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Probability models for blackjack poker

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ABSTRACT

For simplicity in calculation, previous analyses of blackjack poker have employed models which employ sampling with replacement. In order to assess what degree of error this may induce, the purpose here is to calculate results for a typical hand where sampling without replacement is employed. It is seen that significant error can result when long runs are required to complete the hand. The hand examined is itself of particular interest, as regards both its outstanding expectations of high yield and certain implications for pair splitting of two nines against the dealer's seven. Theoretical and experimental methods are used in order to determine whether the calculation can be truncated after the dealer's fifth card without significant loss of accuracy. Less than one tenth of one percent difference is observed between the fifth card truncated expectation and the experimentally obtained expectation.

1. Introduction

A Serendipitous Experiment: A blackjack player chooses to stand on two 9's against the dealer's up card 7. Completing the dealer's hand multiple times in sequence by playing through a randomly shuffled deck of 52 cards from which two 9's and a 7 are omitted, one finds that the player will usually win rather more often than the dealer. Exactly how strong is the player's hand? Is the decision not to split the pair correct? Results here demonstrate exactly how well this decision turns out.

For Las Vegas blackjack, it is known that in general the dealer has about a 0.6% edge [1], but resorting to the optimal strategy [1,2] for minimal standing numbers together with doubling down and pair splitting [2,3] reverses the edge slightly to the player. The techniques of cooperative card counting in conjunction with the practice of variable betting, as depicted in the popular 2008 movie "21: Bringing Down The House [4]", lead to radical improvement. In this article it is seen that the player's prospects for the constrained hand of a certain *serendipitous experiment* are likely about as good as it gets. For unit bets, the player's expectation of winning on or before the dealer's draw of three extra cards exceeds 41% (see Sections 2–5).

Probability Models: The hard way to see this is to calculate the player's expectation by probabilistic methods. This is accomplished in great detail in Refs. [1,2] using a sampling with replacement approach whose precision in some cases may be questioned. A general guideline from the theory of statistics is that if the population sampled is sufficiently large, equivalent results may be obtained. The question is examined here by comparing results from the two kinds of sampling. A less tedious but more ad hoc approach to the estimation of expectation is examined in Ref. [5].

Not much accuracy in calculation is lost if the hand is assumed to terminate with the dealer possessing no more than five cards. Termination error seems to be a second order effect. The validity of this hypothesis is verified by comparing theoretical truncated expectation with experimentally obtained expectation, founded on the law of large numbers. Agreement to within one tenth of one percent difference is obtained.

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2. Probabilistic methods

Serendipitous Hand (SH) Player: (9, 9) Dealer: $(7, x)$ or $(7, x, y)$

The purpose of this section is to determine sample spaces and calculate probabilities for events associated with (SH). Equally likely chances are expected when a card is drawn. This assumption is used in the analyses of Ref. [1]; where, however, no attention is given to the effects of including initial constraints (essentially, sampling with replacement is used).

Hands terminating with the dealer's hole card: Assume the player chooses to stand on two 9's versus the dealer's 7. Denote by X the random variable whose value x matches that of the dealer's hole card, and denote by random variable Y the value y of the next card drawn, given that x does not terminate the hand. The probability distribution for X, based on *equally likely* chances of drawing any particular card is:

$$\Pr(X = x) = f(x) = \left\{ \begin{array}{l} \frac{16}{49},\ x = 10 \\ \frac{2}{49},\ x = 9 \\ \frac{3}{49},\ x = 7 \\ \frac{4}{49},\ x \neq 7, 9, 10 \end{array} \right\} \quad \text{The dealer cannot win on the hole card.}$$

Either (a) he loses when $x = 10$, with probability 16/49; or (b) he pushes if $x = 11$, with probability 4/49; or (c) the decision defers, with probability 29/49, if $2 \leq x \leq 9$.

