Elimination of Edge Effects Using Spline Wavelets Which Maintain a Uniform Two-Scale Relation

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Elimination of Edge Effects Using Spline Wavelets
Which Maintain a Uniform Two-Scale Relation

by

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A Dissertation Submitted to the Faculty of
Old Dominion University in Partial Fulfillment of the
Requirements for the Degree of

DOCTOR OF PHILOSOPHY
Computational and Applied Mathematics

OLD DOMINION UNIVERSITY
March 24, 1995

Approved by:

Charlie H. Cooke (Director)
Abstract

Elimination of Edge Effects Using Spline Wavelets
Which Maintain a Uniform Two-Scale Relation

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Sang Kyu Yang

Old Dominion University, 1995

Director: Dr. Charlie H. Cooke

Use of the compactly supported B-spline wavelet of Chui and Wang is hindered
by loss of accuracy on decomposition, through truncation of weight sequences
which are countably infinite. Adaptations to finite intervals often encounter sig­
nificant problems with error near boundaries, called edge effects. For multireso­
lution analysis on a finite interval which employ the piecewise linear B-wavelet
the present research provides a frontal approach to decomposition which avoids
truncation of weight sequences, experiences no error at boundaries, and which
exhibits a factor of three increase in computational efficiency, over the usual ap­
proach characterized by truncation of infinite weight sequences. As a further
modest contribution, a simple derivation of the piecewise linear B-spline wavelet
for $L_2(R)$ is given. The simple technique is then applied to the derivation of
supplementary boundary wavelets, which are necessary in order to complete the piecewise linear B-wavelet basis on a finite interval.

There is also presented a modification to the Chui and Quak piecewise-cubic spline multiresolution analysis for the finite interval. The modification is intended to simplify implementation. Boundary scaling functions with multiple nodes at interval endpoints are rejected, in favor of the classical B-spline scaling function restricted to the interval. This necessitates derivation of revised boundary wavelets. In addition, a direct method of decomposition results in significant bandwidth reduction on solving an associated linear systems. Image distortion is reduced by employing natural spline projection. Finally, a hybrid projection scheme is proposed, which particularly for large systems further lowers operation count. Numerical experiments which try the algorithm are performed: The problems of edge detection, data compression, and data smoothing by thresholding in the wavelet transform domain are examined. The cubic B-spline wavelet yields compression ratios as high as 40 to 1 in the numerical experiments.
Dedication

To My Family:

• My Parents, Yang Kil Ho and Choi Mal Sun
• My Aunt, Choi Na Mi
• My Siblings, Yang Sang Chon and Yang Sang Uk
Acknowledgment

I would like to thank my advisor, Professor Charlie H. Cooke, for the guidance, support, and inspiration which he has given throughout the period of research on my dissertation. Dr. Cooke’s encouragement and cooperation will not soon be forgotten.

To Dr. Richard Noren, Dr. John Heinbockel, and Dr. Linda Vahala, I express gratitude for helpful comments when reviewing this dissertation, and for serving on my committee.

I also wish to express my sincere respect to Dr. Philip Robert Wohl and Dr. John Swetits, for enjoyable and unforgettable lectures received during my years in graduate school.
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Chapter 1

Introduction

1.1 Research Context

The central problem of discrete wavelet analysis is the decomposition of a signal in terms of a best basis consisting of wavelet component functions which are localized in both time and frequency. Such localizations are achieved through use of the mathematical operations of dilation and translation applied to a generating function (desirably of compact support) which is called a mother wavelet. The mother wavelet $\psi(t)$ is related to a scaling function, $\phi(t)$, which enables the generation of a function space structure called a multi-resolution analysis.

Whenever a wavelet possesses what is referred to as both inter-scale and intra-scale orthogonal translates, the wavelet is called orthogonal; otherwise, it is called semi-orthogonal. Generally, a scaling function whose translates at each fixed frequency level are orthogonal can be associated with an orthogonal wavelet. In the historical development, higher order orthogonal wavelets were first investigated
by I. Daubechies [12], while a parallel exploration of semi-orthogonal B-spline wavelets was accomplished by C. K. Chui et. al. [9]. The two wavelet classes have an underlying common structure, yet exhibit diversity in application. The picture is beginning to emerge that orthogonal wavelets are generally easier and perhaps more economical to use, while more difficulty is involved in their visualization, analysis, and synthesis.

The concept of a multi-resolution analysis is due to Stephen A. Mallat [20], whose work was motivated by the desire to better understand image analysis. Employing the properties of a multi-resolution analysis, the Hilbert space projection of a fine image can be decomposed into successively coarser images, each having a corresponding orthogonal component of error (the missing fine details at this level). Over several stages this breakdown is called a decomposition of the fine image; the reverse process of progressing back from a coarse to a fine image is called reconstruction. Additional manipulation during the decomposition phase can lead to what is called image compression: the resulting reconstructed image is desirably of comparable quality, yet the decomposition requires considerably less data storage (significant reductions of order 40-to-1 are sometimes possible).

Classically, the development of wavelets and multi-resolution analysis focuses upon function spaces over an infinite domain; viz, $L^2(R)$. However, adaptation of such infinite domain algorithms to problems finite in extent often causes error
growth near boundaries. In image analysis such errors are called edge effects. These errors are usually of significant amplitude and characterized by boundary layer structure.

1.2 Statement of Purpose

The investigations of this dissertation are directed to the area of semi-orthogonal wavelets. The aims and purposes of the dissertation are several: (a) Focusing upon piecewise linear B-spline wavelets, an investigation is made of the errors which arise near boundaries when the classical $L_2(R)$ methods of decomposition are adapted to signals of finite extent. By properly treating the problem as one of finite extent, a so-called frontal method of decomposition is obtained, which is characterized by the complete absence of boundary errors. In the course of development, it becomes necessary to introduce boundary wavelets which near boundaries supplement the classical wavelet translates. (b) The frontal method is then applied to the class of piece-wise cubic B-spline wavelets on the interval, with concomitant development of the supplementary boundary wavelets. Here, the total problem (both decomposition and reconstruction) is explored, with the objective of obtaining algorithms which have economical implementation. The usual moving average schemes for reconstruction of functions in $L_2(R)$ are now replaced by dynamical moving average schemes for reconstruction of functions in
However, the classical pyramid schemes for decomposition must give way to the solving of banded linear systems; a process which is more demanding of resources, but which pays dividends through elimination of boundary errors.

(c) One question frequently posed by the user community regards knowing the circumstances under which one type of wavelet may or may not be superior to another. The question is explored here in the context of an application of wavelets to data smoothing called thresholding in the wavelet transform domain. The cubic wavelet structure referred to in (a-b) is applied to data smoothing in the thresholding context, to determine whether semi-orthogonal wavelets can compete with the successes of orthogonal wavelets in this area. The potential of the B-wavelet as a means to achieve data compression is also explored.

1.3 Basic Literature Survey

Classical approaches to wavelet construction deal with multiresolution analysis (MRA) on the entire real axis [9, 12]. More recently, the construction of suitable wavelets for bounded intervals has become of interest [7, 11, 21], primarily as a means for elimination of edge effects in image analysis. Such constructions are usually directed toward synthesis of orthogonal or semi-orthogonal wavelets.

Meyer [21] has adapted the orthogonal wavelets of Daubechies [12] to the bounded interval. However, his procedure for obtaining orthogonality of bound-
ary scaling functions encounters an ill-conditioned matrix and ensuing numerical instability as matrix size increases. This mishap perhaps motivated Cohen, Daubechies, and Vial [11] to take an approach whereby boundary scaling functions are no longer obtained by restriction to the interval of the customary scaling function. This causes some inconvenience in reconstruction, as various two scale relations are required near boundaries.

Perhaps anticipating numerical instabilities if proceeding otherwise, Chui-Quak [7] construct (and Quak-Weyrich [23] implement) semi-orthogonal spline wavelets for the interval, introducing special boundary scaling functions which possess multiple nodes at an end point. Intuitively, it appears unnatural to harbor the inconvenience of scaling functions possessing two scale relations which vary over the interval. There is little reason to suspect that otherwise numerical instabilities might occur, as for spline wavelets orthogonalization is not a factor.

Consequently, one purpose of the present research is to re-examine (see Chapter Four), for the case of piece-wise cubic B-spline wavelets with compact support, the feasibility of employing an alternative MRA for the interval. The modification is as follows: First, boundary scaling functions which are the restriction to the interval of cubic B-spline translates are retained. The reconstruction process is now simplified, as a two-scale equation which is the interval restriction of the classical two-scale relation applies uniformly. Moreover, by employing nested spaces
of natural cubic splines for the MRA, it is expected that distortion of images will
be diminished. In addition, a direct decomposition scheme allows improved op-
eration count due to reduction in bandwidth of a certain linear system. Finally,
a hybrid projection scheme is introduced, which particularly for large systems
further enhances economy of this operation.

1.4 Outline of Procedure

In brief, the purpose of this dissertation is to investigate means to develop al-
gorithms for multi-resolution analysis on the finite interval which are not troubled
by edge effects. This is accomplished as now outlined.

The aim of Chapter 2 is preliminary mathematical development and review
of relevant background material. This includes discussion of wavelets and multi-
resolution analysis as well as the requirements for function spaces of natural cubic
splines on the interval.

In Chapter Three attention is focused upon piecewise-linear B-spline wavelets
and the development of a decomposition scheme for MRA on the finite interval
which eliminates edge effects. Along the way the need arises for the development
of the boundary wavelets which are necessary for a spanning wavelet basis. Nu-
merical experiments which test the efficiency of the frontal method so developed
are presented.
In Chapter Four the frontal method is extended to MRA whose function spaces of natural cubic splines possess piecewise cubic B-spline wavelets. Appropriate boundary wavelets are developed in a unique way through use of a principle of least interference between boundary and interior wavelets. The MRA so developed is a revision of an MRA previously developed by Chui-Quak [7] and later implemented by Quak-Weyrich [23]. The idea of the revision is to avoid having modified scaling functions near boundaries, and in particular to avoid the complexities of spline spaces having multiple nodes at a boundary. The uniform two-scale relation which results leads to simplification of reconstruction. By further avoidance of the dual wavelet, both decomposition and reconstruction algorithms are simplified, as compared to those presented by Quak-Weyrich [23].

In Chapter Five the revised MRA studied in Chapter Four is used to investigate a data smoothing algorithm which employs thresholding in the wavelet transform domain.

In chapter six, the inherent potential of the cubic B-wavelet to achieve data compression by Quantile thresholding of the wavelet coefficients is explored. Compression ratios up to 40 to 1 are achieved.
Chapter 2
Preliminary Concepts

2.1 Wavelet Background

In this section some preliminary background material concerning wavelets is introduced. As wavelet theory is still undergoing a rapid development, it is difficult to attempt a unified presentation. However, there is recalled the notion of wavelet, scaling function, and multiresolution analysis as first introduced in pioneer work for problems of infinite extent.

First, consider the space $L^2(R)$ of all measurable functions $f$, defined on the real line $R$, that satisfy $\int_{-\infty}^{+\infty} |f(x)|^2 dx < \infty$ where $R = (-\infty, +\infty)$. A function $f$ in $L^2(R)$ is used to represent an analog signal with finite energy, and its Fourier transform

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega x} f(x) dx$$

(2.1)

reveals the spectral information of the signal.
A Wavelet is a function $\psi \in L^2(R)$ whose Fourier transform $\hat{\psi}(\omega)$ satisfies the condition (by the Grossman and Morlet)\cite{22}.

$$\int_0^\infty |\hat{\psi}(t\omega)|^2 \frac{dt}{t} = 1 \quad a.e. \quad (2.2)$$

A Wavelet is a function $\psi \in L^2(R)$ whose Fourier transform $\hat{\psi}(\omega)$ satisfies the condition (by the Littlewood, Paley, and Stein theory)\cite{22}.

$$\sum_{-\infty}^{+\infty} |\hat{\psi}(2^{-j}\omega)|^2 = 1 \quad a.e. \quad (2.3)$$

A Wavelet is a function $\psi \in L^2(R)$ such that $2^j \psi(2^j x - k), j, k \in \mathbb{Z}$ is an orthonormal basis for $L^2(R)$ (by the Franklin and Strömberg)\cite{22}.

Thus, the more the conditions, the narrower is the scope of a "Wavelet".

Generally, the term Wavelets refers to a set of functions of the form $\psi_{ab}(x) = |a|^j \hat{\psi}(\frac{x-k}{a^i})$, i.e., sets of functions formed by dilations and translations of a single function $\psi(x)$ sometimes called, variously, the "Mother Wavelets", "Basic Wavelets", or "Analyzing Wavelets".

