3531 (0.12%)
0.8	0.9862 (0%)	0.9772	0.9845	0.9846 (0%)	0.8170 (0.04%)
0.9	1.0000 (0%)	0.9412	0.9433	1.0000 (0%)	0.9992 (0.03%)

TABLE 38: Adjusted Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.6$.

*Percent that did not converge; **Percent that violated the model constraints; ***Percent that violated model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)	TS_2 (***)
0.1	0.9999 (0%)	0.9999	0.9999	0.9988 (1.02%)	0.6882 (6.58%)
0.2	0.9925 (0%)	0.9929	0.9912	0.9813 (0.12%)	0.5515 (3.07%)
0.3	0.9910 (0%)	0.9138	0.9048	0.8786 (0.01%)	0.3915 (1.24%)
0.4	0.6118 (0.01%)	0.6159	0.6003	0.5824 (0%)	0.2325 (0.51%)
0.5	0.2134 (0.03%)	0.2164	0.2080	0.2025 (0%)	0.1059 (0.09%)
0.6	0.0542 (0.07%)	0.0528	0.0536	0.0523 (0%)	0.0521 (0.06%)
0.7	0.2717 (0.04%)	0.2664	0.2764	0.2619 (0%)	0.1436 (0.01%)
0.8	0.8688 (0%)	0.8403	0.8710	0.8597 (0%)	0.5895 (0%)
0.9	0.9998 (0%)	0.9171	0.9195	0.9998 (0%)	0.9915 (0%)

TABLE 39: Adjusted Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.7$.

*Percent that did not converge; **Percent that violated the model constraints; ***Percent that violated model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)	TS_2 (***)
0.1	1.0000 (0%)	1.0000	1.0000	0.9999 (0.90%)	0.9166 (5.95%)
0.2	0.9999 (0%)	0.9999	0.9999	0.9992 (0.04%)	0.8524 (2.88%)
0.3	0.9962 (0%)	0.9965	0.9949	0.9942 (0%)	0.7467 (1.23%)
0.4	0.9457 (0%)	0.9485	0.9343	0.9344 (0%)	0.5707 (0.21%)
0.5	0.7131 (0%)	0.7201	0.6890	0.6982 (0%)	0.3613 (0.06%)
0.6	0.2637 (0.09%)	0.2687	0.2490	0.2573 (0%)	0.1443 (0.05%)
0.7	0.0558 (0.10%)	0.0524	0.0611	0.0548 (0%)	0.0490 (0%)
0.8	0.3972 (0.05%)	0.3665	0.4040	0.3966 (0%)	0.2498 (0%)
0.9	0.9915 (0%)	0.8859	0.8971	0.9909 (0%)	0.9447 (0%)

TABLE 40: Adjusted Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.8$.

*Percent that did not converge; **Percent that violated the model constraints; ***Percent that violated model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)	TS_2 (***)
0.1	1.0000 (0%)	0.9996	0.9998	1.0000 (0.68%)	0.9949 (5.70%)
0.2	1.0000 (0%)	0.9993	0.9994	1.0000 (0.05%)	0.9870 (2.73%)
0.3	1.0000 (0%)	0.9995	0.9995	0.9999 (0%)	0.9708 (1.00%)
0.4	0.9993 (0%)	0.9977	0.9967	0.9990 (0%)	0.9265 (0.30%)
0.5	0.9866 (0%)	0.9830	0.9731	0.9841 (0%)	0.8184 (0.04%)
0.6	0.8555 (0%)	0.8410	0.7962	0.8496 (0%)	0.5778 (0%)
0.7	0.3912 (0.03%)	0.3692	0.3248	0.3863 (0%)	0.2475 (0%)
0.8	0.0587 (0.15%)	0.0521	0.0832	0.0567 (0%)	0.0507 (0%)
0.9	0.7600 (0%)	0.6242	0.6071	0.7540 (0%)	0.6125 (0%)

Since only LRT and TS_1 performed consistently well in the first simulation experiment, it is appropriate to only compare power calculations of the LRT, TS_1, and TS_2. From the tables, one can see that the LRT achieves higher power levels than

TABLE 41: Adjusted Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.9$.

*Percent that did not converge; **Percent that violated the model constraints; ***Percent that violated model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)	TS_2 (***)
0.1	1.0000 (0%)	0.9656	0.9658	1.0000 (0%)	1.0000 (0%)
0.2	1.0000 (0%)	0.9629	0.9630	1.0000 (0.02%)	1.0000 (2.66%)
0.3	1.0000 (0%)	0.9516	0.9506	1.0000 (0%)	0.9999 (0.91%)
0.4	1.0000 (0%)	0.9471	0.9499	1.0000 (0%)	0.9999 (0.31%)
0.5	0.9999 (0%)	0.9359	0.9369	1.0000 (0%)	0.9985 (0.03%)
0.6	0.9999 (0%)	0.9214	0.9252	0.9999 (0%)	0.9929 (0%)
0.7	0.9909 (0%)	0.8603	0.8843	0.9905 (0%)	0.9378 (0%)
0.8	0.7573 (0%)	0.5274	0.5910	0.7499 (0%)	0.6101 (0%)
0.9	0.0592 (0.22%)	0.0798	0.0804	0.0559 (0%)	0.0512 (0%)

both alternative tests, but TS_1 is not far behind. In 40% of the simulations, the estimated power of TS_1 is greater than the LRT or within 0.001. In 63% of the simulations, the estimated power of TS_1 is greater than the LRT or within 0.01. In 83% of the simulations, the estimated power of TS_1 is greater than the LRT or within 0.05.

Tables 42-50 give the power values for a nominal level $\alpha = 0.05$.

TABLE 42: Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.1$.

*Percent that did not converge; **Percent that violated the model constraints; ***Percent that violated model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)	TS_2 (***)
0.1	0.0589 (0.17%)	0.0510	0.0551	0.0494 (4.15%)	0.0500 (14.03%)
0.2	0.2495 (0.04%)	0.2420	0.2340	0.1655 (1.74%)	0.0614 (11.02%)
0.3	0.6574 (0.02%)	0.6526	0.6389	0.5044 (1.43%)	0.1173 (8.35%)
0.4	0.9223 (0%)	0.9221	0.9151	0.8342 (1.19%)	0.2301 (6.95%)
0.5	0.9925 (0%)	0.9925	0.9912	0.9749 (1.19%)	0.4326 (6.94%)
0.6	0.9998 (0%)	0.9998	0.9998	0.9989 (1.04%)	0.6942 (6.45%)
0.7	1.0000 (0.01%)	1.0000	1.0000	1.0000 (0.97%)	0.9177 (6.21%)
0.8	1.0000 (0%)	0.9994	0.9994	1.0000 (0.99%)	0.9962 (6.09%)
0.9	1.0000 (0%)	0.9653	0.9651	1.0000 (0.83%)	1.0000 (5.81%)

TABLE 43: Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.2$.

*Percent that did not converge; **Percent that violated the model constraints; ***Percent that violated model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)	TS_2 (***)
0.1	0.2542 (0.04%)	0.2507	0.2386	0.1688 (1.82%)	0.0644 (10.88%)
0.2	0.0585 (0.04%)	0.0563	0.0506	0.0507 (0.45%)	0.0498 (7.55%)
0.3	0.2028 (0.04%)	0.2014	0.1899	0.1638 (0.10%)	0.0669 (5.62%)
0.4	0.6059 (0%)	0.6013	0.5875	0.5131 (0.05%)	0.1373 (4.21%)
0.5	0.9071 (0%)	0.9057	0.9010	0.8494 (0.06%)	0.2949 (3.47%)
0.6	0.9923 (0%)	0.9924	0.9912	0.9815 (0.07%)	0.5516 (2.87%)
0.7	0.9998 (0%)	0.9998	0.9998	0.9996 (0.05%)	0.8513 (3.25%)
0.8	1.0000 (0%)	0.9999	0.9999	1.0000 (0.05%)	0.9873 (2.58%)
0.9	1.0000 (0%)	0.9627	0.9631	1.0000 (0.03%)	1.0000 (2.39%)

TABLE 44: Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.3$.

*Percent that did not converge; **Percent that violated the model constraints; ***Percent that violated model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)	TS_2 (***)
0.1	0.6765 (0%)	0.6702	0.6583	0.5205 (1.45%)	0.1152 (8.76%)
0.2	0.2129 (0.03%)	0.2112	0.2008	0.1657 (0.14%)	0.0677 (5.37%)
0.3	0.0588 (0.03%)	0.0580	0.0519	0.0544 (0.03%)	0.0530 (2.83%)
0.4	0.1963 (0.02%)	0.1942	0.1865	0.1771 (0%)	0.0728 (2.04%)
0.5	0.5861 (0%)	0.5843	0.5726	0.5324 (0%)	0.1771 (1.52%)
0.6	0.9138 (0%)	0.9124	0.9095	0.8837 (0.01%)	0.4186 (1.42%)
0.7	0.9961 (0%)	0.9961	0.9958	0.9921 (0.01%)	0.7433 (1.25%)
0.8	1.0000 (0%)	0.9991	0.9994	0.9999 (0%)	0.9687 (0.96%)
0.9	1.0000 (0%)	0.9562	0.9568	1.0000 (0%)	1.0000 (0.70%)

TABLE 45: Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.4$.

*Percent that did not converge; **Percent that violated the model constraints; ***Percent that violated model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)	TS_2 (***)
0.1	0.9296 (0.01%)	0.9287	0.9224	0.8416 (1.37%)	0.2266 (7.31%)
0.2	0.5978 (0.01%)	0.5982	0.5774	0.5097 (0.12%)	0.1384 (4.22%)
0.3	0.2057 (0.05%)	0.2041	0.1936	0.1801 (0%)	0.0764 (2.24%)
0.4	0.0545 (0.09%)	0.0549	0.0502	0.0501 (0%)	0.0473 (1.06%)
0.5	0.2070 (0.02%)	0.2058	0.1971	0.1901 (0%)	0.0823 (0.73%)
0.6	0.6258 (0.01%)	0.6223	0.6166	0.5879 (0%)	0.2328 (0.53%)
0.7	0.9522 (0%)	0.9497	0.9510	0.9407 (0%)	0.5757 (0.26%)
0.8	0.9994 (0%)	0.9973	0.9978	0.9991 (0%)	0.9324 (0.18%)
0.9	1.0000 (0%)	0.9471	0.9475	1.0000 (0%)	0.9999 (0.35%)

TABLE 46: Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.5$.

*Percent that did not converge; **Percent that violated the model constraints; ***Percent that violated model constraints.

ρ_2	**LRT** (*)	**Score**	**Wald**	TS_1 (**)	TS_2 (***)
0.1	0.9946 (0%)	0.9945	0.9935	0.9757 (1.06%)	0.4212 (6.82%)
0.2	0.9030 (0%)	0.9027	0.8936	0.8475 (0.04%)	0.3028 (3.78%)
0.3	0.5915 (0%)	0.5906	0.5749	0.5332 (0.01%)	0.1773 (1.83%)
0.4	0.1946 (0.02%)	0.1942	0.1846	0.1823 (0%)	0.0819 (0.59%)
0.5	0.0559 (0.08%)	0.0562	0.0525	0.0537 (0%)	0.0508 (0.24%)
0.6	0.2219 (0.04%)	0.2218	0.2180	0.2174 (0%)	0.1073 (0.12%)
0.7	0.7182 (0.01%)	0.7105	0.7197	0.7013 (0%)	0.3564 (0.12%)
0.8	0.9876 (0%)	0.9798	0.9580	0.9856 (0%)	0.8189 (0.04%)
0.9	1.0000 (0%)	0.9413	0.9434	1.0000 (0%)	0.9992 (0.03%)

TABLE 47: Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.6$.

*Percent that did not converge; **Percent that violated the model constraints; ***Percent that violated model constraints.

ρ_2	**LRT** (*)	**Score**	**Wald**	TS_1 (**)	TS_2 (***)
0.1	0.9999 (0%)	0.9999	0.9999	0.9990 (1.02%)	0.6938 (6.58%)
0.2	0.9932 (0%)	0.9934	0.9922	0.9828 (0.12%)	0.5580 (3.07%)
0.3	0.9171 (0%)	0.9173	0.9103	0.8832 (0.01%)	0.3978 (1.24%)
0.4	0.6251 (0.01%)	0.6266	0.6132	0.5945 (0%)	0.2370 (0.51%)
0.5	0.2234 (0.03%)	0.2228	0.2176	0.2121 (0%)	0.1091 (0.09%)
0.6	0.0542 (0.07%)	0.0528	0.0536	0.0523 (0%)	0.0521 (0.06%)
0.7	0.2820 (0.04%)	0.2759	0.2858	0.2735 (0%)	0.1488 (0.01%)
0.8	0.8749 (0%)	0.8446	0.8772	0.8659 (0%)	0.5965 (0%)
0.9	0.9998 (0%)	0.9174	0.9197	0.9998 (0%)	0.9922 (0%)

TABLE 48: Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.7$.

*Percent that did not converge; **Percent that violated the model constraints; ***Percent that violated model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)	TS_2 (***)
0.1	1.0000 (0%)	1.0000	1.0000	0.9999 (0.90%)	0.9156 (5.95%)
0.2	0.9999 (0%)	0.9999	0.9999	0.9995 (0.04%)	0.8508 (2.88%)
0.3	0.9969 (0%)	0.9970	0.9961	0.9945 (0%)	0.7446 (1.23%)
0.4	0.9509 (0%)	0.9515	0.9483	0.9383 (0%)	0.5671 (0.21%)
0.5	0.7278 (0%)	0.7266	0.7246	0.7127 (0%)	0.3584 (0.06%)
0.6	0.2768 (0.09%)	0.2742	0.2788	0.2690 (0%)	0.1427 (0.05%)
0.7	0.0558 (0.10%)	0.0524	0.0611	0.0548 (0%)	0.0490 (0%)
0.8	0.4122 (0.05%)	0.3733	0.4415	0.4077 (0%)	0.2477 (0%)
0.9	0.9919 (0%)	0.8875	0.8994	0.9918 (0%)	0.9437 (0%)

TABLE 49: Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.8$.

*Percent that did not converge; **Percent that violated the model constraints; ***Percent that violated model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)	TS_2 (***)
0.1	1.0000 (0%)	0.9996	0.9998	1.0000 (0.68%)	0.9949 (5.70%)
0.2	1.0000 (0%)	0.9993	0.9994	1.0000 (0.05%)	0.9873 (2.73%)
0.3	1.0000 (0%)	0.9995	0.9995	0.9999 (0%)	0.9713 (1.00%)
0.4	0.9995 (0%)	0.9977	0.9978	0.9991 (0%)	0.9281 (0.30%)
0.5	0.9896 (0%)	0.9837	0.9856	0.9868 (0%)	0.8215 (0.04%)
0.6	0.8693 (0%)	0.8453	0.8649	0.8612 (0%)	0.5823 (0%)
0.7	0.4154 (0.03%)	0.3760	0.4280	0.4074 (0%)	0.2508 (0%)
0.8	0.0587 (0.15%)	0.0521	0.0832	0.0567 (0%)	0.0507 (0%)
0.9	0.7762 (0%)	0.6297	0.6819	0.7703 (0%)	0.6162 (0%)

TABLE 50: Rejection Proportions, $\alpha = 0.05$, $H_0 : \rho_1 = \rho_2 = \rho$, $\rho_1 = 0.9$.

*Percent that did not converge; **Percent that violated the model constraints; ***Percent that violated model constraints.

ρ_2	LRT (*)	Score	Wald	TS_1 (**)	TS_2 (***)
0.1	1.0000 (0%)	0.9660	0.9666	1.0000 (0.95%)	1.0000 (5.25%)
0.2	1.0000 (0%)	0.9632	0.9638	1.0000 (0.02%)	1.0000 (2.66%)
0.3	1.0000 (0%)	0.9526	0.9516	1.0000 (0%)	0.9999 (0.91%)
0.4	1.0000 (0%)	0.9488	0.9507	1.0000 (0%)	0.9999 (0.31%)
0.5	1.0000 (0%)	0.9382	0.9394	1.0000 (0%)	0.9986 (0.03%)
0.6	0.9999 (0%)	0.9266	0.9280	0.9999 (0%)	0.9933 (0%)
0.7	0.9922 (0%)	0.8869	0.8976	0.9913 (0%)	0.9398 (0%)
0.8	0.7768 (0%)	0.6333	0.6822	0.7695 (0%)	0.6150 (0%)
0.9	0.0592 (0.22%)	0.0798	0.0804	0.0559 (0%)	0.0512 (0%)

It can be seen that similar to the adjusted power levels, the LRT achieves higher power than both TS_1 and TS_2. Since the estimated sizes for the LRT are generally larger than the estimated sizes for TS_1 and TS_2, the LRT powers have an advantage when not adjusted for the size levels each test achieved.

When family sizes were smaller, the alternative test TS_1 compared well with the LRT especially in size performance while the Score and Wald did not perform as well as either the LRT or TS_1. When family sizes were larger, both alternative test performed well compared to the LRT in size estimation, but only TS_1 compared well to the LRT in power estimation. The alternative test TS_1 violates the model constraints more often than the LRT does not converge, especially when family sizes are smaller and the upper bound on ρ_{12} is lower. In practice, both sets of alternative estimates and corresponding tests are easy to compute. Generally, TS_1 performed better and does not require restrictions on the data set that TS_2 needs, therefore TS_1 is recommended.

CHAPTER III

INTERCLASS CORRELATIONS: g POPULATIONS

III.1 INTRODUCTION

Testing equality of g intraclass correlations was considered by Naik and Helu (2007) under the general setup of unequal family sizes. However, under the model where brother-sister interclass correlation is nonzero, it is of interest to test the equality of these interclass correlations from g independent populations. In this chapter, we consider g independent populations where sons and daughters in a family can have different intraclass correlations and different interclass correlations over the populations. We consider the problem of testing if these interclass correlations are the same. Suppose there are g independent populations and data on the children of n_i randomly selected families are available from each population. As in the second chapter, the number of boys and girls in each family is allowed to be different. Denote the number of boys and girls in the j^{th} family from the i^{th} population as m_{1ij} and m_{2ij}, respectively, for $j = 1, \ldots, n_i$; $i = 1, \ldots, g$. Let x_{1ijk}, $k = 1, \ldots, m_{1ij}$; $j = 1, \ldots, n_i$; $i = 1, \ldots, g$ be the observation on the k^{th} boy of the j^{th} family from the i^{th} population. Likewise, let x_{2ijk}, $k = 1, \ldots, m_{2ij}$; $j = 1, \ldots, n_i$; $i = 1, \ldots, g$ be the observation on the k^{th} girl of the j^{th} family from the i^{th} population.

Assume that the expected value of the son observations in a family is $E(x_{1ijk}) = \mu_{1,i}$, the expected value of the daughter observations in a family is $E(x_{2ijk}) = \mu_{2,i}$, the variance of the son observations is $Var(x_{1ijk}) = \sigma^2_{1,i}$, and the variance of the daughter observations is $Var(x_{2ijk}) = \sigma^2_{2,i}$. Denote the son-son *intraclass* correlation of the i^{th} population as $\rho_{1,i}$, the daughter-daughter *intraclass* correlation of the i^{th} population as $\rho_{2,i}$, and the son-daughter *interclass* correlation of the i^{th} population as $\rho_{12,i}$. Assume for each family in the i^{th} population, $Corr(x_{1ijk}, x_{1ijk'}) = \rho_{1,i}$ for $k \neq k'$; $1 \leq k, k' \leq m_{1ij}$, $Corr(x_{2ijk}, x_{2ijk'}) = \rho_{2,i}$ for $k \neq k'$; $1 \leq k, k' \leq m_{2ij}$, and $Corr(x_{1ijk}, x_{2ijk'}) = \rho_{12,i}$ for all k, k'; $1 \leq k \leq m_{1ij}$ and $1 \leq k' \leq m_{2ij}$.