Hands which terminate after the dealer draws once: If the dealer's hole card is not $x = 10$ or $x = 11$, there is needed the joint probability distribution for the random variables, X and Y, signifying the values (x, y) of the dealer's hole card and the next card drawn:

$$\Pr(X = x, Y = y) = G(x, y) = f(x)g_x(y); \quad 2 \leq x \leq 9,\ 2 \leq y \leq 11.$$

Assuming any card is *equally likely* to be drawn, the joint distribution is determined by

$$f(x) = \left\{ \begin{array}{l} \frac{2}{29},\ x = 9 \\ \frac{3}{29},\ x = 7 \\ \frac{4}{29},\ x = 2, 3, 4, 5, 6, 8 \end{array} \right\} \quad \text{and} \quad g_x(y) = \left(\begin{array}{l} \frac{\delta_x^2}{48},\ y = 9 \\ \frac{\delta_x^3}{48},\ y = 7 \\ \frac{16}{48},\ y = 10 \\ \frac{4}{48},\ y \neq 7, 9, 10 \end{array} \right), \quad 2 \leq x \leq 9;\ 2 \leq y \leq 11,$$

where $\delta_x^n = \left(\begin{array}{l} n, x \neq y \\ n-1, x = y \end{array} \right)$. Event total $N = 29 \times 48 = 1392$ and $\sum_{x=2}^{9} \sum_{y=2}^{11} G(x, y) = 1$ assures that the joint distribution is equivalent to assuming a uniform distribution $P = 1/1392$ on the 1392 sample points associated with the joint distribution (each sample point (x, y) is equally likely).

The sample space of possible events associated with the dealers 1st draw is indicated below. Probabilities are *conditioned* on the assumption that the hand is not terminated by the dealer's hole card. A compressed form of the sample space is given by matrix C.S.S., specified in terms of values (x, y), where one pair of card values represents diverse sample points, whose number appears in the corresponding matrix position of W.C.S.S.

$$\text{C.S.S.} = \begin{bmatrix} 2,2 & 2,3 & 2,4 & 2,5 & 2,6 & 2,7 & 2,8 & 2,9 & 2,10 & 2,11 \\ 3,2 & 3,3 & 3,4 & 3,5 & 3,6 & 3,7 & 3,8 & 3,9 & 3,10 & 3,11 \\ 4,2 & 4,3 & 4,4 & 4,5 & 4,6 & 4,7 & 4,8 & 4,9 & 4,10 & 4,1^* \\ 5,2 & 5,3 & 5,4 & 5,5 & 5,6 & 5,7 & 5,8 & 5,9 & 5,10 & 5,1^* \\ 6,2 & 6,3 & 6,4 & 6,5 & 6,6 & 6,7 & 6,8 & 6,9 & 6,10 & 6,1^* \\ 7,2 & 7,3 & 7,4 & 7,5 & 7,6 & 7,7 & 7,8 & 7,9 & 7,10 & 7,1^* \\ 8,2 & 8,3 & 8,4 & 8,5 & 8,6 & 8,7 & 8,8 & 8,9 & 8,10 & 8,1^* \\ 9,2 & 9,3 & 9,4 & 9,5 & 9,6 & 9,7 & 9,8 & 9,9 & 9,10 & 9,1^* \end{bmatrix} \quad \text{Dealer draws 1}$$

Compressed Sample Space $*$ Ace counts as 1 vs 11.

Weighted Compressed Sample Space: The value of each matrix entry indicates the number of sample points for the corresponding matrix pair in C.S.S.

$$\text{W.C.S.S} = \begin{bmatrix} 12 & 16 & 16 & 16 & 16 & 12 & 16 & 8 & 64 & 16 \\ 16 & 12 & 16 & 16 & 16 & 12 & 16 & 8 & 64 & 16 \\ 16 & 16 & 12 & 16 & 16 & 12 & 16 & 8 & 64 & 16 \\ 16 & 16 & 16 & 12 & 16 & 12 & 16 & 8 & 64 & 16 \\ 16 & 16 & 16 & 16 & 12 & 12 & 16 & 8 & 64 & 16 \\ 12 & 12 & 12 & 12 & 12 & 6 & 12 & 6 & 48 & 12 \\ 16 & 16 & 16 & 16 & 16 & 12 & 12 & 8 & 64 & 16 \\ 8 & 8 & 8 & 8 & 8 & 6 & 8 & 2 & 32 & 8 \end{bmatrix} \qquad \sum_{i,j} w_{ij} = 1392 = 29 \times 48 \text{ simple events.}$$

