Summer 1991

On Shock Capturing for Liquid and Gas Media

Tze Jang Chen

Old Dominion University

Follow this and additional works at: https://digitalcommons.odu.edu/mathstat_etds

Part of the Applied Mechanics Commons, Fluid Dynamics Commons, and the Mathematics Commons

Recommended Citation


https://digitalcommons.odu.edu/mathstat_etds/91

This Dissertation is brought to you for free and open access by the Mathematics & Statistics at ODU Digital Commons. It has been accepted for inclusion in Mathematics & Statistics Theses & Dissertations by an authorized administrator of ODU Digital Commons. For more information, please contact digitalcommons@odu.edu.
On Shock Capturing for Liquid and Gas Media

by

Tze Jang Chen

B.S., National Tsing Hua University, 1979
M.S., National Tsing Hua University, 1983

A Dissertation Submitted to the Faculty of Old Dominion University
in Partial Fulfillment of the Requirements for the Degree of

DOCTOR OF PHILOSOPHY
in
COMPUTATIONAL AND APPLIED MATHEMATICS

OLD DOMINION UNIVERSITY
August 1991

Approved by:

Charlie H. Cooke (Director)
Abstract

On Shock Capturing for Liquid and Gas Media

Tze Jang Chen
Old Dominion University, 1991

Director: Dr. Charlie H. Cooke

The numerical investigation of shock phenomena in gas or liquid media where a specifying relation for internal energy is absent poses special problems. Classically, for gas dynamics the usual procedure is to employ a splitting scheme to remove the source terms from the Euler equations, then up-wind biased shock capturing algorithms are built around the Riemann problem for the system which remains. However, in the case where the Euler equations are formulated in the term of total enthalpy, a technical difficulty associated with equation splitting forces a pressure time derivative to be treated as a source term. This makes it necessary to pose the central Riemann problem for a system where one equation is not a conservation law. In the present research, it is established that successful upwind-biased shock capturing schemes can be applied to the pseudo-conservative system. A shock capturing method developed for this purpose is applied to solving the Riemann problem for pure water with general (UNESCO [8]) equation of state.
A second objective of this research is the development of front tracking methods which possess sub-grid resolution capability. One envisions here continuous tracking of the front, as opposed to discrete (one grid point to the next) tracking, such as is provided by the random choice method. A front tracking scheme employing near-front cells which continuously evolve with time is developed. This scheme is applied to the problem of tracking a material interface in the underwater explosion problem. The Riemann problem for water and for a gas-water interface is analyzed, as a vehicle for applying the Godunov method to shock phenomena in liquid media. A by-product of this study is a mapping theorem which establishes a certain correspondence between solutions of the Riemann problem(s) for the ideal gas and the ideal water (modified-Tait equation of state). As a result, present codes which solve the Riemann problem for gases can readily be adapted for use with water.
To my wife, Shang-Tao

and my parents
Acknowledgements

I would like to first thank my advisor Charlie Cooke, who, without a doubt epitomizes the phrase: "a gentleman and a scholar." His understanding, patience, encouragement, and genuine care will not be soon forgotten. I also would like to thank Dr. John Heinbockel, Dr. Sid Roberts and Dr. Richard Noren, for helpful suggestions and comments.

I wish to express to the faculty and graduate students of the Mathematics Department my appreciation for their longstanding support. Especially, I wish to thank Barbara Jeffrey and Gale Tuckelson for assistance in preparing this manuscript.

I would like to thank Leon Arriola for his kind help and suggestions.

Finally, I want to thank my wife, my brother, and my parents for standing by me through these long and tedious years.
Contents

1 Preliminaries 1

1.1 Introduction ......................................................... 1
1.2 Basic Literature Survey ......................................... 4
1.3 Statement of Purpose ............................................. 9

2 A Shock Capturing Method for Pure Water with General Equation of State 12

2.1 Introduction ......................................................... 12
2.2 Governing Equations ............................................. 14
2.3 Outline of the Method ............................................. 15
2.4 Constituent Equations for Pure Water ...................... 17
2.5 Piecewise Linear Approximation of Fluxes ................ 20

3 Conservation Law/Nonconservation Law Form and Averaging 23

3.1 Averaging for Pseudo-Conservation Law Systems .......... 23
3.2 Eigenvalues and Eigenvectors for the Jacobian Matrix of Water Dynamics in the Pseudo-Conservation Law Form ........................................... 29

3.3 Eigenvalues and Eigenvectors for the Jacobian Matrix of Gas Dynamics in the Conservation Law and Pseudo-Conservation Law Form .................................................................................. 32

4 Computational Experiments .......................................................... 39
    4.1 Computational Experiments .................................................. 39

5 Moving Boundary Problems ......................................................... 53
    5.1 A Local Front Tracking Algorithm with Subcell Resolution .... 53
    5.2 Numerical Experiments ...................................................... 56

6 Solution of the Riemann Problem .................................................. 65
    6.1 The Gas-Water Riemann Problem ...................................... 67
    6.2 Contact Surface Relations .................................................. 70
    6.3 Solution of the Riemann Problem for Water ....................... 82
    6.4 A Water-Adapted Gas Dynamics Riemann Solver ............... 86
    6.5 Shocks in Water ............................................................ 89

7 An Underwater Explosion Problem ............................................. 94
    7.1 Introduction ........................................................................ 94
    7.2 Test Problem: Underwater Release of a Sphere of Gas at High Pressure .................................................................................. 96

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
7.3 The Godunov Method .................................................... 98
7.4 Numerical Results ....................................................... 99

8 Summary and Conclusions ................................................. 111

Bibliography ....................................................................... 113
List of Figures

4.1 Effects of Iterating Source Terms (Pressure) .................. 43
4.2 Effects of Iterating Source Terms (Velocity) .................. 44
4.3 Comparison of Results from the Various Averaging Schemes (Density) ......................................................... 45
4.4 Comparison of Results from the Various Averaging Schemes (Velocity) ............................................................... 46
4.5 Results for Modified-Tait Versus UNESCO Equation of State (Pressure) ............................................................... 47
4.6 Results for Modified-Tait Versus UNESCO Equation of State (Density) ................................................................. 48
4.7 Results for Modified-Tait Versus UNESCO Equation of State (Velocity) ............................................................... 49
4.8 Results for Modified-Tait Versus UNESCO Equation of State (Total- Enthalpy) ...................................................... 50
4.9 Water Adapted Roe Scheme Versus the Scheme of Harten and Godunov (Pressure) ............................................... 51
4.10 Water Adapted Roe Scheme Versus the Scheme of Harten and
   Godunov (Velocity) ......................................................... 52

5.1 Variable Cell Sizes Produced by Contact Surface Motion .... 59
5.2 Expanding Cell Method and Flow Behind a Moving Piston (I) 60
5.3 Expanding Cell Method and Flow Behind a Moving Piston (II) 61
5.4 Grid-Size Sensitivity of the Contracting Cell Method ........ 62
5.5 Density Discontinuity Produced by a Variable Speed Piston . 63
5.6 Velocity Discontinuity Produced by a Variable Speed Piston . 64

6.1 The Gas-Water Riemann Problem .................................... 66

7.1 Data Oscillation in Pressure Characteristic of a Hybrid (RCM)
   Front Tracking Scheme ...................................................... 101
7.2 Data Oscillation in Density Characteristic of a Hybrid (RCM)
   Front Tracking Scheme ...................................................... 102
7.3 Comparison of Results for an Exact Riemann Solver for Water
   and an Approximate Riemann Solver (Density) ................. 103
7.4 Comparison of Results for an Exact Riemann Solver for Water
   and an Approximate Riemann Solver (Pressure) ................. 104
7.5 Comparison of Results for an Exact Riemann Solver for Water
   and an Approximate Riemann Solver (Velocity) ................. 105
7.6 Comparison of Results for an Exact Riemann Solver for Water
   and an Approximate Riemann Solver (Mach Number) .......... 106
7.7 Comparison of the Hybrid Method of Charrier-Tessiaras and the Sub-cell Resolution Scheme (Density) .................................................... 107
7.8 Comparison of the Hybrid Method of Charrier-Tessiaras and the Sub-cell Resolution Scheme (Pressure) ................................................ 108
7.9 Grid Size Effects for the Sub-cell Resolution Scheme ......................... 109
7.10 Manifestation of Time Decay in Shock Strength ............................. 110
Chapter 1

Preliminaries

1.1 Introduction

The investigation of shock phenomena in gas dynamic flows by numerical means has reached an advanced state of the art. Methods for both shock fitting and shock capturing are well understood. However, for liquid media such a fortunate state generally does not exist.

In gas dynamics, simple equations of state (for polytropic gases) and expressions for both internal energy and enthalpy provide leeway in choosing a preferred thermodynamic variable which leads itself to most economical problem solution. However, in liquid media such as pure water or sea water, expressions for internal energy simply are not known. Therefore, enthalpy must be chosen as the basic thermodynamic variable, when the most general equation of state
is employed. This leads to special problems, should one set out to adapt gas dynamic shock capturing techniques to flows involving liquid media.

Basically, in building shock capturing schemes for gas dynamics applications, the central Riemann problem involves the Euler equation in full conservation law form. However, due to an unavoidable technical difficulty concerning equation splitting, with the enthalpy formulation one of the equations which describe the central Riemann problem for liquid media is not a conservation law. Thus, the complete dynamics of shock interaction are not fully incorporated in the Riemann solver for the pseudo-conservation equations.

One objective of the present research is to see whether such problems can be overcome. It is shown that the method of Roe [1] can be adapted to the analysis of shock wave propagation in water, provided an appropriate averaging scheme for pseudo-conservation form equations is formulated. Of course, in order for this averaging scheme to be cognitive of shock presence, the mechanics of the Rankine-Hugoniot jump condition somehow must be incorporated in the calculation.

A second area of investigation concerns the tracking of a moving shock or contact surface in a fluid or liquid media, where the objective is to obtain sub-cell resolution of the tracked front. An example where this phenomena is of interest arises where a spherical charge of radius $R_0$ is detonated underwater. A simplified model, which neglects charge fragmentation, concerns the assumption
that when the ignited charge burns out, the by-products result in a gas bubble at high temperature and pressure. The resulting rapid expansion of the gas results in a shock front which propagates radially outward, forced along by a contact surface which represents the interface between explosion gas and the liquid medium. Numerical simulation of this shock phenomena mandates an accurate tracking of the contact surface, as diverse constitutive relations apply to either side. It is intended to develop a front tracking scheme possessing sub-cell resolution capability, which can be implemented with a shock capturing algorithm in order to numerically investigate the underwater explosion problem.

One means of the tracking the contact surface consists of neglecting source terms, essentially replacing the less-tractable spherical Riemann problem which governs motion of the gas-liquid interface with a Cartesian Riemann problem, whose solution can be analyzed by similarity methods. Of course, for small time the solution of the reduced problem is expected to well-approximate that of the original. One contribution of the present research is to formulate the local describing contact Riemann problem in such a manner that the difficulties are reduced to the solving of a single non-linear equation, which can be feasibly accomplished using the technique of interval bisection, with no problems in obtaining convergence.

The solution of the local contact Riemann problem provides only the velocity and fluid state at the contact surface, valid over some short time interval $\Delta t$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
A time integration then produces contact surface position at the end of the interval. Thus, information is provided which allows numerical shock capturing schemes to operate, separately, over the gas and liquid computational domains. A uniform grid is used away from the contact surface. However, the two cells adjacent to the contact surface are allowed to expand (or contract, as the case may be) to account for contact surface motion. An algorithm for updating these two cells can be patterned after that used by Cooke and Ilwang [2] for the transitional ballistics problem.

1.2 Basic Literature Survey

The Shock Capturing Problem for Liquid Media (Water)

No cases are known where shock capturing methods for water with general equation of state have been investigated. One might expect this is due to the fact that methods of solving the Riemann problem for this general case are yet to be discovered. However, in corresponding circumstances which involve gas dynamics, Roe [1] has developed a piecewise linearization technique wherein an approximating linear Riemann problem can be solved exactly. The original nonlinear Riemann problem is that of solving

$$\frac{\partial U}{\partial t} + A(U) \frac{\partial U}{\partial x} = 0,$$
Chapter 1: Preliminaries

\[ U(x, 0) = \begin{cases} U_L, & z < x_i \\ U_R, & z > x_i \end{cases} \]  \hspace{1cm} (1.2.1)

where \( U \) is an \( n \times 1 \) vector, \( x_i \) is a mesh location, and \( A(U) \) is the \( n \times n \) Jacobian matrix of flux function \( F(U) \). For gas dynamics, exact solutions predicted by similarity methods are obtainable, but at the expense of iteratively solving an associated single non-linear equation [3].

In the interests of computational efficiency, Roe [1] suggests replacing (1.2.1) with a piecewise linearized version, where over small intervals a constant matrix \( A(\hat{U}_i) \) replaces \( A(U) \). To incorporate the Rankine-Hugoniot jump condition for shocks into the piecewise linearized problem, the property

\[ F(U_R) - F(U_L) = A(\hat{U})(U_R - U_L), \]  \hspace{1cm} (1.2.2)

dictates the choice of \( \hat{U} \). If the states \( U_R, U_L \) are separated by a shock of speed \( S \), the Rankine-Hugoniot condition becomes

\[ F(U_R) - F(U_L) = S(U_R - U_L) = A(\hat{U})(U_R - U_L). \]  \hspace{1cm} (1.2.3)

Thus, if a shock is present, its speed will be an eigenvalue of \( A(\hat{U}) \). When a shock capturing method is now built around the piecewise linearization (1.2.2), the shock speed will be correctly represented.
Chapter 1: Preliminaries

The correct \( \hat{U} \) which satisfies (1.2.2) is referred to as an averaged value, as it typically involves some sort of averaging of \( U_R, U_L \). The original averaging scheme developed by Roe [1] is referred to as Roe-averaging. Glaister [4] studies the averaging problem with a slightly different formulation, forcing (1.2.2) only in an approximate sense.