Let the vector of observations on the j^{th} family from the i^{th} population be

$$\mathbf{x}_{ij} = \begin{pmatrix} \mathbf{x}_{1ij} \\ \mathbf{x}_{2ij} \end{pmatrix} = \begin{pmatrix} x_{1ij1} \\ \vdots \\ x_{1ijm_{1ij}} \\ x_{2ij1} \\ \vdots \\ x_{2ijm_{2ij}} \end{pmatrix},$$

with

$$E(\mathbf{x}_{ij}) = \boldsymbol{\mu}_{ij} = \begin{pmatrix} \mu_{1,i}\mathbf{1}_{m_{1ij}} \\ \mu_{2,i}\mathbf{1}_{m_{2ij}} \end{pmatrix},$$

and

$$\begin{aligned} Var(\mathbf{x}_{ij}) &= \boldsymbol{\Sigma}_{ij} \\ &= \begin{pmatrix} \sigma_{1,i}^2\{(1-\rho_{1,i})\mathbf{I}_{m_{1ij}} + \rho_{1,i}\mathbf{J}_{m_{1ij}}\} & \rho_{12,i}\sigma_{1,i}\sigma_{2,i}\mathbf{J}_{m_{1ij},m_{2ij}} \\ \rho_{12,i}\sigma_{1,i}\sigma_{2,i}\mathbf{J}_{m_{2ij},m_{1ij}} & \sigma_{2,i}^2\{(1-\rho_{2,i})\mathbf{I}_{m_{2ij}} + \rho_{2,i}\mathbf{J}_{m_{2ij}}\} \end{pmatrix}, \end{aligned}$$

where $\mathbf{I}_m$ is an identity matrix of order m, $\mathbf{J}_m$ is the $m \times m$ matrix of all ones, and $\mathbf{J}_{m,n}$ is the $m \times n$ matrix of all ones. Note $-\infty < \mu_{1,i} < \infty$ and $-\infty < \mu_{2,i} < \infty$.

If there are both sons and daughters in a family, $m_{1ij} > 0$ and $m_{2ij} > 0$, then the determinant of $\boldsymbol{\Sigma}_{ij}$ is

$$\begin{aligned} |\boldsymbol{\Sigma}_{ij}| &= \sigma_{1,i}^{2m_{1ij}}\sigma_{2,i}^{2m_{2ij}}(1-\rho_{1,i})^{m_{1ij}-1}(1-\rho_{2,i})^{m_{2ij}-1} \\ &\times ((1+(m_{1ij}-1)\rho_{1,i})(1+(m_{2ij}-1)\rho_{2,i}) - m_{1ij}m_{2ij}\rho_{12,i}^2). \end{aligned}$$

Restrictions on the parameters so that $\boldsymbol{\Sigma}_{ij}$ is positive definite are $\sigma_{1,i}^2 > 0$, $\sigma_{2,i}^2 > 0$, $\rho_{1,i} < 1$, $\rho_{2,i} < 1$, and

$$(1+(m_{1ij}-1)\rho_{1,i})(1+(m_{2ij}-1)\rho_{2,i}) > m_{1ij}m_{2ij}\rho_{12,i}^2. \tag{15}$$

If $m_{1ij} > 0$ and $m_{2ij} > 0$, then the inverse of $\boldsymbol{\Sigma}_{ij}$ is

$$\boldsymbol{\Sigma}_{ij}^{-1} = \begin{pmatrix} \frac{1}{\sigma_{1,i}^2}\mathbf{A}_{ij} & \frac{1}{\sigma_{1,i}\sigma_{2,i}}\mathbf{B}_{ij} \\ \frac{1}{\sigma_{1,i}\sigma_{2,i}}\mathbf{B}'_{ij} & \frac{1}{\sigma_{2,i}^2}\mathbf{C}_{ij} \end{pmatrix}$$

where

$$
\begin{aligned}
\mathbf{A}_{ij} &= \frac{1}{1-\rho_{1,i}} \\
&\times \left[\mathbf{I}_{m_{1ij}} - \frac{\rho_{1,i}(1+(m_{2ij}-1)\rho_{2,i}) - m_{2ij}\rho_{12,i}^2}{(1+(m_{1ij}-1)\rho_{1,i})(1+(m_{2ij}-1)\rho_{2,i}) - m_{1ij}m_{2ij}\rho_{12,i}^2}\mathbf{J}_{m_{1ij}}\right], \\
\mathbf{B}_{ij} &= \frac{-\rho_{12,i}}{(1+(m_{1ij}-1)\rho_{1,i})(1+(m_{2ij}-1)\rho_{2,i}) - m_{1ij}m_{2ij}\rho_{12,i}^2}\mathbf{J}_{m_{1ij},m_{2ij}}, \\
\mathbf{C}_{ij} &= \frac{1}{1-\rho_{2,i}} \\
&\times \left[\mathbf{I}_{m_{2ij}} - \frac{\rho_{2,i}(1+(m_{1ij}-1)\rho_{1,i}) - m_{1ij}\rho_{12,i}^2}{(1+(m_{1ij}-1)\rho_{1,i})(1+(m_{2ij}-1)\rho_{2,i}) - m_{1ij}m_{2ij}\rho_{12,i}^2}\mathbf{J}_{m_{2ij}}\right].
\end{aligned}
$$

If there are no sons in a family, $m_{1ij} = 0$, then the determinant of $\mathbf{\Sigma}_{ij}$ is

$$|\mathbf{\Sigma}_{ij}| = \sigma_{2,i}^{2m_{2ij}}(1-\rho_{2,i})^{m_{2ij}-1}(1+(m_{2ij}-1)\rho_{2,i}),$$

and the inverse of $\mathbf{\Sigma}_{ij}$ is

$$\mathbf{\Sigma}_{ij}^{-1} = \frac{1}{\sigma_{2,i}^2(1-\rho_{2,i})}\left[I_{m_{2ij}} - \frac{\rho_{2,i}}{(1+(m_{2ij}-1)\rho_{2,i})}J_{m_{2ij}}\right].$$

If there are no daughters in a family, $m_{2ij} = 0$, then the determinant of $\mathbf{\Sigma}_{ij}$ is

$$|\mathbf{\Sigma}_{ij}| = \sigma_{1,i}^{2m_{1ij}}(1-\rho_{1,i})^{m_{1ij}-1}(1+(m_{1ij}-1)\rho_{1,i}),$$

and the inverse of $\mathbf{\Sigma}_{ij}$ is

$$\mathbf{\Sigma}_{ij}^{-1} = \frac{1}{\sigma_{1,i}^2(1-\rho_{1,i})}\left[I_{m_{1ij}} - \frac{\rho_{1,i}}{(1+(m_{1ij}-1)\rho_{1,i})}J_{m_{1ij}}\right].$$

III.2 THE LIKELIHOOD FUNCTION

Assume that $\mathbf{x}_{ij} \sim N_{m_{1ij}+m_{2ij}}(\boldsymbol{\mu}_{ij}, \mathbf{\Sigma}_{ij})$, $j = 1, \ldots, n_i$; $i = 1, \ldots, g$. Let

$$\boldsymbol{\theta} = (\boldsymbol{\theta}_1', \ldots, \boldsymbol{\theta}_g')',$$

where

$$\boldsymbol{\theta}_i = (\mu_{1,i}, \mu_{2,i}, \sigma_{1,i}^2, \sigma_{2,i}^2, \rho_{1,i}, \rho_{2,i}, \rho_{12,i})'.$$

Then

$$
\begin{aligned}
L(\boldsymbol{\theta}) &= \prod_{i=1}^{g}\prod_{j=1}^{n_i} L_{ij}(\boldsymbol{\theta}) \\
&= \prod_{i=1}^{g}\prod_{j=1}^{n_i} \frac{1}{(2\pi)^{(m_{1ij}+m_{2ij})/2}|\mathbf{\Sigma}_{ij}|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}_{ij}-\boldsymbol{\mu}_{ij})'\mathbf{\Sigma}_{ij}^{-1}(\mathbf{x}_{ij}-\boldsymbol{\mu}_{ij})}
\end{aligned}
$$

and

$$log(L(\boldsymbol{\theta})) = \sum_{i=1}^{g}\sum_{j=1}^{n_i} log(L_{ij}(\boldsymbol{\theta})).$$

If $m_{1ij} > 0$ and $m_{2ij} > 0$, then

$$\begin{aligned} log(L_{ij}(\boldsymbol{\theta})) &= -\frac{m_{1ij}}{2}log(2\pi\sigma_{1,i}^2) - \frac{m_{2ij}}{2}log(2\pi\sigma_{2,i}^2) \\ &- \frac{1}{2}(m_{1ij}-1)log(1-\rho_{1,i}) - \frac{1}{2}(m_{2ij}-1)log(1-\rho_{2,i}) \\ &- \frac{1}{2}log\left[(1+(m_{1ij}-1)\rho_{1,i})(1+(m_{2ij}-1)\rho_{2,i}) - m_{1ij}m_{2ij}\rho_{12,i}^2\right] \\ &- \frac{1}{2}(\mathbf{x}_{ij}-\boldsymbol{\mu}_{ij})'\boldsymbol{\Sigma}_{ij}^{-1}(\mathbf{x}_{ij}-\boldsymbol{\mu}_{ij}). \end{aligned}$$

Since

$$\mathbf{x}_{ij} = \begin{pmatrix} \mathbf{x}_{1ij} \\ \mathbf{x}_{2ij} \end{pmatrix},$$

where $\mathbf{x}_{1ij} = (x_{1ij1}, \ldots, x_{1ijm_{1ij}})'$ and $\mathbf{x}_{2ij} = (x_{2ij1}, \ldots, x_{2ijm_{2ij}})'$, $(\mathbf{x}_{ij}-\boldsymbol{\mu}_{ij})'\boldsymbol{\Sigma}_{ij}^{-1}(\mathbf{x}_{ij}-\boldsymbol{\mu}_{ij})$

$$\begin{aligned} &= \left[(\mathbf{x}_{1ij}-\mu_{1,i}\mathbf{1}_{m_{1ij}})'\,(\mathbf{x}_{2ij}-\mu_{2,i}\mathbf{1}_{m_{2ij}})'\right] \\ &\times \begin{bmatrix} \frac{1}{\sigma_{1,i}^2}\mathbf{A}_{ij} & \frac{1}{\sigma_{1,i}\sigma_{2,i}}\mathbf{B}_{ij} \\ \frac{1}{\sigma_{1,i}\sigma_{2,i}}\mathbf{B}_{ij}' & \frac{1}{\sigma_{2,i}^2}\mathbf{C}_{ij} \end{bmatrix} \begin{bmatrix} (\mathbf{x}_{1ij}-\mu_{1,i}\mathbf{1}_{m_{1ij}}) \\ (\mathbf{x}_{2ij}-\mu_{2,i}\mathbf{1}_{m_{2ij}}) \end{bmatrix} \\ &= (\mathbf{x}_{1ij}-\mu_{1,i}\mathbf{1}_{m_{1ij}})'\frac{1}{\sigma_{1,i}^2}\mathbf{A}_{ij}(\mathbf{x}_{1ij}-\mu_{1,i}\mathbf{1}_{m_{1ij}}) \\ &+ (\mathbf{x}_{2ij}-\mu_{2,i}\mathbf{1}_{m_{2ij}})'\frac{1}{\sigma_{1,i}\sigma_{2,i}}\mathbf{B}_{ij}'(\mathbf{x}_{1ij}-\mu_{1,i}\mathbf{1}_{m_{1ij}}) \\ &+ (\mathbf{x}_{1ij}-\mu_{1,i}\mathbf{1}_{m_{1ij}})'\frac{1}{\sigma_{1,i}\sigma_{2,i}}\mathbf{B}_{ij}(\mathbf{x}_{2ij}-\mu_{2,i}\mathbf{1}_{m_{2ij}}) \\ &+ (\mathbf{x}_{2ij}-\mu_{2,i}\mathbf{1}_{m_{2ij}})'\frac{1}{\sigma_{2,i}^2}\mathbf{C}_{ij}(\mathbf{x}_{2ij}-\mu_{2,i}\mathbf{1}_{m_{2ij}}). \end{aligned}$$

If $m_{1ij} = 0$ and $m_{2ij} > 1$, then

$$\begin{aligned} log(L_{ij}(\boldsymbol{\theta})) &= -\frac{m_{2ij}}{2}log(2\pi\sigma_{2,i}^2) - \frac{1}{2}(m_{2ij}-1)log(1-\rho_{2,i}) \\ &- \frac{1}{2}log(1+(m_{2ij}-1)\rho_{2,i}) - \frac{1}{2}(\mathbf{x}_{2ij}-\mu_{2,i}\mathbf{1}_{m_{2ij}})'\boldsymbol{\Sigma}_{ij}^{-1}(\mathbf{x}_{2ij}-\mu_{2,i}\mathbf{1}_{m_{2ij}}). \end{aligned}$$

And, if $m_{1ij} = 0$ and $m_{2ij} = 1$, then

$$log(L_{ij}(\boldsymbol{\theta})) = -\tfrac{1}{2}log(2\pi\sigma_{2,i}^2) - \tfrac{1}{2\sigma_{2,i}^2}(x_{2ij} - \mu_{2,i})^2.$$

Similarly, if $m_{1ij} > 1$ and $m_{2ij} = 0$, then

$$\begin{aligned} log(L_{ij}(\boldsymbol{\theta})) &= -\frac{m_{1ij}}{2}log(2\pi\sigma_{1,i}^2) - \frac{1}{2}(m_{1ij}-1)log(1-\rho_{1,i}) \\ &- \frac{1}{2}log(1+(m_{1ij}-1)\rho_{1,i}) - \frac{1}{2}(\mathbf{x}_{1ij} - \mu_{1,i}\mathbf{1}_{m_{1ij}})'\boldsymbol{\Sigma}_{ij}^{-1}(x_{1ij} - \mu_{1,i}\mathbf{1}_{m_{1ij}}). \end{aligned}$$

And, if $m_{1ij} = 1$ and $m_{2ij} = 0$, then

$$log(L_{ij}(\boldsymbol{\theta})) = -\tfrac{1}{2}log(2\pi\sigma_{1,i}^2) - \tfrac{1}{2\sigma_{1,i}^2}(x_{1ij} - \mu_{1,i})^2.$$

The likelihood function $L(\boldsymbol{\theta})$ or the log-likelihood function $log(L(\boldsymbol{\theta}))$ can be maximized to obtain $\hat{\boldsymbol{\theta}}$, the maximum likelihood estimator (MLE) of $\boldsymbol{\theta}$. Let $\hat{\boldsymbol{\theta}}_i = \left(\hat{\mu}_{1,i}, \hat{\mu}_{2,i}, \hat{\sigma}_{1,i}^2, \hat{\sigma}_{2,i}^2, \hat{\rho}_{1,i}, \hat{\rho}_{2,i}, \hat{\rho}_{12,i}\right)'$ for $i = 1, \ldots, g$. Our interest is to test the hypothesis that the g interclass correlation coefficients are equal, that is, $H_0 : \rho_{12,1} = \ldots = \rho_{12,g} = \rho_{12}$ (say). Under H_0, $\boldsymbol{\theta}_i = (\mu_{1,i}, \mu_{2,i}, \sigma_{1,i}^2, \sigma_{2,i}^2, \rho_{1,i}, \rho_{2,i}, \rho_{12})'$. The likelihood function, $L(\boldsymbol{\theta})$ or log-likelihood function $log(L(\boldsymbol{\theta}))$ can also be maximized under the null hypothesis $H_0 : \rho_{12,1} = \ldots = \rho_{12,g}$ to obtain $\hat{\boldsymbol{\theta}}_0$.

III.3 LIKELIHOOD RATIO TEST

The likelihood ratio test (LRT) for testing H_0 is to reject H_0 for large values of

$$LRT = 2logL(\hat{\boldsymbol{\theta}}) - 2logL(\hat{\boldsymbol{\theta}}_0). \tag{16}$$

This test statistic has a χ^2 asymptotic distribution with $g-1$ degree of freedom.

The maximum likelihood procedures used to find the MLEs in the LRT need fairly good initial values of the parameters which could be chosen as the alternative estimates proposed in the following sections. The other two asymptotic tests, a modified Wald's test and Rao's Score test, investigated in Chapter II for the one population case are not investigated here since the LRT and the alternative tests proposed were more favorable.

III.4 CANONICAL TRANSFORMATION

A canonical transformation can be applied here similarly as was done in the previous chapter. The transformation simplifies the distribution of the data. The transformed

data can be used in alternative estimators of the model parameters as will be shown in the following sections.

Recall, $\mathbf{x}_{ij} = \begin{pmatrix} \mathbf{x}_{1ij} \\ \mathbf{x}_{2ij} \end{pmatrix}$ is distributed with mean $\boldsymbol{\mu}_{ij} = \begin{pmatrix} \mu_{1,i}\mathbf{1}_{m_{1ij}} \\ \mu_{2,i}\mathbf{1}_{m_{2ij}} \end{pmatrix}$ and covariance matrix

$$\boldsymbol{\Sigma}_{ij} = \begin{pmatrix} \sigma_{1,i}^2\{(1-\rho_{1,i})\mathbf{I}_{m_{1ij}} + \rho_{1,i}\mathbf{J}_{m_{1ij}}\} & \rho_{12,i}\sigma_{1,i}\sigma_{2,i}\mathbf{J}_{m_{1ij},m_{2ij}} \\ \rho_{12,i}\sigma_{1,i}\sigma_{2,i}\mathbf{J}_{m_{2ij},m_{1ij}} & \sigma_{2,i}^2\{(1-\rho_{2,i})\mathbf{I}_{m_{2ij}} + \rho_{2,i}\mathbf{J}_{m_{2ij}}\} \end{pmatrix}.$$

Let

$$\boldsymbol{\Gamma}_{ij,(m_{1i}+m_{2i},m_{1i}+m_{2i})} = \begin{pmatrix} \boldsymbol{\Gamma}_{1ij} & \mathbf{0}_{m_{1ij},m_{2ij}} \\ \mathbf{0}_{m_{2ij},m_{1ij}} & \boldsymbol{\Gamma}_{2ij} \end{pmatrix},$$

where

$$\boldsymbol{\Gamma}_{1ij} = \begin{pmatrix} \frac{1}{m_{1ij}} & \frac{1}{m_{1ij}} & \frac{1}{m_{1ij}} & \cdots & \frac{1}{m_{1ij}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{m_{1ij}(m_{1ij}-1)}} & \frac{1}{\sqrt{m_{1ij}(m_{1ij}-1)}} & \frac{1}{\sqrt{m_{1ij}(m_{1ij}-1)}} & \cdots & \frac{-(m_{1ij}-1)}{\sqrt{m_{1ij}(m_{1ij}-1)}} \end{pmatrix},$$

$$\boldsymbol{\Gamma}_{2ij} = \begin{pmatrix} \frac{1}{m_{2ij}} & \frac{1}{m_{2ij}} & \frac{1}{m_{2ij}} & \cdots & \frac{1}{m_{2ij}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{m_{2ij}(m_{2ij}-1)}} & \frac{1}{\sqrt{m_{2ij}(m_{2ij}-1)}} & \frac{1}{\sqrt{m_{2ij}(m_{2ij}-1)}} & \cdots & \frac{-(m_{2ij}-1)}{\sqrt{m_{2ij}(m_{2ij}-1)}} \end{pmatrix},$$

and $\mathbf{0}_{m,n}$ is the $m \times n$ matrix of all zeros.

