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**ON THE THERMAL STRESSES DUE TO
A UNIFORM HEAT FLOW PAST A CIRCULAR
HOLE WITH A RADIAL EDGE CRACK**

by

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**A Dissertation Submitted to the Faculty of Old Dominion University
in Partial Fulfilment of the Requirements for the Degree of**

**DOCTOR OF PHILOSOPHY
in
COMPUTATIONAL AND APPLIED MATHEMATICS**

December, 1987

Approved by,

J. L. Tweed (Director)

ABSTRACT

On The Thermal Stresses Due To A Uniform Heat Flow Past a Circular Hole with a Radial Edge Crack

George F. Edmonds
Old Dominion University, 1987
Director: Dr. John Tweed

The problem solved in this dissertation is that of finding the stresses in an isotropic, linear, thermoelastic solid when a uniform heat flow is disturbed by the presence of an insulated circular hole with a radial edge crack. By superimposing a Mellin transform solution of the equations of thermoelasticity on a Michell series solution the author reduces the problem to a pair of singular integral equations which are then solved numerically. The stress intensity factors and crack formation energies, quantities of interest to workers in fracture mechanics, are then calculated.

To John Tweed for all his help and guidance and to my wife for her patience.

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CHAPTER 1

INTRODUCTION

The problem in this dissertation is that of finding the stress in an infinite, isotropic, homogeneous and linear thermoelastic solid when a uniform heat flow

$$q = q_0(\cos \theta_0 \, \underline{i} + \sin \theta_0 \, \underline{j}) \quad (1.1)$$

is disturbed by the presence of an insulated circular hole $0 \leq R \leq B$, $0 \leq \theta \leq 2\pi$ with a radial edge crack $B \leq R \leq C$, $\theta = 0$.

The problem of determining the stresses near the tip of a crack at the edge of a circular hole in an infinite isothermal elastic solid appears to have been first investigated by Bowie [1]. Using an complex mapping technique Bowie obtained an approximate solution for the case of a pressurized crack. Thereafter, using Naylor's finite Mellin transform [3], Tweed [2] found an exact solution for the case in which the crack is subject to a longitudinal shear load. Tweed and Rooke [4] then showed that, in the plane strain case, the stress intensity factor and the crack formation energy are related to the solution of a weakly singular Fredholm equation. Using this technique, they calculated the stress intensity factors and crack formation energy to a high degree of accuracy.

Tweed and Rooke [5] have also solved the problem when the hole and the crack are subject to arbitrary in-plane loads. Via superposition, they have shown that this case is equivalent to that in which the hole is traction free and the crack is subject to loads of the form:

$$\sigma_{\theta\theta}(R, 0) = \sigma_{\theta\theta}(R, 2\pi) = -p_0 f_1\left(\frac{R}{B}\right), \quad B < R < C, \quad (1.2)$$

$$\sigma_{R\theta}(R, 0) = \sigma_{R\theta}(R, 2\pi) = -p_0 f_2\left(\frac{R}{B}\right), \quad B < R < C. \quad (1.3)$$

Here they reduce the problem to a pair of singular integral equations which are solved using numerical methods. A full discussion of the early work on crack problems of linear elasticity can be found in Sneddon and Lowengrub [6].

When a uniform heat flow in an infinite thermoelastic solid is disturbed by the presence of an insulated circular hole and radial edge crack, the resulting thermal stresses force the crack to open and may cause it to spread. The strategy for analysing this situation involves calculating the stresses at the proposed crack site when the crack is not present for, by linearity, these stresses may be interpreted as the crack opening loads in an equivalent isothermal elastic problem. Once these loads are determined the technique of Tweed and Rooke [5] can be used to calculate the stress intensity factors and the crack formation energy which are used to predict crack growth. Hence, a major portion of this dissertation is concerned with determining these loads.

In Chapter 2 a special case of Michell's solution as well as a Mellin transform solution of the equations of thermoelasticity are described. The special case of the Michell solution is due to Tweed who first derived it in reference [7].

In Chapter 3 the solution of the equations of thermoelasticity for the case in which a uniform heat flow is disturbed by an insulated circular hole is given. The author's problem then reduces to that of determining pertur-

bations in this solution due to the presence of the crack. The problem of determining these perturbations is resolved into two component problems:

PROBLEM 1.

In the absence of a crack and under the assumption that the hole and proposed crack site are thermally insulated determine the thermoelastic stresses occurring at the proposed site as a result of the disturbed heat flow.

PROBLEM 2.

Solve the equations of isothermal elasticity in the region external to the hole $0 \leq R \leq B$, $0 \leq \theta \leq 2\pi$ and the crack $\theta = 0$, $B \leq R < C$ when the crack is opened by the stress loadings obtained in Problem 1.

These problems are solved in Chapters 4 and 5, respectively. In Chapter 6, the numerical procedures used are described and the numerical results are given.

There are three appendices. Appendix A contains sums of series used to calculate the shear load function $s_{r\theta}(r, 0) = f_2(r)$ on the crack. Appendix B contains a table of integrals needed to evaluate the load functions and the crack formation energy. Appendix C contains the author's computer program STRESS, which was used to produce the numerical results described in Chapter 6.

The numbering of equations in the text is a standard one. Equations are numbered consecutively within a section by $[m.n]$ where m is the section number, n refers to the n^{th} equation in the section. Equations are referenced by $(k.m.n)$ where k is the number of the chapter in which the equation occurs.

CHAPTER 2

SOLUTIONS OF THE EQUATIONS OF THERMOELASTICITY

1. General Results

We consider an isotropic, homogeneous thermoelastic solid which occupies the (R, θ) plane and is initially at temperature T_0 at which time it is free of stress and strain. If the solid is subjected to a uniform heat flow

$$\underline{q} = q_0 \{ \cos \theta_0 \underline{i} + \sin \theta_0 \underline{j} \} \quad (1.1)$$

it then acquires a displacement field:

$$\underline{U}(R, \theta) = U_R(R, \theta) \underline{1}_R + U_\theta(R, \theta) \underline{1}_\theta \quad (1.2)$$

and a temperature field

$$T = T_0 + \Psi(R, \theta) \quad (1.3)$$

which produce a strain field $e_{RR}(R, \theta)$, $e_{R\theta}(R, \theta)$, $e_{\theta\theta}(R, \theta)$ and a corresponding stress field $\sigma_{RR}(R, \theta)$, $\sigma_{R\theta}(R, \theta)$, $\sigma_{\theta\theta}(R, \theta)$.

As usual, we shall let κ denote the thermal conductivity, α the coefficient of linear expansion, ν Poisson's ratio, μ the shear modulus, E Young's modulus, and $\Phi(R, \theta)$ the Airy stress function. The strains are then related to the displacements by the equations:

$$e_{RR} = \frac{\partial U_R}{\partial R}, \quad (1.4)$$

$$e_{R\theta} = \frac{1}{2} \left(\frac{1}{R} \frac{\partial U_R}{\partial \theta} + \frac{\partial U_\theta}{\partial R} - \frac{U_\theta}{R} \right), \quad (1.5)$$

$$e_{\theta\theta} = \frac{1}{R} \frac{\partial U_\theta}{\partial \theta} + \frac{U_R}{R} \quad (1.6)$$

and the stress-strain relations are given by

$$e_{RR} = \frac{1+\nu}{E} \left[(1-\nu)\sigma_{\theta\theta} - \nu\sigma_{RR} \right] + (1+\nu)\alpha\Psi, \quad (1.7)$$

$$e_{R\theta} = \frac{1+\nu}{E} \sigma_{R\theta}, \quad (1.8)$$

$$e_{\theta\theta} = \frac{1+\nu}{E} \left[(1-\nu)\sigma_{\theta\theta} - \nu\sigma_{RR} \right] + (1+\nu)\alpha\Psi. \quad (1.9)$$

We can show [10] that, in the absence of body forces, the equations of equilibrium:

$$\frac{\partial \sigma_{RR}}{\partial R} + \frac{1}{R} \frac{\partial \sigma_{R\theta}}{\partial \theta} + \frac{\sigma_{RR} - \sigma_{\theta\theta}}{R} = 0, \quad (1.10)$$

$$\frac{\partial \sigma_{R\theta}}{\partial R} + \frac{1}{R} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2}{R} \sigma_{R\theta} = 0 \quad (1.11)$$

are satisfied if and only if the stresses are derived from an Airy stress function $\Phi(R, \theta)$ in the following manner:

$$\sigma_{RR} = \frac{1}{R} \frac{\partial \Phi}{\partial R} + \frac{1}{R^2} \frac{\partial^2 \Phi}{\partial \theta^2}, \quad (1.12)$$

$$\sigma_{R\theta} = -\frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \Phi}{\partial \theta} \right), \quad (1.13)$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \Phi}{\partial R^2}, \quad (1.14)$$

where $\Phi(R, \theta)$ satisfies the biharmonic equation

$$\nabla^4 \Phi(R, \theta) = 0 \quad (1.15)$$

while the temperature field must satisfy Laplace's equation

$$\nabla^2 \Psi(R, \theta) = 0, \quad (1.16)$$

and, for our problem, be periodic with period 2π in θ .

Before going on to determine the temperature field and the stress field we find it convenient to introduce the dimensionless quantities:

$$\begin{aligned} r &= \frac{R}{B}, \quad c = \frac{C}{B}, & \Theta(r, \theta) &= \frac{1}{T_0} \Psi(R, \theta) \\ u(r, \theta) &= \frac{EU_R(R, \theta)}{(1 + \nu)p_0 B}, & v(r, \theta) &= \frac{EU_\theta(R, \theta)}{(1 + \nu)p_0 B} \\ s_{rr}(r, \theta) &= \frac{\sigma_{RR}(R, \theta)}{p_0}, & s_{r\theta}(r, \theta) &= \frac{\sigma_{R\theta}(R, \theta)}{p_0}, \\ s_{\theta\theta}(r, \theta) &= \frac{\sigma_{\theta\theta}(R, \theta)}{p_0}, & \phi(r, \theta) &= \frac{\Phi(R, \theta)}{p_0 B^2} \end{aligned} \quad (1.17)$$

in which B is a characteristic length, and p_0 is a constant having dimensions of stress. In our problem B is taken to be the radius of the hole while p_0 is given by

$$p_0 = \frac{E\alpha B q_0}{2\kappa(1 - \nu)}. \quad (1.18)$$

2. Michell's Solution of the Equations of Thermoelasticity

By using separation of variables it is not difficult to show [7] that the temperature is given by

$$\begin{aligned}\Theta(r, \theta) = & A_0 + B_0 \ln r + \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n}) \cos n\theta \\ & + \sum_{n=1}^{\infty} (A'_n r^n + B'_n r^{-n}) \sin n\theta,\end{aligned}\tag{2.1}$$

the Airy stress function by

$$\begin{aligned}\phi(r, \theta) = & \{a_0 + b_0 \ln r + c_0 r^2 + d_0 r^2 \ln r\} \\ & + \{a'_0 + b'_0 \ln r + c'_0 r^2 + d'_0 r^2 \ln r\} \theta \\ & + \{a_1 r + b_1 r^{-1} + c_1 r^3 + d_1 r \ln r\} \begin{Bmatrix} \cos \theta \\ \sin \theta \end{Bmatrix} \\ & + \{a'_1 r + d'_1 r \ln r\} \begin{Bmatrix} \theta \cos \theta \\ \theta \sin \theta \end{Bmatrix} \\ & + \sum_{n=2}^{\infty} \{a_n r^n + b_n r^{-n} + c_n r^{2+n} + d_n r^{2-n}\} \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix},\end{aligned}\tag{2.2}$$

and the corresponding stresses by

$$\begin{aligned}
s_{rr}(r, \theta) &= \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \\
&= \{b_0 r^{-2} + 2c_0 + d_0 [1 + 2 \ln r]\} \\
&\quad + \{b'_0 r^{-2} + 2c'_0 + d'_0 [1 + 2 \ln r]\} \theta \\
&\quad + \{-2b_1 r^{-3} + 2c_1 r + d_1 r^{-1}\} \left\{ \frac{\cos \theta}{\sin \theta} \right\} \\
&\quad + \{2a'_1 r^{-1} + 2d'_1 r^{-1} \ln r\} \left\{ -\frac{\sin \theta}{\cos \theta} \right\} \\
&\quad + d'_1 r^{-1} \left\{ \frac{\theta \cos \theta}{\sin \theta} \right\} - \sum_{n=2}^{\infty} \{n(n-1)a_n r^{n-2} \\
&\quad + n(n+1)b_n r^{-n-2} + (n+1)(n-2)c_n r^n \\
&\quad + (n-1)(n+2)d_n r^{-n}\} \left\{ \frac{\cos n \theta}{\sin n \theta} \right\},
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
s_{r\theta}(r, \theta) &= -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \phi}{\partial \theta} \right) \\
&= \{a'_0 r^{-2} - b'_0 r^{-2} [1 - \ln r] - c'_0 - d'_0 [1 + \ln r]\} \\
&\quad + \{2b_1 r^{-3} - 2c_1 r - d_1 r^{-1}\} \left\{ -\frac{\sin \theta}{\cos \theta} \right\} - d'_1 r^{-1} \left\{ \frac{\cos \theta - \theta \sin \theta}{\sin \theta + \theta \cos \theta} \right\} \\
&\quad + \sum_{n=2}^{\infty} \{n(n-1)a_n r^{n-2} - n(n+1)b_n r^{-n-2} \\
&\quad + n(n+1)c_n r^n - n(n-1)d_n r^{-n}\} \left\{ -\frac{\sin n \theta}{\cos n \theta} \right\},
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
s_{\theta\theta}(r, \theta) &= \frac{\partial^2 \phi}{\partial r^2} \\
&= \{-b_0 r^{-2} + 2c_0 + d_0[2 \ln r + 3]\} \\
&+ \{-b'_0 r^{-2} + 2c'_0 + d'_0[2 \ln r + 3]\} \theta \\
&+ \{2b_1 r^{-3} + 6c_1 r + d_1 r^{-1}\} \left\{ \frac{\cos \theta}{\sin \theta} \right\} + d'_1 r^{-1} \left\{ \frac{\theta \cos \theta}{\sin \theta} \right\} \\
&+ \sum_{n=2}^{\infty} \{n(n-1)a_n r^{n-2} + n(n+1)b_n r^{-n-2} \\
&+ (n+2)(n+1)c_n r^n + (n-1)(n-2)d_n r^{-n}\} \left\{ \frac{\cos n\theta}{\sin n\theta} \right\}.
\end{aligned} \tag{2.5}$$

We note that the stresses are single-valued if and only if $b'_0 = c'_0 = d'_0 = d'_1 = 0$.