Of course, the proper averaging scheme depends upon what primitive variables are chosen in formulating the Euler equations (2.2.1). For the liquid medium studied here, enthalpy becomes the preferred thermodynamic variable; hence, the Riemann problem corresponding to (2.2.1) is not posed fully in conservation law form, as is the case where the Roe and Glaister-averaging are developed. Thus, one purpose of the present research is to see how such difficulty can be overcome. In so doing, the proper averaging scheme is derived. Comparison with Glaister’s work and some numerical experimentation leads to the surprising result that Glaister’s averaging is more generally applicable than its original derivation warrants: it serves as well as for water as does the newly developed averaging.

The Underwater Explosion Problem

Calculation of the shock wave propagation induced by an expanding gas bubble by numerical means has been carried out previously. However, published results appear unsatisfactory, as will now be indicated.

In a study by Flores and Holt [5], the random choice method (RCM) of
Glimm [6] is used, both locally to track the gas-water contact surface, and globally, as a shock capturing method, separately in gas and in water. Although the analytical portions of [5] appear sound, the numerical results must be discredited, for two reasons:

(a) Either a scaling error has occurred in the plotting of the velocity profiles, or else some coding error has destroyed the accuracy of the resulting calculations. This conclusion is based upon the fact that, if data from the plots (Holt’s Figures 2-4) is used to calculate (at time 0.05) the flow mach number on the gas side, and as the gas/water contact surface is approached from the left, a supersonic value is obtained. However, this is physically impossible; only a rarefaction wave appears in this region, connecting a zero velocity state to a supposedly supersonic flow state (with no shock in evidence).

(b) Flores and Holt’s Figure 5, which depicts that contact surface velocity approaches zero at time 0.2, actually contradicts the data of Holt’s Figure 4, from which it can be inferred that the contact surface velocity, initially increasing, does not reverse this trend, during the interval [0.0,0.2]. Calculations from the present research indicate that this trend reversal occurs much later.

On the other hand, The Flores-Holt [5] derivation of equations for solving the Riemann problem at the gas-water interface (which determines the contact surface speed) appears sound. However, mention is made that in certain circumstances binomial expansions may be necessary, in the face of a non-convergent
iteration of the several nonlinear equations involved. Thus, one contribution of the present research is to reformulate the gas-water contact surface problem such that its solution can be reduced to the iteration of a single nonlinear equation, whose solution by the method of interval bisection experiences no problems with convergence. Another technique for tracking the gas-water contact surface is developed, which shall be referred to as the method of sub-cell resolution (MSR).

The underwater explosion problem is also attempted in a paper of Charrier and Tessieras [6]. Here is described a hybrid scheme which employs the RCM method for contact front tracking, whereas the Lax-Wendroff two step method is used to either side of the front. The method appears unsatisfactory, from the point of view that severe oscillation is experienced in the vicinity of both the shock front and the contact surface (see Figures 7.1, 7.2). However, contact surface location appears accurate, as will be seen when results are compared with those of the present research, where no oscillations are experienced.

It is to be remarked that in both efforts [5,6] the contact surface location is resolved only to gridpoint accuracy. How fast or how slow the contact surface moves between grid points is taken into account only obliquely, in solving the contact Riemann problem. For cases where contact surface velocity is small compared to shock propagation velocity, it would appear that significant accumulation of error is possible.
Therefore, in the present research interest will be directed to sub-cell resolution (continuous tracking) of the front. A method of Cooke-Hwang [2] developed for the transitional ballistics problem will be adapted to the continuous tracking of the contact surface. Knowledge of when the contact front passes a grid location will be used deterministically rather than stochastically as in the Random Choice Method.

1.3 Statement of Purpose

The purposes of this dissertation are two-fold: First, it is intended to develop a first-order accurate shock capturing scheme which can be used to investigate shock phenomena which occur in liquid media (water) when the most general (UNESCO) [7] equation of state is required. This will be accomplished through adaptation of Roe's shock capturing scheme from gas dynamics [1], where the appropriateness of averaging techniques will be considered.

Secondly, for the underwater explosion problem, a method for continuously tracking the gas-water contact surface is to be developed. Thus, sub-grid resolution of front position is involved. Moreover, for obtaining the front velocity, a method for reducing solution of the local Riemann problem at the gas-water contact surface to the iterative solution of a single nonlinear equation is developed. This iteration is accomplished, feasibly, with the method of interval bisection; no convergence problems are experienced. In tandem with front tracking, the
shock capturing method of Godunov [8] is employed, to numerically simulate the underwater explosion problem.

In Chapter 2, a shock capturing method for (pure) water with general equation of state is outlined.

In Chapter 3, the averaging problem associated with adaptation of Roe’s scheme [1] to liquid media and pseudo-conservation law equations is considered. There is also presented a derivation of the eigenvalues and eigenvectors for the Jacobian matrix of the pseudo-conservation law equations associated with the enthalpy form of the Euler equations. Both the UNESCO and modified-Tait [10] equation of state for water are considered. The eigenvalues and eigenvectors in both conservation law and pseudo-conservation law form for gas dynamics are also derived.

In Chapter 4 some computational experiments are presented, which are intended to assess effectiveness of the shock capturing scheme developed in Chapters 2, 3. The Riemann problem for the hydrodynamic shock tube is solved, using both the UNESCO and modified-Tait [10] equations of state for water. Comparison of results is given, for cases where the Chen and Cooke-, Roe [1]-, and Glaister [4]-averaging are used. It is found experimentally that the Glaister-averaging actually has a wider applicability than its original derivation warrants: being as useful for liquid media as for gas dynamics.

Determination of the proper averaging scheme now opens the way to synthe-
sis of second-order accurate shock capturing schemes for water. Some tentative results when the TVD scheme of Harten [11] is used with the modified-Tait equation of state for water are also listed.

In Chapter 5 there is a developed a front tracking algorithm for the gas-water contact interface problem, which possesses sub-cell resolution capability. This scheme is patterned after that of Cooke and Hwang [2] for tracking a moving projectile in the transitional ballistics problem. The algorithm assumes the interface moves at constant velocity during a short time interval, with the contact surface velocity and pressure externally supplied.

The problem of handling the externally supplied interface velocity is undertaken in Chapter 6. The Riemann problem at the gas-water contact surface is formulated in such a way that its solution is reduced to solving numerically by interval bisection a single nonlinear equation. The researches of Smoller [12] concerning the Riemann problem in polytropic gas dynamics are extremely useful in this effort.

Finally, in the Chapter 7 the results of Chapters 5, 6 are used to develop a shock capturing scheme which is applied to the underwater explosion problem previously discussed. Comparison with previously known results is presented.
Chapter 2

A Shock Capturing Method for Pure Water with General Equation of State

2.1 Introduction

Previous numerical studies of underwater shock phenomena, for the most part, have employed the simplified Tait equation of state [10]. With this equation of state, Holl [13] has established that, to good approximation, a functional form equivalence exists between the shock relations for the ideal gas and water. Thus, the Riemann problems for the two media can be solved, for the most part, using similar methods [5]. In this research, the problem of numerically calculating shock wave propagation in liquid media with general equation of state is considered. This gives rise to the difficulty that techniques for exact solution of the Riemann problem may be unknown.

A further complicating factor is the assumption (true particularly for water) that a relation specifying the dependence of internal energy upon the variables of
state is unknown. However, when a relation between enthalpy and the variables of state is available, an enthalpy formulation for the Euler equations of compressible flow can be useful. Unfortunately, there appears in the energy equation a term involving the time derivative of pressure, which most conveniently can be treated as a source term. When source terms are removed by splitting, the resulting equations do not all represent conservation laws, although a divergence form is exhibited.

Due to the absence of energy conservation, it is clear that the pseudo-system does not fully describe shock dynamics. Still, it can be used as a base upon which to build effective shock capturing algorithms. Demonstration of this fact is the major contribution of the present research. It is shown that the piecewise linearization technique of Roe [1] can be employed. However, the averaging scheme which aids piecewise linearization must be carefully formulated. Although our aims and methods of accomplishing this are similar to those of Glaister [4], the results are applicable under entirely different and unexpected circumstances.

For quality assurance, a well-known problem for the ideal gas is investigated. Next, the shock capturing method is applied to the Riemann problem for pure water, with UNESCO [8] equation of state. As it has been suggested [10] that in some circumstances the modified-Tait equation of state for sea water can approximate the state of pure water, some comparison of results between the two equations of state is given.
2.2 Governing Equations

The one-dimensional equations of motion for an inviscid, non-heat-conducting, flow of a gas or liquid, where specific enthalpy, \( h \), rather than specific internal energy, \( e \), is the preferred thermodynamic variable, can be written in the form

\[
U_t + F(U)_R + W(U) = 0, \tag{2.2.1}
\]

Here

\[
U = \begin{bmatrix}
\rho \\
\rho u \\
\rho H
\end{bmatrix}, \tag{2.2.2}
\]

with

\[
F = \begin{bmatrix}
\rho u \\
P + \rho u^2 \\
\rho Hu
\end{bmatrix}, \tag{2.2.3}
\]

and

\[
W = \begin{bmatrix}
-(\alpha - 1) \frac{e u}{R} \\
-(\alpha - 1) \frac{e u^2}{R} \\
-(\alpha - 1) \frac{e h u}{R} - P_t
\end{bmatrix}. \tag{2.2.4}
\]

The unknowns are \( P \), pressure; \( \rho \), density; \( u \), velocity; \( h \), specific enthalpy; and \( H \), specific total enthalpy \((H = h + u^2/2)\). The independent variables are
Chapter 2: Shock Capturing for Water with General Equation of State

$t$, time, and $x$, streamwise distance. The values 0, 1, 2 for the parameter $\alpha$ yield, respectively, the Euler equations in a Cartesian, Cylindrical, or Spherical Coordinate system.

2.3 Outline of the Method

The form (2.2.1-2.2.4) for the Euler equations is rarely employed when studying gas dynamic or other flow problems where a specifying relation for internal energy is known. However, in circumstances where the constituent equations involve enthalpy rather than internal energy, this form is necessary. In the present research, the problem of adapting shock capturing techniques which have been useful in gas dynamics to the numerical calculation of shocked flows involving pure water, with general equation of state, is attempted. Constituent relations $h = h(T)$ and $P = P(\rho, T)$, though complicated, exist; however, expressions for internal energy are unknown.

Here, the usual approach is followed: the source term in equation (2.2.1) is removed through Sod's operator splitting technique [14], and effective shock capturing methods are determined for the equation

$$U_t + F(U)_x = 0. \quad (2.3.5)$$

In the second stage of the splitting, pressure effects are incorporated by solving...
Chapter 2: Shock Capturing for Water with General Equation of State

\( U_t + W(U) = 0, \quad (2.3.6) \)

The enthalpy form (2.2.1-2.2.4) leads to some special problems in development of shock capturing methods. Success of the classical approach relies upon the fact that in many cases (2.3.5) still represents a set of conservation laws, upon which thirty years of research have focused. In the present case, removal of the pressure time derivative by splitting results in one equation which does not represent a conservation law. Thus, (2.3.5) shall be called pseudo-conservative. Recomposition during the second stage of the splitting appears to compensate for this malefaction. However, two criteria are essential: First, shock recognition must be effectively accomplished when building an upwind-biased shock capturing method on the basis of the pseudo-conservative system (2.3.5). Second, for best results the solving of (2.3.6) requires iteration. Fortunately, this iteration involves only a scalar equation, and requires few repetitions.

As regards the first criteria, in adapting Roe's approach [1], determination of an appropriate averaging scheme is fundamental to success of the method. This will be accomplished in Chapter 3 and 4. In turn, if the correct averaging scheme has been formulated, second-order methods such as Harten's TVD scheme [11] can be attempted. Use of the classical Godunov method [9] is inhibited by lack of a Riemann solver for general equations of state.
2.4 Constituent Equations for Pure Water

In this paper there is used the UNESCO equation of state, which applies in the temperature range -2 to 40°C and the pressure range 0 to 1000 bars. (Actually, salinity $S$ is set to zero in the international equation of state for seawater, whose details are elaborated in [8]). For higher ranges of pressure, the modified-Tait equation of state [10] has been established as a means to predict properties of seawater, and in certain circumstances has been shown to be a good fit to the state of pure water [7].

Modified-Tait Equation of State

The modified-Tait equation of state has been used [10] for predicting the hydrodynamic properties of seawater at the front of a propagating shockwave. This equation has the form

$$P = B[(\rho/\rho_0)^N - 1]. \quad (2.4.7)$$

Here $B = B(T)$ is essentially constant, and $\rho_0$ is the density where $P = P_0$ (usually, $P_0 = 0$). For pure water, the values $B = 2959$ bars (2922 atm) and $N = 7.415$ give the correct speed of sound [14] (4865 ft/sec) at 20°C and 1 atm.

