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A system equivalence related to Dulac's extension of Bendixson's negative theorem for planar dynamical systems

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Abstract

Bendixson's Theorem [H. Ricardo, A Modern Introduction to Differential Equations, Houghton-Mifflin, New York, Boston, 2003] is useful in proving the non-existence of periodic orbits for planar systems

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x, y), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = G(x, y) \tag{1}$$

in a simply connected domain D, where F, G are continuously differentiable. From the work of Dulac [M. Kot, Elements of Mathematical Ecology, 2nd printing, University Press, Cambridge, 2003] one suspects that system (1) has periodic solutions if and only if the more general system

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = B(x, y)F(x, y), \quad \frac{\mathrm{d}y}{\mathrm{d}\tau} = B(x, y)G(x, y) \tag{2}$$

does, which makes the subcase (1) more tractable, when suitable non-zero $B(x, y)$ which are $C^1(D)$ can be found. Thus, Bendixson's Theorem can be applied to system (2), where otherwise it is unfruitful in establishing the non-existence of periodic solutions for system (1). The object of this note is to give a simple proof justifying this Dulac-related postulate of the equivalence of systems (1) and (2).

Keywords: Periodic solutions of planar systems; Bendixson–Dulac Theorem; Parameterized system equivalence

1. Bendixson's Theorem

Let us begin by stating Bendixson's Theorem [1]. If F, G in system (1) above are $C^1(D)$ functions in a simply connected domain D, and the divergence of the tangent vector field $\nabla.(F, G)$ is non-zero and of one algebraic sign in D, then system (1) has no periodic orbits in domain D.

Proof. The proof is accomplished by the method of contradiction, using Green's theorem for the plane. Assume there is a periodic solution of $\frac{\mathrm{d}x}{\mathrm{d}t} = F(x, y)$, $\frac{\mathrm{d}x}{\mathrm{d}t} = F(x, y)$, in D, which traverses a simple closed curve Γ, with $\nabla.(F, G)$

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of one sign within Γ. Then

$$\iint_A \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \mathrm{d}A = \oint_\Gamma [-G\mathrm{d}x + F\mathrm{d}y] = \oint_\Gamma [-GF + FG)\mathrm{d}t = 0,$$

where A is the simply connected interior of Γ.

This statement expresses a contradiction, since $\nabla.(F, G)$ is of one sign and non-zero in D. Therefore, the assumption of a periodic solution in D is false.

System (1) has no periodic solution in D. □

2. Dulac's extension of Bendixson's Theorem

Dulac's extension of Bendixson's Theorem is discussed in Ref. [2].

Dulac's extension is needed whenever one suspects there are no periodic solutions in D, but $\nabla.(F, G)$ is not of one sign. One now seeks a suitable non-zero $B(x, y)$ such that $\nabla.(BF, BG)$ is of one sign in D; then use Bendixson's Theorem to conclude that system (2) has no periodic orbit in D. But, by the Dulac postulate established below, systems (1) and (2) are equivalent, so neither does system (1) have a periodic orbit in D.

To establish the Dulac-related postulate in a simple way, observe that under the reparameterization $t = t(\tau)$ defined on a trajectory Γ of (2) by

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = B[x(\tau), y(\tau)], \quad \text{or } t = \int_0^\tau B[x(s), y(s)]\mathrm{d}s, \tag{3}$$

system (2) transforms into system (1). Hence, system (1) has a periodic orbit in D iff system (2) does. But system (2) does not have a periodic orbit in D, according to Bendixson's Theorem.

Clearly, the reparameterization (3) exists and is well-defined, although the time-scaling is trajectory dependent.

3. Conclusions

A simple proof of a Dulac-related postulate for equivalence between the trajectories of systems (1) and (2) has been established. The reparameterization argument given here seems much cleaner than the alternative argument that systems (1) and (2) have equivalent first integrals because their slope functions along trajectories are identical, when $B(x, y)$ is non-zero and can be cancelled. But, one must now argue that first integrals always exist, because the Dictionary of Mathematics says an integrating factor for such systems always exists. A proof of this has not been seen by the author; so the present simple proof was intended to circumvent having to find an alternative in the literature.

Actually, as Dulac established, it is not necessary to know that systems (1) and (2) are equivalent: rather, the essential ingredient is only the certainty that there exists a differentiable function $B(x, y)$ with divergence $\nabla.(BF, BG)$ having one algebraic sign in a simply connected region D. Then, for any simple closed curve Γ in D, by Green's Theorem for the plane:

$$\iint_A \frac{\partial(BF)}{\partial x} + \frac{\partial(BG)}{\partial y} \mathrm{d}A = \oint_\Gamma B[-G\mathrm{d}x + F\mathrm{d}y] = \oint_\Gamma B\left[-G\frac{\mathrm{d}x}{\mathrm{d}t} + F\frac{\mathrm{d}y}{\mathrm{d}t}\right]\mathrm{d}t.$$

Thus, if it is assumed that $\Gamma : (x(t), y(t))$ is a periodic solution of (1) which lies within D and has simply connected interior A, the above relation gives a contradiction. The right member vanishes ($-GF+FG = 0$), but the left member does not; it is either a positive or negative non-zero number, depending upon the sign of $\nabla.(BF, BG)$.

Equivalence of systems (1) and (2) does not appear to have been established previously, but the above proof for Dulac's extended theorem appears in [3].

References

[1] H. Ricardo, A Modern Introduction to Differential Equations, Houghton-Mifflin, New York, Boston, 2003.
[2] M. Kot, Elements of Mathematical Ecology, 2nd printing, University Press, Cambridge, 2003.
[3] A.C. King, J. Billingham, S.R. Otto, Differential Equations; Linear, Nonlinear, Ordinary, & Partial, Cambridge University Press, 2003.