First draw events:

$$\begin{array}{cc} \text{PUSH} & \text{DEFER} \\ \begin{bmatrix} & & & & & & & 8 \\ & & & & & & 16 & \\ & & & & & 12 & & \\ & & & & 16 & & & \\ & & & 16 & & & & \\ & & 12 & & & & & \\ & 16 & & & & & & \\ 8 & & & & & & & \end{bmatrix} & \begin{bmatrix} 12 & 16 & 16 & 16 & 16 & 12 & \\ 16 & 12 & 16 & 16 & 16 & & \\ 16 & 16 & 12 & 16 & & & 16 \\ 16 & 16 & 16 & & & & 16 \\ 16 & 16 & & & & & 16 \\ 12 & & & & & & 12 \\ & & & & & & 16 \\ & & & & & & \end{bmatrix} \end{array}$$

$$P(\text{Push2}|\text{Defer1}) = \frac{104}{29 \times 48} = \frac{13}{29 \times 6} = 0.0747$$

$$P(\text{Defer2}|\text{Defer1}) = \frac{392}{29 \times 48} = \frac{8 \times 49}{29 \times 48} = \frac{49}{29 \times 6} = 0.2816$$

$$P(\text{Both Defer}) = \frac{29}{49} \times \frac{49}{29 \times 6} = 0.1667 \quad \text{(Chop Factor)}$$

$$P(\text{W1 or W2 or WL}) = P(\text{W1}) + P(\text{Not(T1)})P(\text{W2}|\text{Not(T1)}) + P(\text{Not(T1 or T2)})P(\text{WL}|\text{Not(T1 or T2)}).$$

The chop factor is the multiplier of the probability correction that would be lost if the analysis were stopped with a present dealer draw. Not much could be gained by continuing the calculation, provided the chop factor is small enough that the accompanying probability increment $P(\text{WL}|\text{No Early Termination})$ has minimal effect. Here, the chop factor implies the next correction gets chopped over 83% since $P(\text{No Early Termination}) = 0.1667$.

$$\begin{array}{cc} \text{DEALER WIN} & \text{Bust or (Lose Without Bust)} \\ \begin{bmatrix} & & & & & & & 64 & 16 \\ & & & & & & 8 & 64 & 16 \\ & & & & & 16 & 8 & 64 & \\ & & & & 12 & 16 & 8 & & \\ & & & 12 & 12 & 16 & & & \\ & & 12 & 12 & 6 & & & & \\ & 16 & 16 & 16 & & & & & \\ 8 & 8 & 8 & & & & & & \end{bmatrix} & \begin{bmatrix} & & & & & & 16 & & & \\ & & & & & 12 & & & & \\ & & B & & 16 & & & & & \\ & & & 12 & & & & & 64 & \\ & & 16 & & & & L & 8 & 64 & \\ & 12 & & & & & 12 & 6 & 48 & \\ 16 & & & & & 12 & 12 & 8 & 64 & \\ & & & & 8 & 6 & 8 & 2 & 32 & 8 \end{bmatrix} \end{array}$$

$$P(\text{Win2}|\text{Defer1}) = \frac{434}{29 \times 48} = \frac{2 \times 7 \times 31}{29 \times 48} = \frac{7 \times 31}{29 \times 24} = 0.31178 = P(\text{Dealer Wins 2}|\text{Defer1})$$

$$P(\text{Lose2}|\text{Defer1}) = \frac{462}{29 \times 48} = \frac{2 \times 7 \times 3 \times 11}{29 \times 3 \times 16} = \frac{7 \times 11}{29 \times 8} = 0.33146 = P(\text{Player Wins 2}|\text{Defer1})$$

$$P(\text{Defer1} \cap \text{Win2}) = \frac{29}{49} \times \frac{7 \times 31}{29 \times 24} = \frac{31}{7 \times 24} = 0.1845 = P(\text{Dealer wins on 1st draw}).$$

Overall outlook

$$P(\text{Player Wins Before or On 1st Draw}) = P(\text{Dealer Loses On H. Card or 1st Draw})$$

$$P(\text{Dealer Loses H. or 1st Draw}) = P(\text{Dealer Loses H.}) + P(\text{Defer1}) \times P(\text{Lose2}|\text{Defer1})$$

$$P(\text{Player Wins On or Before 1st Draw}) = \frac{16}{49} + \frac{29}{49} \times \frac{7 \times 11}{29 \times 8} = 0.3265 + 0.1964 = 0.5229!$$

$$P(\text{Dealer Wins On or Before 1st Draw}) = 0 + 0.1845 = 0.1845 \quad \text{(Assume unit bets)}$$

$$\textit{Players Expectation} \text{ (To Win On or Before First Draw)} = 0.5229 - 0.1845 = \mathit{0.3384}!$$

Comment: Ref. [3] recommends splitting 9's against all up cards from 2 through 9, with *the exception of* 7. The stated reasons: Two 9's give 18, which beats the dealer's potential 17, if a 10 is drawn. Our calculation rigorously justifies his expectations.