The speed of sound is given by

$$a^2 = \frac{N(P + B)}{\rho}. \quad (2.4.8)$$
UNESCO Equation of State

The UNESCO equation of state for pure water is

\[ \rho(T, P) = \rho(T, 0) / \left[ 1 - P / K(T, P) \right]. \]  \hspace{1cm} (2.4.9)

Here, the secant bulk modulus (bars) is

\[ K(T, P) = K(T, 0) + A_w P + B_w P^2, \]  \hspace{1cm} (2.4.10)

where \( \rho(T, 0) = \rho_w \) and \( K(T, 0) = K_w \). Also,

\[ \rho_w = a_0 + a_1 T + a_2 T^2 + a_3 T^3 + a_4 T^4 + a_5 T^5, \]  \hspace{1cm} (2.4.11)

where

\[ a_0 = 999.842594, \]
\[ a_1 = 6.793952 \times 10^{-2}, \]
\[ a_2 = -9.095290 \times 10^{-3}, \]
\[ a_3 = 1.001685 \times 10^{-4}, \]
\[ a_4 = -1.120083 \times 10^{-6}, \]
\[ a_5 = 6.536332 \times 10^{-9}, \]

and

\[ K_w = e_0 + e_1 T + e_2 T^2 + e_3 T^3 + e_4 T^4, \]  \hspace{1cm} (2.4.12)
where

\[ e_0 = 19652.21, \]
\[ e_1 = 148.4206, \]
\[ e_2 = -2.327105, \]
\[ e_3 = 1.360477 \times 10^{-2}, \]
\[ e_4 = -5.155288 \times 10^{-5}. \]

Further,

\[ A_w = h_0 + h_1 T + h_2 T^2 + h_3 T^3, \] (2.4.13)

where

\[ h_0 = 3.239908, \]
\[ h_1 = 1.43713 \times 10^{-3}, \]
\[ h_2 = 1.16092 \times 10^{-4}, \]
\[ h_3 = -5.77905 \times 10^{-7}, \]

and

\[ B_w = k_0 + k_1 T + k_2 T^2, \] (2.4.14)

where

\[ k_0 = 8.50935 \times 10^{-5}, \]
\[ k_1 = -6.12293 \times 10^{-6}, \]
\[ k_2 = 5.27870 \times 10^{-8}. \]
Finally, if \( T \) is absolute temperature and \( h \) is kcal/mole, then

\[
h = 11.2T + 0.003585T^2, \quad (2.4.15)
\]

and the speed of sound can be written

\[
a(T) = 1498 - 2.4(T - 25), \quad (2.4.16)
\]

where the units now are: \( T(°C) \) and \( a \) (m/sec).

### 2.5 Piecewise Linear Approximation of Fluxes

In this section the averaging problem which determines an approximate Riemann solver for equations (2.3.5) is considered. Chapter 4 contains a derivation of averaging equations for the pseudo-conservative system. A similar course of reasoning to that used by Roe [1] and Glaister [4] is employed, and the resulting Chen-Cooke averaging scheme is somewhat familiar. However, numerical results are presented in the sequel which give new perspective, by establishing that the averaging schemes of Roe-Glaister, unexpectedly, also have applicability to the splitting represented by the equations (2.3.5, 2.3.6).

Let \( A(U) \) be the Jacobian matrix of the flux function (2.2.3), and let \( x_k; k = 1,M \) be a uniform partition of a computational domain for solutions of (2.2.1-2.2.4), with \( \Delta x = x_{j+1} - x_j \), and \( \Delta_{j+1/2} U = U_{j+1} - U_j \). Piecewise linearization as a technique for solving (2.3.5) leads to a sequence of linear Riemann problems.
Chapter 2: Shock Capturing for Water with General Equation of State

\[ U_t + A(\bar{U})U_x = 0, U(x,t_n) = \begin{cases} U_j, & x < x_{j+1/2}; \\ U_{j+1}, & x > x_{j+1/2} \end{cases} \]  
(2.5.17)

where \( \bar{U} = f(U_j, U_{j+1}) = f(U_L, U_R) \) must be appropriately chosen.

With \( F \) the flux function (2.2.3), the Rankine-Hugoniot jump condition for nonlinear system (2.3.5) has the form

\[ \Delta F = S\Delta U. \]  
(2.5.18)

In expanded form, and if (2.3.5) were a linear system, the Rankine-Hugoniot jump condition becomes

\[ F_R - F_L = A(U_R - U_L) = S(U_R - U_L) \]  
(2.5.19)

where \( S \) is the speed of a discontinuity connecting the states \( U_L, U_R \).

The fundamental premise of Roe's method is that for nonlinear systems the relation (2.5.19) can be forced, by the proper choice of \( \bar{U} \), where \( A(\bar{U}) \) replaces \( A \). (Glaister [4] chooses to force (2.5.19) only to first order accuracy.) Thus, solving (2.5.17) exactly provides a good approximation \( F_{j+1/2}^* \) to the flux on the cell interface \( x_{j+1/2} \). Moreover, if there is a shock between \( U_L, U_R \), its presence will be recognized: As the shock speed will be an eigenvalue of \( \bar{A} \), solutions of (2.5.17) will exhibit a discontinuity whose speed is that of the shock.

The numerical solution of equation (2.3.5) now proceeds via the upwind-biased shock capturing scheme

\[ U_j^{n+1} = U_j^n + \frac{\Delta t}{\Delta x}(F_{j+1/2}^* - F_{j-1/2}^*) \]  
(2.5.20)

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Here, one expression for the flux $F^*$ is

$$F^*_{j+1/2} = (F(U_j) + F(U_{j+1}) - \sum_i \alpha_i |\lambda_k| R_k)/2,$$

(2.5.21)

where the eigenvalues $\lambda$ and eigenvectors $R$ are evaluated on the average value $\tilde{U}_{j+1/2}$. The constants $\alpha$ result from the usual eigenvector projection. The proper average value to use is derived in the next chapter.
Chapter 3

Conservation Law/Nonconservation Law Form and Averaging

3.1 Averaging for Pseudo-Conservation Law Systems

Glaister [4] has analyzed the averaging problem as a technique for piecewise linearization, where (2.3.5) is a fully conservative system, and where both the perfect gas law and a general equation of state are considered. Perhaps surprisingly, he arrives at identical expressions for the averages $\bar{U}$, although expressions for eigenvalues and eigenvectors are radically different. Hence, the averaging appears to be uninfluenced by the particular expression used for pressure. Thus, in view of the extremely complicated general equation of state for water, the approach to be used here is the following: An averaging scheme for the pseudo-conservation law system (2.3.5) is indicated, for the case of a perfect gas. Thereafter, this averaging can be employed in solving (2.3.5, 2.3.6), regardless of the particular equation of state. Where possible, assessment of the
accuracy of results is made, which appears to indicate the acceptability of the choice.

Consider the pseudo-system (2.3.5), where the primitive variables and flux function are

\[ U = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \rho \\ \rho u \\ \rho H \end{bmatrix}, \quad (3.1.1) \]

and

\[ F = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} \rho u \\ \rho u^2 + P \\ \rho H u \end{bmatrix}. \quad (3.1.2) \]

The Case of a Perfect Gas

Let \( A \) be the Jacobian matrix of system (2.3.5). If the equation of state for a perfect gas is now used, the eigenvalues for \( A \) are given by

\[ \lambda_1 = u, \quad (3.1.3) \]

\[ \lambda_{2,3} = \frac{\gamma - 1}{2\gamma} u \pm \sqrt{\Delta/4}, \quad (3.1.4) \]

where

\[ \Delta = \frac{\gamma - 1}{\gamma} \left( 4H - 2u^2 + \frac{\gamma - 1}{\gamma} u^2 \right). \quad (3.1.5) \]
The corresponding eigenvectors \( e_1, e_2, e_3 \) are defined by the columns of the matrix

\[
R = \begin{bmatrix}
1 & 1 & 1 \\
u & \lambda_2 & \lambda_3 \\
u^2/2 & H & H
\end{bmatrix}.
\]  

(For convenience, in what follows, the \((\sim)\) notation will be suppressed, except in certain instances.) If \( U_L \) and \( U_R \) are two fluid states, their difference can be projected on the eigenvectors of \( \hat{A} \):

\[
\Delta U = U_R - U_L = \Sigma^3 \alpha_i e_i,
\]

where \( \alpha_i = L_i \Delta U \), with \( L_i, i = 1, 2, 3 \) the rows of \( R^{-1} \). There results

\[
\Delta \rho = \alpha_1 + \alpha_2 + \alpha_3,
\]

with

\[
\Delta (\rho u) = \alpha_1 u + \alpha_2 \lambda_2 + \alpha_3 \lambda_3,
\]

and

\[
\Delta (\rho H) = \alpha_1 u^2 + \alpha_2 H + \alpha_3 H.
\]

Forcing \( \Delta F = A(\bar{U}) \Delta U \) requires
Chapter 3: Conservation Law/Nonconservation Law Form and Averaging

\[ \Delta(\rho u) = \alpha_1\lambda_1 + \alpha_2\lambda_2 + \alpha_3\lambda_3, \quad (3.1.11) \]

together with

\[ \Delta(\rho u^2 + P) = \alpha_1\lambda_1^2 + \alpha_2\lambda_2^2 + \alpha_3\lambda_3^2, \quad (3.1.12) \]

and

\[ \Delta(\rho IIu) = \alpha_1\lambda_1 u^2/2 + \alpha_2\lambda_2 II + \alpha_3\lambda_3 II. \quad (3.1.13) \]

From Glaister [4], the assumption that are close to the average state \( U \) allows a linearization process to be employed:

\[ \Delta(\rho u) = \bar{\rho} \Delta u + \bar{u} \Delta \rho, \quad (3.1.14) \]

and

\[ \Delta(\rho u^2) = \bar{u}^2 \Delta \rho + 2\bar{u} \bar{\rho} \Delta u. \quad (3.1.15) \]

From (3.1.14, 3.1.15), one can obtain

\[ \bar{u}^2 \Delta \rho - 2\bar{u} \Delta(\rho u) + \Delta(\rho u^2) = 0. \quad (3.1.16) \]

This implies that

\[ \bar{u} = \frac{\Delta(\rho u) - \sqrt{(\Delta \rho u)^2 - \Delta \rho \Delta(\rho u^2)}}{\Delta \rho}. \quad (3.1.17) \]
From eqns. (3.1.14, 3.1.17), it follows that
\[ \bar{u} = \frac{u_L \sqrt{\rho_L} + u_R \sqrt{\rho_R}}{\sqrt{\rho_L} + \sqrt{\rho_R}}. \tag{3.1.18} \]

Moreover, from
\[ \bar{\rho} = \frac{\Delta(\rho u) - \bar{u} \Delta \rho}{\Delta u}, \tag{3.1.19} \]
one can now obtain:
\[ \bar{\rho} = \sqrt{\rho_L \rho_R}. \tag{3.1.20} \]

Since from (3.1.10) and (3.1.13),
\[ u \Delta(\rho H) - \Delta(\rho Hu) = \alpha_2 H(u - \lambda_2) + \alpha_3 H(u - \lambda_3), \tag{3.1.21} \]
and from (3.1.8) and (3.1.11),
\[ u H \Delta \rho - H \Delta(\rho u) = \alpha_2 H(u - \lambda_2) + \alpha_3 H(u - \lambda_3), \tag{3.1.22} \]
then
\[ \bar{u} \bar{H} \Delta \rho - \bar{H} \Delta(\rho u) = \bar{u} \Delta(\rho H) - \Delta(\rho Hu). \tag{3.1.23} \]

Therefore, one has:
\[ \bar{H} = \frac{\bar{u} \Delta(\rho H) - \Delta(\rho Hu)}{\bar{u} \Delta \rho - \Delta(\rho u)}. \tag{3.1.24} \]

From (3.1.18), (3.1.20) and (3.1.24), one can obtain the following averaging scheme:

**Chen-Cooke Averaging:**
\[ \bar{\rho} = \sqrt{\rho_L \rho_R}, \tag{3.1.25} \]
\[ \tilde{u} = \frac{u_L \sqrt{\rho_L} + u_R \sqrt{\rho_R}}{\sqrt{\rho_L} + \sqrt{\rho_R}}, \quad (3.1.26) \]

and

\[ \tilde{H} = \frac{\sqrt{\rho_R} H_R (u_R - u_L) + \sqrt{\rho_L} H_L (u_R - u_L)}{H_R - H_L}. \quad (3.1.27) \]

**Glaister Averaging [4]:**

For conservation law systems (2.3.5), this consists of (3.1.25, 3.1.26), with (3.1.27) replaced by

\[ \tilde{H} = \frac{H_R \sqrt{\rho_R} + H_L \sqrt{\rho_L}}{\sqrt{\rho_R} + \sqrt{\rho_L}}. \quad (3.1.28) \]

**Roe Averaging [1]:**

For conservation law systems (2.3.5), this consists of (3.1.25, 3.1.26), and

\[ \tilde{E} = \frac{E_R \sqrt{\rho_R} + E_L \sqrt{\rho_L}}{\sqrt{\rho_R} + \sqrt{\rho_L}}. \quad (3.1.29) \]
3.2 Eigenvalues and Eigenvectors for the Jacobian Matrix of Water Dynamics in the Pseudo-Conservation Law Form

Case 1: Eigenvalues and Eigenvectors for the Jacobian Matrix of Water Dynamics (UNESCO Equation of State)

Eigenvalues:

\[ \lambda_1 = u, \]

\[ \lambda_{2,3} = 1/2 \frac{\partial p}{\partial u_2} \pm \sqrt{\Delta}, \]