Let $a = 2^{-j}, b = ai$ then

$$\psi_{ij}(x) = 2^{-\frac{j}{2}} \psi(2^{-j} x - i) \quad \text{where} \quad i, j \in \mathbb{Z} \times \mathbb{Z}. \quad (2.4)$$

Thus, equation (2.4) represents dilation by (both positive and negative) powers of 2 and translations by integers, which can be parameterized by a pair of integers $i, j$ rather than using $a, b$. The wavelets are then referred to as discrete. In this research, discrete wavelets are the general area of focus.
Definition 2.1 Let a function $\phi \in L^2(R)$ generate space

\[ V^0 = \text{clos}_{L^2}(\phi(\cdot - k) : k \in Z), \quad V^j = \text{clos}_{L^2}(\phi_k^j : k \in Z) \quad \phi_k^j(x) = 2^j \phi(2^j x - k) \]

$j, k \in Z$ where $\langle \rangle$ denote the linear span.

Then $\phi$ is said to generate a multiresolution analysis (MRA) if the subspaces $V^j$ have the following properties:

1. $\cdots \subseteq V^{-2} \subseteq V^{-1} \subseteq V^0 \subseteq V^1 \subseteq V^2 \cdots$
2. $\text{clos}_{L^2}(\bigcup_{j \in Z} V^j) = L^2(R)$
3. $\bigcap_{j \in Z} V^j = \{0\}$
4. $f(x) \in V^j \iff f(2x) \in V^{j+1}$ for $j \in Z$, and
5. $\{\phi(\cdot - k)\}_{k \in Z}$ forms a Riesz basis of $V^0$, i.e.

There are constants $A$ and $B$ with $0 < A \leq B < \infty$ such that

\[ A \|\{C_k\}\|^2 \leq \sum_{k=-\infty}^{\infty} C_k \phi(\cdot - k)\|^2 \leq B \|\{C_k\}\|^2 \tag{2.5}\]

for all sequences $\{C_k\} \in l^2$ for which $\|\{C_k\}\|^2 = \sum_{k=-\infty}^{\infty} |C_k|^2 < \infty$

Such a function $\phi \in L^2(R)$ is called a scaling function of the MRA [19].

Definition 2.2 A function $\psi \in L^2(R)$ is a discrete wavelet if it generates the complementary orthogonal subspaces $W^j$ of a multi-resolution analysis; that is,

\[ W^j = \text{clos}_{L^2}(\psi_k^j : k \in Z), \quad \psi_k^j(x) = 2^j \psi(2^j x - k) \quad j, k \in Z \]

with

\[ V^{j+1} = V^j \bigoplus W^j \quad \bigoplus: \text{orthogonal sum} \tag{2.6}\]
These subspaces $W_j, j \in \mathbb{Z}$, are called the wavelet subspaces of $L^2(\mathbb{R})$ relative to the mother wavelet $\psi$. Since $V^j = V^{j-1} \oplus W^{j-1}$ for any $j \in \mathbb{Z}$, $f_N$ has a unique decomposition: $f_N = f_{N-1} + g_{N-1}$, where $f_{N-1} \in V^{N-1}$ and $g_{N-1} \in W^{N-1}$.

By repeating this process, we have

$$f_N = g_{N-1} + g_{N-2} + g_{N-3} + \cdots + g_{N-M} + f_{N-M} \quad (2.7)$$

where $f_j \in V_j$ and $g_j \in W_j$ for any $j$, and $M$. The "decomposition" in (2.7), which is unique, is called a "wavelet decomposition". However, the main idea of a multi-resolution analysis is to have $L^2$-function $f$ as a limit of successive approximations, each of which is a blurred version of $f$, with more and more blurred functions. The successive approximations thus use different resolution analysis. Any wavelet, semi-orthogonal or not, generates a direct sum decomposition of $L^2(\mathbb{R})$, by the general properties of Hilbert Space.

**Theorem 2.1** *(The Projection theorem)* Let $M$ be a closed linear subspace of a Hilbert space $H$. Any $x_0 \in H$ can be written in the form $x_0 = y_0 + z_0$ where $y_0 \in M$ ($y_0$ is the minimum norm approximation in $M$ to $x_0$) and $z_0 \in M^\perp$ (the orthogonal complement of $M$). The elements $y_0, z_0$ are uniquely determined by $x_0$.

**Definition 2.3** A function $\psi \in L^2(\mathbb{R})$ is called an orthogonal wavelet, if the family \{\psi \in L^2(\mathbb{R})\}, as defined in (2.4), is an orthonormal basis of $L^2(\mathbb{R})$; that is,

$$< \psi_{j,k}, \psi_{l,m} > = \delta_{j,l} \cdot \delta_{k,m} \quad j, k, l, m \in \mathbb{Z} \quad (2.8)$$
Definition 2.4 A wavelet $\psi$ in $L^2(R)$ is called a semi-orthogonal wavelet if the Riesz basis $\{\psi_{j,k}\}$ it generates satisfies

$$<\psi_{j,k},\psi_{l,m}>=0, \quad j\neq l \quad j,k,l,m \in \mathbb{Z} \quad (2.9)$$

2.2 B-Splines

Let

$$Q^m(x) = \frac{(-1)^m \Delta^m (x-y)_+^m}{m!} = \sum_{i=0}^{m} \frac{(-1)^i \binom{m}{i} (x-i)^{m-1}}{m!}$$

where

$$(x-y)_+^{m-1} = \begin{cases} (x-y)^{m-1} & \text{if } x \geq y \text{ and } m > 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Delta^m \text{ is } m^{th} \text{ divided difference}$$

This is the usual B-spline associated with the simple (uniformly spaced) knots $0,1,2,\ldots,m$. It belongs to $C^{m-1}(R)$. Associated with $Q^m(x)$, we introduce the normalized version $N^m(x) = mQ^m(x)$.

$$N^2(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 2-x & \text{if } 1 \leq x \leq 2 \end{cases} \quad (2.10)$$
The $n$th derivative is given by

$$N_n^n(x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \delta(x - j)$$

where $\delta(x - j)$ denotes the direct delta distribution with unit mass at the origin.

For the present research, the most significant property of the B-splines is the two-scale identity

$$N^m(x) = \sum_{j=0}^{m} 2^{-m+1} \binom{m}{j} N^m(2x - j)$$

Indeed, for every integer $m$, the scaling function $\phi = N^m(x)$ generates a multi-resolution analysis for $L_2(R)$ [9], whose characterizing B-spline wavelet is semi-orthogonal. In Chapters 3-4 there is considered methods for correctly adapting these MRA to the finite interval in such a way that boundary effects are eliminated.

The two-scale identity can be described in terms of the Fourier transform:

$$\tilde{N}^m(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} N^m(x) dx$$
\[ \tilde{N}^m(\omega) = \frac{1}{2} P(Z) \tilde{N}^m\left(\frac{\omega}{2}\right) \] (2.14)

where \( Z = e^{i\pi}, \quad P(Z) = \sum_{j=0}^{m} 2^{-m+1} \binom{m}{j} Z^j \) (2.15)

**Function Spaces of Natural Splines**

Let \( x_0, x_1, ..., x_N \) be the points of a uniform grid of mesh width \( h \), with \( y_j, j = 0,1,..,N \) sampled data values of some function \( f(x) \). It is desired to fit \( f(x) \) over the interval \([x_0, x_N]\) with a cubic spline

\[ P_h(x) = \sum_{j=-1}^{N+1} c_j \phi_j(x) \] (2.16)

where \( \phi_j(x), j = -1,0,1,...,N+1 \) are cubic B-spline translates at some fixed frequency level. This will require the solution of a tri-diagonal system of linear equations which is under-determined: boundary conditions are needed, for \( c_{-1}, c_{N+1} \). It is well-known, as can be verified by using the above formula for second derivatives of a B-spline, that the boundary conditions which force \( P_h(x) \) to be a (unique) natural spline function are

\[ c_{-1} = 2c_0 - c_1 \quad \text{and} \quad c_{N+1} = 2c_N - c_{N-1}. \] (2.17)

This concept will have application in Chapter 4.
Chapter 3

Multiresolution Analysis with Frontal Decomposition

3.1 Introduction

Use of the compactly supported B-spline wavelet of Chui and Wang [4, 9] is hindered by loss of accuracy on decomposition, through truncation of weight sequences which are countably infinite. Adaptations to finite intervals often encounter problems at boundaries. For multi-resolution analysis on a finite interval which employ the piece-wise linear B-wavelet, there is developed in the present chapter a frontal approach to decomposition which avoids truncation of weight sequences, experiences no problems at boundaries, and which for piece-wise linear spline MRA exhibits a factor of three increase in computational efficiency. The boundary wavelets which complete the linear B-wavelet basis on a finite interval are constructed.
The pre-ondelettes of Battle [1] represent one of the first cases studied of what is now called semi-orthogonal wavelets. It having been recognized that for the class of Lemarie functions the pre-ondelette expansion of an arbitrary function has greater computational value than the ondelette expansion [2], Chui et al [4, 9] have vigorously pursued the idea, obtaining means to construct compactly supported, semi-orthogonal B-spline wavelets of arbitrary order P.

Multi-resolution analysis on the finite interval meets boundary problems [25] in adapting wavelet bases developed for $L_2(R)$. Daubechies et al [11, 12] consider the problem when orthonormal wavelets are involved. A workable approach for the case of semi-orthogonal wavelets is presented here. An efficient technique for fully realizing the potential accuracy of these wavelets is developed, and numerical studies of its efficiency are accomplished.

It is well-known that the semi-orthogonal B-wavelets first analyzed by Chui and Wang [4] suffer loss of accuracy on decomposition, due to truncation of infinite weight sequences. For linear B-wavelets these problems turn out to be simultaneously disposed of by a frontal approach to decomposition which is employed in the present work. The usual projection from a fine to a coarser level of approximation is directly obtained by solving banded, symmetric positive definite systems of linear equations.

The frontal method is perhaps not new, but it is not widely used. This is due to
the fact that many wavelets are orthonormal and allow finite weight sequences; viz, the Daubechies wavelet D4 [12]. In such cases pyramid algorithms [11, 17] provide an economical, oblique approach to decomposition. The major contribution of this chapter is to show that the semi-orthogonal, compactly supported, linear B-spline wavelet may be used on the finite interval, with full preservation of accuracy. Moreover, it would appear that the frontal method can be utilized as well for higher order B-wavelets, although it is expected that the computational efficiency may not be as good. This conjecture is explored in Chapter Four, where the frontal approach illustrated here is applied to the piece-wise cubic B-spline wavelet.

One might suppose that use of a method which by-passes the pyramid scheme could result in less efficient decomposition. It may be argued that this is scarcely avoidable, if the inherent potential accuracy of the B-wavelets is to be realized. However, numerical experiments reveal additional merits: for equivalent accuracy on the interior of the computational domain, with piecewise linear spline wavelets, the frontal method for calculating the minimum norm projection on a coarser grid has by a factor of three a smaller floating point operation count than does the pyramid scheme associated with truncated weight sequences. Furthermore, in the case of non-periodic data, accuracy near boundaries is far superior.
3.2 MAPLE Construction of the Linear B-Wavelet

for $L_2(R)$

In order to establish the wavelet concept clearly but firmly, in this section there is indicated how to construct the semi-orthogonal linear B-spline wavelet using the MAPLE program for computer-aided algebraic manipulations. The approach given is similar to that of Michelli which is used in obtaining spline wavelets for the case of non-uniformly spaced knots. In the next section the method is then adapted to the derivation of boundary wavelets which together with the usual B-wavelets derived here for the interior constitute a full basis of the orthogonal complement space $W_0$ on the finite interval.

Consider the tent function

$$N(x) = \begin{cases} 
1 - |x - 1| & \text{for } |x - 1| \leq 1 \\
0 & \text{elsewhere.}
\end{cases}$$

(3.1)

and the multi-resolution analysis (MRA) consisting of subspaces $V_j$ in $L_2(R)$ of piece-wise linear functions which can be generated by the local basis $\phi_{j,k}(x) = N(2^j x - k), k \in \mathbb{Z}$. Let $W_j$ be the orthogonal complement of $V_j$ in $V_{j+1}$, with $V_{j+1} = V_j + W_j$ the direct sum of $V_j$ and $W_j$. The objective of this section is to give a simple derivation of a compactly supported B-wavelet $\psi(x)$ which has the property that
\[ \psi_j = \psi(2^j x - k), \quad k \in \mathbb{Z} \] is a non-orthogonal local basis for \( W_j \).