In order to determine the displacement field it is necessary to integrate the stress displacement relations

$$\frac{\partial u}{\partial r} = (1 - \nu)s_{rr} - \nu s_{\theta\theta} + \frac{E\alpha T_0}{p_0} \Theta(r, \theta), \tag{2.6}$$

$$\frac{\partial v}{\partial \theta} = [(1 - \nu)s_{\theta\theta} - \nu s_{rr}]r + \frac{E\alpha T_0}{p_0} r \Theta(r, \theta) - u, \tag{2.7}$$

$$s_{r\theta} = \frac{1}{2} \left[\frac{1}{r} \frac{\partial u}{\partial \theta} + r \frac{\partial}{\partial r} \left(\frac{v}{r} \right) \right] \tag{2.8}$$

which are obtained by substituting from equations [1.4-1.6] into [1.7-1.9]. After some effort this yields the following displacements corresponding to single-valued stresses to within an arbitrary translation or rotation:

$$\begin{aligned}
u(r, \theta) = & \{-b_0 r^{-1} + 2(1 - 2\nu)c_0 r + d_0 r[2(1 - 2\nu) \ln r - 1]\} \\
& + \frac{E\alpha T_0}{p_0} r \{A_0 + B_0 [\ln r - 1]\} \\
& + \{b_1 r^{-2} + c_1(1 - 4\nu)r^2 + d_1[(1 - 2\nu) \ln r - 2(1 - \nu)]\} \left\{ \frac{\cos \theta}{\sin \theta} \right\} \\
& + 2(1 - \nu)d_1 \left\{ -\frac{\theta \sin \theta}{\cos \theta} \right\} + a'_1 [2(1 - \nu) \ln r - (1 - 2\nu)] \left\{ -\frac{\sin \theta}{\cos \theta} \right\} \\
& + (1 - 2\nu)a'_1 \left\{ \frac{\theta \cos \theta}{\sin \theta} \right\} + \frac{E\alpha T_0}{p_0} \left\{ \frac{1}{2} A_1 r^2 + B_1 [\ln r - 1] \right\} \left\{ \frac{\cos \theta}{\sin \theta} \right\} \\
& + \frac{E\alpha T_0}{p_0} B_1 \left\{ -\frac{\theta \sin \theta}{\cos \theta} \right\} \\
& + \sum_{n=2}^{\infty} \left[\{-n a_n r^{n-1} + n b_n r^{-n-1} - c_n [n - 2 + 4\nu] r^{n+1} \right. \\
& \left. + d_n [n + 2 - 4\nu] r^{-n+1} \} \right] \left\{ \frac{\cos n\theta}{\sin n\theta} \right\} \\
& + \frac{E\alpha T_0}{p_0} \left\{ \frac{A_n}{n+1} r^{n+1} + \frac{B_n}{1-n} r^{1-n} \right\} \left\{ \frac{\cos n\theta}{\sin n\theta} \right\}, \tag{2.9}
\end{aligned}$$

$$\begin{aligned}
v(r, \theta) = & \{4(1 - \nu)d_0 r \theta - a'_0 r^{-1} + 4(1 - \nu)c'_0 r \ln r\} + \frac{E\alpha T_0}{p_0} B_0 r \theta \\
& + \{b_1 r^{-2} + (5 - 4\nu)c_1 r^2 + (1 - 2\nu)d_1[1 - \ln r]\} \left\{ -\frac{\sin \theta}{\cos \theta} \right\} \\
& + 2(1 - \nu)d_1 \left\{ \frac{\theta \cos \theta}{\sin \theta} \right\} - 2a'_1[\nu + (1 - \nu)\ln r] \left\{ \frac{\cos \theta}{\sin \theta} \right\} \\
& - (1 - 2\nu)a'_1 \left\{ -\frac{\theta \sin \theta}{\cos \theta} \right\} + \frac{E\alpha T_0 B_1}{p_0} \left\{ \frac{\theta \cos \theta}{\sin \theta} \right\} \\
& + \frac{E\alpha T_0}{p_0} \left\{ \frac{1}{2}A_1 r^2 + B_1[1 - \ln r] \right\} \left\{ -\frac{\sin \theta}{\cos \theta} \right\} \\
& - \sum_{n=2}^{\infty} \{n a_n r^{n-1} + n b_n r^{-n-1} + [n + 4 - 4\nu]c_n r^{n+1} \\
& + [n - 4 + 4\nu]d_n r^{-n+1}\} \left\{ -\frac{\sin n \theta}{\cos n \theta} \right\} \\
& + \frac{E\alpha T_0}{p_0} \left\{ \frac{A_n}{n+1} r^{n+1} - \frac{B_n}{1-n} r^{1-n} \right\} \left\{ -\frac{\sin n \theta}{\cos n \theta} \right\}. \tag{2.10}
\end{aligned}$$

3. The Mellin Transform Solution of the Equations of Thermoelasticity

We introduce the Mellin transform pair [9]

$$\begin{aligned}\bar{\phi}(s, \theta) &= M[\phi(r, \theta); \quad r \rightarrow s] \\ &= \int_0^{\infty} \phi(r, \theta) r^{s-1} dr\end{aligned}\tag{3.1}$$

and

$$\begin{aligned}\phi(r, \theta) &= M^{-1}[\bar{\phi}(s, \theta); \quad s \rightarrow r] \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{\phi}(s, \theta) r^{-s} ds.\end{aligned}\tag{3.2}$$

In terms of this transformation it can be shown [10] that, with an appropriate choice for $c = \text{Re}(s)$, the temperature field for the problem described in section 2.1 is given by

$$\overline{r^2\Theta}(s, \theta) = N(s) \cos(s+2)\theta + L(s) \sin(s+2)\theta\tag{3.3}$$

and the Airy stress function by

$$\begin{aligned}\bar{\phi}(s, \theta) &= A(s) \sin s\theta + B(s) \cos s\theta \\ &\quad + C(s) \sin(s+2)\theta + D(s) \cos(s+2)\theta\end{aligned}\tag{3.4}$$

where $A(s)$, $B(s)$, $C(s)$, $D(s)$, $N(s)$, and $L(s)$ are arbitrary functions of the parameter s but are independent of θ . The corresponding stresses and displacements are given by

$$s_{rr}(r, \theta) = r^{-2} M^{-1} \left[\frac{\partial^2 \bar{\phi}(s, \theta)}{\partial \theta^2} - s \bar{\phi}(s, \theta); s \rightarrow r \right], \quad (3.5)$$

$$s_{r\theta}(r, \theta) = r^{-2} M^{-1} \left[(s+1) \frac{\partial \bar{\phi}(s, \theta)}{\partial \theta}; s \rightarrow r \right], \quad (3.6)$$

$$s_{\theta\theta}(r, \theta) = r^{-2} M^{-1} [s(s+1) \bar{\phi}(s, \theta); s \rightarrow r], \quad (3.7)$$

$$\begin{aligned} u(r, \theta) = & -r^{-1} M^{-1} \left[(s+1)^{-1} \left\{ (1-\nu) \frac{\partial^2 \bar{\phi}(s, \theta)}{\partial \theta^2} - s(1+\nu s) \bar{\phi}(s, \theta) \right. \right. \\ & \left. \left. + \frac{E\alpha T_0}{p_0} \bar{\Theta}(s+2, \theta) \right\}; s \rightarrow r \right], \end{aligned} \quad (3.8)$$

and

$$\begin{aligned} v(r, \theta) = & -r^{-1} M^{-1} \left[(s+1)^{-1} (s+2)^{-2} \left\{ (1-\nu) \frac{\partial^3 \bar{\phi}(s, \theta)}{\partial \theta^3} \right. \right. \\ & + [(1-\nu)s^2 + (s+1)(s+2)] \frac{\partial \bar{\phi}(s, \theta)}{\partial \theta} \\ & \left. \left. + \frac{E\alpha T_0}{p_0} \frac{\partial \bar{\Theta}(s+2, \theta)}{\partial \theta} \right\}; s \rightarrow r \right]. \end{aligned} \quad (3.9)$$

CHAPTER 3

STATEMENT OF THE PROBLEM

1. The Problem

Consider a two-dimensional, linear, isotropic, thermoelastic solid which occupies the (r, θ) plane and contains a circular hole $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$ with a radial edge crack $1 \leq r \leq c$, $\theta = 0$. We assume that the solid is initially unstressed and has uniform constant temperature T_0 . Under conditions of plane strain we shall determine the thermal stresses in the vicinity of the crack tip when a uniform heat flow

$$\underline{q} = q_0(\cos \theta_0 \underline{i} + \sin \theta_0 \underline{j})$$

is disturbed by the crack and the hole which are thermally insulated [Fig. 1]. As in Chapter 2, we denote the temperature at the point (r, θ) by the symbol

$$T(r, \theta) = T_0[1 + \Theta(r, \theta)]. \quad (1.1)$$

Parkus [8] has shown that, for flow past an insulated circular hole, the thermoelastic field is given by

$$\Theta^0(r, \theta) = -\frac{q_0 B}{\kappa T_0} (r + r^{-1}) \cos(\theta - \theta_0), \quad (1.2)$$

$$\phi^0(r, \theta) = \left(\frac{1}{2} r^{-1} + r \ln r\right) \cos(\theta - \theta_0), \quad (1.3)$$

Figure 1. Uniform heat flow past a circular hole with a radial edge crack.

$$\begin{aligned}
u^0(r, \theta) = & \frac{1}{2} \{ r^{-2} + (1 - 2\nu) \ln r - 4(1 - \nu) \} \cos(\theta - \theta_0) \\
& - (1 - \nu) \{ r^2 + 2 \ln r - 2 \} \cos(\theta - \theta_0), \tag{1.4}
\end{aligned}$$

$$\begin{aligned}
v^0(r, \theta) = & \frac{1}{2} \{ r^{-2} + (1 - 2\nu) [2 - 2 \ln r] \} \sin(\theta - \theta_0) \\
& - (1 - \nu) \{ r^2 + 2 - 2 \ln r \} \sin(\theta - \theta_0), \tag{1.5}
\end{aligned}$$

$$s_{rr}^0(r, \theta) = (r^{-1} - r^{-3}) \cos(\theta - \theta_0), \tag{1.6}$$

$$s_{r\theta}^0(r, \theta) = (r^{-1} - r^{-3}) \sin(\theta - \theta_0), \tag{1.7}$$

and

$$s_{\theta\theta}^0 = (r^{-1} - r^{-3}) \cos(\theta - \theta_0). \tag{1.8}$$

In the presence of the crack the above solution is perturbed. Hence, our problem may be stated as follows:

PROBLEM

Solve the dimensionless, plane-strain equations of linear thermoelasticity in the region $r > 1$, $0 < \theta < 2\pi$ subject to the boundary conditions:

$$1. \text{ At infinity } \Theta(r, \theta) = \Theta^0(r, \theta) + O(r^{-1}) \text{ and } \phi(r, \theta) = \phi^0(r, \theta) + O(r \ln r)$$

$$2. s_{rr}(1, \theta) = s_{r\theta}(1, \theta) = 0, \quad 0 < \theta < 2\pi,$$

$$3. s_{\theta\theta}(r, 2\pi) - s_{\theta\theta}(r, 0) = 0, \quad r > 1,$$

$$s_{r\theta}(r, 2\pi) - s_{r\theta}(r, 0) = 0, \quad r > 1,$$

$$4. s_{\theta\theta}(r, 0) = s_{r\theta}(r, 0) = 0, \quad 1 < r < c,$$

$$u(r, 2\pi) - u(r, 0) = 0, \quad c < r < \infty,$$

$$v(r, 2\pi) - v(r, 0) = 0, \quad c < r < \infty,$$

$$\lim_{r \rightarrow 1^+} \left[\frac{\partial u(r, 2\pi)}{\partial r} - \frac{\partial u(r, 0)}{\partial r} \right] < \infty,$$

$$\lim_{r \rightarrow 1^+} \left[\frac{\partial v(r, 2\pi)}{\partial r} - \frac{\partial v(r, 0)}{\partial r} \right] < \infty,$$

$$5. \frac{\partial \Theta(1, \theta)}{\partial r} = 0, \quad 0 < \theta < 2\pi,$$

$$6. \frac{\partial \Theta(r, 2\pi)}{\partial \theta} - \frac{\partial \Theta(r, 0)}{\partial \theta} = 0, \quad r > 1,$$

$$7. \frac{\partial \Theta}{\partial \theta}(r, 0) = 0, \quad 1 < r < c,$$

$$\Theta(r, 2\pi) - \Theta(r, 0) = 0, \quad c < r < \infty.$$

2. Resolution into Problem 1 and Problem 2

Using linearity, the solution of this problem is obtained by superimposing the solutions to problems 1 and 2 below.

PROBLEM 1

Solve the dimensionless, plane-strain equations of linear thermoelasticity in the region $r > 1$, $0 < \theta < 2\pi$ subject to the boundary conditions:

$$1. \text{ At infinity } \Theta(r, \theta) = \Theta^0(r, \theta) + O(r^{-1}) \text{ and}$$

$$\phi(r, \theta) = \phi^0(r, \theta) + O(r \ln r),$$

$$2. \ s_{rr}(1, \theta) = s_{r\theta}(1, \theta) = 0, \quad 0 < \theta < 2\pi,$$

$$3. \ s_{\theta\theta}(r, 2\pi) - s_{\theta\theta}(r, 0) = 0, \quad r > 1,$$

$$s_{r\theta}(r, 2\pi) - s_{r\theta}(r, 0) = 0, \quad r > 1,$$

$$4. \ u(r, 2\pi) - u(r, 0) = 0, \quad r > 1,$$

$$v(r, 2\pi) - v(r, 0) = 0, \quad r > 1,$$

$$5. \ \frac{\partial \Theta(1, \theta)}{\partial r} = 0, \quad 0 < \theta < 2\pi,$$

$$6. \ \frac{\partial \Theta(r, 2\pi)}{\partial \theta} - \frac{\partial \Theta(r, 0)}{\partial \theta} = 0, \quad r > 1,$$

$$7. \ \frac{\partial \Theta(r, 0)}{\partial \theta} = 0, \quad 1 < r < c,$$

$$\Theta(r, 2\pi) - \Theta(r, 0) = 0, \quad c < r < \infty.$$

Calculate: $s_{\theta\theta}(r, 0) = f_1(r) \cos \theta_0$ and

$$s_{r\theta}(r, 0) = f_2(r) \sin \theta_0 \quad \text{for} \quad 1 < r < c.$$

PROBLEM 2

Solve the dimensionless, plane-strain equations of linear isothermal elasticity in the region $r > 1$, $0 < \theta < 2\pi$ subject to the boundary conditions:

$$1. \text{ At infinity } \phi(r, \theta) = O(\ln r),$$

$$2. \quad s_{rr}(1, \theta) = s_{r\theta}(1, \theta) = 0, \quad 0 < \theta < 2\pi,$$

$$3. \quad s_{\theta\theta}(r, 2\pi) - s_{\theta\theta}(r, 0) = 0, \quad r > 1,$$

$$s_{r\theta}(r, 2\pi) - s_{r\theta}(r, 0) = 0, \quad r > 1,$$

$$4. \quad s_{\theta\theta}(r, 0) = -f_1(r) \cos \theta_0, \quad 1 < r < c,$$

$$s_{r\theta}(r, 0) = -f_2(r) \sin \theta_0, \quad 1 < r < c,$$

$$u(r, 2\pi) - u(r, 0) = 0, \quad c < r < \infty,$$

$$v(r, 2\pi) - v(r, 0) = 0, \quad c < r < \infty,$$

$$\lim_{r \rightarrow 1^+} \left[\frac{\partial u(r, 2\pi)}{\partial r} - \frac{\partial u(r, 0)}{\partial r} \right] < \infty,$$

$$\lim_{r \rightarrow 1^+} \left[\frac{\partial v(r, 2\pi)}{\partial r} - \frac{\partial v(r, 0)}{\partial r} \right] < \infty.$$

CHAPTER 4

The Solution of Problem 1

1. The Temperature Field

We solve problem 1 by perturbing the solution [3.1.2] - [3.1.8] for flow past an insulated circular hole. By inspection we see that the horizontal component of this solution already satisfies the conditions of problem 1 and hence only the vertical component need be perturbed. In addition, by symmetry, we find that the perturbations in the temperature and stress function must be anti-symmetric with respect to the rays $\theta = 0$ and $\theta = \pi$. We begin by examining the temperature field which is assumed to be of the form:

$$\Theta(r, \theta) = -\frac{B q_0}{\kappa T_0} (r + r^{-1}) \cos(\theta - \theta_0) + \Theta^1(r, \theta). \quad (1.1)$$

It is clear that this field satisfies conditions 1, 5, 6, and 7 if $\Theta^1(r, \theta)$ is given by the mixed boundary value problem:

$$\nabla^2 \Theta^1(r, \theta) = 0, \quad r > 1, \quad 0 < \theta < \pi$$

$$\begin{aligned} i. \quad & \Theta^1(r, \theta) = 0(r^{-1}) \quad \text{at infinity,} \\ ii. \quad & \frac{\partial \Theta^1(1, \theta)}{\partial r} = 0, \quad 0 < \theta < \pi, \\ iii. \quad & \Theta^1(r, \pi) = 0, \quad r > 1, \\ iv. \quad & \frac{\partial \Theta^1(r, 0)}{\partial \theta} = \frac{B q_0 \sin \theta_0}{\kappa T_0} (r + r^{-1}), \quad 1 < r < c, \\ & \Theta^1(r, 0) = 0, \quad r > c. \end{aligned} \quad (1.2)$$

The boundary value problem (1.2) is solved by making use of the Naylor transform [3] defined as follows:

$$\begin{aligned}\bar{\Theta}^1(s, \theta) &= H_1[\Theta^1(r, \theta); r \rightarrow s] \\ &= \int_1^\infty \Theta^1(r, \theta)(r^{s-1} - r^{-s-1}) dr\end{aligned}\quad (1.3)$$

whose inverse is

$$\begin{aligned}\Theta^1(r, \theta) &= H_1^{-1}[\bar{\Theta}^1(s, \theta); s \rightarrow r] \\ &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \bar{\Theta}^1(s, \theta) r^{-s} ds,\end{aligned}\quad (1.4)$$

$|Re(s)| = |\gamma| < 1.$

Since

$$\begin{aligned}H_1\left[r \frac{\partial}{\partial r} \left(r \frac{\partial \Theta^1(r, \theta)}{\partial r}\right); r \rightarrow s\right] &= s^2 H_1[\Theta^1(r, s); r \rightarrow s] \\ &\quad - 2 \frac{\partial \Theta^{(1)}}{\partial r}(r, \theta), \quad |Re s| < 1,\end{aligned}\quad (1.5)$$

using boundary condition (ii), the equation $\nabla^2 \Theta^1 = 0$ transforms into

$$\frac{d^2 \bar{\Theta}^1(s, \theta)}{d\theta^2} + s^2 \bar{\Theta}^1(s, \theta) = 0, \quad |Re(s)| < 1, \quad (1.6)$$

whose general solution is

$$\bar{\Theta}^1(s, \theta) = s^{-1}[A(s) \cos s \theta + B(s) \sin s \theta], \quad (1.7)$$

where $A(s)$ and $B(s)$ are arbitrary functions of the transform variable s .