Eigenvectors:

\[ \varepsilon_1 = \begin{bmatrix} 1 \\ u \\ -\frac{\partial p_1 + \partial p_2}{\partial u_3} \end{bmatrix}, \]

\[ \varepsilon_{2,3} = \begin{bmatrix} 1 \\ u + 1/2 \frac{\partial p}{\partial u_2} \pm \sqrt{\Delta} \\ H \end{bmatrix}, \]

here

\[ \Delta = 1/4 \left( \frac{\partial p}{\partial u_2} \right)^2 + u \frac{\partial p}{\partial u_2} + H \frac{\partial p}{\partial u_3} + \frac{\partial p}{\partial u_1}. \]
Furthermore, let \( R = (e_1, e_2, e_3) \) be the matrix whose columns are the eigenvectors and the elements \( Z_{ij} \) of the matrix \( R^{-1} \) are given by

\[
Z_{11} = \frac{2\sqrt{\Delta} H}{\text{det} R},
\]

\[
Z_{12} = 0,
\]

\[
Z_{13} = \frac{-2\sqrt{\Delta}}{\text{det} R},
\]

\[
Z_{21} = \frac{-\alpha_3}{\text{det} R},
\]

\[
Z_{22} = \frac{1}{2} \frac{\partial P}{\partial u_2} + \frac{1}{2} \frac{\partial P}{\partial u_3} + \sqrt{\Delta},
\]

\[
Z_{23} = 0,
\]

\[
Z_{31} = \frac{\beta_3}{\text{det} R},
\]

\[
Z_{32} = \frac{1}{2} \frac{\partial P}{\partial u_2} + \frac{1}{2} \frac{\partial P}{\partial u_3} - \sqrt{\Delta},
\]

\[
Z_{33} = \frac{1}{2} \frac{\partial P}{\partial u_2} + \frac{1}{2} \frac{\partial P}{\partial u_3} + \Delta,
\]

here

\[
aa = uu + \frac{\partial P}{\partial u_1} + \frac{\partial P}{\partial u_2} u \left( u + \frac{\partial P}{\partial u_2} - \sqrt{\Delta} \right),
\]

and

\[
bb = uu + \frac{\partial P}{\partial u_1} + \frac{\partial P}{\partial u_2} u \left( u + \frac{\partial P}{\partial u_2} + \sqrt{\Delta} \right),
\]

also

\[
\text{det} R = 2\sqrt{\Delta} \left[ \frac{\partial P}{\partial u_1} + \frac{\partial P}{\partial u_2} u + II \right].
\]
Case 2: Eigenvalues and Eigenvectors for the Jacobian Matrix for Water Dynamics (Modified-Tait Equation of State)

For the modified Tait equation of state the eigenvalues and eigenvectors are:

**Eigenvalues:**

\[ \lambda_1 = u, \quad (3.2.35) \]

\[ \lambda_2 = u + a, \quad (3.2.36) \]

and

\[ \lambda_3 = u - a. \quad (3.2.37) \]

**Eigenvectors:**

\[ e_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad (3.2.38) \]

\[ e_{2,3} = \begin{bmatrix} 1 \\ \lambda_{2,3} \\ H \end{bmatrix}, \quad (3.2.39) \]

Furthermore, let \( R = (e_1, e_2, e_3) \) be the matrix whose columns are the eigenvectors....
tors; then the elements $Z_{ij}$ of the matrix $R^{-1}$ are given by

\begin{align*}
Z_{11} &= -H/\text{det}R, \\
Z_{12} &= 0, \\
Z_{13} &= 1/\text{det}R, \\
Z_{21} &= -(u-a)/\text{det}R, \\
Z_{22} &= \frac{1}{2a}/\text{det}R, \\
Z_{23} &= 0, \\
Z_{31} &= -(u+a)/\text{det}R, \\
Z_{32} &= -\frac{1}{2a}/\text{det}R, \\
Z_{33} &= 0,
\end{align*}

and

\[ \text{det}R = -2a. \]

3.3 Eigenvalues and Eigenvectors for the Jacobian Matrix of Gas Dynamics in the Conservation Law and Pseudo-Conservation Law Form

The conservation law form of one dimensional, compressible flow of an ideal gas can be written as,

\[ U_t + F(U)_x = W(U), \tag{3.3.40} \]
Here

\[ U = \begin{bmatrix} \rho \\ \rho u \\ \rho E \end{bmatrix}, \quad (3.3.41) \]

with

\[ F = \begin{bmatrix} \rho u \\ P + \rho u^2 \\ (P + \rho E)u \end{bmatrix}, \quad (3.3.42) \]

and

\[ W = - (\alpha - 1) \begin{bmatrix} \frac{\rho u}{R} \\ \frac{\rho u^2}{R} \\ \frac{\rho E u}{R} \end{bmatrix}. \quad (3.3.43) \]

The unknowns are \( P \), pressure; \( \rho \), density; \( u \), velocity; \( e \) specific energy; and \( E \) specific total energy \((E = e + u^2/2)\). The independent variables are \( t \), time, and \( R \), streamwise distance. The constant \( \alpha \) is a constant which determines the types of coordinates system: \( \alpha \) is equal to 1 for Cartesian coordinates, 2 for cylindrical coordinates and 3 for spherical coordinates.
Chapter 3: Conservation Law/Nonconservation Law Form and Averaging

The equations governing one dimension ideal compressible flow in pseudo-conservation law (total enthalpy) form are,

\[ U_t + F(U)_R = W(U). \]  \hspace{1cm} (3.3.44)

Here

\[ U = \begin{bmatrix} \rho \\ \rho u \\ \rho H \end{bmatrix}, \]  \hspace{1cm} (3.3.45)

with

\[ F = \begin{bmatrix} \rho u \\ P + \rho u^2 \\ \rho H u \end{bmatrix}, \]  \hspace{1cm} (3.3.46)

and

\[ W = \begin{bmatrix} -(\alpha - 1) \frac{\rho u}{R} \\ -(\alpha - 1) \frac{\rho u^2}{R} \\ -(\alpha - 1) \frac{\rho H u}{R} - P_t \end{bmatrix}. \]  \hspace{1cm} (3.3.47)

The unknowns are \( P \), pressure; \( \rho \), density; \( u \), velocity; \( h \) specific enthalpy; and \( H \), specific total enthalpy \((H = h + u^2/2)\). The independent variables are \( t \), time, and \( R \), streamwise distance.
Case 1: Conservation Law Form (and Ideal Gas Law)

Eigenvalues:

\[ \lambda_1 = u, \quad (3.3.48) \]

\[ \lambda_{2,3} = u \pm a, \quad (3.3.49) \]

Eigenvectors:

\[
\begin{bmatrix}
1 \\
u \\
\frac{u^2}{2}
\end{bmatrix}
\]

\[ e_1 = \begin{bmatrix}
1 \\
u + a \\
\frac{a^2}{\gamma-1} + \frac{u^2}{2} + ua
\end{bmatrix}, \quad (3.3.51) \]

\[
\begin{bmatrix}
1 \\
u - a \\
\frac{a^2}{\gamma-1} + \frac{u^2}{2} - ua
\end{bmatrix}
\]

\[ e_2 = \begin{bmatrix}
1 \\
u + a \\
\frac{a^2}{\gamma-1} + \frac{u^2}{2} + ua
\end{bmatrix}, \quad (3.3.52) \]

Furthermore, let \( R = (e_1, e_2, e_3) \) be the matrix whose columns are the eigenvectors; then, the elements \( Z_{ij} \) of the matrix \( R^{-1} \) are given by
\[ Z_{11} = 2a\left(\frac{a^2}{\gamma - 1} - \frac{3u^2}{2}\right)/\det R, \]
\[ Z_{12} = 2ua/\det R, \]
\[ Z_{13} = -2a/\det R, \]
\[ Z_{21} = (\frac{u^2a}{2} - \frac{a^2u}{\gamma - 1})/\det R, \]
\[ Z_{22} = (\frac{a^2}{\gamma - 1} - ua)/\det R, \]
\[ Z_{23} = a/\det R, \]
\[ Z_{31} = (\frac{a^2u}{\gamma - 1} - \frac{u^2a}{2})/\det R, \]
\[ Z_{32} = -(\frac{a^2}{\gamma - 1} + ua)/\det R, \]
\[ Z_{33} = a/\det R, \]

here

\[ \det R = \frac{2a^3}{\gamma - 1}. \]

**Case 2: Non-Conservation Law Form (and Ideal Gas)**

**Eigenvalues:**

\[ \lambda_1 = u, \quad (3.3.53) \]

\[ \lambda_{2,3} = u\left(1 - \frac{\gamma - 1}{2\gamma}\right) \pm \sqrt{\frac{\Delta}{4}}, \quad (3.3.54) \]

where
\[ \Delta = \frac{\gamma - 1}{\gamma} [-u^2\frac{(\gamma + 1)}{\gamma} + 4H]. \]  \hspace{1cm} (3.3.55)

**Eigenvectors:**

- \( e_1 = \begin{bmatrix} 1 \\ u \\ \frac{u^2}{2} \end{bmatrix} \) \hspace{1cm} (3.3.56)
- \( e_2 = \begin{bmatrix} 1 \\ \lambda_2 \\ H \end{bmatrix} \) \hspace{1cm} (3.3.57)
- \( e_3 = \begin{bmatrix} 1 \\ \lambda_3 \\ H \end{bmatrix} \) \hspace{1cm} (3.3.58)

Furthermore, let \( R = (e_1, e_2, e_3) \) be the matrix whose columns are the eigenvectors; then the elements \( Z_{ij} \) of the matrix \( R^{-1} \) are given by
Chapter 3: Conservation Law/Nonconservation Law Form and Averaging

\[
Z_{11} = H(\lambda_2 - \lambda_3)/\det R,
\]
\[
Z_{12} = 0,
\]
\[
Z_{13} = (\lambda_3 - \lambda_2)/\det R,
\]
\[
Z_{21} = (-uH + \frac{u^2}{2}\lambda_3)/\det R,
\]
\[
Z_{22} = (H - \frac{u^2}{2})/\det R,
\]
\[
Z_{23} = (u - \lambda_3)/\det R,
\]
\[
Z_{31} = (-uH + \frac{u^2}{2}\lambda_2)/\det R,
\]
\[
Z_{32} = -(H - \frac{u^2}{2})/\det R,
\]
\[
Z_{33} = (\lambda_2 - u)/\det R,
\]

here

\[
\det R = (H - \frac{u^2}{2})\sqrt{\Delta}.
\]  \hspace{1cm} (3.3.59)
Chapter 4

Computational Experiments

4.1 Computational Experiments

Several numeral experiments have been accomplished, whose purpose is to assess effectiveness of the shock-capturing scheme previously discussed.

Case 1: The Ideal Gas

The Riemann problem for the ideal gas, whose solution is well-known, is now considered. Results for the case in which $\rho_L = P_L = 1$, $u_L = 0$ and $\rho_R = P_R = 0.1$, $u_R = 0$ are reported in [2].

The first question to be settled concerns the necessity of iterating the second stage of the splitting, or incorporation of the source term. Figures 4.1, 4.2 show significant differences in comparison of results for the pressure and velocity profiles, with and without iteration. As far as accuracy in general, the results obtained by iteration agree well with those of [12], in terms of shock location
and strength. Three point shock resolution is about what is to be expected from first order accuracy. Thus, there has been obtained a viable algorithm for numerically solving the Euler equations with total enthalpy the preferred thermodynamic variable.

The next issue concerns relative effectiveness of the averaging schemes proposed in section 4.1. Three schemes are compared: Chen averaging equations (3.1.25 - 3.1.27); Glaister averaging equations (3.1.25, 3.1.26, 3.1.28), and an averaging which we refer to as Roe averaging, equations (3.1.25, 3.1.26, 3.1.29) together with an arithmetic (rather than the square root) average for density. Figures 4.3, 4.4 show a comparison of results for the three schemes. Indeed, there does not appear to be a great deal of difference.

Thus, the Glaister averaging can be more widely used than warranted by the original derivation. A priori, this would not have been expected. As equations (3.1.28) is more convenient for calculation than equations (3.1.27), Glaister averaging is used in the calculations reported in the sequel.

**Case 2: Pure Water**

For pure water, the Riemann problem to be considered is characterized by $P_L = 997.286$, $P_R = 0.955$ bar, with $\rho_L = 1037.8$, $\rho_R = 997.09$ kg/m$^3$, zero initial velocity, and temperature $25^\circ C$. For the modified-Tait equation of state, $\rho_0 = 997.04796$ kg/m$^3$, $N = 7.2$, $B = 2996$ bar.

Figures 4.5 - 4.8 show comparison of results for the solution at time 0.0002
seconds, for the modified-Tait versus UNESCO equation of state. In Figure 4.6, density variations across the contact surface are too slight to be visible, although contact surface position is apparent in Figure 4.8. The maximum fluid velocity is so small, in comparison to the speed of sound in water, that the backward moving rarefaction wave and the forward moving shock appear to be moving at approximately the same speeds.

Using the correspondence $\tilde{P} = P + B, \gamma = N$, the Riemann problem (for the modified Tait equation of state) has been solved using a gas dynamics Riemann solver. The results indicate that the contact surface speed (the maximum velocity), together with shock strengths, are reasonably what is to be expected. However, for this problem it is clear that the UNESCO equation of state gives the steeper wave, although in general there is not much difference in results. As the general equation of state is not valid at pressures much above 1000 bar, comparisons at the high pressures experienced in underwater explosions can not be made.