Transform the family scores by making a Srivastava type transformation to create $\mathbf{y}_{ij}$, the transformed vector of family scores,

$$\begin{aligned} \mathbf{y}_{ij} = \begin{pmatrix} \mathbf{y}_{1ij} \\ \mathbf{y}_{2ij} \end{pmatrix} &= \boldsymbol{\Gamma}_{ij} \begin{pmatrix} \mathbf{x}_{1ij} \\ \mathbf{x}_{2ij} \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{\Gamma}_{1ij} & \mathbf{0}_{m_{1ij},m_{2ij}} \\ \mathbf{0}_{m_{2ij},m_{1ij}} & \boldsymbol{\Gamma}_{2ij} \end{pmatrix} \begin{pmatrix} \mathbf{x}_{1ij} \\ \mathbf{x}_{2ij} \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{\Gamma}_{1ij}\mathbf{x}_{1ij} \\ \boldsymbol{\Gamma}_{2ij}\mathbf{x}_{2ij} \end{pmatrix}. \end{aligned}$$

Now, the expected value and variance of the vector of transformed son scores from the j^{th} family of the i^{th} population are as follows

$$E(\mathbf{y}_{1ij}) = \begin{pmatrix} \mu_{1,i} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

and

$$\begin{aligned} Var(\mathbf{y}_{1ij}) &= \sigma^2_{1,i}\mathbf{\Gamma}_{1ij}((1-\rho_{1,i})\mathbf{I}_{m_{1ij}} + \rho_{1,i}\mathbf{J}_{m_{1ij}})\mathbf{\Gamma}'_{1ij} \\ &= \begin{pmatrix} \frac{1}{2}\sigma^2_{1,i}(1+(m_{1ij}-1)\rho_{1,i}) & 0 & \cdots & 0 \\ 0 & \sigma^2_{1,i}(1-\rho_{1,i}) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma^2_{1,i}(1-\rho_{1,i}) \end{pmatrix}. \end{aligned}$$

The covariance between the vector of transformed son scores and the vector of transformed daughter scores from the j^{th} family in the i^{th} population is

$$\begin{aligned} Cov(\mathbf{y}_{1ij}, \mathbf{y}_{2ij}) &= \sigma_{1,i}\sigma_{2,i}\rho_{12,i}\mathbf{\Gamma}_{1ij}\mathbf{J}_{m_{1ij},m_{2ij}}\mathbf{\Gamma}'_{2ij} \\ &= \sigma_{1,i}\sigma_{2,i}\rho_{12,i} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{m_{1ij},m_{2ij}}. \end{aligned}$$

Similarly, the expected value and variance of the vector of transformed daughter scores from the j^{th} family of the i^{th} population are

$$E(\mathbf{y}_{2ij}) = \begin{pmatrix} \mu_{2,i} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

and

$$\begin{aligned} Var(\mathbf{y}_{2ij}) &= \sigma^2_{2,i}\mathbf{\Gamma}_{2ij}((1-\rho_{2,i})\mathbf{I}_{m_{2ij}} + \rho_{2,i}\mathbf{J}_{m_{2ij}})\mathbf{\Gamma}'_{2ij} \\ &= \begin{pmatrix} \frac{1}{2}\sigma^2_{2,i}(1+(m_{2ij}-1)\rho_{2,i}) & 0 & \cdots & 0 \\ 0 & \sigma^2_{2,i}(1-\rho_{2,i}) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma^2_{2,i}(1-\rho_{2,i}) \end{pmatrix}. \end{aligned}$$

Note that only the first transformed son score and the first transformed daughter score in a family, namely y_{1ij1} and y_{2ij1}, are correlated. Also, the vector $\begin{pmatrix} y_{1ij1} \\ y_{2ij1} \end{pmatrix}$ is bivariate normal with mean $\begin{pmatrix} \mu_{1,i} \\ \mu_{2,i} \end{pmatrix}$ and variance covariance matrix

$$\begin{pmatrix} \frac{1}{m_{1ij}}\sigma_{1,i}^2(1+(m_{1ij}-1)\rho_{1,i}) & \sigma_{1,i}\sigma_{2,i}\rho_{12,i} \\ \sigma_{1,i}\sigma_{2,i}\rho_{12,i} & \frac{1}{m_{2ij}}\sigma_{2,i}^2(1+(m_{2ij}-1)\rho_{2,i}) \end{pmatrix}$$

and are independent of $y_{1ij2}, \ldots, y_{1ijm_{1ij}} \sim N(0, \sigma_{1,i}^2(1-\rho_{1,i}))$ and $y_{2ij2}, \ldots, y_{2ijm_{2ij}} \sim N(0, \sigma_{2,i}^2(1-\rho_{2,i}))$.

In terms of $\mathbf{x}_{1ij}$ and $\mathbf{x}_{2ij}$, the first transformed son score, y_{1ij1}, is the average of all the observed boy scores in the family. As well, the first transformed daughter score, y_{2ij1}, is the average of all the observed girl scores in the family. That is,

$$y_{1ij1} = \tfrac{1}{m_{1ij}} \textstyle\sum_{k=1}^{m_{1ij}} x_{1ijk}, \quad y_{2ij1} = \tfrac{1}{m_{2ij}} \textstyle\sum_{k=1}^{m_{2ij}} x_{2ijk}.$$

Hence,

$$\bar{y}_{1i1} = \tfrac{1}{n_i} \textstyle\sum_{j=1}^{n_i} y_{1ij1} = \tfrac{1}{n_i} \sum_{j=1}^{n_i} \tfrac{1}{m_{1ij}} \sum_{k=1}^{m_{1ij}} x_{1ijk},$$

$$\bar{y}_{2i1} = \tfrac{1}{n_i} \textstyle\sum_{j=1}^{n_i} y_{2ij1} = \tfrac{1}{n_i} \sum_{j=1}^{n_i} \tfrac{1}{m_{2ij}} \sum_{k=1}^{m_{2ij}} x_{2ijk}.$$

The average of the first transformed son scores is a population average of the mean family son scores. Similarly, the average of the first transformed daughter scores is a population average of the mean family daughter scores. Further, the average of all the first transformed son scores and the average of all the first transformed daughter scores can be written in terms of the observed familial data as follows

$$\bar{y}_{11} = \tfrac{1}{g} \textstyle\sum_{i=1}^{g} \bar{y}_{1i1} = \tfrac{1}{g} \sum_{i=1}^{g} \tfrac{1}{n_i} \sum_{j=1}^{n_i} \tfrac{1}{m_{1ij}} \sum_{k=1}^{m_{1ij}} x_{1ijk},$$

$$\bar{y}_{21} = \tfrac{1}{g} \textstyle\sum_{i=1}^{g} \bar{y}_{2i1} = \tfrac{1}{g} \sum_{i=1}^{g} \tfrac{1}{n_i} \sum_{j=1}^{n_i} \tfrac{1}{m_{2ij}} \sum_{k=1}^{m_{2ij}} x_{2ijk}.$$

One can also see that

$$\begin{aligned} \sum_{k=2}^{m_{1ij}} y_{1ijk}^2 &= \left(x_{1ij2}, \ldots, x_{1ijm_{1ij}}\right)' \boldsymbol{\Gamma}_{ij}' \boldsymbol{\Gamma}_{ij} \left(x_{1ij2}, \ldots, x_{1ijm_{1ij}}\right) \\ &= \left(x_{1ij2}, \ldots, x_{1ijm_{1ij}}\right)' (I_{m_{1ij}} - \frac{1}{m_{1ij}} J_{m_{1ij}}) \left(x_{1ij2}, \ldots, x_{1ijm_{1ij}}\right) \\ &= \sum_{k=2}^{m_{1ij}} x_{1ijk}^2 - \frac{1}{m_{1ij}} (\sum_{k=2}^{m_{1ij}} x_{1ijk})^2 \end{aligned}$$

and

$$\begin{aligned}\sum_{k=2}^{m_{2ij}} y_{2ijk}^2 &= \left(x_{2ij2},\ldots,x_{2ijm_{2ij}}\right)' \mathbf{\Gamma}_{ij}'\mathbf{\Gamma}_{ij}\left(x_{2ij2},\ldots,x_{2ijm_{2ij}}\right)\\ &= \left(x_{2ij2},\ldots,x_{2ijm_{2ij}}\right)'\left(I_{m_{2ij}} - \frac{1}{m_{2ij}}J_{m_{2ij}}\right)\left(x_{2ij2},\ldots,x_{2ijm_{2ij}}\right)\\ &= \sum_{k=2}^{m_{2ij}} x_{2ijk}^2 - \frac{1}{m_{2ij}}\left(\sum_{k=2}^{m_{2ij}} x_{2ijk}\right)^2.\end{aligned}$$

That is, the sum of squares of the "left-over" transformed son scores, $y_{1ik2},\ldots,y_{1ijm_{1ij}}$, for a family can be written in terms of the second through last son of the family, and the sum of squares of the "left-over" transformed daughter scores, $y_{2ij2},\ldots,y_{2ijm_{2ij}}$, for a family can be written in terms of the second through last daughter of the family.

In order to simplify the transformed model, let

$$\mathbf{\Gamma}_{ij}'\mathbf{\Sigma}_{ij}\mathbf{\Gamma}_{ij} = \begin{bmatrix} \eta_{1ij}^2 & \mathbf{0}_{m_{1ij}-1}' & \sigma_{12,i} & \mathbf{0}_{m_{2ij}-1}' \\ \mathbf{0}_{m_{1ij}-1} & \gamma_{1,i}^2\mathbf{I}_{m_{1ij}-1} & \mathbf{0}_{m_{1ij}-1} & \mathbf{0}_{m_{1ij}-1,m_{2ij}-1} \\ \sigma_{12,i} & \mathbf{0}_{m_{1ij}-1}' & \eta_{2ij}^2 & \mathbf{0}_{m_{2ij}-1}' \\ \mathbf{0}_{m_{2ij}-1} & \mathbf{0}_{m_{1ij}-1,m_{2ij}-1} & \mathbf{0}_{m_{2ij}-1} & \gamma_{2,i}^2\mathbf{I}_{m_{2ij}-1} \end{bmatrix},$$

where

$$\begin{aligned}\eta_{1ij}^2 &= \sigma_{1,i}^2(1+(m_{1ij}-1)\rho_{1,i})/m_{1ij},\\ \eta_{2ij}^2 &= \sigma_{2,i}^2(1+(m_{2ij}-1)\rho_{2,i})/m_{2ij},\\ \gamma_{1,i}^2 &= \sigma_{1,i}^2(1-\rho_{1,i}),\\ \gamma_{2,i}^2 &= \sigma_{2,i}^2(1-\rho_{2,i}),\\ \sigma_{12,i} &= \sigma_{1,i}\sigma_{2,i}\rho_{12,i}.\end{aligned}$$

Note $\eta_{1ij}^2 = \sigma_{1,i}^2 - a_{1ij}\gamma_{1,i}^2$ and $\eta_{2ij}^2 = \sigma_{2,i}^2 - a_{2ij}\gamma_{2,i}^2$ where $a_{1ij} = 1 - m_{1ij}^{-1}$ and $a_{2ij} = 1 - m_{2ij}^{-1}$. Additionally, there is a 1-1 transformation from the old parameters to a new set of parameters. Namely,

$$\begin{aligned}\xi_{1,i} &= \frac{\sigma_{1,i}^2}{\gamma_{1,i}^2},\\ \xi_{2,i} &= \frac{\sigma_{2,i}^2}{\gamma_{2,i}^2},\\ \xi_{12,i} &= \frac{\sigma_{12,i}^2}{\gamma_{1,i}^2\gamma_{2,i}^2}.\end{aligned}$$

III.5 ALTERNATIVE ESTIMATORS FOR g POPULATIONS

From the distribution of the transformed familial data, alternative estimators can be developed similar to those in the second chapter that do not require maximization of the non-linear constraints, as is the case for finding the MLEs used in the LRT. In the previous chapter, two sets of alternative estimators were proposed. In the second set of alternative estimators, the estimators $\tilde{\rho}_{1a}$ and $\tilde{\rho}_{2a}$ require the average number of sons in the data set and the average number of daughters in the data set to both be greater than 2, but the alternative estimators $\tilde{\mu}_{1a}$ and $\tilde{\mu}_{2a}$ are easier to compute than the first set of alternative estimators. For familial data from several populations, a combination of the alternative estimators from Chapter II is proposed that does not require the restrictions on the data set and has simpler estimates of the means.

Let n_{1i} equal the number of families in the i^{th} population with $m_{1ij} > 0$, n_{2i} equal the number of families in the i^{th} population with $m_{2ij} > 0$, and n_{12i} equal the number of families in the i^{th} population with $m_{1ij} > 0$ and $m_{2ij} > 0$. Similar to the first set of alternative estimators proposed in Chapter II, $y_{1ij2}, \ldots, y_{1ijm_{1ij}} \sim N(0, \gamma_{1,i}^2)$ and an unbiased and consistent estimator of $\gamma_{1,i}^2$ is

$$\tilde{\gamma}_{1,i}^2 = \frac{\sum_{j=1}^{n_{1i}} \sum_{k=2}^{m_{1ij}} y_{1ijk}^2}{\sum_{j=1}^{n_{1i}} (m_{1ij} - 1)}.$$

Additionally, $y_{1ij2}, \ldots, y_{2ijm_{2ij}} \sim N(0, \gamma_{2,i}^2)$ and $\gamma_{2,i}^2$ can be estimated by

$$\tilde{\gamma}_{2,i}^2 = \frac{\sum_{j=1}^{n_{2i}} \sum_{k=2}^{m_{2ij}} y_{2ijk}^2}{\sum_{j=1}^{n_{2i}} (m_{2ij} - 1)}.$$

Since

$$\begin{pmatrix} y_{1ij1} \\ y_{2ij1} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_{1,i} \\ \mu_{2,i} \end{pmatrix}, \begin{pmatrix} \eta_{1ij}^2 & \sigma_{12,i} \\ \sigma_{12,i} & \eta_{2ij}^2 \end{pmatrix} \right),$$

take

$$\tilde{\sigma}_{12,i} = \sum_{j=1}^{n_{12i}} \frac{(y_{1ij1} - \bar{y}_{1i1}^*)(y_{2ij1} - \bar{y}_{2i1}^*)}{n_{12i} - 1},$$

where $\bar{y}_{1i1}^* = \frac{1}{n_{12i}} \sum_{j=1}^{n_{12i}} y_{1ij1}$ and $\bar{y}_{2i1}^* = \frac{1}{n_{12i}} \sum_{j=1}^{n_{12i}} y_{2ij1}$. Then, $\tilde{\sigma}_{12,i}$ is an unbiased estimate of $\sigma_{12,i}$.

Seeing that, $\sigma^2_{1,i} = \eta^2_{1ij} + a_{1ij}\gamma^2_{1,i} = \frac{1}{n_{1i}}(\sum_{j=1}^{n_{1i}} \eta^2_{1ij}) + \frac{1}{n_{1i}}\gamma^2_{1,i}(\sum_{j=1}^{n_{1i}} a_{1ij})$ and since $y_{1ij2}, \ldots, y_{1ijm_{1ij}}$ $(j = 1, \ldots, n_i)$ were used for estimating $\gamma^2_{1,i}$, Srivastava (1984) proposed using y_{1ij1} $(i = 1, \ldots, n_i)$ to estimate $\sigma^2_{1,i}$. Consider $E\left[\sum_{j=1}^{n_{1i}}(y_{1ij1} - \bar{y}_{1i1})^2\right]$, where $\bar{y}_{1i1} = \frac{1}{n_{1i}}\sum_{j=1}^{n_{1i}} y_{1ij1}$. Which is

$$
\begin{aligned}
E\left(\sum_{j=1}^{n_{1i}} y^2_{1ij1} - n_{1i}\bar{y}^2_{1i1}\right) &= \sum_{j=1}^{n_{1i}}(\eta^2_{1ij} + \mu^2_{1,i}) - n_{1i}\left(\frac{1}{n^2_{1i}}\sum_{j=1}^{n_{1i}}\eta^2_{1ij} + \mu^2_{1,i}\right) \\
&= \left(1 - \frac{1}{n_{1i}}\right)\sum_{j=1}^{n_{1i}}\eta^2_{1ij} \\
&= \left(1 - \frac{1}{n_{1i}}\right)n_{1i}\sigma^2_{1,i} - \left(1 - \frac{1}{n_{1i}}\right)\gamma^2_{1,i}\left(\sum_{j=1}^{n_{1i}} a_{1ij}\right) \\
&= (n_{1i} - 1)\sigma^2_{1,i} - \left(1 - \frac{1}{n_{1i}}\right)\gamma^2_{1,i}\left(\sum_{j=1}^{n_{1i}} a_{1ij}\right).
\end{aligned}
$$

Now, estimate $\sigma^2_{1,i}$ by

$$
\tilde{\sigma}^2_{1,i} = \frac{1}{n_{1i} - 1}\sum_{j=1}^{n_{1i}}(y_{1ij1} - \bar{y_{1i1}})^2 + \frac{1}{n_{1i}}\tilde{\gamma}^2_{1i}(\sum_{j=1}^{n_{1i}} a_{1ij}).
$$

Similarly, one can estimate $\sigma^2_{2,i}$ by

$$
\tilde{\sigma}^2_{2,i} = \frac{1}{n_{2i} - 1}\sum_{j=1}^{n_{2i}}(y_{2ij1} - \bar{y_{2i1}})^2 + \frac{1}{n_{2i}}\tilde{\gamma}^2_{2,i}(\sum_{j=1}^{n_{2i}} a_{2ij}).
$$

From these, other estimates are

$$
\begin{aligned}
\tilde{\rho}_{1,i} &= 1 - (\tilde{\gamma}^2_{1,i}/\tilde{\sigma}^2_{1,i}), \\
\tilde{\rho}_{2,i} &= 1 - (\tilde{\gamma}^2_{2,i}/\tilde{\sigma}^2_{2,i}), \\
\tilde{\rho}_{12,i} &= \frac{\tilde{\sigma}_{12,i}}{\tilde{\sigma}_{1,i}\tilde{\sigma}_{2,i}}.
\end{aligned}
$$

The means $\mu_{1,i}$ and $\mu_{2,i}$ can be estimated similarly to the second set of alternative estimators in the one population case, (9) and (10). Take

$$
\begin{aligned}
\tilde{\mu}_{1,i} &= \frac{1}{n_{1i}}\sum_{j=1}^{n_{1i}}\left[y_{1ij1} - \frac{1}{\sqrt{m_{1ij}}}\sum_{k=2}^{m_{1ij}} y_{1ijk}\right] \\
&= \frac{1}{n_{1i}}\sum_{j=1}^{n_{1i}}\tilde{y}_{1ij1}
\end{aligned}
$$

and

$$\begin{aligned}\tilde{\mu}_{2,i} &= \frac{1}{n_{2i}}\sum_{j=1}^{n_{2i}}\left[y_{2ij1} - \frac{1}{\sqrt{m_{2ij}}}\sum_{k=2}^{m_{2ij}} y_{2ijk}\right] \\ &= \frac{1}{n_{2i}}\sum_{j=1}^{n_{2i}} \tilde{y}_{2ij1}.\end{aligned}$$

III.6 VARIANCE OF ALTERNATIVE ESTIMATORS

The variance of the alternative estimators will be assessed in this section which will be used in construction of alternative tests to the LRT for testing the null hypothesis that the interclass correlations $\rho_{12,i}$ from several populations are equal, that is, $H_0 : \rho_{12,1} = \ldots = \rho_{12,g}$. In order to determine the variance of the alternative estimators, $\tilde{\rho}_{12,i}$, consider the following asymptotic distributions.

$$n_{12i}^{1/2}\begin{bmatrix} \tilde{\sigma}^2_{1,i} - \sigma^2_{1,i} \\ \tilde{\sigma}^2_{2,i} - \sigma^2_{2,i} \\ \tilde{\sigma}_{12,i} - \sigma_{12,i}\end{bmatrix} \to N(0, \Sigma_{12,i}),$$

where

$$\Sigma_{12,i} = 2\begin{bmatrix} c^2_{1i}\sigma^4_1 & \sigma^2_{12,i} & \lambda_{1i}\sigma^2_{1,i}\lambda_{12i} \\ \sigma^2_{12,i} & c^2_{2i}\sigma^4_2 & \lambda_{2i}\sigma^2_{2,i}\sigma_{12,i} \\ \lambda_{1i}\sigma^2_{1,i}\sigma_{12,i} & \lambda_{2i}\sigma^2_{2,i}\sigma_{12,i} & \frac{1}{2}\left(\sigma^2_{12,i} + \sigma^2_{1,i}\sigma^2_{2,i}\lambda_{1i}\lambda_{2i}\right)\end{bmatrix},$$

with

$$\begin{aligned}
c_{1i}^2 &= 1 - 2(1-\rho_{1,i})n_{1i}^{-1}\sum_{j=1}^{n_{1i}} a_{1ij} \\
&\quad + (1-\rho_{1,i})^2\left[n_{1i}^{-1}\sum_{j=1}^{n_{1i}} a_{1ij}^2 + (\bar{m}_{1i}-1)^{-1}(n_{1i}^{-1}\sum_{j=1}^{n_{1i}} a_{1ij})^2\right], \\
c_{2i}^2 &= 1 - 2(1-\rho_{2,i})n_{2i}^{-1}\sum_{j=1}^{n_{2i}} a_{2ij} \\
&\quad + (1-\rho_{2,i})^2\left[n_{2i}^{-1}\sum_{j=1}^{n_{2i}} a_{2ij}^2 + (\bar{m}_{2i}-1)^{-1}(n_{2i}^{-1}\sum_{j=1}^{n_{2i}} a_{2ij})^2\right], \\
\lambda_{1i} &= 1 - (1-\rho_{1,i})\frac{1}{n_{12i}}\sum_{j=1}^{n_{12i}} a_{1ij}, \\
\lambda_{2i} &= 1 - (1-\rho_{2,i})\frac{1}{n_{12i}}\sum_{j=1}^{n_{12i}} a_{1ij}, \\
\bar{m}_{1i} &= n_{1i}^{-1}\sum_{j=1}^{n_{1i}} m_{1ij}, \\
\bar{m}_{2i} &= n_{2i}^{-1}\sum_{j=1}^{n_{2i}} m_{2ij}.
\end{aligned}$$

Using the delta method, the asymptotic variance of $\tilde{\rho}_{12,i}$ is

$$AV(\tilde{\rho}_{12,i}) = \frac{1}{n_{12,i}}\left[\rho_{12,i}^4 + \rho_{12,i}^2\left(\frac{1}{2}c_{1i}^2 - 2\lambda_{1i} + \frac{1}{2}c_{2i}^2 - 2\lambda_{2i} + 1\right) + \lambda_{1i}\lambda_{2i}\right].$$

Let $\tilde{AV}(\tilde{\rho}_{12,i})$ be the estimated $AV(\tilde{\rho}_{12,i})$ obtained by substituting the estimators in this section for the unknown parameters.

