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Nested balanced incomplete block designs

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Abstract

If the blocks of a balanced incomplete block design (BIBD) with v treatments and with parameters (v, b_1, r, k_1) are each partitioned into sub-blocks of size k_2, and the $b_2 = b_1k_1/k_2$ sub-blocks themselves constitute a BIBD with parameters (v, b_2, r, k_2), then the system of blocks, sub-blocks and treatments is, by definition, a nested BIBD (NBIBD). Whist tournaments are special types of NBIBD with $k_1 = 2k_2 = 4$. Although NBIBDs were introduced in the statistical literature in 1967 and have subsequently received occasional attention there, they are almost unknown in the combinatorial literature, except in the literature of tournaments, and detailed combinatorial studies of them have been lacking. The present paper therefore reviews and extends mathematical knowledge of NBIBDs. Isomorphism and automorphisms are defined for NBIBDs, and methods of construction are outlined. Some special types of NBIBD are defined and illustrated. A first-ever detailed table of NBIBDs with $v \leqslant 16$, $r \leqslant 30$ is provided; this table contains many newly discovered NBIBDs. © 2001 Elsevier Science B.V. All rights reserved.

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1. Definitions and historical background

In standard notation (as in [37,62]), a balanced incomplete block design (BIBD) with parameters (v, b, r, k) is an arrangement of v treatments (sometimes called 'varieties' or 'points') in b blocks, each of size k, where $k < v$, such that

(i) each treatment appears exactly $r = bk/v$ times overall,

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(ii) each treatment occurs no more than once per block, and
(iii) each unordered pair of its treatments appears in exactly $\lambda = r(k-1)/(v-1)$ blocks, the parameter λ often being referred to as the 'concurrence parameter' of the BIBD.

A BIBD is 'unreduced' if its blocks are the distinct k-subsets of the treatments, each such subset occurring just once. Thus an unreduced BIBD has

$$b = \binom{v}{k}, \qquad r = \binom{v-1}{k-1}.$$

The 'complement' of a BIBD $\mathscr{D}$ with parameters (v, b, r, k) is the BIBD whose parameters are $(v, b, b-r, v-k)$ and each of whose blocks contains only those treatments that are absent from the corresponding block of $\mathscr{D}$. The complement of an unreduced BIBD is also unreduced.

A BIBD is 'resolvable' [2] if its set of blocks can be partitioned into subsets such that each subset is a 'replicate' or 'resolution class' or 'parallel class', i.e. such that each subset contains each treatment exactly once. Following [10,11], we say that a resolvable BIBD (RBIBD) has been 'resolved' if it is presented with its blocks arranged in replicates. A BIBD is 'α-resolvable' if its set of blocks can be partitioned into subsets each containing each treatment exactly α times; we refer to such a subset as an 'α-resolution class'. We say that an α-resolvable BIBD has been 'α-resolved' if it is presented with its blocks arranged in α-resolution classes. A BIBD with v treatments is 'almost resolvable' [40, p. 954] or 'near resolvable' [2, p. 88] if its set of blocks can be partitioned into 'near-resolution classes', i.e. into subsets each lacking one of the treatments but containing each of the other treatments exactly once. If an almost resolvable BIBD is presented with its blocks partitioned in such a way, we say that it has been 'almost resolved'.

If the blocks of a BIBD $\mathscr{D}_1$ with parameters (v, b_1, r, k_1) are each partitioned into sub-blocks of size k_2, where k_2 (>1) is a submultiple of k_1, and the $b_2 = b_1k_1/k_2$ sub-blocks themselves constitute a BIBD $\mathscr{D}_2$ with parameters (v, b_2, r, k_2), then, following [42], we define the system of blocks, sub-blocks and treatments to be a 'nested BIBD' (NBIBD) with parameters $(v, b_1, b_2, r, k_1, k_2)$ satisfying $vr = b_1k_1 = b_2k_2$. The nesting here is thus that of blocks of size k_2 within blocks of size k_1, not (as for some other 'nested' designs in the combinatorial literature) of designs within designs. We refer to $\mathscr{D}_1$ and $\mathscr{D}_2$ as the 'component BIBDs' of the NBIBD, with concurrence parameters λ_1 and λ_2, respectively. To avoid cumbersome notation, we henceforth use the concurrence parameters only when they are needed for formal proofs and constructions.