**Case 3: TVD Results for Pure Water**

Once an averaging technique which allows Roe's scheme to be adapted to a liquid media is known, the way is open to use the second-order accurate TVD scheme of Harten [11]. Figures (4.9, 4.10) indicate only slightly sharper results when this method is adapted for use in water by employing Glaister averaging. However, the trouble was not taken to use a second-order equation splitting when
incorporating source terms. Therefore, the potential second-order accuracy is not fully realized.
Figure 4.1:
Effects of Iterating Source Terms (Pressure)
Iterated (o) and Non-iterated (x) Source Term at t = 0.25

Figure 4.2: Effects of Iterating Source Terms (Velocity)

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Figure 4.3: Comparison of Results from the Various Averaging Schemes (Density)

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Figure 4.4:
Comparison of Results from the Various Averaging Schemes (Velocity)
Figure 4.5:
Results for Modified-Tait Versus UNESCO Equation of State (Pressure)
Comparison Tait(o) and UNESCO(x) Equation of State

Figure 4.6:
Results for Modified-Tait Versus UNESCO Equation of State (Density)
Figure 4.7: Results for Modified-Tait Versus UNESCO Equation of State (Velocity)
Chapter 4: Computational Experiments

Figure 4.8:
Results for Modified-Tait Versus UNESCO Equation of State (Total-Enthalpy)

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Chapter 4: Computational Experiments

Comparison: ROE(x), GODUNOV(+) and TVD(o) Results, \( t = 2E-4 \)

Figure 4.9:
Water Adapted Roe Scheme Versus the Scheme of Harten and Godunov (Pressure)
Chapter 4: Computational Experiments

Comparison: ROE(x), GODUNOV(+) and TVD(o) Results, $t = 2E-4$

Figure 4.10:
Water Adapted Roe Scheme Versus the Scheme of Harten and Godunov (Velocity)
Chapter 5

Moving Boundary Problems

5.1 A Local Front Tracking Algorithm with Sub-cell Resolution

The assumptions made in deriving a front tracking algorithm with subcell resolution capability are as follows: During time interval $\Delta t$ the contact surface velocity is constant and externally supplied. The two cells adjacent to the contact surface are allowed to contract (the left cell) or expand (the right cell) as the contact front moves; away from the front a uniform mesh width $\Delta x$ may be chosen. A cell changing in size is never allowed to be longer than $2\Delta x$, and never shorter than $\Delta x$ (see Figure 5.1). Thus, the expanding and contracting cells together temporarily replace three cells of uniform size $\Delta x$, as the front moves across the middle one of the cells.

Taking into account the equation splitting process, and neglecting source terms $W$, there is derived different cell update formulas for the left (expanding)
Expanding Cell Algorithm

By integrating equation (3.3.40) over the expanding cell indicated in Figure 5.1, and applying Green's theorem for the plane, there results

\[ \int_{\Gamma} U \, dx = \int_{\Gamma} F(U) \, dt, \]  

(5.1.1)

where \( \Gamma \) is the curve bounding the domain of integration.

Consider the mean value relations

\[ U_i^n \Delta x = \int_{z_{i-1/2}}^{z_{i+1/2}} U(x, t_n) \, dx, \]  

(5.1.2)

and

\[ U_i^{n+1}(\Delta x + \epsilon) = \int_{z_{i-1/2}}^{z_{i+1/2}} U(x, t_{n+1}) \, dx, \]  

(5.1.3)

Equation (5.1.1) becomes

\[ U_i^{n+1} = \frac{1}{\Delta x + \epsilon} \left[ U_i^n \Delta x - \int_{\Gamma} F(U) \, dt + \int_{\text{wall}} U \, dx \right], \]  

(5.1.4)

where the wall term is taken along the moving contact surface.
Equation (5.1.4) may also be written as,

\[ U_i^{n+1} = \frac{1}{\Delta x + \epsilon} [U_i^n \Delta x - \int_{wall} F(U) dt + \int_{t_n}^{t_{n+1}} F(U(x_{i-1/2}, t)) dt + \int_{wall} U dx]. \]  

(5.1.5)

By using a trapezoidal approximation on wall integrals, there results

\[ U_i^{n+1} = \frac{U_i^n \Delta x - [(F^w + F_w)/2 - U_B(U_w + U_w)/2 - F_{i-1/2}^n] \Delta t}{\Delta x + \epsilon}. \]  

(5.1.6)

Here \( F^w \) refers to the usual Godunov flux at the fixed cell boundary. The 'w' subscript or superscript denotes the values for the flux at the current or the advanced time level, respectively. Also, \( (\cdot)_{i-1/2} \) represents the Godunov flux which emerges from solving the local Riemann problem at \( x_{i-1/2} \).

**Contracting Cell Algorithm**

Similarly, for the contracting cell one may obtain:

\[ U_i^{n+1} = \frac{2U_i^n \Delta x + [(F^w + F_w)/2 - U_B(U_w + U_w)/2 - F_{i+1/2}^n] \Delta t}{2 \Delta x - \epsilon}. \]  

(5.1.7)

Equations (5.1.6, 5.1.7) must be numerically solved by iteration. Source terms may be incorporated using the Euler approximation

\[ \int \int_D WdA = W_i \Delta A, \]  

(5.1.8)
where $D$ is the domain of integration. Of course, equations (5.1.6, 5.1.7) will change slightly, for the case in which the contact surface is initially somewhere on the interior of the middle cell of Figure 5.1.

### 5.2 Numerical Experiments

The shock tube is a device in which shock waves are generated by rupture of a diaphragm separating a stagnant high pressure fluid from a stagnant low pressure fluid. Upon sudden rupture of the diaphragm, a shock wave is established in front of a contact surface, which moves at a constant speed, and a rarefaction wave recedes in the opposite direction. If one uses the constant contact surface speed from the shock tube problem as the piston speed, and calculates the flow in back of (or else, in front of) the contact surface, by using one of the expanding cell or the contracting cell algorithms, of course, one should obtain the same trailing rarefaction wave, or leading shock wave, respectively, as in the shock tube problem. Thus, the parts of the solution respectively in the front or behind the moving front provide good test problems for analyzing the effectiveness of the expanding or contracting cell algorithms.
Case 1: Front Tracking Algorithm for the Perfect Gas

The purpose of this example is to use the well known contact surface speed from Chapter 4 to study the applicability of the expanding cell algorithms when used with the specific internal energy versus the total enthalpy form of the conservation equations, in order to determine for future research which is the more suitable form.

Consider a gas which undergoes an expansion behind an impulsively started piston, which moves at uniform speed. Initially, conditions behind the piston are $P_L = 1.0$, $\rho_L = 1.0$ and $(\rho H)_L = 3.5$, $(\rho E)_L = 2.5$. The piston speed, in order to be consistent with the contact surface speed from the Riemann problem studied in Chapter 4, is set at 0.928. The numerical results are listed in Figures 5.2, 5.3. Obviously, the conservation law form for the expanding cell algorithm produces better accuracy than does the pseudo-conservation law (total enthalpy) form.

Case 2: Front Tracking Algorithm for Water in Cartesian Coordinates

The two-fold objectives of this experiment are: To use the results concerning contact surface speed obtained in Chapter 4 to assess the effectiveness of the contracting cell algorithm; and to determine the sensitivity of the method to grid size effects.

Consider an impulsively started compressing piston moving at a constant velocity, as in the Riemann problem for water studied in Chapter 4, with initially undisturbed water. Since the speed of sound in the water is much larger than in
gas, smaller time and space increments are to be expected. The initial conditions are: $\rho_R = 997.09 \text{ kg/m}^3$, $T = 298^\circ \text{K}$, and the piston speed is $U_p = 31.286 \text{ m/sec}$.

By employing different grid sizes and a common time period, some idea of grid effects can be gained. According to the pressure data of Figure 5.4, the expected step-behavior of the true solution in the front of the compressing piston can be achieved only if the grid is sufficiently fine.

**Case 3: Non-Uniform Front Velocity (Water)**

The next issue is to study the applicability of the contracting cell algorithm in spherical coordinates. In spherical coordinates the contact surface velocity for the Riemann problem is not a constant, as has been the case for the shock tube problem in Cartesian coordinates, but usually has a time decay proportional to $t^{-3/5}$. Thus, we shall consider, in water, a compressing piston whose velocity decay is given by

$$U_p = \begin{cases} 
69(10^6t)^{-0.6}, & t > 0, \\
69, & t = 0 
\end{cases}$$

Corresponding numerical results for velocity and density distributions are listed in Figures 5.5, 5.6. Although the exact solution is unknown, the appearance of the shock depicted in Figures 5.5, 5.6 indicates textbook behavior.
Chapter 5: Moving Boundary Problem

Three Uniform Cells

Expanding Cell

Contracting Cell

Figure 5.1:
Variable Cell Sizes Produced by Contact Surface Motion
Figure 5.2:
Expanding Cell Method and Flow Behind a Moving Piston (I)
Figure 5.3:
Expanding Cell Method and Flow Behind a Moving Piston (II)
Figure 5.4:
Grid-Size Sensitivity of the Contracting Cell Method

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Chapter 5: Moving Boundary Problem

Figure 5.5:
Density Discontinuity Produced by a Variable Speed Piston
Chapter 5: Moving Boundary Problem

Figure 5.6:
Velocity Discontinuity Produced by a Variable Speed Piston
Chapter 6
Solution of the Riemann Problem

In this chapter there is begun the study of the underwater explosion problem. First, efforts will be directed to obtaining a method for predicting contact surface velocity at a gas/water interface. Methods of solving the Riemann problems for (i) gas, (ii) water, and (iii) for a gas-water interface will be indicated. As well, it is shown that, to good approximation, solutions of the Riemann problem for water (and the modified-Tait equation of state) map into solutions of the Riemann problem for the ideal gas under a simple transformation. This means that existing computer codes for solving the Riemann problem for gases can be simply adapted for use in water.
Chapter 6: Solution of the Riemann Problem

Figure 6.1:
The Gas-Water Riemann Problem.
6.1 The Gas-Water Riemann Problem

Consider an imaginary plane diaphragm, separating a water state (1) from a gas state (4). If the diaphragm is suddenly ruptured, a rarefaction wave or else a shock wave moves into the gas, separating it into states (3) and (4), whereas, a rarefaction wave or else a shock wave also moves into the water, separating it into states (1) and (2) (See Figure 6.1). State (2), which is water, is separated from state (3), which is gas, by a contact surface (c), across which pressure and velocity are continuous.

Now the Riemann problem which governs the nature of the solution after rupture of the diaphragm can be shown to have a self-similar solution (which is constant on straight lines through the initial discontinuity). The theory of self-similar solutions to the Euler equations guarantees that states (1-4) will have constant properties, the state boundaries will be linear, and constant states must be separated either by shock waves, across which the Rankine-Hugoniot relations apply, or else by simple (rarefaction) waves, across which isentropic flow relations hold.

The Gas-Water Riemann Problem (R):

Given states (1) and (4), find the possible states (2) and (3) such that the total solution (1-4) satisfies the Euler equations of compressible flow, with gas constitutive relations to one side, and water constitutive relations on the other side of the contact surface (c).
Notes:

1. The velocities in states (1), (4) are not necessarily zero.

2. When the velocities in states (1), (4) are zero, (R) represents the hydrodynamic shock tube problem with gas driver.

The solution to the Riemann problem (R) has been studied by Flores and Holt [5], who reduce it to the simultaneous solution of several nonlinear equations. For some cases difficulty in obtaining convergence is experienced, when these equations are solved by iterative means.

In the present chapter it is shown that this Riemann problem can be reformulated in such a way that its solution can be reduced to the solving of one nonlinear equation. This solution can be accomplished by the method of interval bisection, and no difficulties with convergence are encountered.

Some terminology is necessary. According to Figure 6.1, we shall refer to the left-propagating wave front separating states (3,4) as a 1-wave; whereas the contact surface separating states (2,3) is a 2-wave; and the right-propagating wave front separating states (1,2) is a 3-wave. Given a particular left state, $U_L$, the set of corresponding right states, $U_R$, which are connected to this left state by either a shock, a contact surface, or an expansion front will be referred as a 1-, 2-, or 3-family. By searching for pressure, density, and velocity relations across 1-, 2-, or 3-rarefaction and 1-, 2-, 3-shocks, the Riemann problem (R) can be expressed in a form readily amenable to numerical solution.
For each particular family, generic left and right states $U_L$ and $U_R$ are assumed. If $P_3 < P_4$, the wave front between states $(3,4)$ is a 1-rarefaction; if $P_3 > P_4$, the wave front between states $(3,4)$ is a 1-shock. Similarly, if $P_2 > P_1$, the wave front connecting states $(1,2)$ is a 3-shock; otherwise, it is a 3-rarefaction. As there must be a gas-water separation, the 2-family will always be a contact surface.

**The Rankine-Hugoniot Jump Conditions**

The Rankine-Hugoniot jump conditions across a shock wave in an ideal gas may be written as

$$\rho_1 u_1 v_1 - \rho_0 u_0 v_0 = P_0 - P_1, \quad (6.1.1)$$

$$\rho_1 v_1 = \rho_0 v_0 = m, \quad (6.1.2)$$

and

$$\frac{2}{\gamma - 1} C_1^2 + v_1^2 = \frac{2}{\gamma - 1} C_0^2 + v_0^2. \quad (6.1.3)$$

Here $v = u - W$, where $W$ represents the speed of the shock, and subscripts $(o)$ and (1) denote conditions on the left and right sides of the shock, respectively.

For water, the equation (6.1.1) is modified to

$$\rho_1 u_1 v_1 - \rho_0 u_0 v_0 = \bar{P}_0 - \bar{P}_1, \quad (6.1.4)$$

whereas (6.1.3) is inapplicable. However, the modified-Tait equation applies across either a shock or a rarefaction wave, in the form

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
\[
\frac{\bar{P}_1}{\bar{P}_0} = (\frac{\rho_1}{\rho_0})^N, \tag{6.1.5}
\]

where \(N\) is a constant, and \(\bar{P} = P + B\).