If \( \psi(x) \in W_0 \subset V_1 \), then to be compactly supported \( \psi(x) \) must be a finite linear combination of the basis functions:

\[ \psi(x) = \sum_{k=1}^{M} C_k N(2^j x - k + 1), \quad (3.2) \]

The constants \( C_k \) and \( M \) will be chosen by the following criterion: (a) \( \psi(x) \) is to be minimally supported, and (b) orthogonal to every function in \( V_0 \).

Condition (b) is fulfilled provided the \( C_k \) are chosen such that \( \psi(x) \) is orthogonal to those \( N(x - l) \), \( l = -1, 0, 1, \ldots, L \) with which it shares support. Thus, fulfilling conditions (a-b) requires finding the minimal integer \( M \) for which a resulting linear homogeneous system of \( L + 2 \) equations in \( M \) unknowns has a non-trivial solution. The mathematics involved is such that the system can be set up and solved using a symbolic manipulation package such as MATHCAD or MAPLE which has the capability to integrate piecewise defined polynomial functions.

As \( N(x) \) satisfies the lattice two-scale relation

\[ N(x) = \frac{1}{2} N(2x) + N(2x - 1) + \frac{1}{2} N(2x - 2) \quad (3.3) \]

it can be anticipated that the integer \( M \) should be odd. Still, no such apriori assumption is needed in the construction. By trial and error, the cases \( M = 1, 2, 3, \) or 4 prove unfruitful, as only a trivial solution exists. However, upon choosing

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$M = 5, L = 2$ the following system of equations results from condition (b):

\[
6C_1 + C_2 = 0 
\]  
(3.4)

\[
6C_1 + 10C_2 + 6C_3 + C_4 = 0 
\]  
(3.5)

\[
C_2 + 6C_3 + 10C_4 + 6C_5 = 0 
\]  
(3.6)

\[
C_4 + 6C_5 = 0 
\]  
(3.7)

Because system (3.4)-(3.7) has as solution constant multiples of the particular solution indicated:

\[
C_1 = \frac{1}{12}, \enspace C_2 = -\frac{6}{12}, \enspace C_3 = \frac{10}{12}, \enspace C_4 = -\frac{6}{12}, \enspace C_5 = \frac{1}{12} 
\]  
(3.8)

condition (a) is now satisfied.

A simple calculation shows the B-wavelet so derived is symmetric about the center of its support and has non-orthogonal translates which are linearly independent. Moreover, a theorem of Chui [9] based on the symbols for the functions of eqns. (3.2,3.3,3.8) guarantees that the function (3.2,3.8) can be used to generate a Riesz basis for $L_2(R)$. As this solution for $\psi(x)$ agrees with the semi-orthogonal, compactly supported, linear B-spline wavelet which results when the formulas of Chui [9] are used, we do not pursue further proof that there results a genuine wavelet with the properties stated.
3.3 Construction of Boundary Wavelets

One merit of the previous derivation of the B-wavelet is simplicity of approach. A second merit is that it can easily be extended to the derivation of boundary wavelets, which are needed for multi-resolution analysis on a finite interval. For instance, suppose that a left boundary is encountered at the position \( x = 1 \). The linear B-wavelet is now not sufficient to generate a basis for \( W_0 \), as it is not orthogonal to the half-tent function associated with \( x = 1 \). However, its translates serve as an interior basis, whereas additional boundary wavelets are required for a full basis. At the left boundary there is derived by the above methods the function \( \psi_{L(x)} \) of eqn. (3.2) whose coefficients are determined by

\[
C_1 = 0 \\
5C_2 + 6C_3 + C_4 = 0 \\
C_2 + 6C_3 + 10C_4 + 6C_5 = 0 \\
C_4 + 6C_5 = 0
\]

System (3.9-3.12) has as solution constant multiples of the particular solution indicated:

\[
C_1 = 0, \quad C_2 = -\frac{12}{12}, \quad C_3 = \frac{11}{12}, \quad C_4 = -\frac{6}{12}, \quad C_5 = \frac{1}{12}
\]
If instead of a left boundary a right boundary at \( x = 2 \) is involved, the boundary wavelet \( \psi_R(x) \) would require the coefficients

\[
C_1 = \frac{1}{12}, \quad C_2 = -\frac{6}{12}, \quad C_3 = \frac{11}{12}, \quad C_4 = -\frac{12}{12}, \quad C_5 = 0
\]

(3.14)

It is remarked that the term "boundary wavelet" is somewhat singular, as dilates see all the action; the translates do not belong here. Thus, the terminology is used loosely in the case of a finite interval, however necessary may be the construction. Figure 3.1 shows the appearance of the left boundary wavelet.

### 3.4 Frontal Decomposition on the Finite Interval

Let \( x_0, x_1, ..., x_N, N=2^n \) be the points of a fine grid of mesh-width \( h \) and let

\[
\phi_n(x) = \begin{cases} 
1 - \frac{|x-x_n|}{h} & \text{for } |x-x_n| \leq h \\
0 & \text{elsewhere.}
\end{cases}
\]

(3.15)

If \( y_j, j = 0, 1, ..., N \) are sampled data values from some function \( f(x) \), then

\[
P_n f = \sum_{j=0}^{N} y_j \phi_j(x),
\]

(3.16)

is a projection on \( V_1 \), the space of piecewise linear functions defined on \([x_0, x_N]\) which are continuous over \((x_j, x_{j+1})\). It is to be remarked that imposing for odd
indices the second-difference restrictions
\[ c_j = \frac{1}{2} (c_{j-1} + c_{j+1}) \]  
(3.17)

will force a \( V_1 \) projection
\[ P_{2h} f = \sum_{j=0}^{N} c_j \phi_j(x), \]
(3.18)
also to be in \( V_0 \), the space of piecewise linear functions defined on \([x_0, x_N]\) which are continuous over intervals \((x_{2k}, x_{2k+2})\).

The most desirable feature of a multiresolution analysis is provision of means to calculate for the finite interval \([x_0, x_N]\) the relation between the projection \( P_h f \) on the fine grid \( V_1 \) and its best \( L_2[x_0, x_N] \) approximation \( P_{2h} f \) on the coarse grid \( V_0 \):
\[ P_h f = P_{2h} f + W_h(x), \]
(3.19)
In particular, the function \( W_h(x) \) is to be expressed in terms of a suitably translated and scaled form of a B-wavelet basis, where for a full basis supplementary boundary wavelets may be necessary. Discussion of these boundary wavelets appears previously in Section 3.3.

**The Periodic Case**

To simplify the presentation, at first there will be considered the case for which the function \( f(x) \) is periodic, of period \( T = x_N - x_0 \), with \( f(x_0) = f(x_N) \). Now, periodic wrap-around at the ends [12] allows the problem to be analyzed without
need for boundary wavelets: boundary half-tent functions fill out to full tent functions, and B-wavelet translates which intersect the boundary are similarly extended, through addition of superfluous data at the ends of the interval. In this case, with \( M = \frac{N}{2} \)

\[
W_h(x) = \sum_{j=1}^{M} d_j \psi_{2j-1}(x),
\]

(3.20)

where the indexing associates a B-wavelet translate with each odd-numbered node on the fine mesh. When boundary wavelets are necessary, the indices 1, \( M \) refer to boundary wavelets.

Iteration of the projection (3.19,3.20) over successively coarser grids is referred to as decomposition of the function \( f(x) \) into its wavelet components. The reverse process of building up the function from knowledge of the \( c_j, d_j \) at each level is referred to as reconstruction. For B-wavelets decomposition using the pyramid scheme is hampered by the presence of infinite weight sequences [9]. A frontal decomposition scheme which avoids this difficulty is now presented. Reconstruction proceeds as usual by a reverse pyramid scheme [9], suitably modified to account for boundaries.

By a Hilbert Space approximation theorem, the best approximant in \( V_0 \) is the unique solution of the following constrained optimization problem:

\[
\text{Min} \quad \int_{x_0}^{x_N} (P_h f - P_{2h} f)^2 \, dx,
\]

(3.21)

subject to the restrictions of eqns. (3.17) on the odd indexed variables.
The solution for the even indexed variables \( c_{2k}, k = 0, 1, ..., M = \frac{N}{2} \) is obtained from solving tri-diagonal system

\[
AC = F, \quad (3.22)
\]

where \( a_{11} = a_{MM} = 8, a_{ii} = 16, i = 2, ..., M - 1 \), with \( a_{i,i+1} = a_{i-1,i} = 4 \), and

\[
F_0 = 5y_0 + 6y_1 + y_2, \quad (3.23)
\]

\[
F_N = y_{N-2} + 6y_{N-1} + 5y_N, \quad (3.24)
\]

\[
F_{2k} = y_{2k-2} + 6y_{2k-1} + 10y_{2k} + 6y_{2k+1} + y_{2k+2}. \quad (3.25)
\]

**Data Compression**

Whether or not the periodicity assumption is made, the projection \( P_{2h}f \) can be determined by solving problem (3.21), and the decomposition process can be continued, without knowledge of a wavelet basis of \( W_0 \). The cost is inefficiency in storage of the high frequency portion \( W_h(x) \), which storage requirement is cut in half by the data compression technique of eqn. (3.20). Thus, the fundamentals of multi-resolution analysis can be accomplished on the finite interval with or without benefit of a wavelet expansion.

The coefficients \( d_j \) of the local wavelet translate \( \psi_j \) of eqn. (3.20) now can be found (see Sections 5-6) by solving a tri-diagonal Toeplitz system

\[
BD = G. \quad (3.26)
\]
The numbers characterizing the diagonals are given by $B = T[1, 10, 1]$, where the number 10 associates with the main diagonal, and

$$G_j = 12(P_h f - P_{2h} f) j = 1, 2, \ldots, M$$

(3.27)

Systems (3.22) and (3.26) are positive definite symmetric and readily solvable employing the usual band Cholesky technique.

**The Non-periodic Case**

Here, the major change is that system (3.26) is no longer Toeplitz, as $B_{11} = 11$ and $B_{MM} = 11$; otherwise, $B_{ij}$ remains as before.

### 3.5 Signal Separation Using the Frontal Method

In this section the results of a numerical signal separation experiment are reported. The base signal

$$S(x) = \begin{cases} 
20\sin(x) & \text{for } 0 \leq x \leq x_1 \\
20\sin(x_1) - 8x & \text{for } x_1 \leq x \leq x_2 \\
20\sin(x_1) - 8x_2 + 8x & \text{elsewhere.}
\end{cases}$$

(3.28)

is corrupted by addition of a high frequency oscillation

$$g(x) = 5\sin(20\pi x)$$

(3.29)
and the result is sampled at intervals of $\frac{\pi}{64}$, far under the Nyquist rate for the high frequency oscillation. Thus, the sampled mixed signal $S + g$ contains little information concerning the true nature of $g(x)$.

The base signal is now separated by projecting the sampled signal on the space $V_1$ of piecewise linear functions appropriate to the sampling interval and calculating a coarser projection on $V_0$ by the frontal method previously described. Figures 3.2-3.4 show respectively the aliased signal, the mixed signal, and the piecewise linear extracted signal, which exhibits a maximum nine-tenths of one percent difference from the original signal $S(x)$ relative to maximum amplitude. This is due to the fact that the aliased signal always oscillates in sign from one grid point to the next; hence, it is contained in $W_0$, and is almost perfectly separated by decomposition. However, it is remarked that this is simply an illustrative example of frontal decomposition, and does not necessarily represent advocacy of a signal separation technique.

3.6 Numerical Experimentation with Weight Sequence Truncation

When needs of accuracy require that a large number of coefficients be retained after truncation of weight sequences, the frontal decomposition can be more ef-
ficient than the corresponding oblique scheme. In this section there is reported some numerical experimentation whose aim is to clarify this issue.

For special circumstances the error due to truncation of weight sequences has been analyzed by Chui [9], who obtains so-called crude estimates. For the case of the linear B-wavelet,

$$|E_K(W)C| \leq 2.7320509(0.26795)^{K+1}||W||_2$$  \hspace{1cm} (3.30)

where $W$ is the weight sequence and $2K + 1$ is the number of terms kept in the truncated Laurent expansions.

It is an interesting exercise to compare the accuracy obtainable for frontal versus oblique calculations of the best minimum norm projection where truncated weight sequences are used. This corresponds to truncating a certain Laurent expansion after $2K + 1$ terms [9], in which case the projection coefficients are calculated as

$$C_j = \sum_{k=-K}^{K} W_k C_{2j+k},$$  \hspace{1cm} (3.31)

The weight sequence $W_j$ truncated at $K=20$ is depicted by Figure 3.5. Rapid decay allows truncation at $K = 10$ with reasonable accuracy, meaning that accuracy on the interior which is acceptably close to the accuracy of the frontal method. However, this is equivalent to forty-one flops per data point processed. The accuracy increase going from 21 to 41 weights (41 to 81 flops) is slight.
With reference to solving eqns. (3.22-3.25), the floating point operation count for calculating \( F \) is five flops per data point, with eight flops per data point required to solve the linear system, or a total of thirteen flops per data point, with \( M+1 \) data points involved in calculating the best projection. Thus, the efficiency increases essentially by a factor of three when choosing the frontal method, where the extra work at boundaries which is associated with the weight truncated scheme has not been considered. The efficiency factor should be roughly the same had the expense of data compression (full decomposition) been considered.