3. Sampling with replacement

Initial Conditions: Player (9,9) vs Dealer $\{(7, x) \text{ or } (7, x, y);\ x \neq 10,\ 11\}$

Hand terminates with dealer's hole card: The dealer cannot win; probabilities are (i) $P(\text{Push}) = 4/52$; or (ii) $P(\text{Dealer loses}) = 16/52$; or (iii) $P(\text{Defer}) = 32/52$.

Simple events associated with dealer's 1st draw:

$$
\text{C.S.S.} = \begin{bmatrix}
2,2 & 2,3 & 2,4 & 2,5 & 2,6 & 2,7 & 2,8 & 2,9 & 2,10 & 2,11 \\
3,2 & 3,3 & 3,4 & 3,5 & 3,6 & 3,7 & 3,8 & 3,9 & 3,10 & 3,11 \\
4,2 & 4,3 & 4,4 & 4,5 & 4,6 & 4,7 & 4,8 & 4,9 & 4,10 & 4,1^* \\
5,2 & 5,3 & 5,4 & 5,5 & 5,6 & 5,7 & 5,8 & 5,9 & 5,10 & 5,1^* \\
6,2 & 6,3 & 6,4 & 6,5 & 6,6 & 6,7 & 6,8 & 6,9 & 6,10 & 6,1^* \\
7,2 & 7,3 & 7,4 & 7,5 & 7,6 & 7,7 & 7,8 & 7,9 & 7,10 & 7,1^* \\
8,2 & 8,3 & 8,4 & 8,5 & 8,6 & 8,7 & 8,8 & 8,9 & 8,10 & 8,1^* \\
9,2 & 9,3 & 9,4 & 9,5 & 9,6 & 9,7 & 9,8 & 9,9 & 9,10 & 9,1^*
\end{bmatrix}
$$

$$
\text{W.C.S.S.} = \begin{bmatrix}
16 & 16 & 16 & 16 & 16 & 16 & 16 & 16 & 64 & 16 \\
16 & 16 & 16 & 16 & 16 & 16 & 16 & 16 & 64 & 16 \\
16 & 16 & 16 & 16 & 16 & 16 & 16 & 16 & 64 & 16 \\
16 & 16 & 16 & 16 & 16 & 16 & 16 & 16 & 64 & 16 \\
16 & 16 & 16 & 16 & 16 & 16 & 16 & 16 & 64 & 16 \\
16 & 16 & 16 & 16 & 16 & 16 & 16 & 16 & 64 & 16 \\
16 & 16 & 16 & 16 & 16 & 16 & 16 & 16 & 64 & 16 \\
16 & 16 & 16 & 16 & 16 & 16 & 16 & 16 & 64 & 16
\end{bmatrix} \qquad |\text{W.C.S.S.}| = 52 \times 32
$$

1st draw events:

$$|\text{Defer2}| = 13 \times 32$$

$$
\begin{bmatrix}
16 & 16 & 16 & 16 & 16 & 16 & \\
16 & 16 & 16 & 16 & 16 & & \\
16 & 16 & 16 & 16 & & & 16 \\
16 & 16 & 16 & & & & 16 \\
16 & 16 & & & & & 16 \\
16 & & & & & & 16 \\
 & & & & & & 16
\end{bmatrix} \qquad P(\text{DEF2}|\text{DEF1}) = \frac{13 \times 32}{52 \times 32} = \frac{1}{4} = 0.25
$$

$$P(\text{DEF1 and DEF2}) = P(\text{DEF1})P(\text{DEF2}|\text{DEF1})$$

$$P(\text{Both Defer}) = \frac{32}{52} \times \frac{1}{4} = \frac{2}{13} = 0.1538.$$

Comparison to the sampling without replacement result (0.1667 vs 0.1538) gives 7.785% *error in the chop factor*.