Across an expansion wave, the Riemann invariants, valid for both gas and water, may be used:

\[
u_1 \pm \frac{2}{\gamma - 1} C_1 = u_0 \pm \frac{2}{\gamma - 1} C_0, \tag{6.1.6}
\]

The (+) refers to 1-waves and the (-) to 3-waves. For water, \(N\) replaces \(\gamma\), with the speed of sound given by

\[C^2 = \frac{N\bar{P}}{\bar{\rho}}.\]

### 6.2 Contact Surface Relations

In this section there is derived the one parameter families of relation \(U_R/U_L = f(x)\) which determine the set of right states, \(U_R\), which are connected to a given left state, \(U_L\), through a 1-, 2-, or 3-wave. These relations can be used to solve the gas-water Riemann problem completely. Likewise, expressions for contact surface velocity and thermodynamic state emerge. For water the modified-Tait equation of state

\[P = B\left(\frac{\rho}{\rho_0}\right)^N - 1, \tag{6.2.7}\]
of course, leads to the relation

$$\frac{\bar{p}}{\bar{p}_o} = \left( \frac{\rho}{\rho_o} \right)^N,$$  \hspace{1cm} (6.2.8)

which holds across a rarefaction or a shock wave.

It is assumed the gas is ideal, so that the equation of state is given by

$$P = \rho RT.$$  \hspace{1cm} (6.2.9)

This leads to

$$\frac{P}{P_o} = \left( \frac{\rho}{\rho_o} \right)^\gamma,$$  \hspace{1cm} (6.2.10)

across a rarefaction wave, whereas the Rankine-Hugoniot relations (6.1.1-6.1.3) hold across shock waves.

As previously mentioned, there are three families of functions which are intermediaries in describing the relationship between the gas state (4) and the water state (1). The first and third family describes the gas and water media, with shock or rarefaction wave depending upon the relative pressure distribution of the two sides.

The relations across rarefaction-waves for the first and the third families will now be derived. For completeness, the procedure for the gas dynamics case, although well-known, is also included.
1. Rarefaction Waves for Gas:

Since the Riemann invariant \( u + \frac{2}{\gamma - 1} C \) is constant in a 1-rarefaction wave, one has,

\[
    u_4 + \frac{2}{\gamma - 1} C_4 = u_3 + \frac{2}{\gamma - 1} C_3. \tag{6.2.11}
\]

From the theory of the perfect gas, one has the following identities

\[
    \frac{P_3}{P_4} = \left(\frac{\rho_3}{\rho_4}\right)^{1/\gamma}, \tag{6.2.12}
\]

and

\[
    \left(\frac{C_3}{C_4}\right)^2 = \left(\frac{P_3}{P_4}\frac{\rho_4}{\rho_3}\right) = \left(\frac{P_3}{P_4}\right)^{(\gamma - 1)/\gamma}, \tag{6.2.13}
\]

so that

\[
    \frac{P_3}{P_4} = \left(\frac{C_3}{C_4}\right)^{2(\gamma - 1)/\gamma} = \left(\frac{\rho_3}{\rho_4}\right)\gamma. \tag{6.2.14}
\]

Hence,

\[
    \frac{u_3 - u_4}{C_4} = \frac{2}{\gamma - 1} \left(1 - \frac{C_3}{C_4}\right). \tag{6.2.15}
\]

Also, the rarefaction wave satisfies

\[
    0 < \frac{P_3}{P_4} \leq 1. \tag{6.2.16}
\]
One can introduce a parameter \( x_1 \) by

\[
x_1 = -\ln\left(\frac{P_3}{P_4}\right) \geq 0.
\]

(6.2.17)

For the 1-rarefaction wave in the gas, one can now write the relations

\[
T = e^{x_1},
\]

(6.2.18)

\[
\frac{P_3}{P_4} = e^{-\tau_1},
\]

(6.2.19)

and

\[
\frac{u_3 - u_4}{C_4} = \frac{2}{\gamma - 1} \left(1 - \frac{C_3}{C_4}\right) = \frac{2}{\gamma - 1} \left(1 - e^{-\tau_1}\right),
\]

(6.2.20)

where

\[
\tau = \frac{\gamma - 1}{2\gamma}.
\]

(6.2.21)

2. Rarefaction Waves for Water

Similarly, for water the 3-family rarefaction wave satisfies \((x_3 \leq 0)\),
\[ \frac{P_1}{P_2} = e^{z_3}, \quad (6.2.22) \]

and

\[ \frac{\bar{P}_1}{\bar{P}_2} = \frac{1 + B_1}{e^{-z_3} + B_1}, \quad (6.2.23) \]

where

\[ \bar{P} = P + B, \quad (6.2.24) \]

and

\[ B_1 = \frac{B}{\bar{P}_1}. \quad (6.2.25) \]

Also,

\[ \frac{\rho_1}{\rho_2} = \left( \frac{\bar{P}_1}{\bar{P}_2} \right)^{1/N} = \left( \frac{1 + B_1}{e^{-z_3} + B_1} \right)^{1/N}, \quad (6.2.26) \]

and

\[ \frac{u_1 - u_2}{C_2} = \frac{2}{N - 1} \left[ \left( \frac{1 + B_1}{e^{-z_3} + B_1} \right)^{\bar{\tau}} - 1 \right], \quad (6.2.27) \]

with

\[ \bar{\tau} = \frac{N - 1}{2N}. \quad (6.2.28) \]
3. Shocks in Gas

In order to calculate a one-parameter family of shock waves, or those states $U_R$ which are connected to a given state $U_L$ through a shock, one can define the constants

$$\pi = \frac{P_4}{P_3}, \quad (6.2.29)$$

and

$$Z = \frac{\rho_4}{\rho_3}, \quad (6.2.30)$$

Now

$$\left(\frac{C_4}{C_3}\right)^2 = \frac{\pi}{Z}, \quad (6.2.31)$$

and

$$\frac{\rho_3}{\rho_4} = \frac{v_4}{v_3} = \frac{1}{Z}. \quad (6.2.32)$$

From the Rankine-Hugoniot jump condition (6.1.3),

$$\frac{2}{\gamma - 1} C_3^2 + v_3^2 = \frac{2}{\gamma - 1} C_4^2 + v_4^2 = \frac{2}{\gamma - 1} Z C_3^2 + \frac{v_3^2}{Z^2}, \quad (6.2.33)$$

or

$$\left(\frac{v_3}{C_3}\right)^2 = \frac{2Z}{\gamma - 1} \frac{Z - \pi}{(1 - Z^2)}. \quad (6.2.34)$$

Also, consider

$$P_3 + mv_3 = P_4 + mv_4, m = \rho v, \quad (6.2.35)$$

and

$$P = C^2 \rho / \gamma, v_3 = u_3 - W, v_4 = u_4 - W, \quad (6.2.36)$$
where W is the shock speed. From (6.2.35) there is obtained

$$\frac{C_3^2 \rho_3}{\gamma} + \rho_3 v_3^2 = \frac{C_4^2 \rho_4}{\gamma} + \rho_4 v_4^2,$$

and

$$\frac{C_3^2}{\gamma} + v_3^2 = Z \left( \frac{C_4^2}{\gamma} + v_4^2 \right). \quad (6.2.37)$$

Then, one can write

$$\frac{C_3^2}{\gamma} + v_3^2 = Z \left( \frac{C_4^2}{\gamma Z} + \frac{v_4^2}{Z} \right), \quad (6.2.38)$$

or,

$$\frac{Z C_3^2}{\gamma} (1 - \pi) = v_3^2 (1 - Z). \quad (6.2.39)$$

Comparing this with (6.2.37) gives

$$Z = \frac{1 + \pi \beta}{\pi + \beta}, \quad (6.2.40)$$

where

$$\beta = \frac{\gamma + 1}{\gamma - 1}.$$ 

Rewrite (6.3.39) as

$$\frac{v_3}{C_3} = \pm \left( \frac{\beta - 1}{\beta - Z} \right)^{1/2}, \quad (6.2.41)$$

where the (+) sign is for 1-shocks. Using

$$v = u - W, \quad (6.2.42)$$
one obtains the shock speed for a 1-shock:

\[ W = u_3 + C_3 \left( \frac{(\beta - 1)Z}{\beta - Z} \right)^{1/2}. \]  

(6.2.43)

Since \( v_4 = v_3/Z \), then \( u_4 - W = (u_3 - W)/Z \),

and

\[ u_3 - u_4 = \frac{Z - 1}{Z} (W - u_4) = \left( \frac{\beta - 1}{Z(\beta - Z)} \right)^{1/2} (Z - 1). \]  

(6.2.44)

Using (6.2.40) and the last equation can be simplified to

\[ u_3 - u_4 = C_4 \sqrt{\frac{2}{\gamma(\gamma - 1) \sqrt{1 + \pi \beta}}}. \]  

(6.2.45)

Introducing a new variable, \( x_1 \leq 0 \),

\[ e^{-x_1} = \pi = P_4/P_3, \]  

(6.2.46)

and

\[ x_1 = -ln \pi. \]  

(6.2.47)

This gives the relations

\[ \frac{u_3 - u_4}{C_4} = \frac{2\sqrt{\beta}}{\gamma - 1} \sqrt{1 + e^{-x_1}} \]  

(6.2.48)
and

\[
\frac{\rho_3}{\rho_4} = \frac{\beta + e^{x_1}}{1 + \beta e^{x_1}}. 
\]  
(6.2.49)

With \( x_1 \leq 0 \), the 1-family of right states connecting a given left state through a shock can now be written as:

\[
P_3/P_4 = e^{x_1}, 
\]  
(6.2.50)

\[
\frac{\rho_3}{\rho_4} = \frac{\beta + e^{x_1}}{1 + \beta e^{x_1}}. 
\]  
(6.2.51)

\[
\frac{u_3 - u_4}{C_4} = \frac{2\sqrt{\gamma} \left( 1 - e^{-x_1} \right)}{\gamma - 1 \sqrt{1 + \beta e^{-x_1}}}.
\]  
(6.2.52)

4. **Shocks in Water**

Similarly, with \( x_3 \leq 0 \) the 3-family shock wave relations for water can be written as:

\[
\frac{P_3}{P_2} = e^{x_3}, 
\]  
(6.2.53)

\[
\frac{\bar{P}_3}{\bar{P}_2} = \frac{1 + B_1}{e^{-x_3} + B_1},
\]  
(6.2.54)
Chapter 6: Solution of the Riemann Problem

\[ \frac{\rho_1}{\rho_2} = \left( \frac{1 + B_1}{e^{-z_2} + B_1} \right)^{1/N}, \quad (6.2.55) \]

\[ \frac{u_1 - u_2}{C_2} = \frac{1}{\sqrt{N}} \left[ \left( \frac{P_2}{P_1} \right) \left( \frac{1 - \rho_3}{\rho_1} \right) \right]^{1/2} - \frac{1}{\sqrt{N}} \left[ \left( \frac{P_2}{P_1} \right) \left( \frac{1 - \rho_3}{\rho_1} \right) \right]^{1/2}. \quad (6.2.56) \]

For clarity, the derivation of equation (6.2.56) and its counterpart occurring in equation (6.3.79), will be deferred until section 6.5.

For the 2-family, one should have the contact surface relations

\[ P_2 = P_3, \quad \frac{\rho_2}{\rho_3} = \varepsilon^x, \quad u_2 = u_3. \quad (6.2.57) \]

Thus, there emerges the combined one-parameter families for the gas/water Riemann problem which are listed below.

**Combined One-Parameter Relations**

**1-family**

\[ \frac{P_3}{P_4} = e^{-z_1}, \quad (6.2.58) \]
\[
\frac{\rho_3}{\rho_4} = f_1(x_1) = \begin{cases} 
\frac{e^{-\gamma_1}}{x_1}, & x_1 \geq 0 \\
\frac{\beta \pm e^{x_1}}{1 + \beta e^{x_1}}, & x_1 \leq 0.
\end{cases}
\]

(6.2.59)

\[
\frac{u_3 - u_4}{C_4} = h_1(x_1) = \begin{cases} 
\frac{2}{\gamma_1} (1 - e^{-x_1}), & x_1 \geq 0 \\
\frac{2 \sqrt{\gamma_1}}{\gamma_1} \frac{1 - e^{-x_1}}{\sqrt{1 + \beta e^{-x_1}}}, & x_1 \leq 0.
\end{cases}
\]

(6.2.60)

2-family

\[
\frac{P_2}{P_3} = 1,
\]

(6.2.61)

\[
\frac{P_2}{P_3} = e^{z_2},
\]

(6.2.62)

\[
u_2 = u_3.
\]

(6.2.63)

3-family

\[
\frac{P_1}{P_2} = e^{z_3},
\]

(6.2.64)

\[
\frac{\overline{P}_1}{\overline{P}_2} = \frac{1 + B_1}{e^{-z_3} + B_1},
\]

(6.2.65)
\[ \frac{\rho_1}{\rho_2} = f_3(x_3) = \left( \frac{1 + B_1}{e^{-x_3} + B_1} \right)^{1/N}, \quad (6.2.66) \]

\[ \frac{u_1 - u_2}{C_2} = h_3(x_3), \]

where

\[ h_3(x_3) = \begin{cases} 
\frac{2}{N-1} \left[ (\frac{f_3}{f_1^*})^{(N-1)/2} \right]^{1/2}, & x_3 > 0 \\
\frac{1}{N-1} \left[ (\frac{f_3}{f_1^*})^{(N-1)/2} \right]^{1/2}, & x_3 < 0.
\end{cases} \quad (6.2.67) \]

From the above relations, one has

\[ u_1 - u_4 = C_2 h_3(x_3) + C_4 h_1(x_1), \quad (6.2.68) \]

and

\[ \frac{C_2}{C_1} = \left( \frac{1 + B_1}{e^{-x_3} + B_1} \right)^{\tau}, \quad (6.2.69) \]

\[ \frac{P_1}{P_4} = \frac{P_1 P_2}{P_2 P_4} = e^{x_3 - x_1}, \quad (6.2.70) \]

\[ \frac{\rho_1}{\rho_4} = \frac{\rho_1 \rho_2 \rho_3}{\rho_2 \rho_3 \rho_4} = f_3(x_3) e^{x_3} f_1(x_1), \quad (6.2.71) \]

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
\[ x_2 = \ln\left(\frac{\rho_1}{\rho_4} (f_3(x_3)f_1(x_1))^{-1}\right), \quad (6.2.72) \]

\[ x_1 = x_3 + \ln\left(\frac{P_4}{P_1}\right). \quad (6.2.73) \]

Thus,

\[ \frac{u_1 - u_4}{C_1} = \frac{C_4}{C_1} h_1(x_1) + h_3(x_3) \left( \frac{1 + B_1}{e^{-s_2} + B_1} \right)^{-\frac{n-1}{n}}. \quad (6.2.74) \]

By substituting (6.2.73) into (6.2.74), the ensuing equation can be iteratively solved by the method of interval bisection, to obtain \( x_3 \). No convergence problems are experienced. The values \( x_1, x_2 \) follow from (6.2.72, 6.2.73). Thus, the gas/water Riemann problem has been solved. Equations (6.2.67) provide contact surface velocity. Equation (6.2.64) yields the contact surface pressure.