**Boundary Treatment**

It is clear that eqn. (3.31) calls for special treatment at boundaries, as insufficient data is present to verify the entire sum. One way to solve this problem is to extend the data periodically \( K \) points on either side of boundaries. This works well if the data is from a periodic function; otherwise, a boundary layer of error develops locally, which appears to have little effect on interior accuracy.

**Periodic Data**

Support for conclusions previously stated is provided by numerical experimentation: The base signal now considered is

\[
S(x) = 20 \sin(x),
\]

(3.32)
whereas the corrupting signal

\[ g(x) = 2\sin(63x) \]  

(3.33)
is chosen such that both signals have least common period 2\(\pi\), with sampling interval still \(\frac{\pi}{64}\). The method of truncated weight sequence is used to project \(S + g\) on the next coarser grid. Percent error relative to maximum amplitude of the base signal is depicted in Figure 3.6, for the cases \(K = 5, 10\). Here, error is defined as the difference between \(S(x)\) and the calculated minimum norm projection of \(S + g\). Clearly, \(g\) has been disposed of, as previously was the case with the frontal scheme. Whereas truncation with \(K = 5\) produces an unacceptably large (2.5\%) error, truncation at \(K = 10\) yield errors which are comparable to those for the frontal scheme. Using \(K = 20\) gives negligible improvement.

**Non-Periodic Data**

The corrupting signal is now changed back to

\[ g(x) = 2\sin(20\pi x), \]  

(3.34)
in which case \(S + g\) is no longer periodic. However, the frequency of \(g\) has been only slightly perturbed, so that \(g\) will still be disposed of by projection. For \(K = 10\), Figure 3.7 shows the development of a boundary error for output from the method of truncated weight sequences, which is being compared with output.
from the frontal method. Thus, the problem of potentially large boundary errors is well disposed of by the frontal scheme.

3.7 Chapter Summary

A simple derivation of the piece-wise linear B-spline wavelet and accompanying boundary wavelets has been presented, together with a frontal technique for decomposition over a finite interval, for functions contained in the MRA generated by a piecewise linear B-spline local basis. The technique avoids the use of pyramid schemes and the necessity of truncating weight sequences of infinite length, which procedure theoretically lowers the accuracy of the B-wavelet. Numerical experiments indicate excellent results for the frontal method of decomposition, as regards both accuracy and efficiency of the algorithm. The problem of boundary error is well disposed of by the frontal scheme.
Figure 3.1:
Piece-wise Linear Spline Boundary Wavelet
Figure 3.2: Perturbing Signal

Aliased Signal: $5\sin(2\pi T)$
Figure 3.3:  
Base Signal Plus Perturbation
Figure 3.4: Extraction of the Base Signal by Frontal Projection
Figure 3.5: Weight Sequence for the Oblique Method of Projection
Comparison of % Error: Weight Sequence Length 11 vs. 21

Figure 3.6:
Error Comparison for Weight Sequences of Length K=5 and K=10

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Figure 3.7: Boundary Error Comparison for Frontal Versus Oblique Method
Chapter 4

Multiresolution Analysis on the Interval with Cubic Spline Wavelet and Uniform Two-scale Relation

4.1 Introduction

The aim of this chapter is to present a modification to the Chui-Quak [7] spline multiresolution analysis for the finite interval. Boundary scaling functions with multiple nodes at interval endpoints are rejected, in favor of the classical B-spline scaling function restricted to the interval. This necessitates derivation of revised boundary wavelets. In addition, a direct method of decomposition results in bandwidth reduction on solving some associated linear systems, and image distortion is reduced by employing natural spline projection.

The revised boundary wavelets do not markedly differ in appearance from those of Chui-Quak [7, 23]. As numerical results appear equivalent to those of [23],

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whereas operation count is improved and numerical instability is not experienced, the effort is deemed successful.

It is remarked that in the derivation of boundary wavelets a certain amount of latitude occurs, as wavelets for a spline space are generally not unique. This latitude is disposed of by a principle of minimal interference with interactions on the interior: The inward portion of a boundary wavelet is made to conform to the shape of the inner wavelet insofar as possible. Whether this is the best solution remains an open question, which may or may not be significant.

It is clear that the present methods could be used to obtain similar MRA for general spline spaces on the interval. Here, attention has been restricted to that space most likely to be useful in the applications. Cubic splines are smooth enough to give pleasing images, yet the computational requirements are not prohibitive.

Finally, a hybrid projection scheme is proposed, which particularly for large systems further lowers operation count. Numerical experiments which test the algorithm are performed.

4.2 Preliminary Considerations

Quak and Weyrich [23] review the axioms of Mallat [19] required of MRA for $L_2(R)$, and stipulate the requirements for MRA adapted to an interval. For MRA on $L_2[a, b]$, the sequence of function subspaces can be infinite only in the direction
corresponding to mesh refinement. There is some initial space $V_J$ with sufficient nodes to contain entirely the support of at least one inner wavelet, and a sequence of nested spaces

$$\cdots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset V_J$$

The spaces $V_J$ are related by

$$V_{J+1} = V_J \oplus W_J,$$

where $W_J$ is the orthogonal complement in $V_{J+1}$ of $V_J$. The most fundamental requirement of an MRA is that there is a scaling function $\phi(x)$ such that, for finitely many $k$, the scaling function translates satisfy

$$\phi(x) \in V_0 \Leftrightarrow \phi(2^J x - k) \in V_J,$$

and the linear span of these translates covers $V_J$.

In the sequel attention is directed to the interval $[0, 8]$. $V_0$ is the space of natural cubic splines with (at most) simple nodes at $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. Thus, $V_J$ is the space of natural cubic splines with nodes at $\{\frac{k}{2^J}: k = 0, 1, \ldots, 2^{J+3}\}$.

Generally, scaling functions inhabit an approximation space, while wavelets are members of the orthogonal complement. If $\phi(x) \in V_0$ is a scaling function, a finite number of translates, $\phi(2^J x - k)$, together with an appropriate number of boundary scaling functions, constitute a basis for $V_J$. In this research, a boundary
scaling function is simply the interval restriction of some translate of the interior scaling function.

When the $V_j$ are spaces of natural cubic splines, one can choose as interior scaling function for $V_0$ the cardinal B-spline [24] $N(x)\equiv 4[0,1,2,3,4]; (t-x)^3$. Here, $[4]_t$ is the 4th divided difference of $(t-x)^3$. The corresponding two scale relation is

$$N(x) = \sum_{k=-2}^{2} p_k N(2x - k - 2), \quad (4.4)$$

where $p_k$ is an element of $[9]$

$$P_k = \frac{1}{8}[1,4,6,4,1] \quad (4.5)$$

On the interval, $\psi(x) \in W_0$ is an inner wavelet if it possesses no support at the endpoints, and some finite collection $\psi(2^J x - k)$, together with any required boundary wavelets $\psi^B(2^J x)$ constitute a basis for $W_J$. The inner wavelet used in the sequel is the compactly supported Chui-Wang cubic B-wavelet [6]. For each space $W_J$: $J \geq 0$, three boundary wavelets are required at each end of the interval, together with $2^{J+3} - 6$ inner wavelet translates. Translates of a boundary wavelet are of no interest.
4.3 Maple Derivation of Boundary Wavelets

Boundary wavelets are derived initially for the space $W_0$. For $J = 1, 2, ..., N$, a $W_J$ wavelet is a (possibly translated) scaled version of some $W_0$ wavelet. Wavelets for the ultra-coarse spaces $\{W_J: J = -1, -2, -3\}$ all have support on at least one boundary: no inner wavelet exists. Thus, a unique wavelet derivation is required for each space. In the literature these spaces usually have been neglected [7, 23].

The space $W_0$ allows two inner wavelets, whose supports are $[0,7]$ and $[1,8]$. Wavelets for this space are designated as

$$\{\psi_{j-1}(x) : j = 1, \ldots, 8\}$$

(4.6)

according to the indices of the nodes which disappear when a $V_1$ function is projected on $V_0$, where the indices $\{j = 4, 5\}$ correspond to inner wavelets. Subsequently, scaled versions of these wavelets which are in $V_J$ will be referred to as

$$\{\psi^J_k(x) : k = 1, 3, \ldots, N - 1\}$$

(4.7)

where $N = 2^J + 3$, and the first three and last three index values refer to boundary wavelets.

The necessity for a boundary wavelet arises when it is observed that left (right) translations of $\psi_4$ ($\psi_5$) are no longer orthogonal to $V_0$. Indeed, the only requirement for constructing boundary wavelets in $W_0$ is that (a) they are orthogonal to
\( \Psi_0 \), and (b) the collection (4.5) is linearly independent and spans \( W_0 \).

Linear independence can be arranged as follows [23]: Let \( \Psi_k = \psi_{2k+1}(x) \) and choose the left boundary wavelets \( \Psi_0, \Psi_1, \Psi_2 \) to have support \([0, 4], [0, 5], [0, 6], \) whereas the right boundary wavelets \( \Psi_5, \Psi_6, \Psi_7 \) satisfy \( \Psi_{7-l}(x) = \Psi_l(8-x) \): \( l = 0, 1, 2 \).

To satisfy condition (a), the constants \( C^i_k \) in the equations

\[
\Psi_{j(x)} = \sum_{k=-3}^{2j+4} C^i_k N(2x - k) \quad j = 0, 1, 2
\]

are chosen such that \( \Psi_j \) is orthogonal to every function \( N(x - i) \) with which it shares support. For fixed \( j \), this leads to solving a homogeneous system of \( 7 + j \) equations in \( 8 + 2j \) unknowns.

Latitude in solving these systems is removed by the principle of minimal interference with interior interaction: The innermost profile of a boundary wavelet is chosen insofar as possible to conform to the corresponding profile of the Chui-Wang [6, 9] inner wavelet. This choice is made possible because \( j + 1 \) of the associated determining equations for boundary wavelet \( \Psi_j \) will be identical to those required for the inner wavelet. By substituting inner wavelet coefficients, for each \( j \) this reduces to finding the unique solution of a system involving six inhomogeneous equations in six unknowns. Set up and solution of these equations is accomplished using the Maple symbolic algebraic manipulation package.
The determining equations for the boundary wavelets are of the form $A\tilde{x}=\tilde{b}_i$, $i = 1, 2, 3$. The first boundary wavelet has support $[0,4]$, with

$$A = \begin{bmatrix} 209 & 431 & 31 & 31 & 1 & 0 \\ 80640 & 20160 & 1920 & 20160 & 80640 & 0 \\ 73 & 337 & 17387 & 247 & 559 & 31 \\ 5376 & 2240 & 80640 & 2520 & 26880 & 20160 \\ 25 & 103 & 6091 & 337 & 9241 & 247 \\ 5376 & 1344 & 26880 & 1120 & 40320 & 2520 \\ 1 & 80640 & 31 & 559 & 247 & 9241 & 337 \\ 80640 & 20160 & 26880 & 2520 & 40320 & 1120 \\ 0 & 0 & 1 & 80640 & 20160 & 26880 & 2520 \\ 0 & 0 & 0 & 0 & 1 & 80640 & 20160 \end{bmatrix}$$

$$\tilde{x}^t = \begin{bmatrix} x_1, x_2, x_3, x_4, x_5, x_6 \end{bmatrix}$$

$$\tilde{b}_1^t = \begin{bmatrix} 0, \frac{31}{812861200}, \frac{12989}{203212800}, \frac{95161}{135475200}, \frac{141719}{203212800}, \frac{50011}{812861200} \end{bmatrix}$$

The second boundary wavelet has support $[0,5]$, and

$$\tilde{b}_2^t = \begin{bmatrix} 0, \frac{247}{101606400}, \frac{818449}{203212800}, \frac{923851}{2257200}, \frac{6729679}{203212800}, \frac{141719}{203212800} \end{bmatrix}$$

The third boundary wavelet has support $[0,6]$, and

$$\tilde{b}_3^t = \begin{bmatrix} 0, \frac{337}{45158400}, \frac{890143}{67737600}, \frac{121687}{1254400}, \frac{923851}{2257200}, \frac{95161}{135475200} \end{bmatrix}$$

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The systems of six inhomogeneous equation in six unknowns were solved by using the Maple symbolic algebraic manipulation package. Solutions for the boundary wavelet coefficients which result are in Table 4.1. Of course, the coefficients for boundary wavelets associated with the other (right) end of the interval can be obtained by a principle of reflection.