Applying condition (iii) we find that

$$\bar{\Theta}^1(s, \theta) = s^{-1} A(s) \frac{\sin(\theta - \pi)s}{\sin \pi s} \quad (1.8)$$

or

$$\Theta^1(r, \theta) = H_1^{-1} \left[s^{-1} A(s) \frac{\sin(\theta - \pi)s}{\sin \pi s}; s \rightarrow r \right]. \quad (1.9)$$

Thus, to satisfy condition (iv), $A(s)$ must satisfy the dual integral equations

$$H_1^{-1}[s^{-1} A(s); s \rightarrow r] = 0, \quad c < r < \infty, \quad (1.10)$$

$$H_1^{-1}[A(s) \cot \pi s; s \rightarrow r] = \frac{Bq_0 \sin \theta_0}{\kappa T_0} (r + r^{-1}), \quad (1.11)$$

$$1 < r < c.$$

Dual equations of this type have been discussed by Tweed [2] from who's work it follows that

$$\Theta^1(r, 0) = H(c - r) \int_r^c p(t) dt, \quad r > 1, \quad (1.12)$$

where $H(r)$ is the heaviside function,

$$A(s) = \int_1^c p(t) (t^{-s} - t^s) dt \quad (1.13)$$

and

$$\begin{aligned} p(t) &= \frac{B q_0 \sin \theta_0}{\pi \kappa T_0 \sqrt{(c-t)(ct-1)}} \cdot \\ &\quad \cdot \int_1^c \sqrt{(c-x)(cx-1)} \left\{ \frac{1}{x-t} + \frac{1}{1-xt} \right\} \left\{ 1 + \frac{1}{x^2} \right\} dx \\ &= \frac{B q_0 \sin \theta_0 \sqrt{c}}{2 \kappa T_0 \sqrt{(c-t)(ct-1)}} \cdot \left\{ c^2 + 1 - 2ct - t^{-1}(c^2 + 1 - 2ct^{-1}) \right\}. \end{aligned} \quad (1.14)$$

It follows from equation [1.1] that

$$\begin{aligned} \Theta(r, 0) &= \frac{-B q_0}{\kappa T_0} (r + r^{-1}) \cos \theta_0 \\ &\quad - \frac{B q_0 \sin \theta_0}{2 \kappa T_0} H(c-r) \int_r^c \frac{[c^2 + 1 - 2ct - t^{-1}(c^2 + 1 - 2ct^{-1})] dt}{\sqrt{(c-t)(t-c^{-1})}} \\ &= \frac{-B q_0}{\kappa T_0} (r + r^{-1}) \cos \theta_0 \\ &\quad - \frac{B q_0 \sin \theta_0}{\kappa T_0} (1 + r^{-1}) H[(c-r)(r-c^{-1})] \sqrt{(c-r)(r-c^{-1})}. \end{aligned} \quad (1.15)$$

To find an expression for $\Theta^1(r, \theta)$ we recall that the solution to the Dirichlet problem:

$$\nabla^2 V(x, y) = 0, \quad -\infty < x < \infty, \quad y > 0,$$

- (i) $V(x, y) \rightarrow 0$ as $\sqrt{x^2 + y^2} \rightarrow \infty$,
(ii) $V(x, 0) = w(x)$, $-\infty < x < \infty$,

is given by

$$V(x, y) = \text{Im} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{w(t) dt}{t - z}. \quad (1.17)$$

and, therefore, in view of [1.15], we see that

$$\begin{aligned} \Theta^1(r, \theta) &= \frac{-Bq_0 \sin \theta_0}{\kappa T_0} \text{Im} \left\{ \frac{1}{\pi} \int_{c^{-1}}^c \frac{\sqrt{(c-t)(t-c^{-1})} (1+t^{-1}) dt}{t-z} \right\} \\ &= \frac{-Bq_0 \sin \theta_0}{\kappa T_0} \text{Im} \left\{ (z+1) \left[\frac{\sqrt{(z-c)(z-c^{-1})}}{z} - 1 \right] \right. \\ &\quad \left. + \frac{1}{z} + \frac{c^{-1}+c}{2} \right\} \\ &= \frac{-Bq_0 \sin \theta_0}{\kappa T_0} \text{Im} \left\{ \sqrt{r_1 r_2} e^{i \left(\frac{\theta_1 + \theta_2}{2} \right)} - r e^{i\theta} \right. \\ &\quad \left. + r^{-1} e^{-i\theta} - 1 + \frac{c+c^{-1}}{2} + \frac{\sqrt{r_1 r_2}}{r} e^{i \left(\frac{\theta_1 + \theta_2}{2} \right) - i\theta} \right\} \\ &= \frac{-Bq_0 \sin \theta_0}{\kappa T_0} \left\{ \sqrt{r_1 r_2} \sin \left(\frac{\theta_1 + \theta_2}{2} \right) - (r + r^{-1}) \sin \theta \right. \\ &\quad \left. + \frac{\sqrt{r_1 r_2}}{r} \sin \left(\frac{\theta_1 + \theta_2}{2} - \theta \right) \right\} \end{aligned} \quad (1.18)$$

where

$$z = r e^{i\theta}, \quad z - c = r_1 e^{i\theta_1}, \quad \text{and} \quad z - c^{-1} = r_2 e^{i\theta_2}.$$

Thus,

$$\begin{aligned}
 \Theta(r, \theta) = & \frac{-Bq_0 \sin \theta_0}{\kappa T_0} (r + r^{-1}) \cos \theta \cos \theta_0 \\
 & - \frac{Bq_0 \sin \theta_0}{\kappa T_0} \sqrt{r_1 r_2} \left\{ \sin \left(\frac{\theta_1 + \theta_2}{2} \right) \right. \\
 & \left. + \frac{1}{r} \sin \left(\frac{\theta_1 + \theta_2}{2} - \theta \right) \right\}. \tag{1.19}
 \end{aligned}$$

2. The Mechanical Field

We next investigate the mechanical part of the solution. We assume that the stress function is of the form

$$\phi(r, \theta) = \phi^0(r, \theta) + \hat{\phi}(r, \theta). \quad (2.1)$$

Because of symmetry and linearity of the problem conditions 1, 3, and 4b will be satisfied if $\hat{\phi}(r, \theta)$ is the result of superimposing the solutions of the following two problems [4]:

Problem 1.1

Solve the equations of plane strain linear isothermal elasticity in the region $r > 1$, $0 < \theta < \pi$ subject to the conditions:

- (i) At infinity $\phi'(r, \theta) = 0(r)$,
- (ii) $u'(r, 0) = u'(r, \pi) = 0$, $1 < r < \infty$,
- (iii) $s'_{\theta\theta}(r, 0) = s'_{\theta\theta}(r, \pi) = 0$, $1 < r < \infty$.

Problem 1.2

Solve the equations of plane strain linear thermoelasticity in the region $0 \leq r < \infty$, $0 < \theta < \pi$ for the case where

- (i) At infinity $\phi''(r, \theta) = 0(r^{-2})$,
- (ii) $u''(r, \theta)$, $v''(r, \theta)$, $s''_{ij}(r, \theta)$ are all bounded at the origin,
- (iii) $s''_{\theta\theta}(r, 0) = s''_{\theta\theta}(r, \pi) = 0$, $0 < r < \infty$,
- (iv) $u''(r, \pi) = 0$, $0 < r < \infty$,
- (v) $\Theta''(r, 0) = \Theta^1(r, 0)$, $r > 0$
 $\Theta''(r, \pi) = 0$, $r > 0$.

It is easy to show that the Michell solution to problem 1.1 is given by:

$$\phi'(r, \theta) = \sum_{n=1}^{\infty} \{b_n r^{-n} + d_n r^{2-n}\} \sin n\theta, \quad (2.2)$$

$$s'_{rr}(r, \theta) = - \sum_{n=1}^{\infty} \{n(n+1)b_n r^{-n-2} + (n-1)(n+2)d_n r^{-n}\} \sin n\theta, \quad (2.3)$$

$$s'_{r\theta}(r, \theta) = \sum_{n=1}^{\infty} \{n(n+1)b_n r^{-n-2} + n(n-1)d_n r^{-n}\} \cos n\theta, \quad (2.4)$$

$$s'_{\theta\theta}(r, \theta) = \sum_{n=1}^{\infty} \{n(n+1)b_n r^{-n-2} + (n-1)(n-2)d_n r^{-n}\} \sin n\theta, \quad (2.5)$$

$$\begin{aligned} u'(r, \theta) &= b_1 r^{-2} \sin \theta \\ &+ \sum_{n=1}^{\infty} \{n b_n r^{-n-1} + d_n [n+2-4\nu] r^{-n+1}\} \sin n\theta, \end{aligned} \quad (2.6)$$

$$\begin{aligned} v'(r, \theta) &= -b_1 r^{-2} \cos \theta \\ &- \sum_{n=1}^{\infty} \{n b_n r^{-n-1} + d_n [n-4+4\nu] r^{-n+1}\} \cos n\theta. \end{aligned} \quad (2.7)$$

We use Mellin transforms to solve problem 1.2. From Chapter 2, we have the solution:

$$\begin{aligned}\bar{\phi}''(s, \theta) &= \bar{A}(s) \sin s \theta + \bar{B}(s) \cos s \theta \\ &+ \bar{C}(s) \sin(s+2)\theta + \bar{D}(s) \cos(s+2)\theta,\end{aligned}\tag{2.8}$$

$$\overline{r^2 s''_{rr}}(s, \theta) = \frac{\partial^2 \bar{\phi}''(s, \theta)}{\partial \theta^2} - s \bar{\phi}''(s, \theta),\tag{2.9}$$

$$\overline{r^2 s''_{r\theta}}(s, \theta) = (s+1) \frac{\partial \bar{\phi}''(s, \theta)}{\partial \theta},\tag{2.10}$$

$$\overline{r^2 s''_{\theta\theta}}(s, \theta) = s(s+1) \bar{\phi}''(s, \theta),\tag{2.11}$$

$$\overline{ru''}(s, \theta) = -(s+1)^{-1} \left\{ (1-\nu) \frac{\partial^2 \bar{\phi}''(s, \theta)}{\partial \theta^2} - s(1+\nu s) \bar{\phi}''(s, \theta) \right\}$$

$$- \frac{E \alpha T_0}{p_0} (s+1)^{-1} \bar{\Theta}^1(s+2, \theta),\tag{2.12}$$

$$\begin{aligned}\overline{rv''}(s, \theta) &= -(s+1)^{-1} (s+2)^{-1} \left\{ (1-\nu) \frac{\partial^3 \bar{\phi}''(s, \theta)}{\partial \theta^3} \right. \\ &+ \left. [(1-\nu)s^2 + (s+1)(s+2)] \frac{\partial \bar{\phi}''(s, \theta)}{\partial \theta} \right\} \\ &- \frac{E \alpha T_0}{p_0} (s+1)^{-1} (s+2)^{-1} \frac{\partial \bar{\Theta}^1(s+2, \theta)}{\partial \theta}.\end{aligned}\tag{2.13}$$

Condition (iii) implies that

$$\bar{B}(s) = -\bar{D}(s) \quad \text{and} \quad \bar{A}(s) = -\bar{C}(s).$$

Thus,

$$\begin{aligned} \bar{\phi}''(s, \theta) &= \bar{A}(s)[\sin s \theta - \sin(s+2) \theta] \\ &+ \bar{B}(s)[\cos s \theta - \cos(s+2) \theta] \end{aligned} \quad (2.14)$$

and condition (iv) implies that

$$\bar{A}(s) = -\bar{B}(s) \cot \pi s.$$

This gives the solution

$$\bar{\phi}''(s, \theta) = \frac{\bar{B}(s)}{\sin \pi s} \{ \sin(s+2)(\theta - \pi) - \sin s(\theta - \pi) \}, \quad (2.15)$$

$$\overline{r^2 s''_{rr}}(s, \theta) = \frac{(s+1)\bar{B}(s)}{\sin \pi s} \{ s \sin s(\theta - \pi) - (s+4) \sin(s+2)(\theta - \pi) \}, \quad (2.16)$$

$$\overline{r^2 s''_{r\theta}}(s, \theta) = \frac{(s+1)\bar{B}(s)}{\sin \pi s} \{ (s+2) \cos(s+2)(\theta - \pi) - s \cos s(\theta - \pi) \}, \quad (2.17)$$

$$\overline{r^2 s''_{\theta\theta}}(s, \theta) = \frac{s(s+1)\bar{B}(s)}{\sin \pi s} \{ \sin(s+2)(\theta - \pi) - \sin s(\theta - \pi) \}, \quad (2.18)$$

$$\begin{aligned} \overline{ru''}(s, \theta) &= (s+1)^{-1} \left\{ \frac{(s+1)\bar{B}(s)}{\sin \pi s} [(s+4-4\nu) \sin(s+2)(\theta - \pi) \right. \\ &\quad \left. - s \sin s(\theta - \pi)] \right\} - \frac{E\alpha T_0}{p_0} r^{-1} \frac{\bar{\Theta}''(s+2, \theta)}{s+1}, \end{aligned} \quad (2.19)$$

$$\begin{aligned}
\overline{rv''}(s, \theta) = (s+1)^{-1} & \left\{ \frac{(s+1)\bar{B}(s)}{\sin \pi s} [s \cos s(\theta - \pi) \right. \\
& \left. - (s-2+4\nu) \cos(s+2)(\theta - \pi)] \right\} \\
& - \frac{E \alpha T_0}{p_0} r^{-1} \frac{\partial \bar{\Theta}^1(s+2, \theta)}{\partial \theta} \cdot (s+1)^{-1} (s+2)^{-1},
\end{aligned} \tag{2.20}$$

and we see that to satisfy conditions (i) and (ii) we must have

$-2 < \operatorname{Re}(s) < -1$. On setting $\bar{B}(s) = \frac{1}{2}(s+1)^{-1}A(s)$, this leads, via superposition, to the stress function

$$\begin{aligned}
\phi(r, \theta) = & \phi^0(r, \theta) + \sum_{n=1}^{\infty} [b_n r^{-n} + d_n r^{2-n}] \sin n\theta \\
& + M^{-1} \left[\frac{A(s)}{2(s+1) \sin \pi s} \{ \sin(s+2)(\theta - \pi) \right. \\
& \left. - \sin s(\theta - \pi) \}; r \right], \quad -2 < \operatorname{Re}(s) < -1,
\end{aligned} \tag{2.21}$$

and, hence, the stresses are given by:

$$\begin{aligned}
s_{rr}(r, \theta) = & (r^{-1} - r^{-3}) \cos(\theta - \theta_0) \\
& - \sum_{n=1}^{\infty} \{ n(n+1)b_n r^{-n-2} + (n-1)(n+2)d_n r^{-n} \} \sin n\theta \\
& + r^{-2} M^{-1} \left[\frac{A(s)}{2 \sin \pi s} \{ s \sin(\theta - \pi) s \right. \\
& \left. - (s+4) \sin(\theta - \pi)(s+2) \}; r \right],
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
s_{r\theta}(r, \theta) &= (r^{-1} - r^{-3}) \sin(\theta - \theta_0) \\
&+ \sum_{n=1}^{\infty} \{n(n+1)b_n r^{-n-2} + n(n-1)d_n r^{-n}\} \cos n\theta \\
&+ r^{-2} M^{-1} \left[\frac{A(s)}{2 \sin \pi s} \{(s+2) \cos(s+2)(\theta - \pi) \right. \\
&\left. - s \cos s(\theta - \pi)\}; r \right], \tag{2.23}
\end{aligned}$$