III.7 ALTERNATIVE TESTS

Here we propose two alternative tests for testing the null hypothesis that the interclass correlations $\rho_{12,i}$ from several populations are equal, $H_0 : \rho_{12,1} = \ldots = \rho_{12,g}$. Let

$$\boldsymbol{\rho}_{12} = (\rho_{12,1}, \ldots, \rho_{12,g})'.$$

The null hypothesis $H_0 : \rho_{12,1} = \ldots = \rho_{12,g}$ can be written as $H_0 : \mathbf{C}\boldsymbol{\rho}_{12} = \mathbf{0}$, where

$$\mathbf{C} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix}_{g-1,g}.$$

The two test statistics we propose are

$$T_1 = (\mathbf{C}\tilde{\boldsymbol{\rho}}_{12})'[\mathbf{C}\mathbf{V}_1\mathbf{C}']^{-1}(\mathbf{C}\tilde{\boldsymbol{\rho}}_{12}), \tag{17}$$

where

$$\mathbf{V}_1 = diag(\tilde{AV}(\tilde{\rho}_{12,1}), \ldots, \tilde{AV}(\tilde{\rho}_{12,g})),$$

and

$$T_0 = (\mathbf{C}\tilde{\boldsymbol{\rho}}_{12})'[\mathbf{C}\mathbf{V}_0\mathbf{C}']^{-1}(\mathbf{C}\tilde{\boldsymbol{\rho}}_{12}), \tag{18}$$

where

$$\begin{aligned} \mathbf{V}_0 &= diag(\tilde{AV}(\tilde{\rho}_{12}), \ldots, \tilde{AV}(\tilde{\rho}_{12})) \\ \tilde{\rho}_{12} &= \frac{1}{g}\sum_{j=1}^{g} \tilde{\rho}_{12,i}. \end{aligned}$$

Both T_1 and T_0 have asymptotic chi-square distributions with $g-1$ degrees of freedom. T_1 and T_0 are simpler to implement than the LRT which requires an iterative maximization procedure that does not always converge. However, the alternative estimates $\tilde{\rho}_{12,i}$ do not always satisfy the constraints of the this familial correlation model (15), as was also noted in the one population case.

III.8 SIMULATION EXPERIMENTS AND RESULTS

For testing $H_0 : \rho_{12,1} = \ldots = \rho_{12,g}$, the three tests shown here are expected to behave similarly for large sample sizes, since they all have asymptotic chi-square distributions with $g-1$ degrees of freedom. In order to compare the tests, two small sample simulation experiments were conducted. The first simulation experiment had equal family sizes with 4 boys and 4 girls per family. The second experiment simulated family sizes from the U.S. birth distribution proposed by Brass (1958). For

both simulation experiments, only positive values of the familial correlations, $\rho_{12,i}$, are considered because the model constraints (15) restrict possible negative values based on the other parameter values and a family's size. Each size table for the $\alpha = 0.01$ level gives the percentage of the simulations with convergence problems and the percentage of simulations with parameter violations; these percentages for the $\alpha = 0.01$ level also apply for the other size tables, the $\alpha = 0.05$ and $\alpha = 0.10$ levels, since the different sizes are estimated from the same run. The power tables also give the percentage of simulations with convergence problems and the percentage of simulations that violate the model constraints.

For the first experiment, $n_i = 50$ family score vectors for $g = 3$ populations are simulated as multivariate normal random vectors. The family size for each vector is 8 children consisting of 4 sons and 4 daughters. The choices of parameters are $\mu_{1,1} = 0$, $\mu_{1,2} = 0$, $\mu_{1,3} = 0$, $\sigma^2_{1,1} = 1$, $\sigma^2_{2,1} = 2$, $\sigma^2_{1,2} = 0.5$, $\sigma^2_{2,2} = 1.5$, $\sigma^2_{1,3} = 1.5$, $\sigma^2_{2,3} = 2.5$, and $\rho_{1,i}$ and $\rho_{2,i}$ take on values from 0.1 to 0.9. The interclass correlations, $\rho_{12,i}$, are set between 0 and the smallest population lower bound on $\rho_{12,i}$: $min_i\sqrt{((\rho_{1,i} + (1-\rho_{1,i})/4)(\rho_{2,i} + (1-\rho_{2,i})/4))}$. For each choice of parameters, 5,000 simulations were run and estimated size and power values were computed for testing, $H_0 : \rho_{12,1} = \ldots = \rho_{12,g}$.

Table 51 gives parameter values for five different choices of $\rho_{1,i}$ and $\rho_{2,i}$. Tables 52-54 give the estimated size values for the 5 choices of parameters in Table 51 when

$$\rho_{12,i} \quad = min_i \frac{\sqrt{((\rho_{1,i}+(1-\rho_{1,i})/4)(\rho_{2,i}+(1-\rho_{2,i})/4))}}{2}$$

which is given in the table.

TABLE 51: Parameter Values for 3 Sample Simulations.

Simulation	$\rho_{1,1}$	$\rho_{2,1}$	$\rho_{1,2}$	$\rho_{2,2}$	$\rho_{1,3}$	$\rho_{2,3}$
1	0.1	0.9	0.3	0.7	0.5	0.5
2	0.2	0.8	0.4	0.6	0.8	0.2
3	0.5	0.3	0.4	0.4	0.3	0.5
4	0.3	0.7	0.3	0.7	0.3	0.7
5	0.5	0.5	0.5	0.5	0.5	0.5

All three tests tend to be slightly larger than the assumed level. The alternative test T_0 is closest to the assumed level in 13 of the 15 cases. The alternative test T_1 comes close to the assumed level but is only closer than the LRT in 4 of the 15 cases.

TABLE 52: Sizes, $\alpha = 0.01, H_0 : \rho_{12,1} = \rho_{12,2} = \rho_{12,3} = Midpoint$.
*Percent that did not converge; **Percent that violated the model constraints.

Simulation	$\rho_{12,i}$	LRT (*)	T_1 (**)	T_0
1	0.274146	0.0111 (12.11%)	0.0118 (0%)	0.0108
2	0.291548	0.0108 (0.18%)	0.0106 (0%)	0.0100
3	0.272431	0.0126 (0%)	0.0116 (0%)	0.0110
4	0.303367	0.0134 (0.03%)	0.0152 (0%)	0.0132
5	0.312500	0.0122 (0%)	0.0128 (0%)	0.0118

TABLE 53: Sizes, $\alpha = 0.05, H_0 : \rho_{12,1} = \rho_{12,2} = \rho_{12,3} = Midpoint$.

Simulation	$\rho_{12,i}$	LRT	T_1	T_0
1	0.274146	0.0530	0.0556	0.0526
2	0.291548	0.0534	0.0548	0.0532
3	0.272431	0.0552	0.0554	0.0540
4	0.303367	0.0578	0.0598	0.0586
5	0.3125	0.0584	0.0602	0.0566

Tables 55-57 give the estimated sizes for $\alpha = 0.01, 0.05$, and 0.10, respectively, for the parameter choices in Simulation 3 from Table 51 for values of $\rho_{12,i}$ within bound.

One can see that the alternative tests, T_1 and T_0 still tend to be slightly larger than the assumed level. The estimated sizes for the LRT generally perform well, but notably smaller than the assumed level for $\rho_{12,i} = 0.05$. The alternative test T_0 is closest to the assumed level in 14 of the 24 cases.

Table 58 gives estimated power calculations adjusted to the level each test attained in size estimation for the nominal level of $\alpha = 0.05$. The rejection proportions are based on the 95^{th} percentiles of the test statistics from the size simulation for the parameter choices in Simulation 3 from Table 51. The 95^{th} percentile was 6.21131 for the LRT, 6.25925 for T_1, and 6.15925 for T_2 from the simulation of $H_0 : \rho_{12,1} = \rho_{12,2} = \rho_{12,3}$ for the specified parameters.

It can be seen that, the LRT has the highest power in 3 of the cases, T_0 has the highest power in 2 of the 6 cases, and all 3 tests tie in the other case. When the estimated power for the LRT is higher, T_1 and T_0 are typically not far behind. The estimated powers for T_1 are less than 0.01 below the power levels of the LRT for 5

TABLE 54: Sizes, $\alpha = 0.10, H_0 : \rho_{12,1} = \rho_{12,2} = \rho_{12,3} = Midpoint.$

Simulation	$\rho_{12,i}$	LRT	T_1	T_0
1	0.274146	0.1040	0.1078	0.1048
2	0.291548	0.1040	0.1050	0.1038
3	0.272431	0.1152	0.1140	0.1112
4	0.303367	0.1115	0.1114	0.1086
5	0.3125	0.1122	0.1128	0.1088

TABLE 55: Sizes, $\alpha = 0.01, H_0 : \rho_{12,1} = \rho_{12,2} = \rho_{12,3} = \rho_{12}$.
*Percent that did not converge; **Percent that violated the model constraints.

$\rho_{12,i}$	LRT (*)	T_1 (**)	T_0
0.05	0.0048 (0%)	0.0110 (0%)	0.0106
0.10	0.0076 (0%)	0.0126 (0%)	0.0118
0.15	0.0117 (0.23%)	0.0144 (0%)	0.0130
0.20	0.0098 (0%)	0.0104 (0%)	0.0102
0.25	0.0112 (0%)	0.0116 (0%)	0.0110
0.30	0.0098 (0%)	0.0102 (0%)	0.0082
0.35	0.0098 (0.07%)	0.0128 (0%)	0.0112
0.40	0.0085 (0.41%)	0.0106 (0%)	0.0104

of the 6 cases. The estimated powers for T_0 are either higher or less than 0.01 below the power levels of the LRT for 4 of the 6 cases.

Table 59 gives the estimated power calculations for the nominal level of $\alpha = 0.05$ for the parameter choices in Simulation 3 from Table 51.

It can be seen that the LRT has the highest power in 5 of the 6 cases, but T_1 and T_0 are not far behind. The estimated powers for T_1 are less than 0.01 below the power levels of the LRT and the estimated powers for T_0 are also less than 0.01 below the power levels of the LRT for 4 of the 6 cases.

For the second experiment, $n_i = 50$ family score vectors from $g = 3$ populations are simulated as multivariate normal random vectors. The family size for each vector is simulated from a truncated negative binomial distribution with the number of children ranging from 1 to 15. The mean of the negative binomial distribution is taken as 2.84 and the success probability as 0.483 which is the estimated distribution of U.S. births as proposed by Brass (1958). The choice of parameters is the same

TABLE 56: Sizes, $\alpha = 0.05, H_0 : \rho_{12,1} = \rho_{12,2} = \rho_{12,3} = \rho_{12}$.

$\rho_{12,i}$	LRT	T_1	T_0
0.05	0.0270	0.0544	0.0528
0.10	0.0468	0.0614	0.0594
0.15	0.0547	0.0590	0.0568
0.20	0.0512	0.0518	0.0498
0.25	0.0548	0.0556	0.0528
0.30	0.0502	0.0524	0.0490
0.35	0.0613	0.0648	0.0596
0.40	0.0512	0.0538	0.0512

TABLE 57: Sizes, $\alpha = 0.10, H_0 : \rho_{12,1} = \rho_{12,2} = \rho_{12,3} = \rho_{12}$.

$\rho_{12,i}$	LRT	T_1	T_0
0.05	0.0600	0.1036	0.1004
0.10	0.1010	0.1146	0.1130
0.15	0.1107	0.1170	0.1136
0.20	0.1112	0.1108	0.1096
0.25	0.1088	0.1080	0.1030
0.30	0.1090	0.1116	0.1076
0.35	0.1086	0.1136	0.1096
0.40	0.1030	0.1056	0.1042

as in the first simulation experiment for the 3 groups. For each case of parameters, 5,000 simulations were run.

The same five different choices of $\rho_{1,i}$ and $\rho_{2,i}$ used in the first simulation experiment (Table 51) are used in this experiment with unbalanced family sizes. Tables 60-62 give the estimated size values for the $\alpha = 0.01, 0.05$, and 0.10 levels of for the 5 choices of parameters in Table 51 when

$$\rho_{12,i} \quad = min_i \frac{\sqrt{((\rho_{1,i}+(1-\rho_{1,i})/b_i)(\rho_{2,i}+(1-\rho_{2,i})/g_i))}}{2}$$

where b_i is the maximum number of sons in a family from the i^{th} population and g_i is the maximum number of daughters in a family from the i^{th} population.

The following observations can be made. The LRT sizes are larger than the assumed levels. The alternative test T_1 is closest to the assumed level in 5 of the 15 cases, and the alternative test T_0 is closest to the assumed level in 9 of the 15 cases.

TABLE 58: Adjusted Rejection Proportions, $\alpha = 0.05, H_0 : \rho_{12,1} = \rho_{12,2} = \rho_{12,3} = \rho_{12}$.
*Percent that did not converge; **Percent that violated the model constraints.

$\rho_{12,1}$	$\rho_{12,2}$	$\rho_{12,3}$	LRT (*)	T_1 (**)	T_0
0.15	0.20	0.25	0.1354 (0%)	0.1342 (0%)	0.1366
0.10	0.20	0.30	0.3760 (0%)	0.3690 (0%)	0.3722
0.05	0.20	0.35	0.7532 (0.04%)	0.5524 (0%)	0.5440
0.25	0.20	0.25	0.0758 (0%)	0.0750 (0%)	0.0760
0.10	0.25	0.10	0.2500 (0%)	0.2500 (0%)	0.2500
0.05	0.35	0.05	0.8872 (0.01%)	0.8788 (0%)	0.8780

TABLE 59: Rejection Proportions, $\alpha = 0.05, H_0 : \rho_{12,1} = \rho_{12,2} = \rho_{12,3} = \rho_{12}$.
*Percent that did not converge; **Percent that violated the model constraints.

$\rho_{12,1}$	$\rho_{12,2}$	$\rho_{12,3}$	LRT (*)	T_1 (**)	T_0
0.15	0.20	0.25	0.1348 (0%)	0.1336 (0%)	0.1300
0.10	0.20	0.30	0.4080 (0%)	0.4058 (0%)	0.3996
0.05	0.20	0.35	0.7695 (0.02%)	0.7658 (0%)	0.7628
0.25	0.20	0.25	0.0804 (0%)	0.0814 (0%)	0.0794
0.10	0.25	0.10	0.3000 (0%)	0.2800 (0%)	0.2600
0.05	0.35	0.05	0.8954 (0%)	0.8866 (0%)	0.8834

When comparing T_1 to only the LRT, T_1 is closer to the assumed level in 11 of the 15 cases. Similarly when comparing T_0 to only the LRT, T_0 is closer to the assumed level in 14 of the 15 cases.

Tables 63-65 give the estimated sizes for $\alpha = 0.01, 0.05$, and 0.10, respectively, for the parameter choices in Simulation 3 from Table 51 when values of $\rho_{12,i}$ are within bound.

It can be observed that all three tests generally estimate the sizes well except for $\rho_{12,i} = 0.4$. The LRT tends to be larger than the assumed level while the alternative tests vary in direction. The alternative test T_1 is closest to the assumed level in 11 of the 24 cases, and the alternative test T_0 is closest to the assumed level in 9 of the 24 cases. When comparing each alternative test to only the LRT, both tests are closer than the LRT in 20 of the 24 cases.

Table 66 gives estimated power calculations adjusted to the level each test attained in size estimation for the nominal level of $\alpha = 0.05$. The rejection proportions

TABLE 60: Sizes, $\alpha = 0.01, H_0 : \rho_{12,1} = \rho_{12,2} = \rho_{12,3} = Midpoint.$
*Percent that did not converge; **Percent that violated the model constraints.

Simulation	LRT (*)	T_1 (**)	T_0
1	0.0114 (2.71%)	0.0112 (9.18%)	0.0100
2	0.0141 (1.23%)	0.0154 (8.33%)	0.0126
3	0.0162 (0.75%)	0.0096 (6.69%)	0.0080
4	0.0129 (1.82%)	0.0130 (5.87%)	0.0106
5	0.0127 (2.04%)	0.0186 (2.16%)	0.0154

TABLE 61: Sizes, $\alpha = 0.05, H_0 : \rho_{12,1} = \rho_{12,2} = \rho_{12,3} = Midpoint.$

Simulation	LRT	T_1	T_0
1	0.0653	0.0492	0.0458
2	0.0636	0.0580	0.0554
3	0.0717	0.0488	0.0452
4	0.0565	0.0556	0.0522
5	0.0550	0.0560	0.0528

are based on the 95^{th} percentiles of the test statistics from the size simulation for the parameter choices in Simulation 3 from Table 51. For the specified parameters, the 95^{th} percentile was 6.80493 for the LRT, 5.88769 for T_1, and 5.74912 for T_2 from the simulation of $H_0 : \rho_{12,1} = \rho_{12,2} = \rho_{12,3}$.

The adjusted power estimates in the unbalanced case show T_1 to have higher powers for 3 of the 5 cases. T_0 also has higher adjusted powers than the LRT for these same 3 cases. For the other 2 cases, the LRT adjusted powers are higher but not by much.

Table 67 gives the estimated power calculations for the nominal level of $\alpha = 0.05$ for the parameter choices in Simulation 3 from Table 51.

The unadjusted power estimates in the unbalanced case show the LRT to have higher powers than both alternative tests. The alternative test T_1 has estimated powers closer to the LRT than the alternative test T_0. Since the estimated sizes of the LRT were mostly larger than the estimated sizes of both alternative tests, the LRT has an advantage when not adjusted for the size values attained for a nominal level of α.