It is clear that the inner coefficients of each boundary wavelet match as many as possible with the corresponding coefficients of the inner wavelet, resulting from the principle of least interference. Figures 4.1-4.4 show wavelet profiles. It may be seen that there is no marked difference in appearance between the present boundary wavelets and those of Chui-Quak [7, 23]. As a result of numerical experiments reported in the sequel, there also appears to be no difference in performance.
Table 4.1: Natural Spline Wavelets

<table>
<thead>
<tr>
<th>i</th>
<th>First Boundary Wavelet</th>
<th>Second Boundary Wavelet</th>
<th>Third Boundary Wavelet</th>
<th>Inner Wavelet</th>
</tr>
</thead>
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<td>657311485 / 196700112</td>
<td>-419725 / 24587514</td>
<td>0</td>
</tr>
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<td>-138630091 / 393400224</td>
<td>346803 / 196700112</td>
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<td>-21954047 / 196700112</td>
<td>-168239 / 49175028</td>
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<td>7917027 / 17841280</td>
<td>743093 / 17841280</td>
<td>1 / 40320</td>
</tr>
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<td>-1181603407 / 196700112</td>
<td>-385603003 / 196700112</td>
<td>-31 / 10080</td>
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<td>1202163647 / 262268160</td>
<td>3696559267 / 786804490</td>
<td>559 / 13440</td>
</tr>
<tr>
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<td>-247 / 1260</td>
<td>-337 / 560</td>
<td>-247 / 1260</td>
</tr>
<tr>
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<td>559 / 13440</td>
<td>9241 / 20160</td>
<td>9241 / 20160</td>
</tr>
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<td>5</td>
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<td>-247 / 1260</td>
<td>-337 / 560</td>
</tr>
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<td>1 / 40320</td>
<td>559 / 13440</td>
<td>9241 / 20160</td>
</tr>
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<td>0</td>
<td>-31 / 10080</td>
<td>-247 / 1260</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>1 / 40320</td>
</tr>
</tbody>
</table>
4.4 Frontal Decomposition

Let $V_j$ be the space of natural cubic splines with at most simple nodes at $\{x_k: k = 0, 1, \ldots, N\}$.

Here, $N = 2^{j+3}$, $h$ is the node spacing, and let $\{\phi^{j}_{k(x)}=N(2^{j}x - k + 4): k = -1, \ldots, N + 1\}$ be the cardinal B-spline basis for $V_j$.

If $y_j, j = 0, 1, \ldots, N$ are sampled data values from some function $f(x)$, then

$$P_h f = \sum_{k=-1}^{N+1} C_k \phi_k^j(x),$$

(4.9)

is a projection on $V_j$, provided the following requirements are satisfied:

$$C_0 = y_0,$$

(4.10)

$$C_{i-1} + 4C_i + C_{i+1} = 6y_i \quad i = 1, 2, \ldots, N - 1,$$

(4.11)

$$C_N = y_N,$$

(4.12)

together with the conditions for a natural spline fit

$$C_{-1} = 2C_0 - C_1$$

(4.13)

$$C_{N+1} = 2C_N - C_{N-1}$$

(4.14)

The most desirable feature of a multiresolution analysis is provision of means to calculate the relation between the projection $P_h f$ on the fine grid $V_j$ and its best natural spline approximation $P_{2h} f$ in the coarser space $V_{j-1}$:

$$P_h f = P_{2h} f + W_h(x),$$

(4.15)
where

\[ P_{2h}f = \sum_{k=-1}^{N} c_{2k} \phi_{k}^{j-1}(x). \]  \hspace{1cm} (4.16)

Here, even indices are used to emphasize the disappearance of odd-indexed nodes under projection, and the natural spline conditions are

\[ c_{-2} = 2c_0 - c_2 \]  \hspace{1cm} (4.17)

\[ c_{N+2} = 2c_N - c_{N-2} \]  \hspace{1cm} (4.18)

In particular, the function \( W_k(x) \) is to be expressed in terms of a scaled version applicable to \( W_{j-1} \) of the B-wavelet basis for \( W_0 \) derived in the previous section:

\[ W_k(x) = \sum_{k=1}^{N} d_k \Psi_{2k-1}^j(x), \]  \hspace{1cm} (4.19)

where the indexing associates a wavelet translate with each odd-indexed node on the fine mesh. Boundary wavelets are encountered for the first three and last three index values.

Iteration of the projection (4.20,4.21) over successively coarser grids is referred to as decomposition of the function \( f(x) \) into its wavelet components. There results

\[ P_h f = P_{2h}f + \sum_{k=0}^{j-1} W_{2^k}f(x), \]  \hspace{1cm} (4.20)

The reverse process of building up the function from knowledge of the \( c_k, d_k \) at each level is referred to as reconstruction.
For B-wavelets in $L_2(R)$, decomposition using pyramid schemes usually leads to concomitant development of a boundary layer of error near interval endpoints, caused by truncation of weight sequences [9]. The point of developing MRA for the finite interval is to avoid this difficulty. A frontal decomposition scheme for which no edge effects are present is now presented. Reconstruction proceeds as usual by a reverse pyramid scheme [9], suitably modified to account for boundaries and presence of boundary wavelets.

By the theory of approximation in Hilbert space, the best approximant in $V_{j-1}$ to a given natural spline $P_hf$ is the unique solution of the following constrained optimization problem:

$$\text{Min} \int_{x_0}^{x_N} (P_hf - P_{2hf})^2 dx.$$  \hspace{1cm} (4.21)

The solution for the even indexed variables $c_{2k}, k = 0, 1, ..., M = \frac{N}{2}$ is obtained from solving the banded system
\[ AC = F \]  

(4.22)

where

\[
A = \begin{bmatrix}
451 & 1013 & 61 & 1 & 2520 & 0 & 0 & 0 & 0 \\
630 & 2520 & 1260 & 1 & 2520 & 0 & 0 & 0 & 0 \\
1013 & 41 & 17 & 36 & 1 & 2520 & 0 & 0 & 0 \\
2520 & 45 & 840 & 1 & 2520 & 0 & 0 & 0 & 0 \\
61 & 17 & 302 & 397 & 1 & 2520 & 0 & 0 & 0 \\
1260 & 36 & 315 & 840 & 1 & 2520 & 0 & 0 & 0 \\
1 & 2520 & 1 & 21 & 397 & 840 & 1 & 21 & 2520 & 0 & 0 \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & 0 & \frac{1}{2520} & \frac{1}{21} & 397 & 840 & 302 & 397 & 1 & 21 & 2520 \\
0 & 0 & 0 & \frac{1}{2520} & \frac{1}{21} & 397 & 840 & 302 & 17 & 61 & 2520 \\
0 & 0 & 0 & 0 & \frac{1}{2520} & \frac{1}{21} & 17 & 36 & 45 & 1013 & 2520 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2520} & 61 & 1013 & 451 & 630 & 2520
\end{bmatrix}
\]

\[ C^T = \left[ c_0, c_2, c_4, \ldots, c_{N-4}, c_{N-2}, c_N \right] \]

and
\[ F = \begin{bmatrix}
(\frac{443}{960}C(0) + \frac{9239}{20160}C(1) + \frac{1019}{5040}C(2) + \frac{1679}{40320}C(3) + \frac{31}{10080}C(4)) + \frac{1}{40320}C(5)) \\
(\frac{19}{190}C(0) + \frac{3361}{9064}C(1) + \frac{1207}{2016}C(2) + \frac{18481}{40320}C(3) + \frac{247}{1260}C(4) + \frac{559}{13440}C(5)) + \frac{31}{10080}C(6) + \frac{1}{40320}C(7)) \\
(\frac{1}{320}C(0) + \frac{419}{10080}C(1) + \frac{247}{1260}C(2) + \frac{9241}{20160}C(3) + \frac{337}{560}C(4) + \frac{9241}{20160}C(5) + \frac{247}{1260}C(6)) \\
+ \frac{31}{10080}C(7) + \frac{1}{40320}C(8) \\
(\frac{1}{40320}C(1) + \frac{31}{10080}C(2) + \frac{559}{13440}C(3) + \frac{247}{1260}C(4) + \frac{9241}{20160}C(5) + \frac{337}{560}C(6)) \\
+ \frac{9241}{20160}C(7) + \frac{247}{1260}C(8) + \frac{559}{13440}C(9) + \frac{31}{10080}C(10) + \frac{1}{40320}C(11)) \\
: \\
(\frac{1}{40320}C(N-11) + \frac{31}{10080}C(N-10) + \frac{559}{13440}C(N-9) + \frac{247}{1260}C(N-8) + \frac{9241}{20160}C(N-7)) \\
+ \frac{337}{560}C(N-6) + \frac{9241}{20160}C(N-5) + \frac{247}{1260}C(N-4) + \frac{559}{13440}C(N-3) \\
+ \frac{31}{10080}C(N-2) + \frac{1}{40320}C(N-1)) \\
(\frac{1}{40320}C(N-9) + \frac{31}{10080}C(N-8) + \frac{559}{13440}C(N-7) + \frac{247}{1260}C(N-6) + \frac{9241}{20160}C(N-5)) \\
+ \frac{337}{560}C(N-4) + \frac{9241}{20160}C(N-3) + \frac{247}{1260}C(N-2) + \frac{419}{10080}C(N-1) + \frac{1}{320}C(N)) \\
(\frac{1}{40320}C(N-7) + \frac{31}{10080}C(N-6) + \frac{559}{13440}C(N-5) + \frac{247}{1260}C(N-4) + \frac{18481}{40320}C(N-3)) \\
+ \frac{1207}{2016}C(N-2) + \frac{3361}{9064}C(N-1) + \frac{19}{160}C(N)) \\
(\frac{1}{40320}C(N-5) + \frac{31}{10080}C(N-4) + \frac{1679}{40320}C(N-3) + \frac{1019}{5040}C(N-2) + \frac{9239}{20160}C(N-1)) \\
+ \frac{443}{960}C(N))
\end{bmatrix}
\]
A posteriori there can be enforced the natural spline requirements of equations 4.17 and 4.18. The decomposition is continued by solving the linear system

\[ (P_h - P_{2h})f(x_{2i-1}) = \sum_{j=1}^{N_2} d_j \Psi_j^f(x_{2i-1}) \] (4.23)

where \( P_{2h}f \) is now the best approximant, and \( i=1,2,\ldots,M = \frac{N_2}{2} \). Coarse grid values are readily calculated using the convolution

\[ (P_{2h})f(x_k) = \frac{1}{48}[c_{k-3} + 23(c_{k-1} + c_{k+1}) + c_{k+3}] \] (4.24)

The system matrix \( P \) for eqns. (4.23) has bandwidth 7, and its elements are \( P_{ij} = \Psi_j^f(x_{2i-1}) \). Table 4.2 contains the non-zero elements of columns 1-4. Columns \( j = 5, 6, \ldots, M - 4 \) have zero elements except for \( P_{j-k,j}=P_{4-k,4} \), with \( -3 \leq k \leq 3 \). The elements of the last three columns satisfy \( P_{M-k,M-j}=P_{k+1,j+1}, j = 0, \ldots, 3; k = 0, \ldots, M - 1 \).