$$\begin{aligned}
s_{\theta\theta}(r, \theta) &= (r^{-1} + r^{-3}) \cos(\theta - \theta_0) \\
&+ \sum_{n=1}^{\infty} \{n(n+1)b_n r^{-n-2} + (n-1)(n-2)d_n r^{-n}\} \sin n\theta \\
&+ r^{-2} M^{-1} \left[\frac{s A(s)}{2 \sin \pi s} \{\sin(s+2)(\theta - \pi) \right. \\
&\left. - \sin s(\theta - \pi)\}; r \right], \tag{2.24}
\end{aligned}$$

and the displacements are:

$$\begin{aligned}
u(r, \theta) = & \left\{ \frac{1}{2r^2} + (1 - 2\nu) \ln r - 2(1 - \nu) \right\} \cos(\theta - \theta_0) \\
& - (1 - \nu) \{r^2 + \ln r^2 - 2\} \cos(\theta - \theta_0) + b_1 r^{-2} \sin \theta \\
& + \sum_{n=2}^{\infty} \left\{ n b_n r^{-n-1} + d_n [n + 2 - 4\nu] r^{-n+1} \right\} \sin n\theta \\
& + r^{-1} M^{-1} \left[\frac{A(s)}{2(s+1) \sin \pi s} \{ (s+4-4\nu) \sin(s+2)(\theta - \pi) \right. \\
& \left. - s \sin s(\theta - \pi) \}; r \right] \\
& - \frac{E\alpha T_0}{p_0} r^{-1} M^{-1} \left[\frac{\bar{\Theta}^1(s+2, \theta)}{s+1}; r \right], \tag{2.25}
\end{aligned}$$

$$\begin{aligned}
v(r, \theta) = & \left\{ \frac{1}{2r^2} + (1 - 2\nu)[1 - \ln r] \right\} \sin(\theta - \theta_0) \\
& - (1 - \nu) \{r^2 + 2 - \ln r^2\} \sin(\theta - \theta_0) - b_1 r^{-2} \cos \theta \\
& - \sum_{n=2}^{\infty} \{ n b_n r^{-n-1} + d_n [n - 4 + 4\nu] r^{-n+1} \} \cos n\theta \\
& + r^{-1} M^{-1} \left[\frac{A(s)}{2(s+1) \sin \pi s} \{ s \cos s(\theta - \pi) \right. \\
& \left. - (s - 2 + 4\nu) \cos(s+2)(\theta - \pi) \}; r \right] \\
& - \frac{E\alpha T_0}{p_0} r^{-1} M^{-1} \left[\frac{1}{(s+1)(s+2)} \frac{\partial \bar{\Theta}^1(s+2, \theta)}{\partial \theta}; r \right], \tag{2.26}
\end{aligned}$$

where $-2 < \text{Re}(s) < -1$ and

$$\bar{\Theta}^1(\sigma, \theta) = M[\Theta^1(r, \theta); r \rightarrow \sigma], \quad 0 < \text{Re}(\sigma) < 1. \tag{2.27}$$

Applying condition 4, we find that

$$A(s) = -\frac{E\alpha T_0}{2(1-\nu)p_0} \bar{\Theta}^1(s+2, 0) \quad (2.28)$$

and hence, in view of [2.1.18] and [1.15],

$$A(s) = \sin \theta_0 \int_{c^{-1}}^c Q(t) t^{s+2} dt \quad (2.29)$$

where $0 < \operatorname{Re}(s+2) < 1$, and

$$Q(t) = t^{-2}(t+1) \sqrt{(c-t)(t-c^{-1})}. \quad (2.30)$$

Similarly, on applying condition 2, we find that

$$\begin{aligned} & \sum_{n=1}^{\infty} \{n(n+1)b_n + (n-1)(n+2)d_n\} \sin n\theta \\ &= M^{-1} \left[\frac{A(s)}{2 \sin \pi s} \{s \sin s(\theta - \pi) \right. \\ & \quad \left. - (s+4) \sin(s+2)(\theta - \pi)\}; 1 \right] \end{aligned} \quad (2.31)$$

and

$$\begin{aligned} & \sum_{n=1}^{\infty} \{n(n+1)b_n + n(n-1)d_n\} \cos n\theta \\ &= M^{-1} \left[\frac{A(s)}{2 \sin \pi s} \{s \cos s(\theta - \pi) \right. \\ & \quad \left. - (s+2) \cos(s+2)(\theta - \pi)\}; 1 \right], \end{aligned} \quad (2.32)$$

where $0 < \theta < \pi$, $0 < \operatorname{Re}(s+2) < 1$.

In [2.31] and [2.32] the Fourier coefficients are clearly given by

$$\begin{aligned} & n(n+1)b_n + (n-1)(n+2)d_n \\ &= \frac{2}{\pi} M^{-1} \left[\frac{A(s)}{2 \sin \pi s} \left\{ s \int_0^\pi \sin s(\theta - \pi) \sin n\theta d\theta \right. \right. \\ & \quad \left. \left. - (s+4) \int_0^\pi \sin(s+2)(\theta - \pi) \sin n\theta d\theta \right\}; 1 \right] \quad (2.33) \end{aligned}$$

and

$$\begin{aligned} & n(n+1)b_n + n(n-1)d_n \\ &= \frac{2}{\pi} M^{-1} \left[\frac{A(s)}{2 \sin \pi s} \left\{ s \int_0^\pi \cos s(\theta - \pi) \cos n\theta d\theta \right. \right. \\ & \quad \left. \left. - (s+2) \int_0^\pi \cos(s+2)(\theta - \pi) \cos n\theta d\theta \right\}; 1 \right] \quad (2.34) \end{aligned}$$

Therefore, on making use of the identities

$$\begin{aligned} & \frac{2}{\pi} \int_0^\pi \sin\{(\theta - \pi)(s+k)\} \sin n\theta d\theta \\ &= \frac{(-1)^k}{\pi} \left\{ \frac{1}{s+k-n} - \frac{1}{s+k+n} \right\} \sin \pi s, \end{aligned}$$

$$\begin{aligned} & \frac{2}{\pi} \int_0^\pi \cos\{(\theta - \pi)(s+k)\} \cos n\theta d\theta \\ &= \frac{(-1)^k}{\pi} \left\{ \frac{1}{s+k-n} + \frac{1}{s+k+n} \right\} \sin \pi s, \end{aligned}$$

and

$$\frac{1}{\pi} \int_0^\pi \cos(\theta - \pi)(s + k) d\theta = \frac{(-1)^k}{\pi} \frac{\sin \pi s}{s + k}, \quad k \text{ any integer,}$$

we find that b_n and d_n must satisfy the equations:

$$\begin{aligned} n(n+1)b_n + (n-1)(n+2)d_n \\ = \frac{1}{2\pi} M^{-1} \left[A(s) \left\{ \frac{n}{s-n} + \frac{n}{s+n} \right. \right. \\ \left. \left. - \frac{n+2}{s+2-n} - \frac{n-1}{s+2+n} \right\}; 1 \right], \end{aligned} \quad (2.35)$$

$$\begin{aligned} n(n+1)b_n + n(n-1)d_n \\ = \frac{1}{2\pi} M^{-1} \left[A(s) \left\{ \frac{n}{s-n} - \frac{n}{s+n} \right. \right. \\ \left. \left. + \frac{n}{s+2+n} - \frac{n}{s+2-n} \right\}; 1 \right], \end{aligned} \quad (2.36)$$

$0 < \theta < \pi$, $0 < \operatorname{Re}(s+2) < 1$. It follows that

$$\begin{aligned} n(n+1)b_n = \frac{1}{2\pi} M^{-1} \left[A(s) \left\{ \frac{n}{s-n} \right. \right. \\ \left. \left. - \frac{n(n+1)}{s+n} + \frac{n^2}{s+2+n} \right\}; 1 \right] \end{aligned} \quad (2.37)$$

and

$$n(n-1)d_n = \frac{1}{2\pi} M^{-1} \left[A(s) \left\{ \frac{n^2}{s+n} - \frac{n(n-1)}{s+2+n} + \frac{n}{s+2-n} \right\}; 1 \right] \quad (2.38)$$

Now, by (2.29),

$$\begin{aligned} A(s) &= \sin \theta_0 \int_{c^{-1}}^c Q(t) t^{s+2} dt \\ &= \sin \theta_0 \int_{c^{-1}}^c Q(t) t^{-(s+2)} dt, \quad -2 < \operatorname{Re}(s) < -1, \\ &= \sin \theta_0 \int_{c^{-1}}^c Q(t) t^{-\sigma} dt, \quad 0 < \operatorname{Re}(\sigma) < 1, \end{aligned} \quad (2.39)$$

where

$$Q(t) = \frac{t+1}{t^2} \sqrt{(c-t)(t-c^{-1})} \quad \text{and} \quad \sigma = s+2.$$

Therefore,

$$\begin{aligned} n(n+1)b_n &= \frac{\sin \theta_0}{2\pi} \int_{c^{-1}}^c Q(t) M^{-1} \left[\frac{n}{\sigma-n-2} - \frac{n(n+1)}{\sigma+n-2} + \frac{n^2}{\sigma+n} ; \sigma \rightarrow t \right] dt \end{aligned} \quad (2.40)$$

and

$$\begin{aligned} n(n-1)d_n &= \frac{\sin \theta_0}{2\pi} \int_{c^{-1}}^c Q(t) M^{-1} \left[\frac{n^2}{\sigma+n-2} - \frac{n(n-1)}{\sigma+n} - \frac{n}{\sigma-n} ; \sigma \rightarrow t \right] dt, \end{aligned} \quad (2.41)$$

$0 < \text{Re}(\sigma) < 1, n \geq 1$. It follows that

$$2b_1 = \frac{\sin \theta_0}{2\pi} \int_1^c Q(t) \{3t^{-1} - t^{-3}\} dt \quad (2.42)$$

and, for $n \geq 2$,

$$\begin{aligned} n(n+1)b_n = \frac{\sin \theta_0}{2\pi} \int_1^c Q(t) \{n^2 t^{-n} - n(n+1)t^{-n+2} \\ - n t^{-n-2}\} dt \end{aligned} \quad (2.43)$$

and

$$n(n-1)d_n = \frac{\sin \theta_0}{2\pi} \int_1^c Q(t) \{n^2 t^{-n+2} - n(n-2)t^{-n}\} dt. \quad (2.44)$$

We are now in the position to calculate the quantities $f_1(r)$ and $f_2(r)$ as required. From equation [2.24]

$$\begin{aligned} f_1(r) &= s_{\theta\theta}(r, 0) \sec \theta_0 \\ &= r^{-1} + r^{-3}. \end{aligned} \quad (2.45)$$

In contrast (2.23) yields

$$\begin{aligned} f_2(r) &= r^{-3} - r^{-1} + r^{-2} \csc \theta_0 M^{-1}[A(s) \cot \pi s; r] \\ &\quad + \csc \theta_0 \sum_{n=1}^{\infty} \{n(n+1)b_n r^{-n-2} + n(n-1)d_n\}, \end{aligned} \quad (2.46)$$

$0 < \operatorname{Re}(s+2) < 1$, which needs to be reduced further. On substituting for $A(s)$ from equation [2.37] and using the result

$$M^{-1}[\pi \cot n\pi s; x] = \frac{1}{1-x}, \quad 0 < \operatorname{Re}(s) < 1, \quad (2.47)$$

we obtain

$$\begin{aligned} r^{-2}M^{-1}[A(s) \cot \pi s; r] &= \frac{\sin \theta_0}{\pi} \int_{c-1}^c Q(t) \frac{dt}{1-rt} \\ &= \frac{\sin \theta_0}{\pi} \int_1^c Q(t) \left[\frac{1}{1-rt} + \frac{t}{t-r} \right] dt. \end{aligned} \quad (2.48)$$

Therefore,

$$\begin{aligned} &\sum_{n=1}^{\infty} \{n(n+1)b_n r^{-n-2} + n(n-1)d_n r^{-n}\} \\ &= \frac{\sin \theta_0}{2\pi} \int_1^c Q(t) \{3t^{-1} - t^{-3}\} r^{-3} dt \\ &+ \frac{\sin \theta_0}{2\pi} \int_1^c Q(t) \sum_{n=2}^{\infty} \{n^2 t^{-n} - n(n+1)t^{-n+2} - n t^{-n-2}\} r^{-n-2} dt \\ &+ \frac{\sin \theta_0}{2\pi} \int_1^c Q(t) \sum_{n=2}^{\infty} \{n^2 t^{-n+2} - n(n-2)t^{-n}\} r^{-n} dt \\ &= \frac{\sin \theta_0}{2\pi} \int_1^c Q(t) \left\{ \frac{3}{tr^3} - \sum_1^{\infty} n(rt)^{-n-2} + t^2 \sum_2^{\infty} n^2 (rt)^{-n-2} \right. \\ &\quad \left. - t^4 \sum_{n=2}^{\infty} n(n+1)(rt)^{-n-2} + t^2 \sum_2^{\infty} n^2 (rt)^{-n} - \sum_{n=2}^{\infty} n(n-2)(rt)^{-n} \right\} dt \\ &= \frac{\sin \theta_0}{\pi} \int_1^c Q(t) \left\{ \frac{(t^2-1)^2}{(1-rt)^3} + \frac{(t^2-1)}{(1-rt)^2} \right. \\ &\quad \left. - \frac{(t^2+1)}{1-rt} - \frac{(t^2+1)}{rt} + \frac{t^2+1}{r^3 t} \right\} dt. \end{aligned} \quad (2.49)$$

See Appendix A for details. Thus, [2.46], [2.48] and [2.49] imply that

$$\begin{aligned}
f_2(r) &= r^{-3} - r^{-1} + \frac{1}{\pi} \int_1^c Q(t) \left\{ \frac{t}{t-r} + \frac{(t^2-1)^2}{(1-rt)^3} + \frac{t^2-1}{(1-rt)^2} \right\} dt \\
&\quad - \frac{1}{\pi} \int_1^c M(t) \left\{ \frac{t^2}{1-rt} + \frac{t^2+1}{rt} - \frac{t^2+1}{r^3t} \right\} dt \\
&= (r^{-3} - r^{-1}) \left\{ 1 + \frac{1}{8}(c - c^{-1})^2 + \frac{1}{2} \left(c^{\frac{1}{2}} - c^{-\frac{1}{2}} \right)^2 \right\} \\
&\quad + \left[\frac{1}{2} \left(c^{\frac{1}{2}} - c^{-\frac{1}{2}} \right) - r + \frac{1}{r} \right] \\
&\quad + \frac{1}{\pi} \int_1^c \Delta(t) \frac{1+t}{t^2} \left\{ \frac{(t^2-2)^2}{(1-rt)^3} + \frac{t^2-1}{(1-rt)^2} - \frac{t^2+1}{1-rt} \right\} dt \\
&= (r^{-3} - r^{-1}) \left\{ 1 + \frac{1}{8}(c - c^{-1})^2 + \frac{1}{2} \left(c^{\frac{1}{2}} - c^{-\frac{1}{2}} \right)^2 \right\} \\
&\quad + \left[\frac{1}{2} \left(c^{\frac{1}{2}} - c^{-\frac{1}{2}} \right) - r + \frac{1}{r} \right] + II(r), \tag{2.50}
\end{aligned}$$

$$\Delta(t) = \sqrt{(c-t)(t-c^{-1})}.$$

$II(r)$ is evaluated in appendix B.

Chapter 5

Solution of Problem 2

Problem 2 may be solved by the Mellin transform technique of Tweed and Rooke [4]. In which case it can be shown [5] that the solution is such that for $1 < r < c$,

$$u(r, 0) - u(r, 2\pi) = -4(1 - \nu) \sin \theta_0 \int_r^c \frac{P_2(t) dt}{\sqrt{(c-t)(t-1)}}, \quad (1.1)$$

and

$$v(r, 0) - v(r, 2\pi) = -4(1 - \nu) \cos \theta_0 \int_r^c \frac{P_1(t) dt}{\sqrt{(c-t)(t-1)}}, \quad (1.2)$$

where the quantities $P_i(t)$, $i = 1, 2$ must satisfy the singular integral equations

$$\frac{1}{\pi} \int_1^c \frac{P_i(t)}{\sqrt{(c-t)(t-1)}} \left\{ \frac{1}{t-r} + K_i(r, t) \right\} dt = -f_i(r), \quad (1.3)$$

$1 < r < c$, with subsidiary conditions

$$P_i(1) = 0, \quad i = 1, 2 \quad (1.4)$$

and kernel functions

$$K_1(r, t) = \frac{(1-t^2)^2}{t(1-tr)^3} - \frac{t(1-t^2)}{(1-rt)^2} - \frac{t}{(1-rt)} + \frac{1-t^2}{r^2 t} \quad (1.5)$$

and

$$K_2(r, t) = \frac{(1 - t^2)^2}{t(1 - rt)^3} - \frac{(1 - t^2)}{t(1 - rt)^2} - \frac{t}{(1 - rt)}, \quad (1.6)$$

respectively. Since $f_i(r)$ are known, equations [1.3] maybe solved numerically and the results used to calculate the stress intensity factors and crack formation energy which are quantities of interest to workers in fracture mechanics.