TABLE 62: Sizes, $\alpha = 0.10, H_0 : \rho_{12,1} = \rho_{12,2} = \rho_{12,3} = Midpoint.$

Simulation	**LRT**	T_1	T_0
1	0.1267	0.0988	0.0958
2	0.1193	0.1044	0.1000
3	0.1312	0.0960	0.0918
4	0.1179	0.1082	0.1036
5	0.1090	0.1074	0.1000

TABLE 63: Sizes, $\alpha = 0.01, H_0 : \rho_{12,1} = \rho_{12,2} = \rho_{12,3} = \rho_{12}$.
*Percent that did not converge; **Percent that violated the model constraints.

$\rho_{12,i}$	**LRT (*)**	T_1 (**)	T_0
0.05	0.0118 (0.01%)	0.0074 (2.28%)	0.0054
0.10	0.0100 (0.14%)	0.0114 (2.81%)	0.0084
0.15	0.0165 (0.28%)	0.0110 (4.06%)	0.0084
0.20	0.0139 (0.51%)	0.0126 (6.08%)	0.0096
0.25	0.0185 (1.30%)	0.0128 (9.79%)	0.0106
0.30	0.0210 (2.84%)	0.0156 (14.94%)	0.0108
0.35	0.0266 (5.21%)	0.0182 (22.53%)	0.0136
0.40	0.0273 (9.35%)	0.0248 (31.42%)	0.0200

The percentages of non-convergence and violation constraints observed in this simulation experiment are comparable. In practice, the alternative estimates and corresponding test are easy to compute, although, one would need to check to see if the calculated alternative estimates meet the model constraints before their use. If the model constraints are violated then the LRT can be used.

When family sizes are balanced, both alternative tests compare well to the LRT, but the alternative test T_0 performs better than the T_1. When family sizes are unbalanced, again both alternative tests compare well to the LRT. The alternative test T_0 had better size performance while T_1 performed better in the power calculations. When family sizes were unbalanced, the alternative tests violated the model constraints more often than the LRT failed to converged, but particulary when $\rho_{12,i}$ is close to its upper bound. In practice, the alternative estimates and both corresponding tests are easy to compute. Generally, both alternative tests are recommended, but T_0 showed an advantage when family sizes are equal.

TABLE 64: Sizes, $\alpha = 0.05, H_0 : \rho_{12,1} = \rho_{12,2} = \rho_{12,3} = \rho_{12}$.

$\boldsymbol{\rho_{12,i}}$	**LRT**	$\boldsymbol{T_1}$	$\boldsymbol{T_0}$
0.05	0.0340	0.0420	0.0388
0.10	0.0407	0.0482	0.0446
0.15	0.0603	0.0490	0.0458
0.20	0.0647	0.0472	0.0430
0.25	0.0676	0.0524	0.0464
0.30	0.0712	0.0620	0.0576
0.35	0.0781	0.0660	0.0590
0.40	0.0708	0.0810	0.0740

TABLE 65: Sizes, $\alpha = 0.10, H_0 : \rho_{12,1} = \rho_{12,2} = \rho_{12,3} = \rho_{12}$.

$\boldsymbol{\rho_{12,i}}$	**LRT**	$\boldsymbol{T_1}$	$\boldsymbol{T_0}$
0.05	0.0638	0.0808	0.0784
0.10	0.0854	0.0916	0.0858
0.15	0.1195	0.0946	0.0904
0.20	0.1283	0.0944	0.0890
0.25	0.1230	0.0960	0.0908
0.30	0.1296	0.1138	0.1058
0.35	0.1264	0.1202	0.1136
0.40	0.1267	0.1362	0.1296

TABLE 66: Adjusted Rejection Proportions, $\alpha = 0.05, H_0 : \rho_{12,1} = \rho_{12,2} = \rho_{12,3} = \rho_{12}$.
*Percent that did not converge; **Percent that violated the model constraints.

$\boldsymbol{\rho_{12,1}}$	$\boldsymbol{\rho_{12,2}}$	$\boldsymbol{\rho_{12,3}}$	**LRT** (*)	$\boldsymbol{T_1}$ (**)	$\boldsymbol{T_0}$
0.15	0.20	0.25	0.0641 (0.72%)	0.0732 (6.70%)	0.0706
0.10	0.20	0.30	0.1344 (0.90%)	0.1330 (8.54%)	0.1324
0.05	0.20	0.35	0.2790 (1.57%)	0.2624 (11.43%)	0.2640
0.25	0.20	0.25	0.0520 (0.97%)	0.0600 (8.73%)	0.0534
0.10	0.25	0.10	0.0999 (0.44%)	0.1122 (5.17%)	0.1110
0.05	0.35	0.05	0.3423 (1.41%)	0.3214 (9.69%)	0.3186

TABLE 67: Rejection Proportions, $\alpha = 0.05, H_0 : \rho_{12,1} = \rho_{12,2} = \rho_{12,3} = \rho_{12}$.
*Percent that did not converge; **Percent that violated the model constraints.

$\rho_{12,1}$	$\rho_{12,2}$	$\rho_{12,3}$	LRT (*)	T_1 (**)	T_0
0.15	0.20	0.25	0.0830 (0.94%)	0.0682 (4.72%)	0.0640
0.10	0.20	0.30	0.1841 (0.94%)	0.1370 (8.23%)	0.1282
0.05	0.20	0.35	0.3486 (1.60%)	0.2574 (11.71%)	0.2466
0.25	0.20	0.25	0.0705 (1.50%)	0.0654 (5.11%)	0.0608
0.10	0.25	0.10	0.1318 (0.31%)	0.1044 (5.02%)	0.0960
0.05	0.35	0.05	0.4324 (1.30%)	0.3240 (9.75%)	0.3066

CHAPTER IV

PARENT INTERCLASS CORRELATIONS

IV.1 INTRODUCTION

In this chapter we consider interclass correlations between the parent and sons and between the parent and daughters in a family. The problem of interest is to test the equality of these dependent interclass correlations. Suppose data on a parent and the children of n randomly selected families are available from one population. The number of boys and girls in each family is allowed to be different. Denote the number of sons and daughters in the i^{th} family as m_{1i} and m_{2i}, respectively, for $i = 1, \ldots, n$. Suppose x_{pi}, $i = 1, \ldots, n$ is the observation on the parent of the i^{th} family. Also, suppose x_{1ij}, $j = 1, \ldots, m_{1i}$; $i = 1, \ldots, n$ is the observation on the j^{th} boy of the i^{th} family. Similarly, x_{2ij}, $j = 1, \ldots, m_{2i}$; $i = 1, \ldots, n$ is the observation on the j^{th} girl of the i^{th} family.

Assume that the expected value of the parent observations is $E(x_{pi}) = \mu_p$, the expected value of the son observations is $E(x_{1ij}) = \mu_1$, and the expected value of the daughter observations is $E(x_{2ij}) = \mu_2$. Assume that the variance of the parent observations is $Var(x_{pi}) = \sigma_p^2$, the variance of the son observations is $Var(x_{1ij}) = \sigma_1^2$, and the variance of the daughter observations is $Var(x_{2ij}) = \sigma_2^2$. Denote the parent-son *interclass* correlation as ρ_{p1} and the parent-daughter *interclass* correlation as ρ_{p2}. Assume for each family $Corr(x_{pi}, x_{1ij}) = \rho_{p1}$ and $Corr(x_{pi}, x_{2ij}) = \rho_{p2}$ for all j. Additionally, denote the son-son *intraclass* correlation as ρ_1, the daughter-daughter *intraclass* correlation as ρ_2, and the son-daughter *interclass* correlation as ρ_{12}. Assume for each family $Corr(x_{1ij}, x_{1ij'}) = \rho_1$ for $j \neq j'$; $1 \leq j, j' \leq m_{1i}$, $Corr(x_{2ij}, x_{2ij'}) = \rho_2$ for $j \neq j'$; $1 \leq j, j' \leq m_{2i}$, and $Corr(x_{1ij}, x_{2ij'}) = \rho_{12}$ for all j, j'; $1 \leq j \leq m_{1i}$ and $1 \leq j \leq m_{2i}$.

Let the vector of observations on the i^{th} family be

$$\mathbf{x}_i = \begin{pmatrix} x_{pi} \\ \mathbf{x}_{1i} \\ \mathbf{x}_{2i} \end{pmatrix} = \begin{pmatrix} x_{pi} \\ x_{1i1} \\ \vdots \\ x_{1im_{1i}} \\ x_{2i1} \\ \vdots \\ x_{2im_{2i}} \end{pmatrix}$$

with

$$E(\mathbf{x}_i) = \boldsymbol{\mu}_i = \begin{pmatrix} \mu_p \\ \mu_1 \mathbf{1}_{m_{1i}} \\ \mu_2 \mathbf{1}_{m_{2i}} \end{pmatrix},$$

and

$$\begin{aligned} Var(\mathbf{x}_i) &= \boldsymbol{\Sigma}_i \\ &= \begin{pmatrix} \sigma_p^2 & \rho_{p1}\sigma_p\sigma_1\mathbf{1}'_{m_{1i}} & \rho_{p2}\sigma_p\sigma_2\mathbf{1}'_{m_{2i}} \\ \rho_{p1}\sigma_p\sigma_1\mathbf{1}_{m_{1i}} & \sigma_1^2\{(1-\rho_1)\mathbf{I}_{m_{1i}} + \rho_1\mathbf{J}_{m_{1i}}\} & \rho_{12}\sigma_1\sigma_2\mathbf{J}_{m_{1i},m_{2i}} \\ \rho_{p2}\sigma_p\sigma_2\mathbf{1}_{m_{2i}} & \rho_{12}\sigma_1\sigma_2\mathbf{J}_{m_{2i},m_{1i}} & \sigma_2^2\{(1-\rho_2)\mathbf{I}_{m_{2i}} + \rho_2\mathbf{J}_{m_{2i}}\} \end{pmatrix}, \end{aligned}$$

where $\mathbf{I}_m$ is an identity matrix of order m, $\mathbf{J}_m$ is the $m \times m$ matrix of all ones, and $\mathbf{J}_{m,n}$ is the $m \times n$ matrix of all ones. Note $-\infty < \mu_p < \infty$, $-\infty < \mu_1 < \infty$, and $-\infty < \mu_2 < \infty$.

If a family consists of both sons and daughters, $m_{1i} > 0$ and $m_{2i} > 0$, then the determinant of $\boldsymbol{\Sigma}_i$ is

$$\begin{aligned} |\boldsymbol{\Sigma}_i| &= \sigma_p^2\sigma_1^{2m_{1i}}\sigma_2^{2m_{2i}}(1-\rho_1)^{m_{1i}-1}(1-\rho_2)^{m_{2i}-1}c_i \\ &\times \left[1 - \frac{\rho_{p1}^2}{1-\rho_1}(m_{1i} + \frac{a_i}{c_i}m_{1i}^2) + 2\rho_{p1}\rho_{p2}\rho_{12}\frac{m_{1i}m_{2i}}{c_i} - \frac{\rho_{p2}^2}{1-\rho_2}(m_{2i} + \frac{b_i}{c_i}m_{2i}^2)\right], \end{aligned}$$

where

$$\begin{aligned} a_i &= m_{2i}\rho_{12}^2 - \rho_1(1 + (m_{2i}-1)\rho_2), \\ b_i &= m_{1i}\rho_{12}^2 - \rho_2(1 + (m_{1i}-1)\rho_1), \\ c_i &= (1 + (m_{1i}-1)\rho_1)(1 + (m_{2i}-1)\rho_2) - m_{1i}m_{2i}\rho_{12}^2. \end{aligned}$$

Restrictions on the parameters so that $\boldsymbol{\Sigma}_i$ positive definite are $\sigma_p^2 > 0$, $\sigma_1^2 > 0$, $\sigma_2^2 > 0$, $\rho_1 < 1$, $\rho_2 < 1$,

$$(1 + (m_{1i}-1)\rho_1)(1 + (m_{2i}-1)\rho_2) > m_{1i}m_{2i}\rho_{12}^2, \tag{19}$$

and

$$\frac{\rho_{p1}^2}{1-\rho_1}(m_{1i}+\frac{a_i}{c_i}m_{1i}^2)-2\rho_{p1}\rho_{p2}\rho_{12}\frac{m_{1i}m_{2i}}{c_i}+\frac{\rho_{p2}^2}{1-\rho_2}(m_{2i}+\frac{b_i}{c_i}m_{2i}^2)<1. \qquad (20)$$

If $m_{1i}>0$ and $m_{2i}>0$, then the inverse of $\mathbf{\Sigma}_i$ is

$$\mathbf{\Sigma}_i^{-1}=\begin{pmatrix} \mathbf{D}_i & \mathbf{E}_i & \mathbf{F}_i \\ \mathbf{E}_i' & \mathbf{G}_i & \mathbf{H}_i \\ \mathbf{F}_i' & \mathbf{H}_i' & \mathbf{K}_i \end{pmatrix},$$

where

$$\begin{aligned}
\mathbf{D}_i &= \frac{1}{\sigma_p^2}\left[1+\frac{m_{1i}\rho_{p1}^2 d_i+m_{2i}\rho_{p2}^2 e_i-2m_{1i}m_{2i}\rho_{p1}\rho_{p2}(\rho_{12}-\rho_{p1}\rho_{p2})}{d_ie_i-m_{1i}m_{2i}(\rho_{12}-\rho_{p1}\rho_{p2})^2}\right],\\
\mathbf{E}_i &= \frac{m_{2i}\rho_{p2}(\rho_{12}-\rho_{p1}\rho_{p2})-\rho_{p1}e_i}{\sigma_p\sigma_1(d_ie_i-m_{1i}m_{2i}(\rho_{12}-\rho_{p1}\rho_{p2})^2)}\mathbf{1}'_{m_{1i}},\\
\mathbf{F}_i &= \frac{m_{1i}\rho_{p1}(\rho_{12}-\rho_{p1}\rho_{p2})-\rho_{p2}d_i}{\sigma_p\sigma_2(d_ie_i-m_{1i}m_{2i}(\rho_{12}-\rho_{p1}\rho_{p2})^2)}\mathbf{1}'_{m_{2i}},\\
\mathbf{G}_i &= \frac{1}{\sigma_1^2(1-\rho_1)}\left[\mathbf{I}_{m_{1i}}-\frac{e_i(\rho_1-\rho_{p1}^2)-m_{2i}(\rho_{12}-\rho_{p1}\rho_{p2})^2}{d_ie_i-m_{1i}m_{2i}(\rho_{12}-\rho_{p1}\rho_{p2})^2}\mathbf{J}_{m_{1i}}\right],\\
\mathbf{H}_i &= \frac{-(\rho_{12}-\rho_{p1}\rho_{p2})}{\sigma_1\sigma_2(d_ie_i-m_{1i}m_{2i}(\rho_{12}-\rho_{p1}\rho_{p2})^2)}\mathbf{J}_{m_{1i},m_{2i}},\\
\mathbf{K}_i &= \frac{1}{\sigma_2^2(1-\rho_2)}\left[\mathbf{I}_{m_{2i}}-\frac{d_i(\rho_2-\rho_{p2}^2)-m_{1i}(\rho_{12}-\rho_{p1}\rho_{p2})^2}{d_ie_i-m_{1i}m_{2i}(\rho_{12}-\rho_{p1}\rho_{p2})^2}\mathbf{J}_{m_{2i}}\right],
\end{aligned}$$

and

$$\begin{aligned}
d_i &= 1+(m_{2i}-1)\rho_2-m_{2i}\rho_{p2}^2,\\
e_i &= 1+(m_{1i}-1)\rho_1-m_{1i}\rho_{p1}^2.
\end{aligned}$$

However, these expressions simplify quite a bit if a family consists of only daughters, $m_{1i}=0$. The determinant of $\mathbf{\Sigma}_i$ is then

$$|\mathbf{\Sigma}_i|=\sigma_p^2\sigma_2^{2m_{2i}}(1-\rho_2)^{m_{2i}-1}(1+(m_{2i}-1)\rho_2-m_{2i}\rho_{p2}^2),$$

and the inverse of $\mathbf{\Sigma}_i$ is

$$\mathbf{\Sigma}_i^{-1}=\begin{pmatrix} \frac{1+(m_{2i}-1)\rho_2}{\sigma_p^2(1+(m_{2i}-1)\rho_2-m_{2i}\rho_{p2}^2)} & \frac{-\rho_{p2}}{\sigma_p\sigma_2(1+(m_{2i}-1)\rho_2-m_{2i}\rho_{p2}^2)}\mathbf{1}'_{m_{2i}} \\ \frac{-\rho_{p2}}{\sigma_p\sigma_2(1+(m_{2i}-1)\rho_2-m_{2i}\rho_{p2}^2)}\mathbf{1}_{m_{2i}} & \frac{1}{\sigma_2^2(1-\rho_2)}\left[I_{m_{2i}}-\frac{\rho_2-\rho_{p2}^2}{1+(m_{2i}-1)\rho_2-m_{2i}\rho_{p2}^2}J_{m_{2i}}\right] \end{pmatrix}.$$

Similarly, if a family consists of only sons, $m_{2i}=0$, then the determinant of $\mathbf{\Sigma}_i$ is

$$|\mathbf{\Sigma}_i|=\sigma_p^2\sigma_1^{2m_{1i}}(1-\rho_1)^{m_{1i}-1}(1+(m_{1i}-1)\rho_1-m_{1i}\rho_{p1}^2),$$

and the inverse of $\mathbf{\Sigma}_i$ is

$$\mathbf{\Sigma}_i^{-1} = \begin{pmatrix} \frac{1+(m_{1i}-1)\rho_1}{\sigma_p^2(1+(m_{1i}-1)\rho_1-m_{1i}\rho_{p1}^2)} & \frac{-\rho_{p1}}{\sigma_p\sigma_1(1+(m_{1i}-1)\rho_1-m_{1i}\rho_{p1}^2)}\mathbf{1}'_{m_{1i}} \\ \frac{-\rho_{p1}}{\sigma_p\sigma_1(1+(m_{1i}-1)\rho_1-m_{1i}\rho_{p1}^2)}\mathbf{1}_{m_{1i}} & \frac{1}{\sigma_1^2(1-\rho_1)}\left[I_{m_{1i}} - \frac{\rho_1-\rho_{p1}^2}{1+(m_{1i}-1)\rho_1-m_{1i}\rho_{p1}^2}J_{m_{1i}}\right] \end{pmatrix}.$$

IV.2 LIKELIHOOD FUNCTION

Assume that $\mathbf{x}_i \sim N_{m_{1i}+m_{2i}+1}(\boldsymbol{\mu}_i, \mathbf{\Sigma}_i)$, $i = 1, \ldots, n$. Let

$$\boldsymbol{\theta} = \left(\mu_p, \mu_1, \mu_2, \sigma_p^2, \sigma_1^2, \sigma_2^2, \rho_1, \rho_2, \rho_{12}, \rho_{p1}, \rho_{p2}\right)'$$

then

$$L(\boldsymbol{\theta}) = \prod_{i=1}^n L_i(\boldsymbol{\theta}) = \prod_{i=1}^n \frac{1}{(2\pi)^{(m_{1i}+m_{2i}+1)/2}|\mathbf{\Sigma}_i|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}_i-\boldsymbol{\mu}_i)'\mathbf{\Sigma}_i^{-1}(\mathbf{x}_i-\boldsymbol{\mu}_i)}$$

and

$$log(L(\boldsymbol{\theta})) = \sum_{i=1}^n log(L_i(\boldsymbol{\theta})).$$

If $m_{1i} > 0$ and $m_{2i} > 0$, then

$$\begin{aligned} log(L_i(\boldsymbol{\theta})) &= -\frac{1}{2}log(2\pi\sigma_p^2) - \frac{m_{1i}}{2}log(2\pi\sigma_1^2) - \frac{m_{2i}}{2}log(2\pi\sigma_2^2) \\ &- \frac{1}{2}(m_{1i}-1)log(1-\rho_1) - \frac{1}{2}(m_{2i}-1)log(1-\rho_2) \\ &- \frac{1}{2}log\left[(1+(m_{1i}-1)\rho_1)(1+(m_{2i}-1)\rho_2) - m_{1i}m_{2i}\rho_{12}^2\right] \\ &- \frac{1}{2}log\left[1 - \frac{\rho_{p1}^2}{1-\rho_1}(m_{1i} + \frac{a_i}{c_i}m_{1i}^2) + 2\rho_{p1}\rho_{p2}\rho_{12}\frac{m_{1i}m_{2i}}{c_i}\right. \\ &- \left.\frac{\rho_{p2}^2}{1-\rho_2}(m_{2i} + \frac{b_i}{c_i}m_{2i}^2)\right] \\ &- \frac{1}{2}(\mathbf{x}_i - \boldsymbol{\mu}_i)'\mathbf{\Sigma}_i^{-1}(\mathbf{x}_i - \boldsymbol{\mu}_i). \end{aligned}$$