It is to be remarked that continuance of the decomposition by employing equations 4.23 is a major point of departure from the methods of Quak-Weyrich [23]. Indeed, this step leads to a vast gain in economy of implementation, as eqns. 4.23 are of bandwidth seven, whereas the Quak-Weyrich counterpart is a system whose bandwidth is thirteen.
Table 4.2: System Matrix $P$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$P_{i,1}$</th>
<th>$P_{i,2}$</th>
<th>$P_{i,3}$</th>
<th>$P_{i,4}$</th>
</tr>
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<td>1</td>
<td>$\frac{613572069}{249777920}$</td>
<td>$\frac{-19114505}{1348800768}$</td>
<td>$\frac{6604091}{1348800768}$</td>
<td>$\frac{1}{241920}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{-29086299833}{47208026880}$</td>
<td>$\frac{-1475969333}{5901003360}$</td>
<td>$\frac{-111788779}{2360401344}$</td>
<td>$\frac{197}{40320}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1039839173}{47208026880}$</td>
<td>$\frac{-11642629}{245875140}$</td>
<td>$\frac{-1302921391}{5245336320}$</td>
<td>$\frac{-1273}{26880}$</td>
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<tr>
<td>4</td>
<td>$\frac{1}{241920}$</td>
<td>$\frac{197}{40320}$</td>
<td>$\frac{-1273}{26880}$</td>
<td>$\frac{-15023}{60480}$</td>
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<tr>
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<td>$0$</td>
<td>$0$</td>
<td>$\frac{1}{241920}$</td>
<td>$\frac{197}{40320}$</td>
</tr>
<tr>
<td>7</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$\frac{1}{241920}$</td>
</tr>
</tbody>
</table>
4.5 Reconstruction by Dynamic Moving Averages

The reconstruction problem, which is the inverse of decomposition, requires an algorithm which combines the coefficients $c^l_k$, $d^l_k$ obtained from solving equations (4.22) and (4.23), to regain the coefficients \{\(c^l_l: l = -1, 0, 1, \ldots, N+1\}\} of equation (4.9). Viewing equation (4.15), there will be individual contributions

\[
C_l^j = a_l^j + b_l^j
\]  

(4.25)

due to the separate basis function expansions for \(P_{2h}f\) and \(W_hf\). It is an easy exercise to show that \(P_{2h}f\) contributes as follows:

\[
a_{2k}^j = \sum_{j=-1}^{1} p_{-2j} c_{k+j}^l, \quad k = 0, 1, \ldots, \frac{N}{2} \]  

(4.26)

\[
a_{2k+1}^j = \sum_{j=0}^{1} p_{-2j} c_{k+j}^l, \quad k = 0, 1, \ldots, \frac{N}{2} - 1 \]  

(4.27)

Here \(c_{l-1}^l = c_{2l}\), where \(c_{2l}\) comes from solving equation (4.16), and the \(P_l\) are defined by eqns. (4.4) and (4.5).

Likewise, the contributions from \(W_h(x)\) are given by:

\[
b_{2k}^j = \sum_{j=-2}^{3} q_{-2j}^{k+j} d_{k+j}^l \]  

(4.28)

and

\[
b_{2k+1}^j = \sum_{j=-3}^{3} q_{-2j}^{k+j+1} d_{k+j+1}^l \]  

(4.29)
Here, \( d_l^{j-1} = d_l \) results from solving eqn. (4.23), and \( q_k^l \) is a coefficient from the two-scale relation for wavelet \( \psi_l \) (see Table 4.1), where, for an inner wavelet, the zero index value for \( k \) refers to the center of symmetry. For a boundary wavelet the center of indexing is, again, the 6th nonzero coefficient, but moving from the interior towards the boundary with which it is associated. (It is noted that the so-called third boundary wavelet has six non-zero coefficients to one side of the center of indexing, and five to the other side; whereas, an inner wavelet has five coefficients to either side.)

**Interpretation as a moving average**

By the well-known method of up-sampling (insertion of zeros to get the vector \( \{c_{-1,0}, c_0, 0, c_1, 0, c_2, \ldots, 0, c_N, 0, c_{N+1}, 0\} \)) the two formulas of eqns. (4.26,4.27) can be combined into the moving average scheme

\[
    a_l^j = \sum_{j=-2}^{2} p_{j-j} c_{l+j}
\]

Likewise, by upsampling the \( \{d_k^{j-1}\} \), the two equations (4.28,4.29) can be represented by the single dynamic moving average

\[
    b_l^j = \sum_{j=-6}^{6} q^{l+j}_{j} d_{l+j}^{j-1}
\]

Both equations (4.30, 4.31) require index values \( l=0,1,2,\ldots,N \) with a posteriori calculation (the natural spline condition) of

\[
    c_{l-1}^j = 2c_0^j - c_1^j
\]

56
and

\[ c_{N+1}^{j} = 2c_{N}^{j} - c_{N-1}^{j} \]  \hspace{1cm} (4.33)

In these equations, \( d_{i} = 0 \) if \( i < 1 \) or \( i > N - 1 \). Except for third boundary wavelet(s), \( g_{\pm0} = 0 \).

### 4.6 Algorithm Efficiency

In reference [23] Quak and Weyrich describe two kinds of decomposition algorithms for implementing the Chui-Quak [7] spline wavelets, employing boundary wavelets characterized by multiple nodes at interval endpoints. The first method is encumbered by use of dual functions; thus, a second (improved) decomposition technique which avoids dual functions is formulated [23].

It is believed that the decomposition algorithm formulated by the present authors is more efficient than the improved version. The improved algorithm [23] involves solution of two banded systems of linear equations: the first, a counterpart of system (4.22) has the same bandwidth \( 2m - 1 \) (=7, for splines of order \( m = 4 \)), and comparable manipulation is involved in system setup (see the right-member of eqns.(4.22)). However, their second banded system, a counterpart of system (4.23), has bandwidth \( 4m - 3 = 13 \), comparing unfavorable with the bandwidth of (4.23), which again is seven. Even after considering the flops generated by system (4.24), the present scheme clearly has a significant increase in efficiency.
However, in comparison with the present dynamic moving average scheme their reconstruction algorithm appears somewhat comparable.

4.7 Numerical Experiments

As significant computer aided algebra has been accomplished in deriving the present decomposition algorithm, the first experiment is aimed at verification of accuracy. For cubic splines having two continuous derivatives, one should be able to detect discontinuities in a function and its first or second derivatives by performing a wavelet decomposition. Further, use of boundary wavelets and the frontal decomposition technique should eliminate the boundary errors which arise when countable infinite weight sequences are truncated, in adapting wavelet methods for $L_2(R)$ to the interval.

For the edge detection problem, we use the test functions $f_1(x)$ and $f_2(x)$

$$f_1(x) = \begin{cases} 
\frac{1}{36}(2x-1)(4(2x-1)^2 + 32x - 3) & \text{if } 0 \leq x \leq \frac{1}{2} \\
\frac{1}{6}(2x-1)(2x-2)(2x-3) & \text{if } \frac{1}{2} < x \leq 1
\end{cases}$$

$$f_2(x) = \begin{cases} 
\frac{1}{1+(2x-1)^2} & \text{if } 0 \leq x \leq \frac{1}{2} \\
\frac{2}{\sqrt{\pi}}e^x - 2x & \text{if } \frac{1}{2} < x \leq 1
\end{cases}$$

interpolated on 128 points, with grid spacing $2^{-7}$. Figure 4.5 shows the first function and its wavelet part (a plot of the wavelet coefficients); Figure 4.6 exhibits
corresponding results for the second test function. Although the derivative discontinuity at the center of the interval is not visible in plotting a test function, its detection by observance of wavelet coefficients (see Figures 4.5-4.6) is well-localized and accurately space centered, even though the wavelet coefficients are of small magnitude. Furthermore, the usual boundary layer of error at end points, which is associated with weight sequence truncation [9], is seen to be absent. But, this is the point of using boundary wavelets; hence, the major criteria for success of the present method clearly has been achieved.

**Decomposition Using a Hybrid Scheme**

The only fault of adapting to finite intervals the truncated weight sequences described in [9] (for decomposition on the infinite interval) is an accumulation of error near boundaries. For large systems the work of solving eqns.(4.22) can be lessened by employing a hybrid method: The implicit eqns.(4.22) are solved near the ends of the interval, whereas the usual explicit weight sequencing is used on the interior.

Figure 4.7 shows the results of an experiment where the hybrid projection scheme is employed on a damped sine wave. Again, it is to be noted that near boundary error appears absent, in a case where its presence is a certainty if only truncated weight sequences are used.
4.8 Signal Separation Using the Frontal Method

In this section the results of a numerical signal separation experiment are reported. The base signal

\[ S(x) = \begin{cases} 
20\sin(x) & \text{if } 0 \leq x \leq \frac{\pi}{2} \\
-\frac{104}{11\pi}x + \frac{272}{11} & \text{if } \frac{\pi}{2} \leq x \leq \frac{15\pi}{8} \\
\frac{224}{23\pi}x - \frac{280}{23} & \text{if elsewhere}
\end{cases} \]  

(4.34)

is corrupted by addition of a high frequency oscillation

\[ g(x) = 5\sin(20\pi x) \]  

(4.35)

and the result is sampled at intervals of \( \frac{\pi}{64} \), far under the Nyquist rate for the high frequency oscillation. Thus, the sampled mixed signal \( S + g \) contains little information concerning the true nature of \( g(x) \).

The base signal is now separated by projecting the sampled signal on the space \( V_1 \) of piecewise cubic functions appropriate to the sampling interval and calculating a coarser projection on \( V_0 \) by the frontal method. Figure 4.8 simultaneously shows the mixed signal and the piecewise cubic extracted signal, which exhibits only negligible difference from the original signal \( S(x) \). This is due to the fact that the aliased signal always oscillates in sign from one grid point to the next. Thus, it is essentially contained in \( W_0 \), and is almost perfectly separated by decomposition.

However, it is remarked that this is simply an illustrative example of frontal
decomposition, and does not necessarily represent advocacy of a signal separation technique.

4.9 Chapter Summary

A Maple derivation of new boundary wavelets which accompany the cubic inner spline wavelet of Chui-Quak [7] has been presented, together with a frontal technique for decomposition over a finite interval of functions contained in the MRA generated by a piecewise cubic B-spline local basis. The technique avoids the use of pyramid schemes and the necessity of truncating weight sequences of infinite length, which procedure would appear to lower the accuracy of the B-wavelet. Numerical experiments indicate excellent results for the frontal method of decomposition, as regards both accuracy and efficiency of the algorithm. The problem of boundary error is well disposed of by the present scheme, where boundary scaling functions with multiple nodes [7, 12] at interval endpoints have been avoided by use of more natural scaling functions. Moreover, this approach apparently gives results which are equivalent to those of [23], with improved algorithm efficiency.
Figure 4.1: First Boundary Wavelet
Figure 4.2: Second Boundary Wavelet

Supp[0,5]
Third Boundary Wavelet

Supp[0,6]

Figure 4.3:
Third Boundary Wavelet
Figure 4.4: Inner Wavelet

Supp[0,7]
First Test function

Wavelet Coefficients (j=7)

Figure 4.5:
First Test Function and Wavelet Coefficients
Figure 4.6: Second Test Function and Wavelet Coefficients

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Figure 4.7: Damped Sine Wave and Hybrid Projection

Original Signal \( y = 2^{-x} \sin(4x) \)
Figure 4.8: Mixed Signals and Extracted Signal

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Chapter 5

Soft Threshold De-Noising with Wavelet Coefficients

5.1 Introduction

A question frequently posed by the user community regards knowing the circumstances under which one type of wavelet may or may not be superior to another. The question is explored here in the context of an application of wavelets to data smoothing called thresholding in the wavelet transform domain. The cubic wavelet structure developed in Chapter 4 is now applied to data smoothing in the thresholding context, to determine whether semi-orthogonal wavelets can compete with the successes of orthogonal wavelets in this area.

Donoho and Johnstone [13]-[15] have proposed a very simple procedure for recovering functions from noisy data, which attempts to reject noise by thresholding in the wavelet transform domain. When the wavelet transform utilizes an
orthogonal wavelet, the data smoothing which results by thresholding is claimed to be very good [14].

The objective in this chapter is to report some numerical experiments where the thresholding technique is attempted, for the case in which the wavelet transform utilizes the semi-orthogonal cubic B-spline wavelet and the interval MRA algorithm developed in Chapter 4. The best that can be said is that some degree of data smoothing is obtained; however, the level of variance reduction is insufficient for acclaim to be credited the method. Evidently the property of an orthogonal transform of the data, strongly utilized in theoretical studies of Donoho’s method, is crucial to significant variance reduction.