The opening and sliding mode stress intensity factors k_1 and k_2 , respectively, and the crack formation energy W are defined by the equations

$$k_1 = -\frac{E}{4(1 - \nu^2)} \lim_{R \rightarrow C^-} \sqrt{2(C - R)} \frac{\partial}{\partial R} [U_\theta(R, 0) - U_\theta(R, 2\pi)], \quad (1.7)$$

$$k_2 = -\frac{E}{4(1 - \nu^2)} \lim_{R \rightarrow C^-} \sqrt{2(C - R)} \frac{\partial}{\partial R} [U_R(R, 0) - U_R(R, 2\pi)], \quad (1.8)$$

and

$$W = \frac{\cos \theta_0}{2} \int_B^C p_0 f_1(R/B) [U_\theta(R, 0) - U_\theta(R, 2\pi)] dR \\ + \frac{\sin \theta_0}{2} \int_B^C p_0 f_2(R/B) [U_R(R, 0) - U_R(R, 2\pi)] dR. \quad (1.9)$$

It is well known [6] that the stress intensity factor k_0 and the crack formation energy W_0 of a crack of length $2(C - B)$ in an infinite elastic solid which is subject to a uniform all round tension p_0 are given by $k_0 = p_0(C - B)^{\frac{1}{2}}$ and $W_0 = \pi(1 - \nu^2)p_0^2(C - B)^2/E$, respectively. Therefore, upon substituting [1.1] - [1.2] into [1.7] - [1.9] we have

$$\frac{k_1}{k_0} = \frac{-\sqrt{2}}{(c-1)} P_1(c) \cos \theta_0, \quad (1.10)$$

$$\frac{k_2}{k_0} = \frac{-\sqrt{2}}{(c-1)} P_2(c) \sin \theta_0 \quad (1.11)$$

and

$$\frac{W}{W_0} = \frac{W_1}{W_0} \cos \theta_0 + \frac{W_2}{W_0} \sin \theta_0, \quad (1.12)$$

where

$$\frac{W_1}{W_0} = \frac{-2}{\pi(c-1)^2} \int_1^c \frac{P_1(t)}{\sqrt{(c-t)(t-1)}} \int_1^t f_1(r) dr dt, \quad (1.13)$$

$$\frac{W_2}{W_0} = \frac{-2}{\pi(c-1)^2} \int_1^c \frac{P_2(t)}{\sqrt{(c-t)(t-1)}} \int_1^t f_2(r) dr dt. \quad (1.14)$$

CHAPTER 6

NUMERICAL PROCEDURES

1. The Stress Intensities and The Crack Formation Energies

Following Erdogan and Gupta [11] we set

$$x_k = \cos \frac{(2k-1)\pi}{2n}, \quad t_k = \frac{1}{2}(c-1)x_k + \frac{1}{2}(c+1),$$

$r_j = \frac{1}{2}(c-1)\cos \frac{j\pi}{n} + \frac{1}{2}(c+1)$, $k = 1, 2, \dots, n$, $j = 1, 2, \dots, n-1$, and replace the singular integral equations [5.1.3] by the linear algebraic system

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n P_i(t_k) \left\{ \frac{1}{t_k - r_j} + K_i(r_j, t_k) \right\} &= -f_i(r_j), \\ \frac{1}{n} \sum_{k=1}^n (-1)^k P_i(t_k) \left(\frac{1-x_k}{1+x_k} \right)^{\frac{1}{2}} &= 0, \end{aligned} \quad (1.1)$$

$j = 1, 2, \dots, n-1$; $i = 1, 2$. Having solved the system for the $P_i(t_k)$ we calculate k_i/k_0 and from the Chebyshev-Lagrange interpolation formula

$$\frac{k_i}{k_0} = \frac{\sqrt{2}}{n(c-1)} \sum_{k=1}^n (-1)^k \left(\frac{1+x_k}{1-x_k} \right)^{\frac{1}{2}} P_i(t_k), \quad (1.2)$$

and W_1/W_0 from the Gaussian quadrature formula

$$\frac{W_i}{W_0} = \frac{-2}{n(c-1)} \sum_{k=1}^n P_i(t_k) \int_1^{t_k} f_i(r) dr, \quad (1.3)$$

$i = 1, 2$, as described by Tweed and Rooke [5].

2. The Computation of the Energy Weight Functions

By [4.2.45]

$$\begin{aligned}\int_1^t f_1(r) dr &= \int_1^t (r^{-1} + r^{-2}) dr \\ &= \ln t + \frac{1}{2}(1 - t^{-2}).\end{aligned}\quad (2.1)$$

Similarly, by [4.2.50] we have

$$\begin{aligned}\int_1^t f_2(r) dr &= \left[1 + \frac{1}{\pi} \int_1^c \Delta(\rho) \rho^{-3} (\rho + 1) (\rho^2 + 1) d\rho \right] \int_1^t (r^{-3} - r^{-1}) dr \\ &\quad + \frac{1}{\pi} \int_1^t \int_1^c \Delta(\rho) \frac{1+\rho}{\rho^2} \left\{ \frac{\rho}{\rho-r} + \frac{1}{1-r\rho} \right\} d\rho dr \\ &\quad + \frac{1}{\pi} \int_1^t \int_1^c \Delta(\rho) \frac{1+\rho}{\rho^2} \left\{ \frac{(\rho^2-1)^2}{(1-r\rho)^3} + \frac{\rho^2-1}{(1-r\rho)^2} - \frac{\rho^2+1}{1-r\rho} \right\} d\rho dr \\ &= \left\{ 1 + \frac{1}{8}(c - c^{-1})^2 + \frac{1}{2}(c^{\frac{1}{2}} - c^{-\frac{1}{2}})^2 \right\} \left\{ \frac{1}{2}(1 - t^{-2}) - \ln t \right\} \\ &\quad + \left\{ \frac{1}{2}(c^{\frac{1}{2}} - c^{-\frac{1}{2}})(t - 1) - \frac{1}{2}(t^2 - 1) + \ln t \right\} \\ &\quad + \frac{1}{\pi} \int_1^c \Delta(\rho) \frac{1+\rho}{\rho^2} \left\{ \frac{(\rho^2-1)^2}{2\rho} \cdot \frac{1}{(1-\rho t)^2} + \frac{\rho^2-1}{\rho(1-\rho t)} \right. \\ &\quad \left. + \frac{1-\rho^2}{2\rho} + \frac{\rho^2+1}{\rho} \ln \left| \frac{\rho t - 1}{\rho - 1} \right| \right\} d\rho \\ &= \left\{ 1 + \frac{1}{8}(c - c^{-1})^2 + \frac{1}{2}(c^{\frac{1}{2}} - c^{-\frac{1}{2}})^2 \right\} \left\{ \frac{1}{2}(1 - t^{-2}) - \ln t \right\} \\ &\quad + \left\{ \frac{1}{2}(c^{\frac{1}{2}} - c^{-\frac{1}{2}})^2(t - 1) - \frac{1}{2}(t^2 - 1) + \ln t \right\} + III(t).\end{aligned}\quad (2.2)$$

$III(t)$ is evaluated in Appendix B.

3. Numerical Results

The results of the computations described above are displayed in figure 2 which shows the variation of $k_1/(k_0 \cos \theta_0)$, $-k_2/(k_0 \sin \theta_0)$, W_1/W_0 , and W_2/W_0 with the relative crack length $L = (C - B)/B$.

When $L \ll 1$, $f_1(r) \approx 2$ and $f_2(r) = 0$ so that the crack behaves like an edge crack in a half-space under uniaxial tension $2p_0 \cos \theta_0$. Numerically we find that as L approaches zero k_1 and k_2 approach the expected values $1.12 (2p_0 \cos \theta_0)(C - B)^{\frac{1}{2}}$ and zero, respectively.

Similarly, when $L \gg 1$, the crack and the hole behave like a line crack of length $C + B$ in a uniform heat flow

$$\underline{g} = q_0(\cos \theta_0 \underline{i} + \sin \theta_0 \underline{j})$$

Again we find numerically that k_1 and k_2 approach their theoretical limiting values [12] of zero and $-(p_0/8B)\sqrt{2} \sin \theta_0(C + B)^{\frac{3}{2}}$, respectively.

FIGURE 2. The variation of $K_1/(K_0 \cos \theta_0)$, $-K_2/(K_0 \sin \theta_0)$, W_1/W_0 , and W_2/W_0 with the relative crack length $(C-B)/B$.

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Appendix A

Sums of Series Involved in Calculating $f_2(r)$

We wish to evaluate

$$\begin{aligned}
 S = & \frac{3}{tr^3} - \sum_1^{\infty} n(rt)^{-n-2} + t^2 \sum_{n=2}^{\infty} n^2(rt)^{-n-2} \\
 & - t^4 \sum_{n=2}^{\infty} n(n+1)(rt)^{-n-2} + t^2 \sum_{n=2}^{\infty} n^2(rt)^{-n} \\
 & - \sum_{n=2}^{\infty} n(n-2)(rt)^{-n}.
 \end{aligned} \tag{A1}$$

Now

$$\begin{aligned}
 \sum_{n=1}^{\infty} n(rt)^{-n-2} &= \sum_{n=1}^{\infty} n z^{n+2} \quad , \quad z = \frac{1}{rt} \\
 &= z^3 \sum_{n=0}^{\infty} n z^{n-1} = z^3 \frac{d}{dz} \left(\sum_0^{\infty} z^n \right) \\
 &= z^3 \frac{d}{dz} \frac{1}{1-z} = \frac{z^3}{(1-z)^2} \\
 &= \frac{z^2}{(z-1)^2} + \frac{z}{z-1} + z \\
 &= \frac{1}{(1-rt)^2} + \frac{1}{1-rt} + \frac{1}{rt}.
 \end{aligned} \tag{A2}$$

The following sums are obtained in a similar manner.

$$\sum_{n=2}^{\infty} n^2 (rt)^{-n-2} = \frac{-2}{(1-rt)^3} - \frac{1}{(1-rt)^2} - \frac{1}{1-rt} - \frac{1}{rt} - \frac{1}{(rt)^3}, \quad (\text{A3})$$

$$\sum_{n=2}^{\infty} n(n+1)(rt)^{-n-2} = \frac{-2}{(1-rt)^3} - \frac{2}{(rt)^3}, \quad (\text{A4})$$

$$\sum_{n=2}^{\infty} n^2 (rt)^{-n} = \frac{-1}{1-rt} + \frac{3}{(1-rt)^2} - \frac{2}{(1-rt)^3} - \frac{1}{rt}, \quad (\text{A5})$$

$$\sum_{n=2}^{\infty} n(n-2)(rt)^{-n} = \frac{-2}{(1-rt)^2} + \frac{1}{(1-rt)^2} + \frac{1}{1-rt} + \frac{1}{rt} \quad (\text{A6})$$

Thus,

$$S = \frac{(t^2 - 1)^2}{(1-rt)^3} + \frac{t^2 - 1}{(1-rt)^2} - \frac{t^2 + 1}{1-rt} - \frac{t^2 + 1}{rt} + \frac{t^2 + 1}{r^3 t}. \quad (\text{A7})$$

end

Appendix B

Some Useful Integrals

The following integrals are readily obtainable by making the change of variable

$$t = c^{-1} \cos^2 \theta + c \sin^2 \theta, \quad 0 \leq \theta \leq \frac{\pi}{2},$$

where $c > 0$ and $\Delta(t) = \sqrt{(c-t)(t-c^{-1})}$.

$$R_1(c) = \int_1^c \Delta(t) dt = \frac{1}{4} \left(\frac{c^2 - 1}{c} \right)^2 \tan^{-1} \sqrt{c} + \frac{1}{4} \left(c^{\frac{1}{2}} - c^{-\frac{1}{2}} \right)^3, \quad (\text{B1})$$

$$R_2(c) = \int_1^c t^{-1} \Delta(t) dt = \frac{(c+1)^2}{c} \tan^{-1} \sqrt{c} - \left(c^{\frac{1}{2}} - c^{-\frac{1}{2}} \right) - \pi, \quad (\text{B2})$$

$$R_3(c) = \int_1^c t^{-2} \Delta(t) dt = \frac{(c+1)^2}{c} \tan^{-1} \frac{1}{\sqrt{c}} + \left(c^{\frac{1}{2}} - c^{-\frac{1}{2}} \right) - \pi. \quad (\text{B3})$$

$$\begin{aligned} I_1(x) &= \int_1^c \frac{dt}{\Delta(t)(t-x)} \\ &= 2 \int_{\theta_0}^{\frac{\pi}{2}} \frac{\sec^2 \theta d\theta}{(c - c^{-1}) \tan^2 \theta + (c^{-1} - x) \sec^2 \theta} \\ &= 2 \int_{\frac{1}{\sqrt{c}}}^{\infty} \frac{du}{(c - c^{-1}) u^2 - (x - c^{-1})(1 + u^2)} \\ &= \frac{2}{c-x} \int_{\frac{1}{\sqrt{c}}}^{\infty} \frac{du}{u^2 - \frac{x-c^{-1}}{c-x}}, \quad \theta_0 = \sin^{-1} \frac{1}{\sqrt{c+1}}, \quad c^{-1} < x < 1. \end{aligned}$$

Thus,

$$I_1(x) = \int_1^c \frac{dt}{\Delta(t)(t-x)} = \frac{1}{\Delta(x)} \ln \left\{ \frac{\sqrt{c-x} + \sqrt{cx-1}}{\sqrt{c-x} - \sqrt{cx-1}} \right\},$$

$$c^{-1} < x < 1. \quad (\text{B4})$$

Using (B.4) it is not difficult to show that for $c^{-1} < x < 1$,

$$I_2(x) = \int_1^c \frac{\Delta(t)dt}{t-x} = \Delta(x) \ln \left\{ \frac{\sqrt{c-x} + \sqrt{cx-1}}{\sqrt{c-x} - \sqrt{cx-1}} \right\}$$

$$- \frac{c-1}{\sqrt{c}} + (c + c^{-1} - 2x) \tan^{-1} \sqrt{c}, \quad (\text{B5})$$

$$I_3(x) = \int_1^c \frac{\Delta(t)dt}{t(t-x)} = \frac{\Delta(x)}{x} \ln \left\{ \frac{\sqrt{c-x} + \sqrt{cx-1}}{\sqrt{c-x} - \sqrt{cx-1}} \right\}$$

$$+ \frac{2(x+1)}{x} \tan^{-1} \frac{1}{\sqrt{c}} - \pi, \quad (\text{B6})$$

$$I_4(x) = \int_1^c \frac{\Delta(t)dt}{t^2(t-x)} = \frac{\Delta(x)}{x^2} \ln \left\{ \frac{\sqrt{c-x} + \sqrt{cx-1}}{\sqrt{c-x} - \sqrt{cx-1}} \right\}$$

$$- \frac{c-1}{\sqrt{cx}} + \left\{ \frac{2}{x^2} - \frac{c^2+1}{cx} \right\} \tan^{-1} \frac{1}{\sqrt{c}}. \quad (\text{B7})$$