Note $\mathbf{x}_i = \begin{pmatrix} x_{pi} \\ \mathbf{x}_{1i} \\ \mathbf{x}_{2i} \end{pmatrix}$, where $\mathbf{x}_{1i} = (x_{1i1}, \ldots, x_{1im_{1i}})'$ and $\mathbf{x}_{2i} = (x_{2i1}, \ldots, x_{2im_{2i}})'$, therefore

$$
\begin{aligned}
& (\mathbf{x}_i - \boldsymbol{\mu}_i)'\boldsymbol{\Sigma}_i^{-1}(\mathbf{x}_i - \boldsymbol{\mu}_i) \\
= \;& [(x_{pi} - \mu_p)\,(\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}})'\,(\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}})'] \begin{pmatrix} \mathbf{D}_i & \mathbf{E}_i & \mathbf{F}_i \\ \mathbf{E}_i' & \mathbf{G}_i & \mathbf{H}_i \\ \mathbf{F}_i' & \mathbf{H}_i' & \mathbf{K}_i \end{pmatrix} \begin{bmatrix} (x_{pi} - \mu_p) \\ (\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}}) \\ (\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}}) \end{bmatrix} \\
= \;& (x_{pi} - \mu_p)^2\mathbf{D}_i + (x_{pi} - \mu_p)(\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}})'\mathbf{E}_i' \\
+ \;& (x_{pi} - \mu_p)(\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}})'\mathbf{F}_i' + (x_{pi} - \mu_p)\mathbf{E}_i(\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}}) \\
+ \;& (\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}})'\mathbf{G}_i(\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}}) + (\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}})'\mathbf{H}_i'(\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}}) \\
+ \;& (x_{pi} - \mu_p)\mathbf{F}_i(\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}}) + (\mathbf{x}_{1i} - \mu_1\mathbf{1}_{m_{1i}})'\mathbf{H}_i(\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}}) \\
+ \;& (\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}})'\mathbf{K}_i(\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}}).
\end{aligned}
$$

If $m_{1i} = 0$ and $m_{2i} > 1$, then

$$
\begin{aligned}
log(L_i(\boldsymbol{\theta})) \;=\; & -\frac{1}{2}log2\pi\sigma_p^2 - \frac{m_{2i}}{2}log(2\pi\sigma_2^2) - \frac{1}{2}(m_{2i} - 1)log(1 - \rho_2) \\
- \;& \frac{1}{2}log(1 + (m_{2i} - 1)\rho_2 - m_{2i}\rho_{p2}^2) \\
- \;& \frac{1 + (m_{2i} - 1)\rho_2}{2\sigma_p^2(1 + (m_{2i} - 1)\rho_2 - m_{2i}\rho_{p2}^2)}(x_{pi} - \mu_p)^2 \\
+ \;& \frac{\rho_{p2}}{2\sigma_p\sigma_2(1 + (m_{2i} - 1)\rho_2 - m_{2i}\rho_{p2}^2)}(x_{pi} - \mu_p)(\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}})'\mathbf{1}_{m_{2i}} \\
+ \;& \frac{\rho_{p2}}{2\sigma_p\sigma_2(1 + (m_{2i} - 1)\rho_2 - m_{2i}\rho_{p2}^2)}(x_{pi} - \mu_p)\mathbf{1}_{m_{2i}}'(\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}}) \\
- \;& \frac{1}{2\sigma_2^2(1 - \rho_2)}(\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}})'\left[\mathbf{I}_{m_{2i}} - \frac{\rho_2 - \rho_{p2}^2}{1 + (m_{2i} - 1)\rho_2 - m_{2i}\rho_{p2}^2}\mathbf{J}_{m_{2i}}\right] \\
\times \;& (\mathbf{x}_{2i} - \mu_2\mathbf{1}_{m_{2i}}).
\end{aligned}
$$

In the case $m_{1i} = 0$ and $m_{2i} = 1$, then

$$
\begin{aligned}
log(L_i(\boldsymbol{\theta})) \;=\; & -\frac{1}{2}log(2\pi\sigma_p^2\sigma_2^2) - \frac{1}{2}log(1 - \rho_{p2}^2) \\
- \;& \frac{1}{2\sigma_p^2(1 - \rho_{p2}^2)}(x_{pi} - \mu_p)^2 - \frac{1}{2\sigma_2^2(1 - \rho_{p2}^2)}(x_{2i} - \mu_2)^2 \\
+ \;& \frac{\rho_{p2}}{\sigma_p\sigma_2(1 - \rho_{p2}^2)}(x_{pi} - \mu_p)(x_{2i} - \mu_2).
\end{aligned}
$$

Similarly, if $m_{1i} > 1$ and $m_{2i} = 0$, then

$$\begin{aligned}
log(L_i(\boldsymbol{\theta})) &= -\frac{1}{2}log2\pi\sigma_p^2 - \frac{m_{1i}}{2}log(2\pi\sigma_1^2) - \frac{1}{2}(m_{1i}-1)log(1-\rho_1) \\
&- \frac{1}{2}log(1+(m_{1i}-1)\rho_1 - m_{1i}\rho_{p1}^2) \\
&- \frac{1+(m_{1i}-1)\rho_1}{2\sigma_p^2(1+(m_{1i}-1)\rho_1 - m_{1i}\rho_{p1}^2)}(x_{pi}-\mu_p)^2 \\
&+ \frac{\rho_{p1}}{2\sigma_p\sigma_1(1+(m_{1i}-1)\rho_1 - m_{1i}\rho_{p1}^2)}(x_{pi}-\mu_p)(\mathbf{x}_{1i}-\mu_1\mathbf{1}_{m_{1i}})'\mathbf{1}_{m_{1i}} \\
&+ \frac{\rho_{p1}}{2\sigma_p\sigma_1(1+(m_{1i}-1)\rho_1 - m_{1i}\rho_{p1}^2)}(x_{pi}-\mu_p)\mathbf{1}'_{m_{1i}}(\mathbf{x}_{1i}-\mu_1\mathbf{1}_{m_{1i}}) \\
&- \frac{1}{2\sigma_1^2(1-\rho_1)}(\mathbf{x}_{1i}-\mu_1\mathbf{1}_{m_{1i}})'\left[\mathbf{I}_{m_{1i}} - \frac{\rho_1-\rho_{p1}^2}{1+(m_{1i}-1)\rho_1 - m_{1i}\rho_{p1}^2}\mathbf{J}_{m_{1i}}\right] \\
&\times (\mathbf{x}_{1i}-\mu_1\mathbf{1}_{m_{1i}}),
\end{aligned}$$

and if $m_{1i} = 1$ and $m_{2i} = 0$, then

$$\begin{aligned}
log(L_i(\boldsymbol{\theta})) &= -\frac{1}{2}log(2\pi\sigma_p^2\sigma_1^2) - \frac{1}{2}log(1-\rho_{p1}^2) \\
&- \frac{1}{2\sigma_p^2(1-\rho_{p1}^2)}(x_{pi}-\mu_p)^2 - \frac{1}{2\sigma_1^2(1-\rho_{p1}^2)}(x_{1i}-\mu_1)^2 \\
&+ \frac{\rho_{p1}}{\sigma_p\sigma_1(1-\rho_{p1}^2)}(x_{pi}-\mu_p)(x_{1i}-\mu_1).
\end{aligned}$$

The likelihood function $L(\boldsymbol{\theta})$ or $log(L(\boldsymbol{\theta}))$ can be maximized to obtain $\hat{\boldsymbol{\theta}}$, the maximum likelihood estimator (MLE) of $\boldsymbol{\theta}$. Let $\hat{\boldsymbol{\theta}} = \left(\hat{\mu}_p, \hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}_p^2, \hat{\sigma}_1^2, \hat{\sigma}_2^2, \hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_{12}, \hat{\rho}_{p1}, \hat{\rho}_{p2}\right)'$. Suppose we are interested in testing the hypothesis that the two interclass correlation coefficients between the parent and children are equal, that is, $H_0 : \rho_{p1} = \rho_{p2} = \rho_p\ (say)$. Under H_0, $\boldsymbol{\theta} = (\mu_p, \mu_1, \mu_2, \sigma_p^2, \sigma_1^2, \sigma_2^2, \rho_1, \rho_2, \rho_{12}, \rho_p, \rho_p)'$. The likelihood function, $L(\boldsymbol{\theta})$ or $log(L(\boldsymbol{\theta}))$ can also be maximized under the null hypothesis $H_0 : \rho_{p1} = \rho_{p2}$ to obtain $\hat{\boldsymbol{\theta}}_0$.

IV.3 LIKELIHOOD RATIO TEST

The likelihood ratio test (LRT) for testing H_0 is to reject H_0 for large values of

$$LRT = 2logL(\hat{\boldsymbol{\theta}}) - 2logL(\hat{\boldsymbol{\theta}}_0). \tag{21}$$

This test statistic has a χ^2 asymptotic distribution with 1 degree of freedom.

This test depends on the computation of the maximum likelihood estimates which have to be obtained numerically as described above. This procedure requires good

initial values of the parameters, which could be selected as the alternative estimators provided in the following sections. Further, as noted in previous chapters, non-convergence is an issue making it harder to obtain estimates in some situations.

IV.4 CANONICAL TRANSFORMATION

Similar to the previous chapters, we apply a canonical transformation to the familial data reducing the variance-covariance structure of the model. This provides transformed data that can be easily used to estimate the model parameters, as will be seen in the next section.

Recall, $\mathbf{x}_i = \begin{pmatrix} x_{pi} \\ \mathbf{x}_{1i} \\ \mathbf{x}_{2i} \end{pmatrix}$ has mean $\boldsymbol{\mu}_i = \begin{pmatrix} \mu_p \\ \mu_1 \mathbf{1}_{m_{1i}} \\ \mu_2 \mathbf{1}_{m_{2i}} \end{pmatrix}$ and variance matrix

$$\mathbf{\Sigma}_i = \begin{pmatrix} \sigma_p^2 & \rho_{p1}\sigma_p\sigma_1 \mathbf{1}'_{m_{1i}} & \rho_{p2}\sigma_p\sigma_2 \mathbf{1}'_{m_{2i}} \\ \rho_{p1}\sigma_p\sigma_1 \mathbf{1}_{m_{1i}} & \sigma_1^2\{(1-\rho_1)\mathbf{I}_{m_{1i}} + \rho_1 \mathbf{J}_{m_{1i}}\} & \rho_{12}\sigma_1\sigma_2 \mathbf{J}_{m_{1i},m_{2i}} \\ \rho_{p2}\sigma_p\sigma_2 \mathbf{1}_{m_{2i}} & \rho_{12}\sigma_1\sigma_2 \mathbf{J}_{m_{2i},m_{1i}} & \sigma_2^2\{(1-\rho_2)\mathbf{I}_{m_{2i}} + \rho_2 \mathbf{J}_{m_{2i}}\} \end{pmatrix}.$$

Let

$$\mathbf{\Gamma}_{i,(m_{1i}+m_{2i}+1 \times m_{1i}+m_{2i}+1)} = \begin{pmatrix} 1 & \mathbf{0}'_{m_{1i}} & \mathbf{0}'_{m_{2i}} \\ \mathbf{0}_{m_{1i}} & \mathbf{\Gamma}_{1i} & \mathbf{0}_{m_{1i},m_{2i}} \\ \mathbf{0}_{m_{2i}} & \mathbf{0}_{m_{2i},m_{1i}} & \mathbf{\Gamma}_{2i} \end{pmatrix},$$

where

$$\mathbf{\Gamma}_{1i} = \begin{pmatrix} \frac{1}{m_{1i}} & \frac{1}{m_{1i}} & \frac{1}{m_{1i}} & \cdots & \frac{1}{m_{1i}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{m_{1i}(m_{1i}-1)}} & \frac{1}{\sqrt{m_{1i}(m_{1i}-1)}} & \frac{1}{\sqrt{m_{1i}(m_{1i}-1)}} & \cdots & \frac{-(m_{1i}-1)}{\sqrt{m_{1i}(m_{1i}-1)}} \end{pmatrix},$$

$$\mathbf{\Gamma}_{2i} = \begin{pmatrix} \frac{1}{m_{2i}} & \frac{1}{m_{2i}} & \frac{1}{m_{2i}} & \cdots & \frac{1}{m_{2i}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{m_{2i}(m_{2i}-1)}} & \frac{1}{\sqrt{m_{2i}(m_{2i}-1)}} & \frac{1}{\sqrt{m_{2i}(m_{2i}-1)}} & \cdots & \frac{-(m_{2i}-1)}{\sqrt{m_{2i}(m_{2i}-1)}} \end{pmatrix},$$

$\mathbf{0}_m$ is the $m \times m$ matrix of all zeros, and $\mathbf{0}_{m,n}$ is the $m \times n$ matrix of all zeros.

Transform the family scores by making a Srivastava type transformation to create $\mathbf{y}_i$, the transformed vector of family scores,

$$\mathbf{y}_i = \begin{pmatrix} \mathbf{y}_{pi} \\ \mathbf{y}_{1i} \\ \mathbf{y}_{2i} \end{pmatrix} = \mathbf{\Gamma}_i \begin{pmatrix} \mathbf{x}_{pi} \\ \mathbf{x}_{1i} \\ \mathbf{x}_{2i} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & \mathbf{0}_{m_{1i}} & \mathbf{0}_{m_{2i}} \\ \mathbf{0}_{m_{1i}} & \mathbf{\Gamma}_{1i} & \mathbf{0}_{m_{1i},m_{2i}} \\ \mathbf{0}_{m_{2i}} & \mathbf{0}_{m_{2i},m_{1i}} & \mathbf{\Gamma}_{2i} \end{pmatrix} \begin{pmatrix} x_{pi} \\ \mathbf{x}_{1i} \\ \mathbf{x}_{2i} \end{pmatrix}$$
$$= \begin{pmatrix} x_{pi} \\ \mathbf{\Gamma}_{1i}\mathbf{x}_{1i} \\ \mathbf{\Gamma}_{2i}\mathbf{x}_{2i} \end{pmatrix}.$$

The expected value and variance of the transformed parent scores are still

$$E(y_{pi}) = \mu_p \text{ and } Var(y_{pi}) = \sigma_p^2.$$

But now, the expected value of the transformed son vector is

$$E(\mathbf{y}_{1i}) = \begin{pmatrix} \mu_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

and the variance matrix of the transformed son vector is

$$\begin{aligned} Var(\mathbf{y}_{1i}) &= \sigma_1^2 \mathbf{\Gamma}_{1i}((1-\rho_1)\mathbf{I}_{m_{1i}} + \rho_1 \mathbf{J}_{m_{1i}})\mathbf{\Gamma}'_{1i} \\ &= \begin{pmatrix} \frac{1}{2}\sigma_1^2(1+(m_{1i}-1)\rho_1) & 0 & \cdots & 0 \\ 0 & \sigma_1^2(1-\rho_1) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma_1^2(1-\rho_1) \end{pmatrix}. \end{aligned}$$

Similarly, the expected value of the transformed daughter vector is

$$E(\mathbf{y}_{2i}) = \begin{pmatrix} \mu_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

and the variance matrix of the transformed daughter vector is

$$\begin{aligned} Var(\mathbf{y}_{2i}) &= \sigma_2^2 \mathbf{\Gamma}_{2i}((1-\rho_2)\mathbf{I}_{m_{2i}} + \rho_2 \mathbf{J}_{m_{2i}})\mathbf{\Gamma}'_{2i} \\ &= \begin{pmatrix} \frac{1}{2}\sigma_2^2(1+(m_{2i}-1)\rho_2) & 0 & \cdots & 0 \\ 0 & \sigma_2^2(1-\rho_2) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma_2^2(1-\rho_2) \end{pmatrix}. \end{aligned}$$

The covariance matrix between the transformed son vector and the transformed daughter vector is

$$\begin{aligned} Cov(\mathbf{y}_{1i}, \mathbf{y}_{2i}) &= \sigma_1\sigma_2\rho_{12}\mathbf{\Gamma}_{1i}\mathbf{J}_{m_{1i},m_{2i}}\mathbf{\Gamma}'_{2i} \\ &= \sigma_1\sigma_2\rho_{12} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{m_{1i},m_{2i}}. \end{aligned}$$

Further, the covariance vector between the transformed son vector and the parent score is

$$Cov(\mathbf{y}_{1i}, y_{pi}) = \begin{pmatrix} \sigma_p\sigma_1\rho_{p1} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

and the covariance between the transformed daughter vector and the parent score is

$$Cov(\mathbf{y}_{2i}, y_{pi}) = \begin{pmatrix} \sigma_p\sigma_1\rho_{p2} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Note that only the transformed parent observation, the first transformed son observation, and the first transformed daughter observation, namely y_{pi}, y_{1i1}, and y_{2i1}, are correlated. Also, $\begin{pmatrix} y_{pi} \\ y_{1i1} \\ y_{2i1} \end{pmatrix}$ is a tri-variate normal with mean $\begin{pmatrix} \mu_p \\ \mu_1 \\ \mu_2 \end{pmatrix}$ and variance-covariance matrix

$$\begin{pmatrix} \sigma_p^2 & \sigma_p\sigma_1\rho_{p1} & \sigma_p\sigma_2\rho_{p2} \\ \sigma_p\sigma_1\rho_{p1} & \frac{1}{m_{1i}}\sigma_1^2(1+(m_{1i}-1)\rho_1) & \sigma_1\sigma_2\rho_{12} \\ \sigma_p\sigma_2\rho_{p2} & \sigma_1\sigma_2\rho_{12} & \frac{1}{m_{2i}}\sigma_2^2(1+(m_{2i}-1)\rho_2) \end{pmatrix}. \tag{22}$$

The observations y_{pi}, y_{1i1}, and y_{2i1} are also independent of $y_{1i2}, \ldots, y_{1im_{1i}} \sim N(0, \sigma_1^2(1-\rho_1))$ and $y_{2i2}, \ldots, y_{2im_{2i}} \sim N(0, \sigma_2^2(1-\rho_2))$.