5.2 An Orthogonal Wavelet

We now consider how well a given noisy data can be approximated by linear combinations of Haar wavelets which are defined by (5.1).

\[ \psi_{m,n}(x) = 2^{-\frac{m}{2}} \psi(2^{-m}x - n) \]  

(5.1)

The \( \psi_{m,n}(x) \) constitute an orthonormal basis for \( L^2(R) \). Here, the Haar function,
\( \psi(x) \), is given by

\[
\psi(x) = \begin{cases} 
1 & \text{if } 0 < x < \frac{1}{2} \\
-1 & \text{if } \frac{1}{2} < x < 1 \\
0 & \text{if otherwise.}
\end{cases}
\]  
(5.2)

The Haar basis has been known since Haar (1910). We will use it here for numerical comparison of data smoothing results from the semi-orthogonal case. Any \( f(x) \in L^2(R) \) can be arbitrarily well approximated by a function with compact support which is piecewise constant on the interval \([n,n+1]\), where \( n=0,1,\ldots,i-1 \). Let us denote such a piecewise constant approximation by \( f^1(x) \). We now represent \( f^1(x) \) as a sum of two pieces, \( f^1(x) = f^0(x) + d^0(x) \), where \( f^0(x) \) is an approximation to \( f^1(x) \) which is piecewise constant over intervals twice as large as previously. The values \( f^0(x) \) are computed by the averages of the two corresponding constant values of \( f^1(x) \) (over adjacent intervals),

\[
f^0_0 = \frac{1}{2}(f^1_0 + f^1_1).
\]  
(5.3)

The function \( d^0(x) \) is piecewise constant with the same stepwidth as \( f^0(x) \) such that

\[
d^0_0 = \frac{1}{2}(f^1_0 - f^1_1)
\]  
(5.4)

and

\[
d^0_1 = \frac{1}{2}(f^1_1 - f^1_0).
\]  
(5.5)
We have therefore written the projection scheme as

\[ f_{2k}^j = \frac{1}{2}(f_{2k}^{j-1} + f_{2k+1}^{j-1}) \]  \hspace{1cm} (5.6)

and

\[ d_{2k}^j = \frac{1}{2}(f_{2k}^{j-1} - f_{2k+1}^{j-1}) \hspace{1cm} d_{2k+1}^j = -d_{2k}^j. \]  \hspace{1cm} (5.7)

It is clear that this is a Hilbert space projection scheme since \( d^0(x) \) and \( f^0(x) \) are orthogonal in the \( L^2(R) \) norm. The equations (5.6) and (5.7) have used a "multiresolution" approach: we have written successive coarser and coarser approximations to \( f(x) \), where at every step the difference between the approximation with resolution \( 2^j \), and the next coarser level, with resolution \( 2^{j-1} \), can be written as a linear combination of the \( \psi_{j,k}(x) \), with coefficients \( d_{2k}^j \).

The reconstruction scheme, which is the reverse of decomposition, is expressed as follows:

\[ f_{2k}^j = (f_{2k}^{j-1} + d_{2k}^{j-1}) \]  \hspace{1cm} (5.8)

and

\[ f_{2k+1}^j = (f_{2k}^{j-1} - d_{2k}^{j-1}). \]  \hspace{1cm} (5.9)

### 5.3 De-Noising by Thresholding

Suppose it is desired to recover an unknown function \( f(t_i) \) defined on \([a, b]\)
from noisy data

\[ d_i = f(t_i) + \sigma Z_i, \quad i=0, 1, 2, \ldots, n-1 \]  \hspace{1cm} (5.10)

where \( t_i = \frac{i}{n} \)

\( Z_i \sim N(0,1) \) Gaussian White Noise

\( \sigma \) Noise Level

Here \( \sigma > 0 \) is a specified noise level and \( Z_i \) is a nuisance term known only to satisfy

\[ |Z_i| \leq 1 \quad \forall i \in I \]

It is to be supposed that the nuisance term is chosen by a clever opponent to de-evaluate performance such that worst case possible error is experienced:

\[ \sup_{|Z_i| \leq 1} \| f_n^*(t_i) - f(t_i) \|_h^2 \]

One interpretation of the term "De-Noising" is that one's goal is to optimize the mean-squared error

\[ E[\| f_n^*(t_i) - f(t_i) \|_h^2] \]

\[ = \sum_{i=0}^{n-1} E[ f_n^*(\frac{i}{n}) - f(\frac{i}{n})]^2 \]

The de-noising process, which is not necessarily optimal, is as follows: First, the interval-adapted cubic B-spline wavelet and natural spline MRA of Chapter 4 are
employed to “wavelet transform” the data by decomposition over several levels of increasing coarseness. Next, the resulting wavelet coefficients are translated towards zero, by applying the soft threshold nonlinearity

\[ \eta_t(y) = sgn(y) |y| - t \]

where

\[ sgn(y) = \begin{cases} 1 & \text{if } y > 0 \\ 0 & \text{if } y = 0 \\ -1 & \text{if } y < 0 \end{cases} \] (5.11)

\[ (|y| - t)_+ = \begin{cases} |y| - t & \text{if } |y| > t \\ 0 & \text{elsewhere} \end{cases} \] (5.12)

coordinate-wise to the wavelets coefficients, with specially chosen values for the threshold \( t \). Upon reversing the pyramid filtering process (by reconstruction), there is recovered the smoothed values \( f_n(x, i) = 0, 1, 2, \ldots, n-1 \).

### 5.4 Numerical Experiments

The outcome of some numerical experiments which implement the de-noising technique are now reported. A graduated sequence of \( t \)-values are employed for the natural B-spline wavelets and for the Haar wavelet as seen in Table 5.2 and 5.4.
The corrupting signal is white (Gaussian) noise from a random number generator, which is added to a sinusoidal input sampled over the interval $[0, 2\pi]$. Numerical results are shown below. Table 5.1 and Table 5.3 show the smoothing accomplished by the natural B-spline wavelets and the Haar function simply projecting from the fine grid to a coarser grid level, with no thresholding. Table 5.2 and Table 5.4 show the effects of thresholding several coarse levels after reconstructing back to the fine level. There is a progression in quality of the resulting data smoothing as the threshold varies, as may be seen from Table 5.2 and Table 5.4. Indeed, as the threshold approaches $t=3.0$, it is seen in Table 5.2 that the standard deviation converges to 0.178734. The corresponding best converged value as $t$ approaches 0.05 is approximately 0.1154 when using the Haar wavelet.

Figures 5.1-5.3 depict projected data, wavelet coefficients, and reconstructed data over several levels. From the appearance of the wavelet transform coefficients, one might suspect that the random noise generator is producing correlated noise in the vicinity of $x = 5.0$.

It is seen that Donoho and Johnstone's soft thresholding method in conjunction with the semi-orthogonal wavelet lead to numerical experiments whose resulting smoothing recovery from an unknown noisy data is, at best, about the 15 % level in variance reduction. Indeed, as a result of simply projecting to the fourth level ($j=4$) of coarseness, better smoothing results than is obtainable by thresholding,
as may be seen from Table 5.1.

**Smoothing a Corrupted Sine Wave:**

\[ d_i = f(t_i) + \sigma Z_i, \quad i=0,1,2,\ldots,n-1 \]

where \( t_i = \frac{i}{n} \)

\( Z_i \sim N(0,1) \) Gaussian White Noise

\( \sigma = 1. \)

\( f(t_i) = \sin(t_i) \)

Wavelet Transform \( \bar{d} = \) A Soft Thresholded Wavelet Coefficient

\[ \text{Variance} \quad \sum_i (\bar{d}_i - f(t_i))^2 / n \]

\[ \text{R.M.S. Deviation} \quad \sqrt{\text{Variance}} \]

\[ \text{R.E.} \quad \frac{\|f(x) - \tilde{f}(x)\|}{\|f(x)\|} \]
Table 5.1: Smoothing by Projection Only

<table>
<thead>
<tr>
<th>Level</th>
<th>R.M.S Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>j=8</td>
<td>0.30634297538607</td>
</tr>
<tr>
<td>j=7</td>
<td>0.37763354502562</td>
</tr>
<tr>
<td>j=6</td>
<td>0.29557825406323</td>
</tr>
<tr>
<td>j=5</td>
<td>0.30936344208145</td>
</tr>
<tr>
<td>j=4</td>
<td>0.16003431015278</td>
</tr>
<tr>
<td>j=3</td>
<td>0.13060963413918</td>
</tr>
</tbody>
</table>

Table 5.2: Reconstruction with Threshold(t)

<table>
<thead>
<tr>
<th>Multi-level Reconstruction Over 5 Levels</th>
</tr>
</thead>
<tbody>
<tr>
<td>Threshold</td>
</tr>
<tr>
<td>-------------</td>
</tr>
<tr>
<td>t=0.5</td>
</tr>
<tr>
<td>t=1</td>
</tr>
<tr>
<td>t=1.5</td>
</tr>
<tr>
<td>t=2</td>
</tr>
<tr>
<td>t=3</td>
</tr>
</tbody>
</table>

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Table 5.3: Smoothing by Projection (Haar) Only

<table>
<thead>
<tr>
<th>Level</th>
<th>R.M.S Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>j=8</td>
<td>0.30634297538607</td>
</tr>
<tr>
<td>j=7</td>
<td>0.87280445385675</td>
</tr>
<tr>
<td>j=6</td>
<td>0.24936540189022</td>
</tr>
<tr>
<td>j=5</td>
<td>0.27797327452273</td>
</tr>
<tr>
<td>j=4</td>
<td>0.31691320650028</td>
</tr>
<tr>
<td>j=3</td>
<td>0.28169075518644</td>
</tr>
</tbody>
</table>

Table 5.4: Reconstruction (Haar) with Threshold (t)

<table>
<thead>
<tr>
<th>Multi-level Reconstruction Over 5 Levels</th>
</tr>
</thead>
<tbody>
<tr>
<td>Threshold</td>
</tr>
<tr>
<td>-----------</td>
</tr>
<tr>
<td>t=0.01</td>
</tr>
<tr>
<td>t=0.035</td>
</tr>
<tr>
<td>t=0.05</td>
</tr>
<tr>
<td>t=0.075</td>
</tr>
<tr>
<td>t=.1</td>
</tr>
<tr>
<td>t=.5</td>
</tr>
</tbody>
</table>

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Figure 5.1: Multi-level Decomposition with Noisy Signal
Figure 5.2: Signal Reconstruction ($|y| > 1$)
Figure 5.3: Wavelet Coefficients (|y| > 1)

Wavelet Coefficients (t = 1)

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Figure 5.4: Multi-level Decomposition (Haar) with Noisy Signal
Figure 5.5: Signal Reconstruction ($t = 0.05$)
Wavelet (Haar) Coefficients (|y| > 0.05)

Figure 5.6: Wavelet (Haar) Coefficients (t = 0.05)
Chapter 6

Data Compression

6.1 Introduction

In the last few years the advent of multimedia computing has initiated a revolution in research concerning image compression. Here, storage and manipulation of image data in raw form can be very expensive. For example, a standard 35mm photograph digitized at 12 μm per pixel requires about 18 MBytes of storage, and one second of NTSC-quality color video requires about 23 MBytes of storage[18]. High definition television(HDTV) is another area where sheer volume of data which must be transported and stored(in real time) boggles the imagination.

It is clear that in order to take advantage of what is becoming cutting edge technology in these areas, some form of data compression is necessary. Indeed, the introduction of the wavelet concept has spurred a revolution in the field of data compression.
In this chapter there is investigated the quality of data compression that can be obtained from lossy compression schemes which employ the interval wavelets derived in chapter 4. Basically, there are two different kinds of schemes for compression: lossless and lossy. In the case of lossless compression, one is interested in reconstructing the data exactly, without loss of information, such as with Huffman Coding[27]. Lossy compression is considered here, where error is permitted, as long as image quality after reconstruction is acceptable. In general, much higher compression ratios can be obtained with lossy schemes than those characterized by lossless compression.

Loss or distortion introduced in reconstruction shall be defined as the R.M.S. or \( l^2 \) norm of the difference between \( f(x) \) and its approximation \( \hat{f}(x) \). For simplicity of comparison, this shall be normalized to the relative difference \( \frac{||f(x)-\hat{f}(x)||}{||f(x)||} \).

6.2 Numerical Experiments

Signals which are of interest in data compression application are usually characterized by a time-varying frequency. One such signal is \( y = \sin(t^2) \), which is to be used for numerical experiments. Interest is focused on a data window of length \( N=2^n \), where \( n=10 \). Using the techniques of chapter four involving multi-resolution analysis on the interval, this data will be decomposed to a level of coarseness \( n=3 \), with quantile thresholding of the wavelet coefficients.
Since distortions up to 5% are considered acceptable in some speech processing applications, an optimal compression method for a deterministic signal is one that yields no more than 5% distortion while maximizing the compression ratio. Though the quality of the restored signal may be criticized in some cases, the data compression results from wavelet signal analysis shown in the Table [6.1-6.2] are, perhaps, amazing. For best case compression, to approximately reconstruct the original 1024 word signal requires only 24 data non-zero values to be stored. This is a compression ratio of about 40 to 1.