### 6.3 Solution of the Riemann Problem for Water

In order to apply Godunov's method to liquid media, the approach of section 6.2 will now be used to develop the one parameter families of shock and rarefaction relations which permit solution of the Riemann problem in water with the modified-Tait equation of state.
1-family

\[ \frac{P_3}{P_4} = e^{-x_1}, \quad (6.3.75) \]

\[ \frac{\tilde{P}_3}{\tilde{P}_4} = \frac{e^{-x_1} + B_4}{1 + B_4}, \quad (6.3.76) \]

\[ \frac{\rho_3}{\rho_4} = f_1(x_1) = \left( \frac{e^{-x_1} + B_4}{1 + B_4} \right)^{1/N}, \quad (6.3.77) \]

where

\[ B_4 = B/P_4. \quad (6.3.78) \]

\[ \frac{u_3 - u_4}{C_4} = h_1(x_1), \]

and

\[ h_1(x_1) = \left\{ \begin{array}{ll}
\frac{2}{N-1} \left[ 1 - \left( \frac{e^{-x_1} + B_4}{1 + B_4} \right)^{1/2} \right], & x_1 \geq 0 \\
\frac{1}{\sqrt{N}} \left( \frac{1 - \frac{\tilde{c}^2_3}{\tilde{c}^2_4}}{1 + B_4} \right)^{1/2}, & \frac{1 - \frac{\tilde{c}^2_3}{\tilde{c}^2_4}}{1 + B_4} \leq 0, \end{array} \right. \quad (6.3.79) \]

with

\[ \tilde{\chi} = \frac{N - 1}{2N}. \quad (6.3.80) \]
2-family

\[ \frac{P_2}{P_3} = 1, \quad (6.3.81) \]

\[ \frac{\rho_2}{\rho_3} = e^{\pi^2}, \quad (6.3.82) \]

\[ u_2 = u_3. \quad (6.3.83) \]

3-family

\[ \frac{P_1}{P_2} = e^{\pi^2}, \quad (6.3.84) \]

\[ \frac{\bar{P}_1}{\bar{P}_2} = \frac{1 + B_1}{e^{-\pi^2} + B_1}, \quad (6.3.85) \]

\[ \frac{\rho_1}{\rho_2} = f_3(x_3) = \left( \frac{1 + B_1}{e^{-\pi^2} + B_1} \right)^{1/N}, \quad (6.3.86) \]

\[ \frac{u_1 - u_2}{C_2} = h_3(x_3), \]

where
Chapter 6: Solution of the Riemann Problem

\[ h_3(x_3) = \begin{cases} \frac{2}{N-1}[\left(\frac{1+B_1}{e^{-x_3}+B_1}\right)^r - 1], & x_3 \geq 0 \\ \frac{1}{\sqrt{N}} \left[ \frac{(\rho_3^2)(1-p_3)}{(\rho_1^2+p_1)(\rho_3^2-1)} \right]^{1/2}, & x_3 < 0. \end{cases} \] (6.3.87)

From the above relations, one has

\[ u_1 - u_4 = C_2 h_3(x_3) + C_4 h_1(x_1), \] (6.3.88)

and

\[ \frac{C_2}{C_1} = \left( \frac{1+B_1}{e^{-x_3}+B_1} \right)^r, \] (6.3.89)

\[ \frac{P_1}{P_4} = \frac{P_1}{P_2} \frac{P_2}{P_4} = e^{x_3-x_1}, \] (6.3.90)

\[ \frac{\rho_1}{\rho_4} = \frac{\rho_1 \rho_2 \rho_3}{\rho_2 \rho_3 \rho_4} = f_3(x_3)e^{x_3}f_1(x_1), \] (6.3.91)

\[ x_2 = \ln\left[ \frac{\rho_1}{\rho_4} (f_3(x_3)f_1(x_1))^{-1} \right], \] (6.3.92)

\[ x_1 = x_3 + \ln\left( \frac{P_4}{P_1} \right). \] (6.3.93)
Then
\[
\frac{u_1 - u_4}{C_1} = \frac{C_1}{C_1} h_1(x_1) + h_3(x_2) \left( \frac{1 + B_1}{e^{x_2} + B_1} \right)^{\frac{1}{g}}.
\] (6.3.94)

By substituting (6.3.93) into (6.3.94), the ensuing equation can be iteratively solved by the method of interval bisection, to obtain \( x_3 \). No convergence problems are experienced. \( x_1, x_2 \) follow from (6.3.92, 6.3.93). Thus, the Riemann problem for water has been solved. Equations (6.3.87) provides contact surface velocity.

**Special Note:**

For water/water situations, use of the same value for \( B \) on the two sides of a contact surface will not permit a density discontinuity in crossing the contact surface! Thus, if it is assumed that \( B \) is constant, then choosing \( x_2 = 0 \) and discarding (6.3.91) prohibits the appearance of a contact discontinuity.

### 6.4 A Water-Adapted Gas Dynamics Riemann Solver

The relations given by equations (6.2.11-6.2.21) and (6.2.29-6.2.52) form the basis for solving the classical Riemann problem for the ideal gas (see Sod [17] for further amplification). Also, relations (6.2.22-6.2.28) and (6.2.53-6.2.57) allow construction of a Riemann solver code for water, which has been accomplished during the course of this research.
However, in liquid media the omission of an expression for internal energy forces the Rankine-Hugoniot jump conditions to be employed in enthalpy form:

**Continuity:**

\[ \rho_1 v_1 = \rho_2 v_2. \]  \hspace{1cm} (6.4.95)

**Momentum:**

\[ P_1 + \rho_1 v_1^2 = P_2 + \rho_2 v_2^2. \]  \hspace{1cm} (6.4.96)

**Energy:**

\[ h_1 + \frac{v_1^2}{2} = h_2 + \frac{v_2^2}{2}. \]  \hspace{1cm} (6.4.97)

Here, as before,

\[ v = u - W. \]  \hspace{1cm} (6.4.98)

Moreover, Holl [13] has shown that when the modified-Tait equation of state is used, the enthalpy rise across a water shock, referred to in (6.4.97), can be well-approximated by the relation

\[ h_2 - h_1 = \frac{N}{N - 1} \left( \frac{\hat{P}_2}{\rho_2} - \frac{\hat{P}_1}{\rho_1} \right). \]  \hspace{1cm} (6.4.99)

Therefore, to good approximation equation (6.4.97) can be replaced with

\[ \frac{1}{N - 1} C_1^2 + \frac{v_1^2}{2} = \frac{1}{N - 1} C_2^2 + \frac{v_2^2}{2}. \]  \hspace{1cm} (6.4.100)
Now, the jump conditions (6.4.95, 6.4.96) hold regardless of whether the medium is gas or liquid. Furthermore, for the case of an ideal gas, equation (6.4.97) is equivalent to

\[
\frac{1}{\gamma - 1} C_1^2 + \frac{v_1^2}{2} = \frac{1}{\gamma - 1} C_2^2 + \frac{v_2^2}{2}.
\]

(6.4.101)

It is noted that the respective formulas for the speed(s) of sound for water and gas are

\[
C^2 = \frac{N\dot{P}}{\rho}, \quad \text{(6.4.102)}
\]

and

\[
C^2 = \frac{\gamma P}{\rho}. \quad \text{(6.4.103)}
\]

The conclusion is that the correspondence

\[
\dot{P} \iff P, N \iff \gamma,
\]

(6.4.104)

transforms gas shock relations to water shock relations. Of course, the same observation holds regarding relations across rarefaction waves. The outcome is that the following theorem has been established:

**Theorem 1.** To good approximation the correspondence \( \dot{P} \iff P, N \iff \gamma \) transforms the solution of the Riemann problem for water, with modified-Tait
equation of state, into the solution of the Riemann problem for the ideal gas, and conversely.

**Lemma 1.** Any computer code which solves the Riemann problem for the ideal gas can be adapted, via equation (6.4.104), to approximation of the solution to the Riemann problem for the ideal water (modified-Tait equation of state).

**Note 1:** For the underwater explosion problem, it is shown in the sequel that the result of applying Theorem 1 and Lemma 1 leads to negligible difference in results, from that of coding the bona-fide water Riemann solver, determined by the relations of section 6.3.

### 6.5 Shocks in Water

In this section equation (6.2.56), and (6.3.79) will be derived. From equation (6.4.96) one has

\[ \rho_1(W - u_1) = \rho_2(W - u_2), \quad (6.5.105) \]

and

\[ u_2 = W(1 - \frac{\rho_1}{\rho_2}) + u_1 \frac{\rho_1}{\rho_2}. \quad (6.5.106) \]

From the modified-Tait equation of state one has

\[ \frac{\rho_2}{\rho_1} = \left( \frac{P_2 + B}{P_1 + B} \right)^{1/N} = \left( \frac{\bar{P}_2}{\bar{P}_1} \right)^{1/N}. \quad (6.5.107) \]
Chapter 6: Solution of the Riemann Problem

Rewrite (6.5.106) as

$$u_2 = W\left[1 - P_{21}^{-1/N}\right] + u_1 P_{21}^{-1/N},$$ (6.5.108)

where $P_{21} = \frac{\bar{P}_2}{\bar{P}_1}$.

From the momentum equation (6.4.96), one has

$$\bar{P}_1 + \rho_1(W - u_1)^2 = \bar{P}_2 + \rho_2(W - u_2)^2,$$ (6.5.109)

therefore

$$\bar{P}_1 + \rho_1(W - u_1) = \bar{P}_2 + \rho_2(W - u_2),$$ (6.5.110)

and

$$u_2 = \frac{\bar{P}_2 - \bar{P}_1}{m} + u_1.$$ (6.5.111)

From (6.5.108) the shock speed $W$ may be written as

$$W = \frac{u_1 - P_{21}^{1/N} u_2}{1 - P_{21}^{1/N}}.$$ (6.5.112)

One may substitute (6.5.111) into (6.5.112), to obtain

$$W = u_1 - \frac{P_{21}^{1/N} (\bar{P}_2 - \bar{P}_1)}{m(1 - P_{21}^{1/N})},$$ (6.5.113)

or
Chapter 6: Solution of the Riemann Problem

\[ W - u_1 = \frac{P_{21}^{1/n}(\bar{P}_2 - \bar{P}_1)}{\rho_1(W - u_1)(1 - P_{21}^{1/n})}. \]  

(6.5.114)

Then

\[ W = u_1 \pm \sqrt{\frac{P_{21}^{1/n}(\bar{P}_2 - \bar{P}_1)}{\rho_1(P_{21}^{1/n} - 1)}}. \]  

(6.5.115)

Here, the (+) sign is used for 3-shocks, and (-) for 1-shocks. Substitute (6.5.115) into (6.5.106), one has

\[ u_1 - u_2 = \frac{\rho_1(P_1 - P_2)}{\rho_2(\rho_1 - \rho_2)} - \frac{\rho_2(P_1 - P_2)}{\rho_1(\rho_1 - \rho_2)}. \]  

(6.5.116)

Since

\[ C_2 = \sqrt{\frac{N\bar{P}_2}{\rho_2}}, \]  

(6.5.117)

then, for 3-shocks one has

\[ \frac{u_1 - u_2}{C_2} = \frac{1}{\sqrt{N}} \left( \frac{\rho_1(P_1 - P_2)}{P_2\rho_2(\rho_1 - \rho_2)} - \frac{\rho_2(P_1 - P_2)}{\rho_1\bar{P}_2(\rho_1 - \rho_2)} \right). \]  

(6.5.118)

For 1-shocks, one can derive the following:

\[ \rho_3(W - u_3) = \rho_4(W - u_4), \]  

(6.5.119)

and

\[ u_3 = W(1 - \frac{\rho_4}{\rho_3}) + u_4\frac{\rho_4}{\rho_3}. \]  

(6.5.120)
From the modified-Tait equation of state one has

\[
\frac{\rho_3}{\rho_4} = \left( \frac{P_3 + B}{P_4 + B} \right)^{1/N} = \left( \frac{\tilde{P}_3}{\tilde{P}_4} \right)^{1/N}.
\] (6.5.121)

Equation (6.5.120) may now be rewritten as

\[
u_3 = W \left[ 1 - P_{34}^{-1/N} \right] + u_1 P_{34}^{-1/N},
\] (6.5.122)

where \(P_{34} = \frac{P_3}{P_4}\).