The following integrals are obtained by differentiating (B5)-(B7) with respect to x :

$$\begin{aligned}
J_1(x) &= \int_1^c \frac{\Delta(t)dt}{(t-x)^2} \\
&= \frac{c+c^{-1}-2x}{2\Delta(x)} \ln \left\{ \frac{\sqrt{c-x} + \sqrt{cx-1}}{\sqrt{c-x} - \sqrt{cx-1}} \right\} \\
&\quad + \frac{c^{\frac{1}{2}} - c^{-\frac{1}{2}}}{1-x} - 2 \tan^{-1} \sqrt{c}, \quad c^{-1} < x < 1, \tag{B8}
\end{aligned}$$

$$\begin{aligned}
J_2(x) &= \int_1^c \frac{\Delta(t)dt}{t(t-x)^2} \\
&= \left\{ \frac{c+c^{-1}-2x}{2x\Delta(x)} - \frac{\Delta(x)}{x^2} \right\} \ln \left\{ \frac{\sqrt{c-x} + \sqrt{cx-1}}{\sqrt{c-x} - \sqrt{cx-1}} \right\} \\
&\quad + \frac{c^{\frac{1}{2}} - c^{-\frac{1}{2}}}{x(1-x)} - \frac{2}{x^2} \tan^{-1} \frac{1}{\sqrt{c}}, \quad c^{-1} < x < 1, \tag{B9}
\end{aligned}$$

$$\begin{aligned}
J_3(x) &= \int_1^c \frac{\Delta(t)dt}{t^2(t-x)^2} \\
&= \left\{ \frac{c+c^{-1}-2x}{2x^2\Delta(x)} - \frac{2\Delta(x)}{x^3} \right\} \ln \left\{ \frac{\sqrt{c-x} + \sqrt{cx-1}}{\sqrt{c-x} - \sqrt{cx-1}} \right\} \\
&\quad + \frac{c^{\frac{1}{2}} - c^{-\frac{1}{2}}}{x^2(1-x)} + \frac{c-1}{\sqrt{cx^2}} \\
&\quad + \left\{ \frac{c^2+1}{cx^2} - \frac{4}{x^3} \right\} \tan^{-1} \frac{1}{\sqrt{c}}, \quad c^{-1} < x < 1. \tag{B10}
\end{aligned}$$

By differentiating (B8) - (B10) with respect to x we obtain:

$$\begin{aligned}
K_1(x) &= \int_1^c \frac{\Delta(t)dt}{(t-x)^3} = \frac{(c^{\frac{1}{2}} - c^{-\frac{1}{2}})^3(1+x)}{4(\Delta(x))^2(1-x)^2} \\
&\quad - \frac{(c-c^{-1})^2}{8(\Delta(x))^3} \ln \left\{ \frac{\sqrt{c-x} + \sqrt{cx-1}}{\sqrt{c-x} - \sqrt{cx-1}} \right\}, \quad c^{-1} < x < 1, \tag{B11}
\end{aligned}$$

$$\begin{aligned}
K_2(x) &= \int_1^c \frac{\Delta(t)dt}{t(t-x)^3} \\
&= \left\{ \frac{(c+c^{-1})x-2}{2x^3\Delta(x)} - \frac{(c-c^{-1})^2}{8x(\Delta(x))^3} \right\} \ln \left\{ \frac{\sqrt{c-x}+\sqrt{cx-1}}{\sqrt{c-x}-\sqrt{cx-1}} \right\} \\
&\quad - \frac{(c^{\frac{1}{2}}-c^{-\frac{1}{2}})(1-2x)}{2x^2(1-x)^2} + \frac{2}{x^3} \tan^{-1} \frac{1}{\sqrt{c}} \\
&\quad - \frac{\{(c+c^{-1})x-2\}\{c^{\frac{1}{2}}-c^{-\frac{1}{2}}\}}{4x^2(\Delta(x))^2(1-x)}, \quad c^{-1} < x < 1, \tag{B12}
\end{aligned}$$

$$\begin{aligned}
K_3(x) &= \int_1^c \frac{\Delta(t)dt}{t^2(t-x)^3} \\
&= \left\{ \frac{3\Delta(x)}{x^4} - \frac{(c-c^{-1})^2}{8x^2[\Delta(x)]^3} - \frac{c+c^{-1}-2x}{x^3\Delta(x)} \right\} \\
&\quad \cdot \ln \left\{ \frac{\sqrt{c-x}+\sqrt{cx-1}}{\sqrt{c-x}-\sqrt{cx-1}} \right\} - \frac{c-1}{\sqrt{c}x^3} \\
&\quad + \frac{(c^{\frac{1}{2}}-c^{-\frac{1}{2}})(c+c^{-1}-2x)}{4x^2(\Delta(x))^2(1-x)} - \frac{(c^{\frac{1}{2}}-c^{-\frac{1}{2}})(4-5x)}{2x^3(1-x)^2} \\
&\quad + \left\{ \frac{6}{x^4} - \frac{c^2+1}{cx^3} \right\} \tan^{-1} \frac{1}{\sqrt{c}}, \quad c^{-1} < x < 1. \tag{B13}
\end{aligned}$$

In evaluating the following integrals we use the results in (B1) to (B13)

The Integral $II(r)$.

$$\begin{aligned}
 II(r) &= \frac{1}{\pi} \int_1^c \Delta(t) \frac{1+t}{t^2} \left\{ \frac{(t^2-1)^2}{(1-rt)^3} + \frac{t^2-1}{(1-rt)^2} - \frac{t^2+1}{1-rt} \right\} dt \\
 &= \frac{1}{\pi} \int_1^c \Delta(t) \frac{(1+t)}{t^2} \frac{(t^2-1)^2}{(1-rt)^3} dt + \frac{1}{\pi} \int_1^c \Delta(t) \frac{(1+t)}{t^2} \frac{(t^2-1)}{(1-rt)^2} dt \\
 &\quad - \frac{1}{\pi} \int_1^c \Delta(t) \frac{(1+t)}{t^2} \frac{(t^2+1)}{(1-rt)} dt = II_3(r) + II_2(r) + II_1(r). \quad (B14)
 \end{aligned}$$

$$\begin{aligned}
 II_1(r) &= -\frac{1}{\pi} \int_1^c \frac{\Delta(t)}{1-rt} \left(t + 1 + \frac{1}{t} + \frac{1}{t^2} \right) dt \\
 &= \frac{1}{\pi r} \int_1^c \frac{\Delta(t)}{t-r^{-1}} \left([t-r^{-1}] + [1+r^{-1}] + \frac{1}{t} + \frac{1}{t^2} \right) dt. \quad (B15)
 \end{aligned}$$

Therefore,

$$II_1(r) = \frac{r^{-1}}{\pi} \{ R_1(c) + (1+r^{-1})I_2(r^{-1}) + I_3(r^{-1}) + I_4(r^{-1}) \}. \quad (B16)$$

Similarly,

$$\begin{aligned}
 II_2(r) &= \frac{r^{-2}}{\pi} \int_1^c \frac{\Delta(t)}{(t-r^{-1})^2} \left\{ t + 1 - \frac{1}{t} - \frac{1}{t^2} \right\} dt \\
 &= \frac{r^{-2}}{\pi} \{ I_2(r^{-1}) + (1+r^{-1})J_1(r^{-1}) \\
 &\quad - J_2(r^{-1}) - J_3(r^{-1}) \} \quad (B17)
 \end{aligned}$$

and

$$\begin{aligned}
II_3(r) &= \int_1^c \Delta(t) \left(\frac{1+t}{1-rt} \right)^3 \left(1 - \frac{1}{t} \right)^2 dt \\
&= \frac{1}{\pi r^3} \int_1^c \Delta(t) \left\{ \frac{1+r}{r-rt} - 1 \right\}^3 \left(1 - \frac{1}{t} \right)^2 dt \\
&= \frac{1}{\pi} \int_1^c \Delta(t) \left\{ \frac{-(1+r)^3}{r^6} \cdot \frac{1}{(t-r^{-1})^3} - \frac{3(1+r)^2}{r^5} \cdot \frac{1}{(t-r^{-1})^2} \right. \\
&\quad \left. - \frac{3(1+r)}{r^4} \cdot \frac{1}{t-r^{-1}} - \frac{1}{r^3} \right\} \left(1 - \frac{1}{t} \right)^2 dt \\
&= \frac{-(1+r)^3}{\pi r^6} \{k_1(r^{-1}) - 2k_2(r^{-1}) + k_3(r^{-1})\} \\
&\quad - \frac{3(1+r)^2}{\pi r^5} \{J_1(r^{-1}) - 2J_2(r^{-1}) + J_3(r^{-1})\} \\
&\quad - \frac{3(1+r)}{\pi r^4} \{I_2(r^{-1}) - 2I_3(r^{-1}) + I_4(r^{-1})\} \\
&\quad - \frac{1}{\pi r^3} \{R_1(c) - 2R_2(c) + R_3(c)\}. \tag{B18}
\end{aligned}$$

The Integral III(t)

$$\begin{aligned}
III(t) &= \frac{1}{\pi} \int_1^c \Delta(\rho) \frac{1+\rho}{\rho^2} \left\{ \frac{(\rho^2-1)^2}{2\rho} \cdot \frac{1}{(1-\rho t)^2} + \frac{\rho^2-1}{\rho} \cdot \frac{1}{1-\rho t} \right. \\
&\quad \left. + \frac{1-\rho^2}{2\rho} + \frac{\rho^2+1}{\rho} \ln \left| \frac{\rho t-1}{\rho-1} \right| \right\} d\rho \\
&= \frac{1}{2\pi} \int_1^c \Delta(\rho) \frac{\rho+1}{\rho} \left\{ (t^{-2}-1) + \frac{2t^{-1}(t^{-2}-1)}{\rho-t^{-1}} \right. \\
&\quad \left. + \frac{(t^{-2}-1)^2}{(\rho-t^{-1})^2} + 2(1+\rho^{-2}) \ln \left| \frac{\rho t-1}{\rho-1} \right| \right\} d\rho \\
&= \frac{1}{2\pi} (t^{-2}-1) [R_1(c) + R_2(c)] + \frac{1}{\pi} t^{-1} (t^{-2}-1) [I_2(t^{-1}) + I_3(t^{-1})] \\
&\quad + \frac{1}{2\pi} (t^{-2}-1)^2 [J_1(t^{-1}) + J_2(t^{-1})] \\
&\quad + \frac{1}{\pi} \int_1^c \Delta(\rho) (1+\rho^{-1})(1+\rho^{-2}) \ln \left| \frac{\rho t-1}{\rho-1} \right| d\rho \\
&= \frac{1}{2\pi} (t^{-2}-1) [R_1(c) + R_2(c)] + \frac{1}{\pi} t^{-1} (t^{-2}-1) [I_2(t^{-1}) + I_3(t^{-1})] \\
&\quad + \frac{1}{2\pi} (t^{-2}-1)^2 [J_1(t^{-1}) + J_2(t^{-1})] + III_1(t); \tag{B19}
\end{aligned}$$

$$\begin{aligned}
III_1(t) &= \frac{1}{\pi} \int_1^c \Delta(\rho)(1+\rho^{-1})(1+\rho^{-2}) \ln \left| \frac{\rho t - 1}{\rho - 1} \right| d\rho \\
&= \frac{4(c^{\frac{1}{2}} - c^{-\frac{1}{2}})}{\pi(c-1)} \int_1^c \ln \left| \frac{\rho t - 1}{\rho - 1} \right| (c - \rho) d\rho \\
&\quad + \frac{1}{\pi} \int_1^c \left\{ \Delta(\rho)(1+\rho^{-1})(1+\rho^{-2}) \right. \\
&\quad \left. - 4(c^{\frac{1}{2}} - c^{-\frac{1}{2}}) \frac{c - \rho}{c - 1} \right\} \ln \left| \frac{\rho t - 1}{\rho - 1} \right| d\rho \\
&= \frac{4}{\pi\sqrt{c}} \int_1^c \ln \left| \frac{\rho t - 1}{\rho - 1} \right| (c - \rho) d\rho \\
&\quad + \frac{1}{\pi} \int_1^c \left\{ \Delta(\rho)(1+\rho^{-1})(1+\rho^{-2}) \right. \\
&\quad \left. - \frac{4}{\sqrt{c}}(c - \rho) \right\} \ln \left| \frac{\rho t - 1}{\rho - 1} \right| d\rho. \tag{B20}
\end{aligned}$$

Hence, after a little manipulation, we find that

$$\begin{aligned}
III(t) &= \frac{1}{2\pi}(t^{-2} - 1)[R_1(c) + R_2(c)] + \frac{1}{\pi}t^{-1}(t^{-2} - 1)[I_2(t^{-1}) + I_3(t^{-1})] \\
&\quad + \frac{1}{2\pi}(t^{-2} - 1)^2[J_1(t^{-1}) + J_2(t^{-1})] \\
&\quad + \frac{2}{\pi\sqrt{c}} \left\{ (c-1)^2 \ln \left| \frac{t-1}{c-1} \right| + (c-t^{-1})^2 \ln \left| \frac{ct-1}{t-1} \right| \right. \\
&\quad \left. - (c-1)(1-t^{-1}) \right\} + \frac{1}{\pi} \int_1^c \left\{ \Delta(\rho)(1+\rho^{-1})(1+\rho^{-2}) \right. \\
&\quad \left. - \frac{4}{\sqrt{c}}(c - \rho) \right\} \ln \left| \frac{\rho t - 1}{\rho - 1} \right| d\rho, \tag{B21}
\end{aligned}$$

where the integral in (B21) may be approximated to a high degree of accuracy using cubic splines.

Appendix C

```
PROGRAM STRESS
IMPLICIT REAL*8(A-G),REAL*8(O-Z)
IMPLICIT INTEGER*4(H-N)

C      Given the length of the crack, C1, and the desired
C      number of Gaussian points, N, STRESS constructs vectors
C      containing the abscissas of the Gaussian points T(j) and
C      the collocation points Y(j). STRESS then calls subroutine
C      SYS1 or SYS2 which returns the opening mode stress
C      intensity and crack energy or the sliding mode intensity
C      and crack energy, respectively. These are stored in data
C      files which are used to construct graphs of the stress
C      intensities and crack energies for various crack lengths
C      using GRAPHIER, a computer graphing package by Golden
C      Graphics Software, Inc.

C*****INPUT*****
C N      Number of Gaussian points.
C C1     Length of crack.
C NN     Control variable: NN =1, k1/k0 is calculated;
C                      =2, w1/w0 is calculated;
C                      =3, k2/k0 is calculated;
C                      =4, w2/w0 is calculated.
C*****OUTPUT*****
C C      Abscissa of crack tip.
C ABSISA N-vector containing Gaussian points on the interval
C      (1,C).
C T      N-vector containing Gaussian points on (-1,1).
C Y      N-vector containing collocation points.
C*****
C
C ROUTINES CALLED:
C SYS1
C SYS2
C
C      DIMENSION ABSISA(64),T(64),Y(64)
C      EXTERNAL SYS1, SYS2
C      COMMON/ALL/N
C      COMMON/GRP1/T,Y
C      COMMON/GRP2/ABSISA
C      COMMON/GRP3/C1,C,CC,ON,TW,TH,FO,EI
C      DATA ON,TW,TH,FO,EI/1.0D00,2.0D00,3.0D00,4.0D00,8.0D00/
C      WRITE(0,*) 'TYPE 1 TO CALCULATE K1'
C      WRITE(0,*) '      2 TO CALCULATE W1'
C      WRITE(0,*) '      3 TO CALCULATE K2'
C      WRITE(0,*) '      4 TO CALCULATE W2'
C      WRITE(0,*) '? '
C      READ(0,*) NN
C      IF(NN.EQ.1) THEN
```

```

OPEN(7,FILE='K1.DAT',STATUS='NEW')
ELSEIF(NN.EQ.2) THEN
OPEN(8,FILE='W1.DAT',STATUS='NEW')}
ELSEIF(NN.EQ.3) THEN
OPEN(9,FILE='K2.DAT',STATUS='NEW')
ELSE
OPEN(10,FILE='W2.DAT',STATUS='NEW')
ENDIF
PI=3.1415926535897932D00
N=64
6000 C1=0.2D00
11 C=C1+1.0D00
CC=DSQRT(C)

C      We construct the abscissas.

DO 2000 J=1,N
ABSISA(J)=DCOS(DBLE(2*J-1)*PI/DBLE(2*N))
T(J)=0.5D00*((C-1.0D00)*ABSISA(J)+(C+1.0D00))
2000 CONTINUE

C      We construct the collocation points.

DO 3000 J=1,N-1
V1=DCOS((DBLE(J))*PI/DBLE(N))
Y(J)=0.5D00*(C1*V1+C+1.0D00)
3000 CONTINUE
IF(NN.EQ.1.OR.NN.EQ.2) THEN
CALL SYS1(NN)
ELSE
CALL SYS2(NN)
ENDIF
IF(C.LT.10.99D00) THEN
C1=C1+0.2D00
GOTO 11
ENDIF
STOP
END