In terms of $\mathbf{x}_{1i}$ and $\mathbf{x}_{2i}$,

$$y_{1i1} = \tfrac{1}{m_{1i}} \textstyle\sum_{j=1}^{m_{1i}} x_{1ij}, \qquad y_{2i1} = \tfrac{1}{m_{2i}} \textstyle\sum_{j=1}^{m_{2i}} x_{2ij}.$$

One can see that the first transformed son score, y_{1i1}, is the average of all the boy scores of the family, as well as the first transformed daughter score, that is y_{2i1} is the average of all the girl scores of the family. Hence,

$$\bar{y}_{1i1} = \tfrac{1}{n} \textstyle\sum_{i=1}^{n} y_{1i1} = \tfrac{1}{n} \textstyle\sum_{i=1}^{n} \tfrac{1}{m_{1i}} \textstyle\sum_{j=1}^{m_{1i}} x_{1ij},$$

and

$$\bar{y}_{2i1} = \tfrac{1}{n} \textstyle\sum_{i=1}^{n} y_{2i1} = \tfrac{1}{n} \textstyle\sum_{i=1}^{n} \tfrac{1}{m_{2i}} \textstyle\sum_{j=1}^{m_{2i}} x_{2ij}.$$

The average of the first transformed son scores is an average of the mean son score for each family. Similarly, the average of the first transformed daughter scores is an average of the mean daughter score for each family. Also,

$$\begin{aligned}
\sum_{j=2}^{m_{1i}} y_{1ij}^2 &= (x_{1i2}, \ldots, x_{1im_{1i}})' \, \boldsymbol{\Gamma}_i' \boldsymbol{\Gamma}_i \, (x_{1i2}, \ldots, x_{1im_{1i}}) \\
&= (x_{1i2}, \ldots, x_{1im_{1i}})' \, (I_{m_{1i}} - \frac{1}{m_{1i}} J_{m_{1i}}) \, (x_{1i2}, \ldots, x_{1im_{1i}}) \\
&= \sum_{j=2}^{m_{1i}} x_{1ij}^2 - \frac{1}{m_{1i}} (\sum_{j=2}^{m_{1i}} x_{1ij})^2
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=2}^{m_{2i}} y_{2ij}^2 &= (x_{2i2}, \ldots, x_{2im_{2i}})' \, \boldsymbol{\Gamma}_i' \boldsymbol{\Gamma}_i \, (x_{2i2}, \ldots, x_{2im_{2i}}) \\
&= (x_{2i2}, \ldots, x_{2im_{2i}})' \, (I_{m_{2i}} - \frac{1}{m_{2i}} J_{m_{2i}}) \, (x_{2i2}, \ldots, x_{2im_{2i}}) \\
&= \sum_{j=2}^{m_{2i}} x_{2ij}^2 - \frac{1}{m_{2i}} (\sum_{j=2}^{m_{2i}} x_{2ij})^2.
\end{aligned}$$

The sum of squares of the "left-over" transformed son scores, $y_{1i2}, \ldots, y_{1im_{1i}}$, for a family can be written in terms of the second through last son of the family. Similarly, the sum of squares of the "left-over" transformed daughter scores, $y_{2i2}, \ldots, y_{2im_{2i}}$, for a family can be written in terms of the second through last daughter of the family.

In order to simplify the transformed model, let the variance matrix of the transformed family vector be

$$\Gamma_i'\Sigma_i\Gamma_i = \begin{bmatrix} \sigma_p^2 & \sigma_{p1} & \mathbf{0}'_{m_{1i}-1} & \sigma_{p2} & \mathbf{0}'_{m_{2i}-1} \\ \sigma_{p1} & \eta_{1i}^2 & \mathbf{0}'_{m_{1i}-1} & \sigma_{12} & \mathbf{0}'_{m_{2i}-1} \\ \mathbf{0}_{m_{1i}-1} & \mathbf{0}_{m_{1i}-1} & \gamma_1^2\mathbf{I}_{m_{1i}-1} & \mathbf{0}_{m_{1i}-1} & \mathbf{0}_{m_{1i}-1,m_{2i}-1} \\ \sigma_{p2} & \sigma_{12} & \mathbf{0}'_{m_{1i}-1} & \eta_{2i}^2 & \mathbf{0}'_{m_{2i}-1} \\ \mathbf{0}_{m_{2i}-1} & \mathbf{0}_{m_{2i}-1} & \mathbf{0}_{m_{1i}-1,m_{2i}-1} & \mathbf{0}_{m_{2i}-1} & \gamma_2^2\mathbf{I}_{m_{2i}-1} \end{bmatrix},$$

where

$$\begin{aligned} \sigma_{p1} &= \sigma_p\sigma_1\rho_{p1}, \\ \sigma_{p2} &= \sigma_p\sigma_2\rho_{p2}, \\ \eta_{1i}^2 &= \sigma_1^2(1+(m_{1i}-1)\rho_1)/m_{1i}, \\ \eta_{2i}^2 &= \sigma_2^2(1+(m_{2i}-1)\rho_2)/m_{2i}, \\ \gamma_1^2 &= \sigma_1^2(1-\rho_1), \\ \gamma_2^2 &= \sigma_2^2(1-\rho_2), \\ \sigma_{12} &= \sigma_1\sigma_2\rho_{12}. \end{aligned}$$

Note $\eta_{1i}^2 = \sigma_1^2 - a_{1i}\gamma_1^2$ and $\eta_{2i}^2 = \sigma_2^2 - a_{2i}\gamma_2^2$ where $a_{1i} = 1 - m_{1i}^{-1}$ and $a_{2i} = 1 - m_{2i}^{-1}$. Also, there is a 1-1 transformation from the old parameters to a new set of parameters. Namely,

$$\begin{aligned} \xi_1 &= \frac{\sigma_1^2}{\gamma_1^2}, \\ \xi_2 &= \frac{\sigma_2^2}{\gamma_2^2}, \\ \xi_{12} &= \frac{\sigma_{12}^2}{\gamma_1^2\gamma_2^2}. \end{aligned}$$

IV.5 ALTERNATIVE ESTIMATORS

The transformed familial data has good distributional properties from which alternative estimators can be constructed that do not require a maximization procedure. Let n_1 = number of families with $m_{1i} > 0$, n_2 = number of families with $m_{2i} > 0$, and n_{12} = number of families with $m_{1i} > 0$ and $m_{2i} > 0$. Recalling that alternative estimators for familial data consisting of only sons and daughters were constructed in the second chapter. These estimators can be applied in this chapter as well. Both sets of alternative estimators performed well in the simulation experiments; a disadvantage of the second set of alternative estimators is the requirement that $m_{1i} > 2$

and $m_{2i} > 2$, but the second set of alternative estimators has better distributional properties that make estimating μ_1 and μ_2 easier. For familial data including a parent observation, our proposed alternative estimators start with a combination of the two previously proposed sets.

An unbiased and consistent estimator of γ_1^2 is

$$\tilde{\gamma}_1^2 = \frac{\sum_{i=1}^{n_1}\sum_{j=2}^{m_{1i}} y_{1ij}^2}{\sum_{i=1}^{n_1}(m_{1i}-1)}.$$

Similarly, γ_2^2 can be estimated by

$$\tilde{\gamma}_2^2 = \frac{\sum_{i=1}^{n_2}\sum_{j=2}^{m_{2i}} y_{2ij}^2}{\sum_{i=1}^{n_2}(m_{2i}-1)}.$$

Take an unbiased estimator of σ_{12} as

$$\tilde{\sigma}_{12} = \sum_{i=1}^{n_{12}} \frac{(y_{1i1}-\bar{y}_{11}^*)(y_{2i1}-\bar{y}_{21}^*)}{n_{12}-1},$$

where $\bar{y}_{11}^* = \sum_{i=1}^{n_{12}} y_{1i1}/n_{12}$ and $\bar{y}_{21}^* = \sum_{i=1}^{n_{12}} y_{2i1}/n_{12}$, since

$$\begin{pmatrix} y_{1i1} \\ y_{2i1} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \eta_{1i}^2 & \sigma_{12} \\ \sigma_{12} & \eta_{2i}^2 \end{pmatrix}\right).$$

Estimate σ_1^2 by

$$\tilde{\sigma}_1^2 = \tfrac{1}{n_1-1}\sum_{i=1}^{n_1}(y_{1i1}-\bar{y_{11}})^2 + \tfrac{1}{n_1}\tilde{\gamma}_1^2(\sum_{i=1}^{n_1} a_{1i}),$$

and estimate σ_2^2 by

$$\tilde{\sigma}_2^2 = \tfrac{1}{n_2-1}\sum_{i=1}^{n_2}(y_{2i1}-\bar{y_{21}})^2 + \tfrac{1}{n_2}\tilde{\gamma}_2^2(\sum_{i=1}^{n_2} a_{2i}).$$

From these, other estimators are

$$\begin{aligned}
\tilde{\rho}_1 &= 1-(\tilde{\gamma}_1^2/\tilde{\sigma}_1^2),\\
\tilde{\rho}_2 &= 1-(\tilde{\gamma}_2^2/\tilde{\sigma}_2^2),\\
\tilde{\rho}_{12} &= \frac{\tilde{\sigma}_{12}}{\tilde{\sigma}_1\tilde{\sigma}_2}.
\end{aligned}$$

Estimate μ_1 and μ_2 using a second transformation as was done in the second set

of alternative estimators in Chapter II, (9) and (10).

$$\begin{aligned}
\tilde{\mu}_1 &= \frac{1}{n_1}\sum_{i=1}^{n_1}\left[y_{1i1} - \frac{1}{\sqrt{m_{1i}}}\sum_{j=2}^{m_{1i}} y_{1ij}\right] \\
&= \frac{1}{n_1}\sum_{k=1}^{n_1}\tilde{y}_{1i1}, \\
\tilde{\mu}_2 &= \frac{1}{n_2}\sum_{i=1}^{n_2}\left[y_{2i1} - \frac{1}{\sqrt{m_{2i}}}\sum_{j=2}^{m_{2i}} y_{2ij}\right] \\
&= \frac{1}{n_2}\sum_{j=1}^{n_2}\tilde{y}_{2i1}.
\end{aligned}$$

Estimate μ_p and σ_p^2 by the standard unbiased estimators

$$\tilde{\mu}_p = \frac{1}{n}\sum_{i=1}^{n} y_{pi} = \bar{y}_p$$

and

$$\tilde{\sigma}_p^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_{pi} - \bar{y}_p)^2.$$

Then σ_{p1} and σ_{p2} can be estimated by

$$\tilde{\sigma}_{p1} = \frac{1}{n_1 - 1}\sum_{i=1}^{n_1}(y_{pi} - \bar{y}_p^{(1)})(y_{1i1} - \bar{y}_{11})$$

and

$$\tilde{\sigma}_{p2} = \frac{1}{n_2 - 1}\sum_{i=1}^{n_2}(y_{pi} - \bar{y}_p^{(2)})(y_{2i1} - \bar{y}_{21}),$$

where

$$\begin{aligned}
\bar{y}_p^{(1)} &= \frac{1}{n_1}\sum_{i=1}^{n_1} y_{pi}, \\
\bar{y}_p^{(2)} &= \frac{1}{n_2}\sum_{i=1}^{n_2} y_{pi}, \\
\bar{y}_{11} &= \frac{1}{n_1}\sum_{i=1}^{n_1} y_{1i1}, \\
\bar{y}_{21} &= \frac{1}{n_2}\sum_{i=1}^{n_2} y_{2i1}.
\end{aligned}$$

Finally, estimates of ρ_{p1} and ρ_{p2} are

$$\tilde{\rho}_{p1} = \frac{\tilde{\sigma}_{p1}}{\tilde{\sigma}_p \tilde{\sigma}_1}$$

and

$$\tilde{\rho}_{p2} = \frac{\tilde{\sigma}_{p2}}{\tilde{\sigma}_p \tilde{\sigma}_1}.$$

These alternative estimators are easier to implement than the MLEs. However, as before, it is possible for the alternative estimators to violate the model constraints, (19) and (20).

IV.6 VARIANCE OF ALTERNATIVE ESTIMATORS

In this section, the asymptotic variance of the alternative estimators is derived in order to construct alternative tests for testing the null hypothesis that the parent-son correlation equals the parent-daughter correlation, $H_0 : \rho_{p1} = \rho_{p2} = \rho_p$.

In order to find the asymptotic distribution of $\tilde{\rho}_{p1}$ and $\tilde{\rho}_{p2}$, the distribution of the sample covariance matrix for

$$\mathbf{y}_i = \begin{pmatrix} y_{pi} \\ y_{1i1} \\ y_{2i1} \end{pmatrix}$$

is needed. Using the distribution of $\mathbf{y}_i$ given in (22) and the distribution of the sample covariance matrix as shown in Appendix A.3, one can find the following asymptotic distributions needed for the variance and covariance of $\tilde{\rho}_{p1}$ and $\tilde{\rho}_{p2}$

$$\sqrt{n_{12i}} \begin{bmatrix} \tilde{\sigma}_{p1} - \sigma_{p1} \\ \tilde{\sigma}_{p2} - \sigma_{p2} \\ \tilde{\sigma}_p^2 - \sigma_p^2 \\ \tilde{\sigma}_1^2 - \sigma_1^2 \\ \tilde{\sigma}_2^2 - \sigma_2^2 \end{bmatrix} \to N(0, \Sigma_p),$$

where

$$\mathbf{\Sigma}_p = 2 \begin{bmatrix} \frac{1}{2}(\sigma_{p1}^2 + \sigma_p^2\sigma_1^2\lambda_1) & \frac{1}{2}(\sigma_{p1}\sigma_{p2} + \sigma_p^2\sigma_{12}) & \sigma_p^2\sigma_{p1} & \lambda_1\sigma_1^2\sigma_{p1} & \sigma_{p2}\sigma_{12} \\ \frac{1}{2}(\sigma_{p1}\sigma_{p2} + \sigma_p^2\sigma_{12}) & \frac{1}{2}(\sigma_{p2}^2 + \sigma_p^2\sigma_2^2\lambda_2) & \sigma_p^2\sigma_{p2} & \sigma_{p1}\sigma_{12} & \lambda_2\sigma_2^2\sigma_{p2} \\ \sigma_p^2\sigma_{p1} & \sigma_p^2\sigma_{p2} & \sigma_p^4 & \sigma_{p1}^2 & \sigma_{p2}^2 \\ \lambda_1\sigma_1^2\sigma_{p1} & \sigma_{p1}\sigma_{12} & \sigma_{p1}^2 & c_1^2\sigma_1^4 & \sigma_{12}^2 \\ \sigma_{p2}\sigma_{12} & \lambda_2\sigma_2^2 & \sigma_{p2}^2 & \sigma_{12}^2 & c_2^2\sigma_2^4 \end{bmatrix},$$

and where

$$c_1^2 = 1 - 2(1-\rho_1)n_{12}^{-1}\sum_{i=1}^{n_{12}} a_{1i} + (1-\rho_1)^2\left[n_{12}^{-1}\sum_{i=1}^{n_{12}} a_{1i}^2 + (\bar{m}_1 - 1)^{-1}(n_{12}^{-1}\sum_{i=1}^{n_{12}} a_{1i})^2\right],$$

$$c_2^2 = 1 - 2(1-\rho_2)n_{12}^{-1}\sum_{i=1}^{n_{12}} a_{2i} + (1-\rho_2)^2\left[n_{12}^{-1}\sum_{i=1}^{n_{12}} a_{2i}^2 + (\bar{m}_2 - 1)^{-1}(n_{12}^{-1}\sum_{i=1}^{n_{12}} a_{2i})^2\right],$$

$$\lambda_1 = 1 - (1-\rho_1)n_{12}^{-1}\sum_{j=1}^{n_{12}} a_{1i},$$

$$\lambda_2 = 1 - (1-\rho_2)n_{12}^{-1}\sum_{j=1}^{n_{12}} a_{1i},$$

$$\bar{m}_1 = n_{12}^{-1}\sum_{j=1}^{n_{12}} m_{1i},$$

$$\bar{m}_2 = n_{12}^{-1}\sum_{j=1}^{n_{12}} m_{2i}.$$

Therefore using the delta method,

$$AV(\tilde{\rho}_{p1}) = \frac{1}{n_1}\left[\rho_{p1}^4 + \rho_{p1}^2\left(\frac{1}{2}c_1^2 - 2\lambda_1 - \frac{1}{2}\right) + \lambda_1\right]$$
$$AV(\tilde{\rho}_{p2}) = \frac{1}{n_2}\left[\rho_{p2}^4 + \rho_{p2}^2\left(\frac{1}{2}c_2^2 - 2\lambda_2 - \frac{1}{2}\right) + \lambda_2\right]$$

and $Cov(\tilde{\rho}_{p1}, \tilde{\rho}_{p2}) =$

$$\frac{1}{n_{12}}\left[\rho_{12} - \rho_{12}\rho_{p1}^2 - \frac{1}{2}\rho_{p1}\rho_{p2} + \frac{1}{2}\rho_{p1}^3\rho_{p2} - \rho_{12}\rho_{p2}^2 + \frac{1}{2}\rho_{p1}\rho_{p2}^3 + \frac{1}{2}\rho_{12}^2\rho_{p1}\rho_{p2}\right].$$

Let $\tilde{AV}(\tilde{\rho}_{p1})$ be the estimated $AV(\tilde{\rho}_{p1})$ obtained by substituting the alternative estimators $\tilde{\rho}_1$ and $\tilde{\rho}_{p1}$ for the unknown parameters. Let $\tilde{AV}(\tilde{\rho}_{p2})$ be the estimated $AV(\tilde{\rho}_{p2})$ obtained by substituting the alternative estimators $\tilde{\rho}_2$ and $\tilde{\rho}_{p2}$ for the unknown parameters. Also, let $\tilde{Cov}(\tilde{\rho}_{p1}, \tilde{\rho}_{p2})$ be the estimated $Cov(\tilde{\rho}_{p1}, \tilde{\rho}_{p2})$ obtained by substituting the alternative estimators $\tilde{\rho}_{12}, \tilde{\rho}_{p1}$, and $\tilde{\rho}_{p2}$ for the unknown parameters.

IV.7 ALTERNATIVE TEST

The test we propose is

$$TS = \left(\frac{\tilde{\rho}_{p1} - \tilde{\rho}_{p2}}{S.E.(\tilde{\rho}_{p1} - \tilde{\rho}_{p2})}\right)^2 \sim \chi_1^2, \quad (23)$$

where

$$S.E.(\tilde{\rho}_1 - \tilde{\rho}_2) = \tilde{AV}(\tilde{\rho}_{p1}) + \tilde{AV}(\tilde{\rho}_{p2}) - 2\tilde{Cov}(\tilde{\rho}_{p1}, \tilde{\rho}_{p2}).$$

TS is easier to implement than the LRT.

IV.8 SIMULATION EXPERIMENT AND RESULTS

Both tests are expected to behave similarly for large sample sizes, since they both have an asymptotic chi-square distribution with 1 degree of freedom. To compare the performance of the tests a small sample simulation experiment was performed. In this experiment, 50 family score vectors consisting of a parent score and children scores were simulated as multivariate normal random vectors. As was done in the previous simulation experiments, the number of children in each family is simulated from a truncated negative binomial distribution ranging from 1 to 15 children per family. The parameters of the negative binomial come from the estimated distribution of U.S. births with a mean of 2.84 and the success probability as 0.483 (Brass 1958). A discrete uniform distribution was used to assign gender to each child. The arbitrary choices of parameters were $\mu_p = 0, \mu_1 = 0, \mu_2 = 0, \sigma_p^2 = 1, \sigma_1^2 = 1.5, \sigma_2^2 = 2, \rho_1 = 0.7, \rho_2 = 0.3$, and $\rho_{12} = 0.1$. Values of ρ_{p1} and ρ_{p2} ranged from 0.1 to 0.5 in increments of 0.05 and 0.1. Only positive values of the interclass correlations were simulated because the model constraints restrict possible negative values based on parameter choices and family size.

For each choice of parameters, 10,000 simulations were run and estimated size and power values were computed for testing $H_0 : \rho_{p1} = \rho_{p2} = \rho_p$. The $\alpha = 0.01$ size table and rejection proportion tables provide the percentage of simulations for which the maximum likelihood procedure did not converge and the percentage of simulations for which the alternative estimates violated the model constraints (19) and (20).

Tables 68-70 give the estimated sizes for $\alpha = 0.01, 0.05$, and 0.10, respectively.

From the tables, we can see that both the LRT and TS estimate the assumed level reasonably well, but the alternative test TS clearly performs better, as TS is closer to the assumed level in 21 of the 24 cases simulated. As ρ_p increases the performance of the LRT decreased, but TS continued to perform well.