The rule for thresholding is given as

\[
    d_j^{\text{quant}} = \begin{cases} 
        0 & \text{if } d_j < \epsilon \\
        d_j & \text{if } d_j \geq \epsilon.
    \end{cases}
\]

Results for reconstruction when the value \( \epsilon = .01 \) is used are shown in Figures 6.2 and 6.3. When reconstructing to the original signal, R.E. and R.M.S. errors converge to \( 6.4 \times 10^{-3} \) and \( 4.5 \times 10^{-3} \), as shown in the Table [6.1-6.2].
Table 6.1: Data Cutoff by Compression on 1024 Points

<table>
<thead>
<tr>
<th>Level</th>
<th>Cutoff($\epsilon = 0.001$)</th>
<th>Cutoff($\epsilon = 0.01$)</th>
<th>Cutoff($\epsilon = 0.1$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>j=9</td>
<td>308</td>
<td>414</td>
<td>512</td>
</tr>
<tr>
<td>j=8</td>
<td>146</td>
<td>174</td>
<td>234</td>
</tr>
<tr>
<td>j=7</td>
<td>66</td>
<td>76</td>
<td>96</td>
</tr>
<tr>
<td>j=6</td>
<td>31</td>
<td>34</td>
<td>37</td>
</tr>
<tr>
<td>j=5</td>
<td>16</td>
<td>16</td>
<td>19</td>
</tr>
<tr>
<td>j=4</td>
<td>9</td>
<td>9</td>
<td>15</td>
</tr>
<tr>
<td>j=3</td>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

Remaining Data: 438, 291, 99

R.M.S.: $4.11 \times 10^{-3}$, $4.07 \times 10^{-3}$, $4.45 \times 10^{-3}$

R.E.: $5.91 \times 10^{-3}$, $5.86 \times 10^{-3}$, $6.41 \times 10^{-3}$
Table 6.2: Data Cutoff by Compression on 1024 Points

<table>
<thead>
<tr>
<th>Level</th>
<th>$Cutoff(\epsilon = 1)$</th>
<th>$Cutoff(\epsilon = 2)$</th>
<th>$Cutoff(\epsilon = 5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$j=9$</td>
<td>512</td>
<td>512</td>
<td>512</td>
</tr>
<tr>
<td>$j=8$</td>
<td>256</td>
<td>256</td>
<td>256</td>
</tr>
<tr>
<td>$j=7$</td>
<td>119</td>
<td>127</td>
<td>128</td>
</tr>
<tr>
<td>$j=6$</td>
<td>48</td>
<td>49</td>
<td>53</td>
</tr>
<tr>
<td>$j=5$</td>
<td>24</td>
<td>27</td>
<td>27</td>
</tr>
<tr>
<td>$j=4$</td>
<td>15</td>
<td>15</td>
<td>16</td>
</tr>
<tr>
<td>$j=3$</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>Remaining Data</td>
<td>36</td>
<td>23</td>
<td>16</td>
</tr>
<tr>
<td>R.M.S.</td>
<td>$4.45 \times 10^{-3}$</td>
<td>$4.45 \times 10^{-3}$</td>
<td>$4.45 \times 10^{-3}$</td>
</tr>
<tr>
<td>R.E.</td>
<td>$6.41 \times 10^{-3}$</td>
<td>$6.41 \times 10^{-3}$</td>
<td>$6.41 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

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Signal Decomposition

Figure 6.1: Multi-level Decomposition $y = \sin(x^2)$
Figure 6.2: Signal Reconstruction (\( \epsilon = 0.1 \))
Figure 6.3: Wavelet Coefficients (\(\epsilon = 0.1\))

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Chapter 7

Summary and Conclusions

1. Piecewise linear and piecewise cubic spline MRA for the finite interval have been developed. These MRA have the merit that the usual edge effects associated with image decomposition are eliminated.

2. Numerical experiments show that edge detection is readily accomplished, by the observance of wavelet coefficients for the cubic spline MRA.

3. Piece-wise Linear B-spline boundary wavelets have been developed, by means of the computer aided algebraic manipulation package Maple.

4. Piecewise Cubic B-spline boundary wavelets have been developed in a unique way through use of a principle of least interference between boundary and interior wavelets.

5. A hybrid scheme is developed, which enhances economy of operation count over the method of decomposition pioneered by the Chui and Wang [6].
particular, economy of operation count, without going to the hybrid schemes, has been improved, by avoidance of the dual wavelet. Further economy arises from an approach whereby the solving a linear system of bandwidth seven replaces the solving of a linear system of bandwidth thirteen.

6. The reconstruction process is simplified, by means of a dynamic moving average scheme.

7. Data smoothing by means of thresholding in the wavelet transform domain is not as successful for semi-orthogonal wavelets as it has been advertised for orthogonal wavelets.

8. On the other hand, very good results are achieved for a data compression scheme which employs these interval wavelets. Compression ratios of 40 to 1 are experienced.

7.1 Directions of Future Research

The boundary wavelets developed for the cubic spline MRA are restricted in coarseness by a certain threshold level, below which the discrete mesh does not afford enough points to support an inner wavelet. To go below this level, a special set of wavelets, all of which have support on the boundaries, would have to be developed. Going in the open-ended direction, it would be interesting to see how
two-dimensional problems might be handled with a tensor product formulation of the B-spline wavelets. It is in two dimensional image processing that the real need for MRA with finite domain arises.
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Appendix A

Natural Spline Inner Products

Consider the cubic B-spline(s) on a fine grid of mesh width $h$ possessing points $x_{-1}, x_0, x_1, \ldots, x_N, x_{N+1}$

$$\phi^h(x) = \begin{cases} \frac{4}{3} + 2x + x^2 + \frac{1}{6}x^3 & \text{if } -2 \leq x \leq -1 \\ \frac{2}{3} - x^2 - \frac{1}{2}x^3 & \text{if } -1 \leq x \leq 0 \\ \frac{2}{3} - x^2 + \frac{1}{2}x^3 & \text{if } 0 \leq x \leq 1 \\ \frac{4}{3} - 2x + x^2 - \frac{1}{6}x^3 & \text{if } 1 \leq x \leq 2 \end{cases} \quad (8.1)$$

and on the next coarser grid

$$\phi^{2h}(x) = \begin{cases} \frac{4}{3} + x + \frac{1}{4}x^2 + \frac{1}{48}x^3 & \text{if } -4 \leq x \leq -2 \\ \frac{2}{3} - \frac{1}{4}x^2 - \frac{1}{18}x^3 & \text{if } -2 \leq x \leq 0 \\ \frac{2}{3} - \frac{1}{4}x^2 + \frac{1}{18}x^3 & \text{if } 0 \leq x \leq 2 \\ \frac{4}{3} - x + \frac{1}{4}x^2 - \frac{1}{48}x^3 & \text{if } 2 \leq x \leq 4 \end{cases} \quad (8.2)$$
Table 8.1: Coarse-Coarse Natural Spline Inner Products

<table>
<thead>
<tr>
<th>Product</th>
<th>Value</th>
<th>Product</th>
<th>Value</th>
<th>Product</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>((\phi_{-2}^{2h},\phi_{-2}^{2h}))</td>
<td>(\frac{1}{126})</td>
<td>((\phi_{0}^{2h},\phi_{0}^{2h}))</td>
<td>(\frac{151}{315})</td>
<td>((\phi_{-2}^{2h},\phi_{2}^{2h}))</td>
<td>(\frac{43}{840})</td>
</tr>
<tr>
<td>((\phi_{0}^{2h},\phi_{2}^{2h}))</td>
<td>(\frac{59}{140})</td>
<td>((\phi_{0}^{2h},\phi_{4}^{2h}))</td>
<td>(\frac{1}{21})</td>
<td>((\phi_{2}^{2h},\phi_{2}^{2h}))</td>
<td>(\frac{1}{2520})</td>
</tr>
<tr>
<td>((\phi_{2}^{2h},\phi_{4}^{2h}))</td>
<td>(\frac{397}{840})</td>
<td>((\phi_{2}^{2h},\phi_{6}^{2h}))</td>
<td>(\frac{1}{12})</td>
<td>((\phi_{4}^{2h},\phi_{4}^{2h}))</td>
<td>(\frac{1}{2520})</td>
</tr>
<tr>
<td>((\phi_{4}^{2h},\phi_{6}^{2h}))</td>
<td>(\frac{397}{840})</td>
<td>((\phi_{4}^{2h},\phi_{8}^{2h}))</td>
<td>(\frac{1}{21})</td>
<td>((\phi_{6}^{2h},\phi_{6}^{2h}))</td>
<td>(\frac{1}{2520})</td>
</tr>
<tr>
<td>((\phi_{6}^{2h},\phi_{8}^{2h}))</td>
<td>(\frac{302}{315})</td>
<td>((\phi_{6}^{2h},\phi_{10}^{2h}))</td>
<td>(\frac{1}{12})</td>
<td>((\phi_{8}^{2h},\phi_{8}^{2h}))</td>
<td>(\frac{1}{2520})</td>
</tr>
<tr>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
</tr>
</tbody>
</table>

Together with the translates

\[
\phi_j^h = \phi_j^h(x - jh) \quad j = -1, 0, 1, 2, \ldots, N+1
\]

\[
\phi_k^{2h} = \phi_k^{2h}(x - 2kh) \quad k = -1, 0, 1, 2, \ldots, \frac{N}{2}+1
\]

In this research, use consistently is made of the B-spline inner products presented in the Tables 8.1 and 8.2.
### Table 8.2: Fine-Coarse Natural Spline Inner Products

<table>
<thead>
<tr>
<th>Inner Products</th>
<th>Value</th>
<th>Inner Products</th>
<th>Value</th>
<th>Inner Products</th>
<th>Value</th>
<th>Inner Products</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\phi_{-2}^h, \phi_{-1}^h)$</td>
<td>$\frac{209}{40320}$</td>
<td>$(\phi_{0}^h, \phi_{-1}^h)$</td>
<td>$\frac{73}{2688}$</td>
<td>$(\phi_{2}^h, \phi_{-1}^h)$</td>
<td>$\frac{25}{2688}$</td>
<td>$(\phi_{4}^h, \phi_{-1}^h)$</td>
<td>$\frac{1}{40320}$</td>
</tr>
<tr>
<td>$(\phi_{-2}^h, \phi_{0}^h)$</td>
<td>$\frac{431}{10080}$</td>
<td>$(\phi_{0}^h, \phi_{0}^h)$</td>
<td>$\frac{337}{1120}$</td>
<td>$(\phi_{2}^h, \phi_{0}^h)$</td>
<td>$\frac{103}{672}$</td>
<td>$(\phi_{4}^h, \phi_{0}^h)$</td>
<td>$\frac{31}{10080}$</td>
</tr>
<tr>
<td>$(\phi_{-2}^h, \phi_{1}^h)$</td>
<td>$\frac{31}{960}$</td>
<td>$(\phi_{0}^h, \phi_{1}^h)$</td>
<td>$\frac{17387}{40320}$</td>
<td>$(\phi_{2}^h, \phi_{1}^h)$</td>
<td>$\frac{6091}{13440}$</td>
<td>$(\phi_{4}^h, \phi_{1}^h)$</td>
<td>$\frac{559}{13440}$</td>
</tr>
<tr>
<td>$(\phi_{-2}^h, \phi_{2}^h)$</td>
<td>$\frac{31}{10080}$</td>
<td>$(\phi_{0}^h, \phi_{2}^h)$</td>
<td>$\frac{247}{1296}$</td>
<td>$(\phi_{2}^h, \phi_{2}^h)$</td>
<td>$\frac{337}{560}$</td>
<td>$(\phi_{4}^h, \phi_{2}^h)$</td>
<td>$\frac{247}{1296}$</td>
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<tr>
<td>$(\phi_{-2}^h, \phi_{3}^h)$</td>
<td>$\frac{1}{40320}$</td>
<td>$(\phi_{0}^h, \phi_{3}^h)$</td>
<td>$\frac{559}{13440}$</td>
<td>$(\phi_{2}^h, \phi_{3}^h)$</td>
<td>$\frac{9241}{20160}$</td>
<td>$(\phi_{4}^h, \phi_{3}^h)$</td>
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</tr>
<tr>
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<td>$(\phi_{0}^h, \phi_{4}^h)$</td>
<td>$\frac{31}{10080}$</td>
<td>$(\phi_{2}^h, \phi_{4}^h)$</td>
<td>$\frac{247}{1296}$</td>
<td>$(\phi_{4}^h, \phi_{4}^h)$</td>
<td>$\frac{337}{560}$</td>
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<tr>
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<tr>
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<td>$0$</td>
<td>$(\phi_{4}^h, \phi_{9}^h)$</td>
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</table>

Use of symmetry principles allows extrapolation of the above inner products to the other end of the interval (N+2,N,N-2, etc). In each table, all inner products in successive columns to the right vanish.
Autobiographical Statement

The author was born on the November 8, 1961 in Seoul, Korea. In December 1988, he received from Christopher Newport University the joint degrees of Bachelor of Science in Mathematics and Bachelor of Science in Applied Physics (with specialty in Micro-Electronics).

From January 1989 to May 1995 he attended Old Dominion University. There he received a Master of Science degree in Computational and Applied Mathematics in May 1991, and a Doctor of Philosophy degree in Computational and Applied Mathematics in May 1995.