```

```

SUBROUTINE SYS1(NN)
IMPLICIT REAL*8 (A-G), REAL*8 (O-Z)
IMPLICIT INTEGER*4 (H-N)

C*****INPUT*****
C NN Control variable: NN=1,  $k_1/k_0$  is calculated,
C =2,  $w_1/w_0$  is calculated.
C T, Y N-vectors from the main program STRESS.
C C1,C Crack length and abscissa of crack tip from the main
C program STRESS.
C IPIVOT N-vector from subroutine FACTOR.
C IFLAG Control variable from subroutine FACTOR.

C*****OUTPUT*****
C W Array of size (N,N) containing the matrix of
C coefficients for the linear system related to  $k_1/k_0$ .
C B N-vector containing the right sides for the system.
C X N-vector containing the solution to the system  $WX=B$ .
C RK1 The stress intensity factor  $k_1/k_0$ .
C W1 The stress energy  $w_1/w_0$ .
C*****
C
C ROUTINES CALLED:
C FACTOR
C SUBST
C INTENS
C CRACEN

DIMENSION T(64),Y(64),B(64),W(64,64),X(64)
DIMENSION ABSISA(64),IPIVOT(64)
COMMON/ALL/N
COMMON/GRP1/T,Y
COMMON/GRP3/C1,C,CC,ON,TW,TH,FO,EI
COMMON/GRP4/W
COMMON/GRP2/ABSISA
COMMON/GRP5/IPIVOT,IFLAG
COMMON/GRP6/B,X
EXTERNAL ZKERN1,CRACEN,INTENS,FACTOR,SUBST
VK3(Z)=DSQRT((1.0D00-Z)/(1.0D00+Z))
RSIDE1(Z)=(Z*Z+1.0D00)/(Z*Z*Z)

C We construct the matrix of coefficients for the
C linear system used to calculate  $k_1/k_0$ .

DO 4000 I=1,N-1
DO 5000 J=1,N
U1=1.0D00/(T(J)-Y(I))
U2=1.0D00/DELE(N)
W(I,J)=U2*(U1+ZKERN1(Y(I),T(J)))
5000 CONTINUE
4000 CONTINUE

```

```

      DO 4010 I=1,N
      H=(-1)**I
      W(N,I)=VK3 (ABSISA(I)) *DELE(H)/DELE(N)
4010 CONTINUE

C      We construct the right side functions for the
C      linear system.

      DO 7000 J=1,N-1
      B(J)= -RSIDE1(Y(J))
7000 CONTINUE
      B(N)=0

C      We calculate the interpolated values for the solution
C      to the integral equation related to  $k_1/k_0$ .

      CALL FACTOR
      IF(IFLAG.EQ.0) THEN
      WRITE(6,1)
1    FORMAT(1X,'IFLAG=0'/)
      ENDIF
      CALL SUBST

C      We calculate  $k_1/k_0$ .

      CALL INTENS(RK1)
      IF(NN.EQ.1) THEN
      WRITE(7,804) C1,RK1
804  FORMAT(1X,F5.2,F10.4)
      RETURN
      ENDIF
      CALL CRACEN(NN,WOWWO)
      WRITE(8,804) C1,WOWWO
      RETURN
      END

```

```

SUBROUTINE SYS2(NN)
  IMPLICIT REAL*8(A-G),REAL*8(O-Z)
  IMPLICIT INTEGER*4(H-N)

C
C*****INPUT*****
C NN      Control variable NN=3, k2/k0 is calculated,
C          =4, w2/w0 is calculated.
C T, Y    N-vectors from the main program SIRESS.
C C1,C    Crack length and abscissa of crack tip from the main
C          program SIRESS.
C IPIVOT  N-vector from subroutine FACTOR.
C IFLAG   Control variable from subroutine FACTOR.}
C*****OUTPUT*****
C W       Array of size (N,N) containing the matrix of
C          coefficients for the linear system related to k2/k0.
C B       N-vector containing the right sides for the system.
C X       N-vector containing the solution to the system WX=B.
C RK2     The stress intensity factor k2/k0.
C W2      The stress energy w2/w0.
C*****

C      ROUTINES CALLED:
C      FACTOR
C      SUBST
C      INTENS
C      CRACEN

      DIMENSION T(64),Y(64),B(64),W(64,64),X(64),IPIVOT(64)
      DIMENSION ABSISA(64)
      COMMON/ALL/N
      COMMON/GRP1/T,Y
      COMMON/GRP3/C1,C,CC,ON,TW,TH,FO,EI
      COMMON/GRP2/ABSISA
      COMMON/GRP4/W
      COMMON/GRP5/IPIVOT,IFLAG
      COMMON/GRP6/B,X
      EXTERNAL ZKERN1,ZKERN2,CRACEN,INTENS,FACTOR,SUBST
      VK3(Z)=DSQRT((1.0D00-Z)/(1.0D00+Z))

C      We construct the matrix of coefficients for the linear
C      system used to calculate k2/k0.

      DO 4111 I=1,N-1
      DO 5111 J=1,N
      U1=1.0D00/(T(J)-Y(I))
      U2=1.0D00/DBLE(N)
      W(I,J)=U2*(U1+ZKERN2(Y(I),T(J)))
5111 CONTINUE
4111 CONTINUE
      DO 4121 I=1,N

```

```

      H=(-1)**I
      W(N,I)=VK3 (ABSISA(I)) *DELE(H)/DELE(N)
4121 CONTINUE

C      We construct the right side functions for the linear
C      system.

      DO 7111 J=1,N-1
      B(J)= -RSIDE2(Y(J),C)
7111 CONTINUE
      B(N)=0

C      We calculate the interpolated values for the solution
C      to the integral equation related to  $k_2/k_0$ .

      CALL FACTOR
      IF (IFLAG.EQ.0) THEN
      WRITE(0,2)
2      FORMAT(1X,'IFLAG=0'/)
      ENDIF
      CALL SUBST

C      We calculate  $k_2/k_0$ .

      CALL INTENS(RK2)
      IF (NN.EQ.3) THEN
      WRITE(9,805) C1,-RK2
805  FORMAT(1X,F5.3,F10.4)
      RETURN
      ENDIF

C      We calculate the crack energy.

      CALL CRACEN(NN,WOVW0)
      WRITE(10,806) C1,WOVW0
806  FORMAT(1X,F5.3,F10.4)
      RETURN
      END

```



```

SUBROUTINE FACTOR
IMPLICIT REAL*8 (A-G), REAL*8 (O-Z)
IMPLICIT INTEGER*4 (H-N)

```

C Reference [13].

C*****INPUT*****

C W An array of size (N,N) containing a matrix A of order N to
C be factored.
C D A double precision vector of length N, to hold row sizes.

C*****OUTPUT*****

C W An (N,N) array containing the LU factorization of P*A
C where the permutation matrix P is specified by IPIVOT.
C IPIVOT An integer vector of length N indicating that row
C IPIVOT(K) was used to eliminate X(K), K=1,2,...,N.
C IFLAG = 1, if an even number of interchanges was carried
C out,
C = -1, if an odd number of interchanges was carried
C out,
C = 0, if the upper triangular factor has one or more
C zero diagonal entries.
C If IFLAG.NE.0 the system A*X=B is solved for X by a
C CALL SUBST(W,IPIVOT,B,N,X)
C The subroutine uses Gaussian elimination with partial
C pivoting.

C*****

```

      DIMENSION B(64),W(64,64),X(64),IPIVOT(64),D(64)
      COMMON/ALL/N
      COMMON/GRP4/W
      COMMON/GRPS/IPIVOT,IFLAG
      COMMON/GRP6/B,X
      IFLAG=1

```

C We initialize IPIVOT and D.

```

      DO 9 I=1,N
      IPIVOT(I)=I
      ROWMAX=0.0D00
      DO 5 J=1,N
5      ROWMAX=DMAX1(ROWMAX,DABS(W(I,J)))
      IF (ROWMAX.LE.1.0D-10) THEN
      IFLAG=0
      ROWMAX=1.0D00
      ENDIF
9      D(I)=ROWMAX
      IF (N.LE.1) THEN
      RETURN
      ENDIF

```

```

C      We factorize the matrix.

DO 20 K=1,N-1

C      We determine the pivot row—the row ISTAR.

COLMAX=DABS(W(K,K))/D(K)
ISTAR=K
DO 13 I=K+1,N
  AWIKOD=DABS(W(I,K))/D(I)
  IF (AWIKOD.GT.COLMAX) THEN
    COLMAX=AWIKOD
    ISTAR=I
  ENDIF
13 CONTINUE
  IF (COLMAX.EQ.0.0D00) THEN
    IFLAG=0
  ELSE
    IF (ISTAR.GT.K) THEN

C      We make K the pivot row by interchanging it with
C      the chosen row ISTAR.

    IFLAG=IFLAG
    I=IPIVOT(ISTAR)
    IPIVOT(ISTAR)=IPIVOT(K)
    IPIVOT(K)=I
    TEMP=D(ISTAR)
    D(ISTAR)=D(K)
    D(K)=TEMP
    DO 15 J=1,N
      TEMP=W(ISTAR,J)
      W(ISTAR,J)=W(K,J)
15    W(K,J)=TEMP
    ENDIF
  ENDIF

C      We eliminate X(K) from rows K+1,...,N.

DO 16 I=K+1,N
  W(I,K)=W(I,K)/W(K,K)
  RATIO=W(I,K)
  DO 19 J=K+1,N
    W(I,J)=W(I,J)-RATIO*W(K,J)
19  CONTINUE
16  CONTINUE
20  CONTINUE
  IF (W(N,N).LE.1.0D-10) THEN
    IFLAG=0
  ENDIF
  RETURN
END

```

```

SUBROUTINE SUBST
IMPLICIT REAL*8 (A-G), REAL*8 (O-Z)
IMPLICIT INTEGER*4 (H-N)
C
C      Reference [13].

C*****INPUT*****
C      W  (N,N) matrix passed to SUBST by SYS1 or SYS2.
C      B  N-vector passed to SUBST by SYS1 or SYS2.

C*****OUTPUT*****
C      X  N-vector satisfying the matrix equation  $W \cdot X = B$ .
C      Given the factorized matrix W, the permutation matrix P,
C      and the right side N-vector forward- and back-
C      substitution are used to solve the system  $W \cdot X = B$ .
C*****

      DIMENSION T(64), Y(64), B(64), W(64,64), X(64), IPIVOT(64)
      COMMON/ALL/N
      COMMON/GRP1/T,Y
      COMMON/GRP4/W
      COMMON/GRP5/IPIVOT,IFLAG
      COMMON/GRP6/B,X
      IF (N.LE.1) THEN
        X(1)=B(1)/W(1,1)
        RETURN
      ENDIF
      IP=IPIVOT(1)
      X(1)=B(IP)
      DO 150 I=2,N
        SUM=C.0D00
        DO 140 J=1,I-1
          SUM=W(I,J)*X(J)+SUM
140      CONTINUE
        IP=IPIVOT(I)
        X(I)=B(IP)-SUM
150      CONTINUE
        X(N)=X(N)/W(N,N)
        DO 2001 I=N-1,1,-1
          SUM=0.0D00
          DO 190 J=I+1,N
            SUM=W(I,J)*X(J)+SUM
190          CONTINUE
          X(I)=(X(I)-SUM)/W(I,I)
2001      CONTINUE
      RETURN
      END

```

```

SUBROUTINE CRACEN(NN,W0VW0)
IMPLICIT REAL*8 (A-G),REAL*8 (O-Z)
IMPLICIT INTEGER*4 (H-N)

C*****INPUT*****
C C1,C   Crack length and abscissa of crack tip.
C N      Number of Gaussian points.
C NN     Control variable NN=2, w1/w0 is calculated,
C        =4, w2/w0 is calculated.
C T      N-vector containing Gaussian points.
C X      N-vector containing solution to the matrix equation
C        WX=B.

C*****OUTPUT*****
C W0VW0   w1/w0 if NN=2, w2/w0 if NN=4.
C*****

      DIMENSION X(64),T(64),Y(64),B(64)
      COMMON/ALL,N
      COMMON/GRP1/T,Y
      COMMON/GRP3/C1,C,CC,CN,TW,TH,FO,EI
      COMMON/GRP6/B,X
      EXTERNAL W2
      W1(Y)=( (CN-CN/(YY*YY))/TW+DLOG(DABS(YY)) )
      U=0.0D00
      DO 1013 I=1,N
      IF(NN.EQ.2) THEN

C          We calculate w1/w0 from the Gaussian quadrature
C          formula.

      U=U+X(I)*W1(T(I))
      ELSE

C          We calculate w2/w0.

      U=U+X(I)*W2(T(I),C)
      ENDIF
1013  CONTINUE
      W0VW0=2.0D00*U/(DBLE(N)*(C-CN)**2)
      RETURN
      END

```

```

SUBROUTINE INTENS(RK)
IMPLICIT REAL*8(A-G),REAL*8(O-Z)
IMPLICIT INTEGER*4(H-N)

```

```

C*****INPUT*****

```

```

C      N      Number of Gaussian points.
C      ABSISA N-vector containing the Gaussian points.
C      X      N-vector containing the solution of linear algebraic
C              system from the subroutine SUBST.
C      C,C1,CC The abscissa of the crack tip, length of the crack,
C              and square root of C.

```

```

C*****OUTPUT*****

```

```

C      RK      k1/k0 if NN=1; K2/K0 if NN=2.
C*****

```

```

      DIMENSION ABSISA(64),B(64),X(64)
      COMMON/ALL/N
      COMMON/GRP2/ABSISA
      COMMON/GRP6/B,X
      COMMON/GRP3/C1,C,CC,CN,TW,TH,FO,EI
      RK=0.0D00

```

```

C      We calculate the stress intensity factor from the
C      Chebyshev-Lagrange interpolation formula.

```

```

      DO 1800 I=1,N
      H1=(-1)**I
      U1=(1.0D00+ABSISA(I))/(1.0D00-ABSISA(I))
      RK=RK+DBLE(H1)*DSQRT(U1)*X(I)
1800  CONTINUE
      Z2=DSQRT(DBLE(2))/(DBLE(N)*(C-1.0D00))
      RK=Z2*RK
      RETURN
      END

```

```

      DOUBLE PRECISION FUNCTION ZKERN1(Z,ZZ)
      DOUBLE PRECISION K1,K2,K3,K4,Z,ZZ
      K1=(1.0D00-ZZ**2)**2/(ZZ*(1.0D00-Z*ZZ)**3)
      K2=ZZ*(1.0D00-ZZ**2)/((1.0D00-Z*ZZ)**2)
      K3=ZZ/(1.0D00-Z*ZZ)
      K4=(1.0D00-ZZ**2)/((Z**2)*ZZ)
      ZKERN1=K1-K2-K3+K4
      RETURN
      END

```

```

      DOUBLE PRECISION FUNCTION ZKERN2(Z,ZZ)
      DOUBLE PRECISION K1,K2,K3,Z,ZZ,U
      K1=(1.0D00-ZZ**2)**2/(ZZ*(1.0D00-Z*ZZ)**3)
      K2=(1.0D00-ZZ**2)/(ZZ*(1.0D00-Z*ZZ)**2)
      K3=ZZ/(1.0D00-Z*ZZ)
      ZKERN2=K1-K2-K3
      RETURN
      END

      SUBROUTINE CURBSPL(T,ENDPT,Q)
      IMPLICIT REAL*8 (A-G),REAL*8 (O-Z)
      IMPLICIT INTEGER*4 (H-N)

C
C      Reference [14].

C*****INPUT*****
C   T      A Gaussian point supplied by the main program STRESS.
C   ENDPT  Abscissa of the crack tip.
C*****WORK AREA*****
C   N      Number of data points.
C   TAU(I), C(1,I), I=1,...,N abscissas and ordinates (determined
C      from the function GG) of the data points.
C   C(2,1), C(2,N) Boundary conditions information--no boundary
C      conditions at TAU(1) and TAU(N) are given.
C   C(J,I), J=1,...,4; I=1,...,N-1 The polynomial coefficients of
C      the cubic interpolating spline with interior knots
C
C      TAU(2), ..., TAU(N-1), precisely, in the interval
C      (TAU(I), TAU(I+1)), the spline F is given by
C
C      
$$F(X)=C(1,I)+H*(C(2,I)+H*(C(3,I)+H*C(4,I)/3.)/2.)$$

C      where  $H=X-TAU(X)$ .
C      A tridiagonal linear system for the unknown slopes S(I)
C      of F at TAU(I), I=1,...,N, is generated and then solved
C      by Gauss elimination, with S(I) ending up in C(2,I).
C      Once the spline coefficients are determined the integral
C      of the original function is approximated by integrating
C      F on each interval and summing from I=1 to N-1.

C*****OUTPUT*****
C   Q      The sum of the integrals of the spline functions.
C*****

      DIMENSION C(4,1000),TAU(1000)
      EXTERNAL GG
      N=1000
      DATA CN/1.0D00/,TW/2.0D00/,TH/3.0D00/

C      We initialize the 4 by N matrix C.

      DO 1014 I=1,4}
      DO 1015 J=1,N