$$|\text{Lose 2}| = 19 \times 32$$

$$
\begin{bmatrix}
 & & & & & & 16 & & & \\
 & & & & & 16 & & & & \\
 & & & & 16 & & & & & \\
 & & & 16 & & & & & 64 & \\
 & & 16 & & & & & 16 & 64 & \\
 & 16 & & & & & 16 & 16 & 64 & \\
16 & & & & & 16 & 16 & 16 & 64 & \\
 & & & & 16 & 16 & 16 & 16 & 64 & 16
\end{bmatrix} \qquad P(\text{Lose2}|\text{DEF1}) = \frac{19 \times 32}{52 \times 32} = \frac{19}{52} = 0.3654
$$

$$|\text{Win 2}| = 16 \times 32$$

$$
\begin{bmatrix}
 & & & & & & & 64 & 16 \\
 & & & & & & 16 & 64 & 16 \\
 & & & & & 16 & 16 & 64 & \\
 & & & & 16 & 16 & 16 & & \\
 & & & 16 & 16 & 16 & & & \\
 & & 16 & 16 & 16 & & & & \\
 & 16 & 16 & 16 & & & & & \\
16 & 16 & 16 & & & & & &
\end{bmatrix} \qquad P(\text{Win2}|\text{DEF1}) = \frac{16 \times 32}{52 \times 32} = \frac{4}{13} = 0.3077
$$

$$|\text{PUSH 2}| = 4 \times 32$$

$$\begin{bmatrix} & & & & & & & 16 \\ & & & & & & 16 & \\ & & & & & 16 & & \\ & & & & 16 & & & \\ & & & 16 & & & & \\ & & 16 & & & & & \\ & 16 & & & & & & \\ 16 & & & & & & & \end{bmatrix} \qquad P(\text{PUSH2}|\text{EF1}) = \frac{4 \times 32}{52 \times 32} = \frac{1}{13} = 0.07692.$$

Overall outlook.

$$P(\text{Player Wins On Or Before 1st Draw}) = P(\text{Dealer Loses H. Card or 1st Draw})$$

$$P(\text{Player Wins On or Before 1st Draw}) = P(\text{Lose1}) + P(\text{Defer1}) \times P(\text{Lose2}|\text{Defer1})$$

$$P(\text{Player Wins On or Before 1st Draw}) = \frac{16}{52} + \frac{32}{52} \times \frac{19}{52} = 0.5324 \quad (0.5229^* \text{ before})$$

$$P(\text{Dealer Wins On 1st Draw}) = P(\text{DEF1 and Win2}) = 8/13 \times 0.3077 = 0.1892\,(0.1845^*)$$

$$\textit{Players Expectation}(\text{To Win On or Before First Draw}) = 0.5324 - 0.1892 = 0.3432.$$

Error comparisons.

Comparison to sampling without replacement result (0.3384 vs 0.3432) indicates player expectation is too large by a percentage error of 0.48.

Also, the chop error caused by neglecting continuations past 3 dealer cards, when predicted by sampling with replacement, is too small by *7.785%* (See comparison (0.1667 vs 0.1538) for likelihood of deferral on two successive cards.)

Conclusion: Sampling with replacement will lead to under-predictions of statistical indicators, especially significant for hands where long runs occur in the process of termination.

4. The serendipitous hand continued to the 4th dealer card

Player stands: (9, 9) Dealer Draws: $(7, x, y, z)$

The computer will be used to calculate hard to visualize 3D events. 2D events which force a decision deferment to a future draw are given by:

$$\text{Defer2}$$

$$\text{DF2} = \begin{bmatrix} 2,2 & 2,3 & 2,4 & 2,5 & 2,6 & 2,7 & \\ 3,2 & 3,3 & 3,4 & 3,5 & 3,6 & & \\ 4,2 & 4,3 & 4,4 & 4,5 & & & 4,1^* \\ 5,2 & 5,3 & 5,4 & & & & 5,1^* \\ 6,2 & 6,3 & & & & & 6,1^* \\ 7,2 & & & & & & 7,1^* \\ & & & & & & 8,1^* \end{bmatrix},$$

which continue to 3D events as $\text{DF2} \times [z] : z = 2, 3, \ldots 10, (11, 1)$. The 3D joint sample space has 18,424 simple events with uniform probability $1/18,424$ (Table 1). Visualization is more difficult, but Microsoft Fortran Power Station 4 allows quick calculation:

Table 1
Computational results: 4th dealer card.