From the momentum equation (6.4.96), one has

\[
\tilde{P}_3 + \rho_3 (W - u_3)^2 = \tilde{P}_4 + \rho_4 (W - u_4)^2,
\] (6.5.123)

therefore

\[
\tilde{P}_3 + m \rho_3 (W - u_3) = \tilde{P}_4 + m \rho_4 (W - u_4),
\] (6.5.124)

and

\[
u_3 = \frac{\tilde{P}_3 - \tilde{P}_4}{m} + u_4.
\] (6.5.125)

From (6.5.108) the shock speed \(W\) may be written as

\[
W = \frac{u_4 - P_{34}^{1/N} u_3}{1 - P_{34}^{1/N}}.
\] (6.5.126)

One may substitute equation (6.5.125) into (6.5.126), to obtain
\[ W = u_4 - \frac{P_{34}^{1/N}(\bar{P}_3 - \bar{P}_4)}{\rho_4 m(1 - P_{34}^{1/N})}, \]  
\[ (6.5.127) \]

or

\[ W - u_4 = \frac{P_{34}^{1/N}(\bar{P}_3 - \bar{P}_4)}{\rho_4 (W - u_4)(1 - P_{34}^{1/N})}. \]  
\[ (6.5.128) \]

Then

\[ W = u_4 \pm \sqrt{\frac{P_{34}^{1/N}(\bar{P}_3 - \bar{P}_4)}{\rho_4 (P_{34}^{1/N} - 1)}}. \]  
\[ (6.5.129) \]

Here, the (-) sign is used for 1-shocks. After substituting (6.5.129) into (6.5.127),

one has

\[ u_3 - u_4 = \sqrt{\frac{\rho_4 (P_4 - P_3)}{\rho_3 (P_4 - P_3)}} - \sqrt{\frac{\rho_2 (P_4 - P_3)}{\rho_4 (P_4 - P_3)}}. \]  
\[ (6.5.130) \]

Since

\[ C_4 = \sqrt{\frac{N\bar{P}_4}{\rho_4}}, \]  
\[ (6.5.131) \]

for 1-shocks, one has the relation

\[ \frac{u_3 - u_4}{C_4} = \frac{1}{\sqrt{N}} \sqrt{\frac{\rho_2^2 (P_4 - P_3)}{\bar{P}_4 \rho_3 (P_4 - P_3)}} - \frac{1}{\sqrt{N}} \sqrt{\frac{\rho_2 (P_4 - P_3)}{\bar{P}_4 (P_4 - P_3)}}. \]  
\[ (6.5.132) \]
Chapter 7
An Underwater Explosion Problem

7.1 Introduction

An explosion is a physical phenomenon resulting from a sudden release of energy. It may come from a detonation such as gunpowder, TNT, a nuclear blast, an earthquake, or lightning, etc. A large explosion may inflict widespread damage and create major disaster. To minimize such effects some technical knowledge of the explosion phenomena becomes essential. It is also helpful in the design and construction of disaster resistant facilities. In this chapter, concern is directed to the area of underwater explosions.

When a spherical charge of explosive submerged in a large body of water is ignited at its center, the following sequence of phenomena occurs. A combustion wave travels from the center toward the surface, converting the solid into gaseous by-products. Fragmentation of the solid shell accompanies the arrival of the detonation wave at the solid boundary. As the gas by-products expand, they
create a shock wave which advances into the water side, and a reflected wave recedes into the gas sphere. The reflected wave, depending on the circumstances, may initially be a shock wave, but it changes rapidly to become a rarefaction wave behind the advancing boundary.

The purpose of this chapter is to give a method to numerically simulate the effects of an underwater explosion for a given spherical charge. This numerical scheme has sub-cell resolution capability in contact surface tracking. The numerical model will be based on the following assumptions:

1. The burnt gases are assumed to be at rest and with a uniform pressure distribution at the instant of detonation. This is not exactly true, because such a model neglects the effects of fragmentation as the phase of solid burn out concludes. Additionally, there will be a higher pressure in the inner layer and a lower pressure in the outer layer, plus an outward motion. However, in simulating long term, far field effects of the explosion, neglect of such phenomena do not represent serious flaws.

2. The underwater explosion is assumed deep enough so that the motion of the shock waves will not be affected by surface interactions.

3. The one-dimensional equation of motion for an inviscid, non-heat-conducting fluid, with gravitational force neglected, are used.

4. It is reasonable to assume that the spherical gas bubble is a perfect gas,
and that the water satisfies the modified Tait equation of state.

In this chapter there is considered an underwater explosion problem which has been previously investigated by Holt and Flores [5] (1981), also Charrier and Tessieras [7] (1986). However, a superior method for tracking the contact surface is employed: the sub-cell resolution technique.

### 7.2 Test Problem: Underwater Release of a Sphere of Gas at High Pressure

Table 7.1 gives the particulars of the underwater explosion which is to be numerically simulated.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Charge Radius</td>
<td>$\frac{1}{3}$ ft</td>
</tr>
<tr>
<td>Depth of Charge Center</td>
<td>1 ft</td>
</tr>
<tr>
<td>Initial Pressure of Explosion Gas</td>
<td>9000 atm</td>
</tr>
<tr>
<td>Initial Explosion Gas Temperature</td>
<td>2500°K</td>
</tr>
<tr>
<td>Specific Heat Ratio of Explosion Gas</td>
<td>1.4</td>
</tr>
<tr>
<td>Initial Water Pressure</td>
<td>1 atm</td>
</tr>
<tr>
<td>Modified Tait Equation of State:</td>
<td>$B = 3268$ atm, $\rho_0 = 1007$ kg/m³, N = 7</td>
</tr>
</tbody>
</table>

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
In what follows, all graphical results are non-dimensionalized as follows: Let $(\sim)$ denote dimensional quantities, and choose $h = 1$ ft as a characteristic length. All lengths are divided by the characteristic length, $R = \frac{h}{h}$. Velocities are divided by the speed of sound in the undisturbed water, $u = \frac{u}{u}$; densities are given by $\rho = \frac{\rho}{\rho}$; pressures are given by $P = \frac{P}{P}$, where $P = P_w + B$; $P_w = 1$ atm in undisturbed water; and time is divided by $t = \frac{t}{t}$. The space step is either $\Delta R = 0.01$ (coarse grid) or $\Delta R = 0.00125$ (fine grid). For stability, the time step is chosen to satisfy the condition

$$\max(|u| + a)\Delta t / \Delta R < 1,$$

where $a$ is local sound speed.
7.3 The Godunov Method

The Godunov method [9] for solving the conservation law equations

\[ U_t + F(U)_R = 0, \quad (7.3.1) \]

can be written as

\[ U_j^{n+1} = U_j^n - \Delta t \frac{\Delta X}{\Delta R} \left[ F_{j+\frac{1}{2}}^* - F_{j-\frac{1}{2}}^* \right]. \quad (7.3.2) \]

The flux \( F_{j+\frac{1}{2}}^* \) is obtained through solving the Riemann problem(s)

\[ U_t + F(U)_R = 0, \quad (7.3.3) \]

\[ U(x,0) = \begin{cases} U_L, & x < x_i \\ U_R, & x > x_i \end{cases} \quad (7.3.4) \]

over a uniform grid of mesh-width \( \Delta R \). The methods of Chapter 6 are used, and (7.3.3, 7.3.4) is solved separately (i) in the gas region, (ii) in the water region, and (iii) for the variable size cells adjacent to the contact surface. Source terms not indicated by (7.3.3) are accounted for by the splitting technique elaborated in Chapter 2. The contact surface position is updated each time step, under the assumption of a constant contact surface velocity over the step, which is obtained by the methods of Chapter 6. Solution of the Riemann problem(s) (7.3.3, 7.3.4) is also accomplished by the methods of Chapter 6.
7.4 Numerical Results

As pointed out in Chapter 1, the underwater explosion problem has been studied by Flores-Holt [5], and Charrier-Tessiaras [7]. The results [5] are suspiciously inconsistent, with supersonic mach numbers occurring in an expansion wave (subsonic) region! Except for vast differences in wave and contact surface speeds, these results are qualitatively similar to the present results.

Figures 7.1, 7.2 show (coarse grid) results of Charrier-Tessiaras [7] for density and pressure distributions, at the time the contact discontinuity between gas and water has advanced from $\frac{5}{3}$ ft to 0.385 ft. Truly, contact surface position is perfectly sharp; however, data scatter (oscillation) near both shock and contact discontinuities seems to be an inherent drawback of the hybrid method.

For the present sub-cell resolution scheme, results depicted by Figures 7.3-7.6 have been calculated by two approaches: (a) a bona-fide Riemann solver is used for the water region, and (b) the Riemann solver for the water side has been adapted from a gas dynamics Riemann solver, according to the transformation of equation (6.4.104). The general quality of the results speak for themselves. Moreover, it is clear that negligible error occurs due to the adaptation of the gas dynamics Riemann solver. From the pressure and density results of Figures 7.7, 7.8, it is clear that Charrier-Tessiaras [7] and the present work, which substantially agree (see Figure 7.7 (coarse grid density profile), and Figure 7.8 (fine vs. coarse grid pressure profile)), are clearly in conflict with Holt- Flores [5] results,
the substance of which is apparently incorrect.

Figure 7.9 shows the effects of mesh refinement, when changing from the coarse to the fine grid. Figure 7.10 exhibits (fine grid) pressure profiles at times $t = 0.05, 0.15$. Clearly, the shock strength is decaying with time, as is the expected physical trend. Results not shown here indicate that the contact surface velocity initially increases, with later a trend reversal. The velocity eventually approaches zero, but after a much longer time period than is indicated by the final figure of [5].
Chapter 7: An Underwater Explosion Problem

Glimm-Lax-Wendroff method, the Contact Discontinuity Located at \( R = 0.385 \)

Figure 7.1:
Data Oscillation in Pressure Characteristic of a Hybrid (RCM) Front Tracking Scheme

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Chapter 7: An Underwater Explosion Problem

The Glimm–Lax–Wendroff method, the Contact Discontinuity is located at \( r = 0.305 \).

**Figure 7.2:**
Data Oscillation in Density Characteristic of a Hybrid (RCM) Front Tracking Scheme
Figure 7.3:
Comparison of Results for an Exact Riemann Solver for Water and an Approximate Riemann Solver (Density)
Chapter 7: An Underwater Explosion Problem

Figure 7.4:
Comparison of Results for an Exact Riemann Solver for Water and an Approximate Riemann Solver (Pressure)
Figure 7.5:
Comparison of Results for an Exact Riemann Solver for Water and an Approximate Riemann Solver (Velocity)
Figure 7.6:
Comparison of Results for an Exact Riemann Solver for Water and an Approximate Riemann Solver (Mach Number)
A Comparison of Glimm-Lax-Wendroff and Godunov method for Density Profile

Figure 7.7:
Comparison of the Hybrid Method of Charrier-Tessiaras and the Sub-cell Resolution Scheme (Density)

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Chapter 7: An Underwater Explosion Problem

Figure 7.8:
Comparison of the Hybrid Method of Charrier-Tessiaras and the Sub-cell Resolution Scheme (Pressure)
Figure 7.9:
Grid Size Effects for the Sub-cell Resolution Scheme

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Density Distributions at Times 0.05, 0.15: 800 Grid Points

Figure 7.10:
Manifestation of Time Decay in Shock Strength
Chapter 8

Summary and Conclusions

1. The method of Roe [1] has been adapted to the problem of obtaining a first order accurate, upwind-biased shock capturing scheme for the medium pure water and general equation of state. With minor modification, the method can be applied to seawater problems.

2. An averaging scheme which aids shock recognition and which allows Roe's piecewise linearization method to be applied to the enthalpy oriented Euler equations (2.2.1-2.2.4) has been derived, and tested. It is seen that the Roe-Glaister averaging can be applied, perhaps unexpectedly, to pseudo-conservative, divergence form systems. Equivalent numerical results are obtained, for Chen-Cooke averaging, as for Roe or Glaister averaging.

3. Availability of these averaging schemes now allows application of the excellent second-order accurate TVD scheme of Harten, by means of Strang's
Chapter 8: Summary and Conclusions

[15] operator splitting, to calculation of shockwaves in water. The method has been demonstrated, through application to typical Riemann problems for both the ideal gas and the medium pure water.

4. A method for predicting contact surface velocity and pressure at a gas/water interface has been derived. Methods of solving the Riemann problems for (i) gas, (ii) water, and (iii) at a gas-water interface are indicated.

5. As well, it is seen that, to good approximate, solutions of the Riemann problem for water map into solutions of the Riemann problem for the ideal gas under a simple transformation. Numerical experiments indicate that this correspondence can be employed as a vehicle to adapt gas dynamics Riemann solver codes for use in water.

6. The sub-cell resolution technique coupled with the Godunov scheme gives very good results for the underwater explosion problem, as compared with Glimm-Lax-Wendroff and the Random Choice method, in one space dimension. Thus, it would be interesting to extend this capability to solution of higher dimensional problems.

7. In the future, one could anticipate using the sub-cell resolution method to adapt Harten's TVD scheme to the analysis of other shock phenomena in water. An interesting application would be simulation of the conical shock tube [14], which has been developed to test the effects of underwater
explosions on sonar transducers.
Bibliography


15. G. STRANG, Construction and Comparison of Difference Schemes, SINUM, 5, (1968), 506-517


Biography

Tze-Jang Chen was born on April 8, 1957 in Taipei, Taiwan, Republic of China. He graduated from National Tsing-Hua University in May, 1979 with a B.S. in Mathematics and graduated from National Tsing-Hua University in May, 1981 with an M.S. in Applied Mathematics. He was awarded a Ph.D in Mathematics from Old Dominion University in August 1991.

He is a co-author of the following papers:


He also has the following work experience:


2. Teaching Assistant at University of South Carolina (1987-1989).