```

```

1015 C(I,J)=0.0D00
1014 CONTINUE
C      We calculate the vectors containing the abscissas and
C      ordinates of the data points.

GG=(ENDPT-ON)/DELE(N-1)
DO 41 I=1,N
TAU(I)=ON+CH*DELE(I-1)
IF(I.EQ.1) THEN
C(1,I)=0.0D00
ELSEIF(I.EQ.N) THEN
C(1,I)=0.0D00
ELSE
C(1,I)=GG(TAU(I),ENDPT,T)
ENDIF
41 CONTINUE
L=N-1

C      We compute and store first differences of TAU sequence
C      and first divided differences of data.

DO 101 M=2,N
C(3,M)=TAU(M)-TAU(M-1)
101 C(4,M)=(C(1,M)-C(1,M-1))/C(3,M)

C      We construct the first equation from boundary
C      condition of form: C(4,1)*S(1)+C(3,1)*S(2)=C(2,1).

C(4,1)=C(3,3)
C(3,1)=C(3,2)+C(3,3)
C(2,1)=(C(3,2)+TW*C(3,1))*C(4,2)*C(3,3)
C(2,1)=(C(2,1)+C(3,2)**2*C(4,3))/C(3,1)

C      We construct the equations corresponding to the
C      interior knots and carry out the forward pass of Gauss
C      elimination, after which the mth equation reads:
C      C(4,M)*S(M)+C(3,M)*S(M+1)=C(2,M).

DO 201 M=2,L
G=-C(3,M+1)/C(4,M-1)
C(2,M)=G*C(2,M-1)+3.0D00*
1(C(3,M)*C(4,M+1)+C(3,M+1)*C(4,M))
201 C(4,M)=G*C(3,M-1)+TW*(C(3,M)+C(3,M+1))

C      We construct the last equation from the second
C      boundary condition of the form:
C      (-G*S(N-1)+C(4,N)*S(N)=C(2,N).

G=C(3,N-1)+C(3,N)
U1=(C(3,N)+TW*G)*C(4,N)*C(3,N-1)
U2=C(3,N)**2*(C(1,N-1)-C(1,N-2))
C(2,N)=(U1+U2/C(3,N-1))/G

```

```

      G=G/C(4,N-1)
      C(4,N)=C(3,N-1)
C      We complete forward pass of Gauss elimination.

      C(4,N)=G*C(3,N-1)+C(4,N)
      C(2,N)=(G*C(2,N-1)+C(2,N))/C(4,N)

C      We carry out back substitution.

      DO 401 J=L,1,-1
401    C(2,J)=(C(2,J)-C(3,J)*C(2,J+1))/C(4,J)
      DO 501 I=2,N

C      We generate cubic coefficients in each interval.

      DTAU=C(3,I)
      DIVDF1=(C(1,I)-C(1,I-1))/DTAU
      DIVDF3=C(2,I-1)+C(2,I)-TW*DIVDF1
      C(3,I-1)=TW*(DIVDF1-C(2,I-1)-DIVDF3)/DTAU
501    C(4,I-1)=(DIVDF3/DTAU)*(6.0D00/DTAU)
      Q=0.0D00

C      We integrate the spline function on each interval then
C      find the sum of these integrals.

      DO 601 I=1,N-1
      DIF=TAU(I+1)-TAU(I)
      Q=Q+DIF*(C(1,I)+(DIF/TW)*(C(2,I)+DIF/3.0D00*C(3,I)))
      Q=Q+(DIF**4)*C(4,I)/24.0D00
601    CONTINUE
      RETURN
      END

      DOUBLE PRECISION FUNCTION RSIDE2(R,C)
      IMPLICIT REAL*8(A-G),REAL*8(O-Z)
      EXTERNAL FL1,FL2,FL3
      DATA ON/1.0D00/,TW/2.0D00/
      CC=DSQRT(C)
      X=ON/R

C      We calculate the right side functions related to
C      k2/k0.

      RSIDE2=(X**3-X)*(ON+(C-ON/C)**2/TW**3+(CC-ON/CC)**2/TW)
      RSIDE2=RSIDE2+(CC-ON/CC)**2/TW-R+X
      RSIDE2=(RSIDE2+FL3(R,C)+FL2(R,C)+FL1(R,C))
      RETURN
      END

```



```

DOUBLE PRECISION FUNCTION FL1(R,C)
IMPLICIT REAL*8 (A-G) ,REAL*8 (O-Z)
EXTERNAL R4,FI4,FI5,FI6
PI=3.1415926535897932D00
X=1.0D00/R
FL1=R4(C)+(1.0D00+X)*FI4(X,C)+FI5(X,C)+FI6(X,C)
FL1=X*FL1/PI
RETURN
END

DOUBLE PRECISION FUNCTION FL2(R,C)
IMPLICIT REAL*8 (A-G) ,REAL*8 (O-Z)
EXTERNAL FI4,FJ1,FJ2,FJ3
PI=3.1415926535897932D00
X=1.0D00/R
FL2=FI4(X,C)+(1.0D00+X)*FJ1(X,C)-FJ2(X,C)-FJ3(X,C)
FL2=X**2*FL2/PI
RETURN
END

DOUBLE PRECISION FUNCTION FL3(R,C)
IMPLICIT REAL*8 (A-G) ,REAL*8 (O-Z)
IMPLICIT INTEGER*4 (H-N)
EXTERNAL R4,FI4,FI5,FI6,FJ1,FJ2,FJ3,FK1,FK2,FK3
DATA ON/1.0D00/,TW/2.0D00/,TH/3.0D00/,FO/4.0D00/
PI=3.1415926535897932D00
X=ON/R
CC=DSQRT(C)
U1=ON+R
FL3=U1**3*X**6*(FK1(X,C)-TW*FK2(X,C)+FK3(X,C))
FL3=FL3+TH*U1**2*X**5*(FJ1(X,C)-TW*FJ2(X,C)+FJ3(X,C))
FL3=FL3+TH*U1*X**4*(FI4(X,C)-TW*FI5(X,C)+FI6(X,C))
U1=((C-ON/C)**2/FO-TH*(CC+ON/CC)**2)*DATAN(CC)
U1=U1+(CC-ON/CC)**3/FO+TH*(CC-ON/CC)
U1=U1+((CC+ON/CC)**2/TW+ON)*PI
FL3=-(FL3+U1*X**3)/PI
RETURN
END

```

```

DOUBLE PRECISION FUNCTION FK3(X,C)
IMPLICIT REAL*8 (A-G) ,REAL*8 (O-Z)
IMPLICIT INTEGER*4 (H-N)
EXTERNAL DEL,FLN
DATA ON/1.0D00/,TW/2.0D00/,TH/3.0D00/
CC=DSQRT(C)
U1=((C-ON/C)/X)**2/((TW*DEL(X,C))**3)
U2=(C+ON/C-TW*X)/(X**3*DEL(X,C))
FK3=(TH*DEL(X,C)/(X**4)-U1-U2)*FLN(X,C)
U1=(CC-ON/CC)*(C+ON/C-TW*X)/((TW*X*DEL(X,C))**2*(ON-X))
U2=(CC-ON/CC)*(TW**2-5.0D00*X)/(TW*X**3*(ON-X)**2)
FK3=FK3+U1-U2
U1=6.0D00/X**4-(C**2+ON)/(C*X**3)
FK3=FK3+U1*DATAN(ON/CC)-(C-ON)/(CC*X**3)
RETURN
END

```

```

DOUBLE PRECISION FUNCTION FK2(X,C)
IMPLICIT REAL*8 (A-G) ,REAL*8 (O-Z)
IMPLICIT INTEGER*4 (H-N)
EXTERNAL DEL,FLN
DATA ON/1.0D00/,TW/2.0D00/
CC=DSQRT(C)
U1=((C+ON/C)*X-TW)/(TW*X**3*DEL(X,C))
U2=(C-ON/C)**2/(((TW*DEL(X,C))**3)*X)
FK2=(U1-U2)*FLN(X,C)+TW*DATAN(ON/CC)/(X**3)
U1=(CC-ON/CC)*(ON-TW*X)/(TW*(X*(ON-X))**2)
U2=((C+ON/C)*X-TW)/((TW*X*DEL(X,C))**2)
FK2=FK2-(U1+U2*(CC-ON/CC)/(ON-X))
RETURN
END

```

```

DOUBLE PRECISION FUNCTION FK1(X,C)
IMPLICIT REAL*8 (A-G) ,REAL*8 (O-Z)
IMPLICIT INTEGER*4 (H-N)
EXTERNAL DEL,FLN
DATA ON/1.0D00/,TW/2.0D00/
CC=DSQRT(C)
FK1=(C-ON/C)**2/((TW*DEL(X,C))**3)*FLN(X,C)
FK1=(CC-ON/CC)**3*(ON+X)/((TW*DEL(X,C)*(ON-X))**2)-FK1
RETURN
END

```

```

DOUBLE PRECISION FUNCTION FJ3(X,C)
IMPLICIT REAL*8 (A-G) ,REAL*8 (O-Z)
IMPLICIT INTEGER*4 (H-N)
EXTERNAL FJ1,FI4,R9,R12
FJ3=2.0D00*(FI4(X,C)-R9(C))-X*R12(C)
FJ3=(FJ1(X,C)-FJ3/X)/(X**2)
RETURN
END

```

```

DOUBLE PRECISION FUNCTION FJ2(X,C)
IMPLICIT REAL*8 (A-G) ,REAL*8 (O-Z)
IMPLICIT INTEGER*4 (H-N)
EXTERNAL FJ1,FI4,R9
FJ2=(FJ1(X,C)-(FI4(X,C)-R9(C))/X)/X
RETURN
END

```

```

DOUBLE PRECISION FUNCTION FJ1(X,C)
IMPLICIT REAL*8 (A-G) ,REAL*8 (O-Z)
IMPLICIT INTEGER*4 (H-N)
EXTERNAL DEL,FLN
DATA ON/1.0D00/,TW/2.0D00/
CC=DSQRT(C)
FJ1=(C+ON/C-TW*X)*FLN(X,C)/(TW*DEL(X,C))
FJ1=FJ1+(CC-ON/CC)/(ON-X)-TW*DATAN(CC)
RETURN
END

```

```

DOUBLE PRECISION FUNCTION FI6(X,C)
IMPLICIT REAL*8 (A-G) ,REAL*8 (O-Z)
IMPLICIT INTEGER*4 (H-N)
EXTERNAL FI5,R12
FI6=(FI5(X,C)-R12(C))/X
RETURN
END

```

```

DOUBLE PRECISION FUNCTION FI5(X,C)
IMPLICIT REAL*8 (A-G) ,REAL*8 (O-Z)
IMPLICIT INTEGER*4 (H-N)
EXTERNAL FI4,R9
FI5=(FI4(X,C)-R9(C))/X
RETURN
END

```

```

DOUBLE PRECISION FUNCTION FI4(X,C)
IMPLICIT REAL*8 (A-G) ,REAL*8 (O-Z)
IMPLICIT INTEGER*4 (H-N)
EXTERNAL DEL,FLN
CC=DSQRT(C)
FI4=DEL(X,C)*FLN(X,C)-(C-1.0D00)/CC
FI4=FI4+(C+1.0D00/C-2.0D00*X)*DATAN(CC)
RETURN
END

```

```

DOUBLE PRECISION FUNCTION R12(C)
IMPLICIT REAL*8 (A-G) ,REAL*8 (O-Z)
IMPLICIT INTEGER*4 (H-N)
PI=3.1415926535897932D00
CC=DSQRT(C)
R12=(C+1.0D00)**2*DATAN(1.0D00/CC)/C
R12=R12+(C-1.0D00)/CC-PI
RETURN
END

```

```

DOUBLE PRECISION FUNCTION R9(C)
IMPLICIT REAL*8 (A-G) ,REAL*8 (O-Z)
IMPLICIT INTEGER*4 (H-N)
PI=3.1415926535897932D00
CC=DSQRT(C)
R9=(C+1.0D00)**2*DATAN(CC)/C
R9=R9-(C-1.0D00)/CC-PI
RETURN
END

```

```

DOUBLE PRECISION FUNCTION R4(C)
IMPLICIT REAL*8 (A-G) ,REAL*8 (O-Z)
IMPLICIT INTEGER*4 (H-N)
PI=3.1415926535897932D00
CC=DSQRT(C)
R4=((C**2-1.0D00)/C)**2*DATAN(CC)
R4=((C-1.0D00)/CC)**3+R4)/4.0D00
RETURN
END

```

```

DOUBLE PRECISION FUNCTION FLN(X,C)
IMPLICIT REAL*8 (A-G) ,REAL*8 (O-Z)
IMPLICIT INTEGER*4 (H-N)
U1=DSQRT(C-X)
U2=DSQRT(C*X-1.0D00)
FLN=DLOG((U1+U2)/(U1-U2))
RETURN
END

```

```

DOUBLE PRECISION FUNCTION DEL(X,C)
DOUBLE PRECISION X,C
DEL=DSQRT((C-X)*(X-1.0D00/C))
RETURN
END

```

```

DOUBLE PRECISION FUNCTION W2(T,C)
IMPLICIT REAL*8(A-G),REAL*8(O-Z)
IMPLICIT INTEGER*4(H-N)
EXTERNAL FIII3
DATA ON/1.0D00/,TW/2.0D00/
X=ON/T
CC=DSQRT(C)
W2=ON+(C-ON/C)**2/8.0D00+(CC-ON/CC)**2/TW
W2=W2*((ON-X**2)/TW-DLOG(T))
W2=W2+(((T-ON)/TW)*((CC-ON/CC)**2-(T+ON))+DLOG(T))
W2=(W2+FIII3(T,C))
RETURN
END

```

```

DOUBLE PRECISION FUNCTION FIII3(T,C)
IMPLICIT REAL*8(A-G),REAL*8(O-Z)
IMPLICIT INTEGER*4(H-N)
EXTERNAL FI4,FI5,FJ1,FJ2,CUBSPL
DATA ON/1.0D00/,TW/2.0D00/
PI=3.1415926535897932D00
X=ON/T
CC=DSQRT(C)
U1=((C-ON/C)**2/4.0D00+(CC+ON/CC)**2)*DATAN(CC)
U2=(CC-ON/CC)*((CC-ON/CC)**2/4.0D00-ON)-PI
FIII3=(X**2-ON)*(U1+U2)/TW
FIII3=FIII3+X*(X**2-ON)*(FI4(X,C)+FI5(X,C))
FIII3=FIII3+(X**2-ON)**2*(FJ1(X,C)+FJ2(X,C))/TW
U1=(C-ON)**2*DLOG(DABS((T-ON)/(C-ON)))
U2=(C-X)**2*DLOG(DABS((C*T-ON)/(T-ON)))
FIII3=(FIII3+TW*(U1+U2-(C-ON)*(ON-X)))/CC
CALL CUBSPL(T,C,Q)
FIII3=(FIII3+Q)/PI
RETURN
END

```

```

DOUBLE PRECISION FUNCTION GG(Y,C,T)
IMPLICIT REAL*8(A-G),REAL*8(O-Z)
IMPLICIT INTEGER*4(H-N)
EXTERNAL DEL
DATA ON/1.0D00/,FO/4.0D00/
X=ON/Y
CC=DSQRT(C)
GG=DEL(Y,C)*(ON+X)*(ON+X**2)-FO*(C-Y)/CC
GG=GG*DLOG(DABS((Y*T-ON)/(Y-ON)))
RETURN
END

```

Autobiographical Statement

The author was born in Tillery, North Carolina on May 10, 1928. He received the Bachelor of Science degree from Hampton Institute, Hampton, Virginia in social science with a minor in mathematics in 1955, a Master of Science degree in mathematics from New York University, New York, New York in 1957, and a Certificate of Advanced Graduate Studies in mathematics and science from Wesleyan University, Middletown, Connecticut in 1965. The author has served as Supervisor of Mathematics Instruction for the public schools of Glastonbury, Connecticut and of Hingham, Massachusetts. He has also served as Chairman of the Mathematics Section of the Curriculum Development and Research Center, Kenya Ministry of Education, Nairobi, Kenya. He is the author of articles in The Arithmetic Teacher and The Mathematics Teacher, journals of the National Council of Teachers of Mathematics, Co-author of Patterns in Mathematics, Houghton Mifflin, Boston, Massachusetts, 1965, and co-author of Harper & Row Mathematics, Harper & Row, Publishers, New York, N.Y., 1982.