Dealer event	# Sample points	Probability
NWIN3	4,308	0.2338254
NDEF3	2,468	0.1339557
NLOSE3	10,196	0.5534086
NPUSH3	*1,452*	*0.0788103*
Totals	18,424	1.0000000

Overall outlook.

$$\begin{aligned} P(\text{Player Wins On or Before 2nd Draw}) &= 0.5229 + P(\text{DEF12})P(\text{PW3}|\text{DEF12}) \\ &= 0.5229 + 1/6 \times 0.5534086 = 0.62514 \end{aligned}$$

$$P(\text{Dealer Wins On or Before 2nd Draw}) = 0.1845 + 1/6 \times 0.2338254 = 0.22347$$

$$\textit{Expectation} : \text{Player expected gain after 2nd draw} = 0.61514 - 0.22347 = 0.40119.$$

This may be compared with the expected gain after 1st draw: 0.3384. Quite a significant error is seen. In the Section 5 the 2468 unfinished hands are continued one further draw by the dealer, with only a small increase in expectation.

5. The serendipitous hand continued to the 5th dealer card

Player Stands: (9,9) Dealer Draws: $(7, x, y, z, w)\mathrm{DF2} \times \{z, w; z, w = 2, 3, \ldots, 10(11, 1)\}$

Table 2
Computational results: 5th dealer card. 500 lines of logic with Fortran Power Station 4.

Dealer event	# Sample points	Probability
NWIN4	26,656	0.2341367
NDEF4	7,704	0.0676692
NLOSE4	71,152	0.6249737
NPUSH4	*8,336*	*0.0732204*
Totals	113,848	1.0000000

$$\begin{aligned} P(\text{Player Wins On or Before 3rd Draw}) &= 0.62514 + P(\mathrm{DEF123})P(\mathrm{PW4}|\mathrm{DEF123}) \\ &= 0.62514 + 1/6 \times 0.1339557 \times 0.6249737 \\ &= 0.63909313157 \\ P(\text{Dealer Wins On or Before 3rd Draw}) &= 0.22347 + 0.02232595 \times 0.234136 \\ &= 0.228697324 \end{aligned}$$

Expectation:

Player expected gain after 3rd draw = 0.410396.

Chop factors: The chop factor after the second draw is:

$$P(\mathrm{DEF1})P(\mathrm{DEF2}|\mathrm{DEF1})P(\mathrm{DEF3}|\mathrm{DEF1 \text{ and } DEF2}) = \frac{1}{6} \times \frac{2468}{18424} = 0.0223.$$

After the third draw the chop factor becomes:

$$P(\mathrm{def1234}) = \frac{1}{6} \times \frac{2468}{18424} \times \frac{7704}{113848} = 0.001509.$$

Due to the indicated rapid convergence to zero of the chop factor $(0.59, 0.1667, 0.0223, 0.00151, \ldots)$, continuing to further draws seems inessential, as the importance of successive corrections is diminishing rapidly (Table 2). This will be analyzed below by means of experimental calculations of expectation.

6. Conclusions

All methods of calculation employed indicate that the serendipitous hand is just that: a very strong hand for the player. Calculation should continue until the dealer holds at least 5 cards, assuming there is not a natural earlier termination. Sampling without replacement seems to be the more accurate method. However, favorable trends of play would in most cases be ascertained using either sampling technique.

Due to the indicated rapid convergence to zero of the chop factor $(0.59, 0.1667, 0.0223, 0.00151, \ldots)$, continuing further seems inessential. Moreover, this is now affirmed by resort to experimental methods and the law of large numbers [6].

Experimental confirmation: One can experimentally estimate expectations by turning to statistical methods applied to repeated trials of the parlor game of Section 1. Using a well-shuffled deck of cards with a seven and two nines removed, assume the player stands on two nines, and simulate the dealer's luck by repeated play against the pair of 9's, until the deck is exhausted. By counting total wins, losses, and pushes after a sequence of N decks have been exhausted, player expectation per unit bet can be estimated from $E = (W - L)/(W + L + P)$, as for constant expectation per unit bet, *(action)* $\times$ *(expectation)* = *total increase*. By the law of large numbers [6], theoretically as n increases $E(n)$ converges to the true expectation. Here $n = \mathit{total}(W + L + P)$ after playing through N decks.

Tabulating results for play, once with a never shuffled factory deck, and eight times for which the deck is well shuffled, totals are: $W = 151$, $L = 51$, and $P = 39$, which yields player expectation Es = 0.410788. This compares to the above best theoretical result Et = 0.410396, giving percent difference $D = 0.09551\%$ (less than one tenth of one per cent!)

For a smaller number of test decks, the statistical estimate is below and farther removed from the best theoretical result. However, recall that the statistical estimate is made from a total of 241 random completed hands, whereas the theoretical estimate is based on 106,144 completed hands and 7704 not completed hands involved with the fifth dealer card, plus the hands completed on dealer cards two, three, and four.

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