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On the existence of strong solutions to autonomous differential equations with minimal regularity

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Abstract

For proving the existence and uniqueness of strong solutions to

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = F(Y),\, Y(0) = C, \tag{1}$$

the most quoted condition seen in elementary differential equations texts is that $F(Y)$ and its first derivative be continuous. One wonders about the existence of a minimal regularity condition which allows unique strong solutions. In this note, a bizarre example is seen where $F(Y)$ is not differentiable at an equilibrium solution; yet unique non-global strong solutions exist at each point, whereas global non-unique weak solutions are allowed. A characterizing theorem is obtained.

Keywords: Strong and weak solutions of an ordinary differential equation; Autonomous differential equation; Global orbit; Semi-orbit

1. Introduction

A *strong solution* of (1) is a function $Y(t)$ which is continuous and differentiable at each point in the domain of definition of $Y(t)$, which when substituted satisfies the differential equation (1). If differentiability is lost at isolated points, the result is, speaking naively, a *weak* solution. Neither the weaker case of a piecewise defined solution which also experiences breaks in continuity, nor the enlarged concept of distributional weak solutions, will be of interest here.

However, we explore further the strong solution concept. The basic implication of a strong solution is that at each point of its domain a two-sided derivative exists; i.e., the usual delta process can be applied, with a unique limit obtained when $\Delta x \to 0$ from either direction. If the domain of the solution function, $Y(t)$, is a half-closed interval, existence of a one-sided limit is acceptable at the closed end-point, $t = b$. A piecewise defined solution which is a continuation away from the questionable end-point by means of a second function, $Z(t)$, can be a strong solution iff $Z(b)$ is defined and $Z(t)$ has a one-sided derivative at b, which matches the one-sided derivative of $Y(t)$.

If the domain of definition of solution $Y(t)$ is $-\infty < t < \infty$, the solution is said to be a global orbit of (1). For any more restricted domain it is called a non-global semi-orbit. An orbit is the geometric path in the (t, Y) phase space of a solution $Y(t)$.

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Theorem 1 (*The Standard Existence and Uniqueness Theorem [1]*).
Hypotheses:

Let R be a rectangular region in the (t, y) plane described by inequalities $0 \le t \le b$ and $c \le y \le d$. Suppose that the point $(0, y_0)$ is inside R, and that $f(y)$ and the partial derivative $\frac{\partial f}{\partial y}(y)$ are continuous functions in R.

Conclusions:

Under the above hypotheses, there is an interval, I, centered at t_0 and a unique function $y(t)$ defined on I such that $y(t)$ is a solution of the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y), \quad y(t_0) = y_0. \tag{2}$$

Ref. [1] quotes this as the "easiest and most widely used result that guarantees existence and uniqueness of solutions" to (2), where IVP (2) is a generalization which includes IVP (1) as a sub-case. As a case of a theorem which guarantees existence and uniqueness with a weaker hypothesis is not known to the author, it is felt that discovery of an example of such an instance should be of interest to the mathematical community. Such an example is discussed in Section 2.

2. A bizarre example

The standard example of existence and non-uniqueness [2] is given by the IVP

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = 3Y^{\frac{2}{3}}, \quad Y(0) = 0 \tag{3}$$

which has the two solutions $Y(t) = 0$ and $Y(t) = t^3$. Note that non-uniqueness occurs at every point on the equilibrium solution, since $Y(t) = (t - \tau)^3$ and the equilibrium solution both satisfy the differential equation, and $Y(\tau) = 0$. By Theorem 1, non-uniqueness occurs only at these points.

Consider now the similar problem

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = -Y^{\frac{1}{3}}, \quad Y(0) = 0. \tag{4}$$

Changing the initial condition to $V(0) = -1$, the solution is

$$V(t) = \left(-\sqrt{1 - \frac{2}{3}t}\right)^3$$

where the domain of the negative square root function is $t \le 1.5$. Since the translate of a solution is also a solution, $H(t) = V(t + 1.5)$ is a solution of (4), as $H(0) = V(1.5) = 0$.

Thus, $H(t)$ and the equilibrium solution $Y(t) = 0$ *appear* to be non-unique solutions of IVP (4). But, when we consider only *strong* solutions, the domain of $V(t)$ is $t < 1.5$, as the negative square root function is not differentiable at $t = 1.5$. $V(t)$ sweeps out a non-global orbit as $t \to -\infty$.

By considering all t-translates of $V(t)$, a non-global, unique strong solution passes through each point (t, Y), $Y < 0$. Even more interesting, the reflection in the t-axis of each of these solutions produces a non-global unique semi-orbit which is the path of a strong solution in the region (t, Y), $Y > 0$! Together with the equilibrium global orbit, every point is covered by a unique strong solution!

In both the upper and the lower half-planes, differentiability breaks down if the time domain of a solution is extended to allow the orbit to contact the equilibrium path, $Y(t) = 0$. So, if the breakdown of differentiability is allowed, there are three possible non-unique weak solutions through every point of the $Y = 0$ axis, obtained by the natural extension of an upper or lower semi-orbit into a global orbit by attaching a portion of the equilibrium solution, $t > a$, a the time of contact.

3. Conclusion

An interesting example of a case where unique solutions of IVP (1) exist, under a weakening of the usual hypothesis that $F(Y)$ is continuously differentiable, has been presented. Such a case has not been observed previously by the author.

It is expected that a weaker theorem might be:

Theorem 2. *Suppose that $F(Y)$ has a unique equilibrium solution $Y(t) = a$, stemming from a k-th order fractional zero of $F(Y)$, $0 < k < 1$. If $F(Y) < 0$ above and $F(Y) > 0$ below the line $Y = a$, semi-orbits starting above and below will approach the equilibrium solution at some finite time, $t = a$, and will be limited in domain to $t < a$, if only strong solutions are allowed.*

Proof. For a fractional order zero, $0 < k < 1$, improper integrals $T = \int_b^a \frac{\mathrm{d}w}{f(w)}$ exist, by a theorem from advanced calculus, so the time for approaching $Y(t) = a$ is finite. However, line $Y(t) = a$ cannot be crossed, because the slope field reverses sign. Likewise, lack of differentiability of $F(Y)$ at $Y = a$ prevents a strong solution from making contact.

To see the non-differentiability, consider the inverse of the function $G(Y) = \int_b^Y \frac{\mathrm{d}w}{f(w)} = t$; namely $Y(t) = G^{-1}(t)$. Since $G(a)$ is finite, but $G'(a) = \infty$, $G(Y)$ has a half-cusp at $Y = a$, so G^{-1} is not differentiable there; hence, $Y(t)$ is not differentiable on $Y = a$. □

When there are several equilibrium points, results of this kind can be obtained in certain strips, $c < Y < d$.

References

[1] Henry Ricardo, A Modern Introduction To Differential Equations, Houghton-Mifflin, New York, Boston, 2003.
[2] Dennis G. Zill, A First Course In Differential Equations, Classic 5th edition, Brooks-Cole, New York, Sydney, Australia, 2001.