Tables 71-74 give estimated power values adjusted to the level each test attained in the size calculations. For each table, the rejection proportions are based on the

TABLE 68: Sizes, $\alpha = 0.01, H_0 : \rho_{p1} = \rho_{p2} = \rho_p$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_p	**LRT** (*)	$\boldsymbol{TS}$ (**)
0.10	0.0083 (3.76%)	0.0123 (2.73%)
0.15	0.0119 (2.31%)	0.0116 (2.81%)
0.20	0.0116 (1.94%)	0.0094 (3.18%)
0.25	0.0144 (1.96%)	0.0111 (3.74%)
0.30	0.0134 (3.63%)	0.0105 (4.83%)
0.35	0.0139 (6.60%)	0.0120 (7.50%)
0.40	0.0185 (12.53%)	0.0099 (10.96%)
0.45	0.0317 (27.38%)	0.0106 (19.90%)

TABLE 69: Sizes, $\alpha = 0.05, H_0 : \rho_{p1} = \rho_{p2} = \rho_p$.

ρ_p	**LRT**	$\boldsymbol{TS}$
0.10	0.0447	0.0486
0.15	0.0529	0.0455
0.20	0.0565	0.0459
0.25	0.0596	0.0468
0.30	0.0569	0.0482
0.35	0.0578	0.0502
0.40	0.0624	0.0449
0.45	0.0788	0.0551

95^{th} percentiles of the test statistics from the size simulations for the value of ρ_{p1}. For example, Table 71 shows the proportion of simulations for which the test statistic LRT is greater than 3.60548 and TS is greater than 3.68883.

The tables show the alternative test TS to perform better than the LRT in 15 of the 20 estimated powers. In the other 5 cases, the estimated powers for TS are not far behind that of the LRT.

Tables 75-78 give the estimated powers for a nominal level $\alpha = 0.05$.

Here the powers of the alternative test TS were not as favorable compared with the LRT, but the alternative test TS was closer to the assumed level in the size calculations and the LRT's estimated size tended to be larger than the corresponding estimated size for TS. Because of this, estimated powers for the LRT are expected to be larger than the estimated powers for TS when not adjusted for the size each

TABLE 70: Sizes, $\alpha = 0.10, H_0 : \rho_{p1} = \rho_{p2} = \rho_p$.

ρ_p	LRT	TS
0.10	0.0880	0.0896
0.15	0.0992	0.0889
0.20	0.1105	0.0946
0.25	0.1103	0.0939
0.30	0.1076	0.0939
0.35	0.1104	0.0972
0.40	0.1147	0.0974
0.45	0.1264	0.1098

TABLE 71: Adjusted Rejection Proportions, $\alpha = 0.05, H_0 : \rho_{p1} = \rho_{p2} = \rho_p$, $\rho_{p1} = 0.1$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_{p2}	LRT (*)	TS (**)
0.1	0.0447 (3.76%)	0.0486 (2.73%)
0.2	0.0754 (2.48%)	0.0684 (3.08%)
0.3	0.1924 (4.48%)	0.1667 (5.55%)
0.4	0.3848 (12.69%)	0.3526 (10.31%)
0.5	0.6413 (41.01%)	0.7337 (22.89%)

test actually attained.

The percentages of non-convergence and violation constraints observed in this simulation experiment are comparable. In practice, the alternative estimators and corresponding test are easy to compute. Generally, the alternative test TS performs better than the LRT in the simulation studies so is recommended.

TABLE 72: Adjusted Rejection Proportions, $\alpha = 0.05, H_0 : \rho_{p1} = \rho_{p2} = \rho_p$, $\rho_{p1} = 0.2$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_{p2}	**LRT** (*)	$\boldsymbol{TS}$ (**)
0.1	0.0835 (2.11%)	0.0906 (2.55%)
0.2	0.0565 (1.94%)	0.0459 (3.18%)
0.3	0.0819 (3.67%)	0.0807 (5.58%)
0.4	0.1875 (12.07%)	0.1985 (9.77%)
0.5	0.3747 (39.25%)	0.4836 (23.01%)

TABLE 73: Adjusted Rejection Proportions, $\alpha = 0.05, H_0 : \rho_{p1} = \rho_{p2} = \rho_p$, $\rho_{p1} = 0.3$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_{p2}	**LRT** (*)	$\boldsymbol{TS}$ (**)
0.1	0.1833 (1.11%)	0.1794 (2.32%)
0.2	0.0880 (1.43%)	0.0881 (3.11%)
0.3	0.0569 (3.63%)	0.0482 (4.83%)
0.4	0.0792 (11.65%)	0.0886 (10.24%)
0.5	0.1760 (38.08%)	0.2477 (23.89%)

TABLE 74: Adjusted Rejection Proportions, $\alpha = 0.05, H_0 : \rho_{p1} = \rho_{p2} = \rho_p$, $\rho_{p1} = 0.4$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_{p2}	**LRT** (*)	$\boldsymbol{TS}$ (**)
0.1	0.3704 (1.05%)	0.3958 (2.81%)
0.2	0.1983 (2.03%)	0.2135 (3.37%)
0.3	0.0954 (4.09%)	0.1068 (5.12%)
0.4	0.0624 (12.53%)	0.0449 (10.96%)
0.5	0.0776 (42.93%)	0.1291 (26.78%)

TABLE 75: Rejection Proportions, $\alpha = 0.05, H_0 : \rho_{p1} = \rho_{p2} = \rho_p$, $\rho_{p1} = 0.1$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_{p2}	**LRT** (*)	$\boldsymbol{TS}$ (**)
0.1	0.0447 (3.76%)	0.4865 (2.73%)
0.2	0.0849 (2.48%)	0.0749 (3.08%)
0.3	0.2107 (4.48%)	0.1761 (5.55%)
0.4	0.4073 (12.69%)	0.3679 (10.31%)
0.5	0.6627 (41.01%)	0.7561 (22.89%)

TABLE 76: Rejection Proportions, $\alpha = 0.05, H_0 : \rho_{p1} = \rho_{p2} = \rho_p$, $\rho_{p1} = 0.2$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_{p2}	LRT (*)	TS (**)
0.1	0.0913 (2.11%)	0.0811 (2.55%)
0.2	0.0565 (1.94%)	0.0459 (3.18%)
0.3	0.0908 (3.67%)	0.0717 (5.58%)
0.4	0.2032 (12.07%)	0.1773 (9.77%)
0.5	0.3949 (39.25%)	0.4508 (23.01%)

TABLE 77: Rejection Proportions, $\alpha = 0.05, H_0 : \rho_{p1} = \rho_{p2} = \rho_p$, $\rho_{p1} = 0.3$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_{p2}	LRT (*)	TS (**)
0.1	0.1970 (1.11%)	0.1623 (2.32%)
0.2	0.0961 (1.43%)	0.0771 (3.11%)
0.3	0.0569 (3.63%)	0.0482 (4.83%)
0.4	0.0867 (11.65%)	0.0770 (10.24%)
0.5	0.1894 (38.08%)	0.2206 (23.89%)

TABLE 78: Rejection Proportions, $\alpha = 0.05, H_0 : \rho_{p1} = \rho_{p2} = \rho_p$, $\rho_{p1} = 0.4$.
*Percent that did not converge; **Percent that violated the model constraints.

ρ_{p2}	LRT (*)	TS (**)
0.1	0.4136 (1.05%)	0.3523 (2.81%)
0.2	0.2284 (2.03%)	0.1804 (3.37%)
0.3	0.1133 (4.09%)	0.0886 (5.12%)
0.4	0.0624 (12.53%)	0.0449 (10.96%)
0.5	0.0918 (42.93%)	0.1024 (26.78%)

CHAPTER V

CONCLUDING REMARKS AND FUTURE WORK

In this thesis we have considered the problem of studying the relationships between various members of the family, namely, son-son, son-daughter, daughter-daughter, parent-son, and parent-daughter when families are allowed to have different numbers of boys and girls. We also considered the analysis of data on families coming from different independent groups. Both estimation and testing of hypothesis problems are considered using methods based on maximum likelihood and certain alternative estimators. The alternative estimators are obtained using canonical transformation of the data. Galton's data is used to illustrate some of the methods that have been developed. Maximum likelihood based tests are compared with the tests developed using transformed data using simulations. In most cases, the alternative tests that we have proposed do quite well compared to the likelihood ratio test. Since the alternative estimators and their corresponding tests are easy to compute we recommend using them in practice.

Our future investigation involves exploring the analysis of familial data when we have very large family sizes and very small number of families, as in gene expression data. Assuming that each blood serum sample is a family and the genes are the children, the problem is to study the correlations between various genes. Recently, numerous papers have explored the use of intraclass correlation in bioinformatics. For example, see Pellis et al. (2003), Dobbin et al. (2005), and Tan et al. (2008). A careful study of application of intraclass and interclass correlations in the situation of high dimensional data is to be done and our future plan is to explore this area of application.

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APPENDIX A

MULTIVARIATE NORMAL DISTRIBUTIONS

In order to determine the variance of the alternative estimators proposed in Chapters II and IV, we need the following results on the distribution of a sample covariance matrix from a multivariate normal random sample.

A.1 SAMPLE COVARIANCE MATRIX DISTRIBUTION

Assume $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n$ are independent multivariate normal random vectors with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. The $p \times p$ sample variance-covariance matrix is defined as

$$\mathbf{S} = \frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{Y}_i - \bar{\mathbf{Y}})(\mathbf{Y}_i - \bar{\mathbf{Y}})'$$

where

$$\bar{\mathbf{Y}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{Y}_i.$$

Suppose $\mathbf{S}_{p\times p} = (\mathbf{S}_1\, \mathbf{S}_2\, \ldots\, \mathbf{S}_p)$ where $\mathbf{S}_i$ is the i^{th} $(p \times 1)$ column vector of $\mathbf{S}$, then

$$vec(\mathbf{S})_{p^2\times 1} = \begin{pmatrix} \mathbf{S}_1 \\ \mathbf{S}_2 \\ \vdots \\ \mathbf{S}_p \end{pmatrix},$$

and $vech(\mathbf{S})$ is the $\frac{p(p+1)}{2} \times 1$ vector of non-redundant elements of $\mathbf{S}$.

For any sample variance-covariance matrix of the same form, the Duplication matrix $\mathbf{D}_p$ is a $p^2 \times \frac{p(p+1)}{2}$ matrix such that

$$\mathbf{D}_p vech(\mathbf{S}_{p\times p}) = vec(\mathbf{S}).$$

From multivariante normal samples, we have

$$\sqrt{n}(vech(\mathbf{S}) - vech(\boldsymbol{\Sigma})) \to N(\mathbf{0}, \boldsymbol{\Delta})$$

where

$$\boldsymbol{\Delta} \;=\; 2(\mathbf{D}_p^{+})(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma})(\mathbf{D}_p^{+})',$$

$$\mathbf{D}_p^+ = (\mathbf{D}_p'\mathbf{D}_p)^{-1}\mathbf{D}_p',$$

and $\otimes$ is the Kronecker product between the two matrices.

Details will be worked out for the 2×2 and 3×3 cases.

A.2 BIVARIATE SAMPLE COVARIANCE MATRIX DISTRIBUTION

For the 2×2 case, $\mathbf{Y}_i = \begin{pmatrix} y_{1i} \\ y_{2i} \end{pmatrix}$ and

$$\mathbf{S}_{2\times 2} = \begin{pmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{S}_1 & \mathbf{S}_2 \end{pmatrix}.$$

We have

$$vec(\mathbf{S})_{4\times 1} = \begin{pmatrix} S_{11} \\ S_{12} \\ S_{12} \\ S_{22} \end{pmatrix}$$

and

$$vech(\mathbf{S})_{3\times 1} = \begin{pmatrix} S_{11} \\ S_{12} \\ S_{22} \end{pmatrix}.$$

We can see that $\mathbf{D}_p vech(\mathbf{S}) = vec(\mathbf{S})$ for

$$\mathbf{D}_{p,4\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then,

$$\mathbf{D}_p'\mathbf{D}_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times 3}.$$

Hence,

$$\mathbf{D}_p^+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{3\times 4},$$

$$\mathbf{\Sigma} \otimes \mathbf{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \otimes \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_{11}\sigma_{11} & \sigma_{11}\sigma_{12} & \sigma_{12}\sigma_{11} & \sigma_{12}\sigma_{12} \\ \sigma_{11}\sigma_{12} & \sigma_{11}\sigma_{22} & \sigma_{12}\sigma_{12} & \sigma_{12}\sigma_{22} \\ \sigma_{12}\sigma_{11} & \sigma_{12}\sigma_{12} & \sigma_{22}\sigma_{11} & \sigma_{22}\sigma_{12} \\ \sigma_{12}\sigma_{12} & \sigma_{12}\sigma_{22} & \sigma_{22}\sigma_{12} & \sigma_{22}\sigma_{22} \end{pmatrix},$$

and

$$\mathbf{\Delta} = \begin{pmatrix} 2\sigma_{11}^2 & 2\sigma_{11}\sigma_{12} & 2\sigma_{12}^2 \\ 2\sigma_{11}\sigma_{12} & \sigma_{11}\sigma_{22} + \sigma_{12}^2 & 2\sigma_{12}\sigma_{22} \\ 2\sigma_{12}^2 & 2\sigma_{12}\sigma_{22} & 2\sigma_{22}^2 \end{pmatrix}.$$

A.3 TRI-VARIATE SAMPLE COVARIANCE MATRIX DISTRIBUTION

For the 3×3 case, $\mathbf{Y}_i = \begin{pmatrix} y_{1i} \\ y_{2i} \\ y_{3i} \end{pmatrix}$ and

$$\mathbf{S}_{3\times 3} = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{pmatrix} = \begin{pmatrix} \mathbf{S}_1 & \mathbf{S}_2 & \mathbf{S}_3 \end{pmatrix}.$$

We have

$$vec(\mathbf{S})_{9\times 1} = \begin{pmatrix} S_{11} \\ S_{12} \\ S_{13} \\ S_{12} \\ S_{22} \\ S_{23} \\ S_{13} \\ S_{23} \\ S_{33} \end{pmatrix}$$

and

$$vech(\mathbf{S})_{6\times 1} = \begin{pmatrix} S_{11} \\ S_{12} \\ S_{13} \\ S_{22} \\ S_{23} \\ S_{33} \end{pmatrix}.$$

We can see that $\mathbf{D}_p vech(\mathbf{S}) = vec(\mathbf{S})$ for

$$\mathbf{D}_{p,9\times 6} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then,

$$\mathbf{D}_p'\mathbf{D}_p = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}_{6\times 6}.$$

Hence,

$$\mathbf{D}_p^+ = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}_{6\times 9},$$

$$\mathbf{\Sigma}\otimes\mathbf{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} \otimes \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_{11}\sigma_{11} & \sigma_{11}\sigma_{12} & \sigma_{11}\sigma_{13} & \sigma_{12}\sigma_{11} & \sigma_{12}\sigma_{12} & \sigma_{12}\sigma_{13} & \sigma_{13}\sigma_{11} & \sigma_{13}\sigma_{12} & \sigma_{13}\sigma_{13} \\ \sigma_{11}\sigma_{12} & \sigma_{11}\sigma_{22} & \sigma_{11}\sigma_{23} & \sigma_{12}\sigma_{12} & \sigma_{12}\sigma_{22} & \sigma_{12}\sigma_{23} & \sigma_{13}\sigma_{12} & \sigma_{13}\sigma_{22} & \sigma_{13}\sigma_{23} \\ \sigma_{11}\sigma_{13} & \sigma_{11}\sigma_{23} & \sigma_{11}\sigma_{33} & \sigma_{12}\sigma_{13} & \sigma_{12}\sigma_{23} & \sigma_{12}\sigma_{33} & \sigma_{13}\sigma_{13} & \sigma_{13}\sigma_{23} & \sigma_{13}\sigma_{33} \\ \sigma_{12}\sigma_{11} & \sigma_{12}\sigma_{12} & \sigma_{12}\sigma_{13} & \sigma_{22}\sigma_{11} & \sigma_{22}\sigma_{12} & \sigma_{22}\sigma_{13} & \sigma_{23}\sigma_{11} & \sigma_{23}\sigma_{12} & \sigma_{23}\sigma_{13} \\ \sigma_{12}\sigma_{12} & \sigma_{12}\sigma_{22} & \sigma_{12}\sigma_{23} & \sigma_{22}\sigma_{12} & \sigma_{22}\sigma_{22} & \sigma_{22}\sigma_{23} & \sigma_{23}\sigma_{12} & \sigma_{23}\sigma_{22} & \sigma_{23}\sigma_{23} \\ \sigma_{12}\sigma_{13} & \sigma_{12}\sigma_{23} & \sigma_{12}\sigma_{33} & \sigma_{22}\sigma_{13} & \sigma_{22}\sigma_{23} & \sigma_{22}\sigma_{33} & \sigma_{23}\sigma_{13} & \sigma_{23}\sigma_{23} & \sigma_{23}\sigma_{33} \\ \sigma_{13}\sigma_{11} & \sigma_{13}\sigma_{12} & \sigma_{13}\sigma_{13} & \sigma_{23}\sigma_{11} & \sigma_{23}\sigma_{12} & \sigma_{23}\sigma_{13} & \sigma_{33}\sigma_{11} & \sigma_{33}\sigma_{12} & \sigma_{33}\sigma_{13} \\ \sigma_{13}\sigma_{12} & \sigma_{13}\sigma_{22} & \sigma_{13}\sigma_{23} & \sigma_{23}\sigma_{12} & \sigma_{23}\sigma_{22} & \sigma_{23}\sigma_{23} & \sigma_{33}\sigma_{12} & \sigma_{33}\sigma_{22} & \sigma_{33}\sigma_{23} \\ \sigma_{13}\sigma_{13} & \sigma_{13}\sigma_{23} & \sigma_{13}\sigma_{33} & \sigma_{23}\sigma_{13} & \sigma_{23}\sigma_{23} & \sigma_{23}\sigma_{33} & \sigma_{33}\sigma_{13} & \sigma_{33}\sigma_{23} & \sigma_{33}\sigma_{33} \end{pmatrix},$$

and $\mathbf{\Delta} =$

$$\begin{pmatrix} 2\sigma_{11}^2 & 2\sigma_{11}\sigma_{12} & 2\sigma_{11}\sigma_{13} & 2\sigma_{12}^2 & 2\sigma_{12}\sigma_{13} & 2\sigma_{13}^2 \\ 2\sigma_{11}\sigma_{12} & \sigma_{11}\sigma_{22}+\sigma_{12}^2 & \sigma_{12}\sigma_{13}+\sigma_{11}\sigma_{23} & 2\sigma_{12}\sigma_{22} & \sigma_{12}\sigma_{23}+\sigma_{13}\sigma_{22} & 2\sigma_{13}\sigma_{23} \\ 2\sigma_{11}\sigma_{13} & \sigma_{12}\sigma_{13}+\sigma_{11}\sigma_{23} & \sigma_{11}\sigma_{33}+\sigma_{13}^2 & 2\sigma_{12}\sigma_{23} & \sigma_{12}\sigma_{33}+\sigma_{13}\sigma_{23} & 2\sigma_{13}\sigma_{33} \\ 2\sigma_{12}^2 & 2\sigma_{12}\sigma_{22} & 2\sigma_{12}\sigma_{23} & 2\sigma_{22}^2 & 2\sigma_{22}\sigma_{23} & 2\sigma_{23}^2 \\ 2\sigma_{12}\sigma_{13} & \sigma_{12}\sigma_{23}+\sigma_{13}\sigma_{22} & \sigma_{12}\sigma_{33}+\sigma_{13}\sigma_{23} & 2\sigma_{22}\sigma_{23} & \sigma_{22}\sigma_{33}+\sigma_{23}^2 & 2\sigma_{23}\sigma_{33} \\ 2\sigma_{13}^2 & 2\sigma_{13}\sigma_{23} & 2\sigma_{13}\sigma_{33} & 2\sigma_{23}^2 & 2\sigma_{23}\sigma_{33} & 2\sigma_{33}^2 \end{pmatrix}.$$

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