As an example of an NBIBD, consider the following NBIBD with treatments $0, 1, 2, 3, 4$ and parameters $(5, 5, 10, 4, 4, 2)$, where, as elsewhere in this paper, each block is within round brackets, and sub-blocks within a block are separated by a vertical bar:

$$(1\ 4 \mid 2\ 3)(2\ 0 \mid 3\ 4)(3\ 1 \mid 4\ 0)(4\ 2 \mid 0\ 1)(0\ 3 \mid 1\ 2)$$

This NBIBD has the further property that each successive block is obtainable from the previous one by means of cyclic substitutions modulo 5; this NBIBD can therefore be specified by a single initial block, and may be written more concisely as

$$(1\ 4 \mid 2\ 3) \bmod 5.$$

By analogy with the definition of a resolvable BIBD, we say that an NBIBD is 'resolvable' if its set of blocks of size k_1 can be partitioned into subsets each of which is a resolution class. We say that a resolvable NBIBD has been 'resolved' if it is presented with its blocks of size k_1 arranged in resolution classes. An NBIBD is 'α-resolvable' if its set of blocks of size k_1 can be partitioned into subsets each containing each treatment exactly α times. We say that an 'α-resolvable' NBIBD has been 'α-resolved' if it is presented with its blocks of size k_1 arranged in α-resolution classes. An NBIBD with v treatments is 'almost resolvable' if its set of blocks of size k_1 can be partitioned into subsets each of which is a near-resolution class. If an almost resolvable NBIBD is presented with its blocks so partitioned, it is 'almost resolved'.

As an example of a resolved NBIBD, we may take the following NBIBD with treatments $0,1,2,3,4,5,6,\infty$ and parameters $(8,14,28,7,4,2)$, where the blocks within square brackets constitute a resolution class, and the treatment ∞ is invariant under the cyclic development of the initial blocks:

$$[(0\ 1 \mid 4\ 2)(3\ 6 \mid 5\ \infty)] \bmod 7.$$

As an example of an almost resolved NBIBD, we may take the following NBIBD with treatments $0,1,\dots,12$ and parameters $(13,39,78,12,4,2)$, where the blocks within the angled brackets constitute a near-resolution class:

$$\langle(1\ 4 \mid 2\ 7)(3\ 12 \mid 6\ 8)(9\ 10 \mid 5\ 11)\rangle \bmod 13$$

The two examples of the previous paragraph are 'whist tournaments' [5–8,21], the literature of which goes back to Moore's 1896 paper [39] and earlier (see [9]). If $v \equiv 0$ or 1, modulo 4, then a whist tournament Wh(v) is a resolved (if $v \equiv 0$) or almost resolved (if $v \equiv 1$) NBIBD with $k_1 = 2k_2 = 4$ and $r = v - 1$; the 4 treatments in a block of size k_1 are 4 players of the card-game 'whist' who are seated at the same table in the current game, and the 2 treatments in a sub-block of size k_2 are 2 players who are partners of one another in the current game. Whist tournaments exist for all $v = 4,8,\dots$ and all $v = 5,9,\dots$, and are also used when the card-game 'bridge' is played without fixed partnerships [15]. More generally, any NBIBD with $k_1 = 2k_2 = 4$ is a 'balanced doubles schedule' (BDS) as considered by Healey [24], who followed [15] by using λ_1 and λ_2 to denote the values that are equal, in the notation of the present paper, to λ_2 and $\lambda_1 - \lambda_2$, respectively. The further generalization to $k_1 = 2k_2$, without the restriction $k_2 = 2$, gives schedules for competitions where the team size may be greater than 2. These schedules may be used as calibration designs where objects are to be weighed or measured in some other way [15,16]. Taking $k_1 = 2k_2 = 8$, $r = v - 1$, $v \equiv 0$ or 1 modulo 8, we have 'pitch tournament' designs [1,22]. In general with $k_1 = 2k_2$, we have 'team tournaments' in the sense of [13].

In 1950, independently of the literature of tournaments, Kleczkowski [32, Table 2] reported using an experimental design based on a resolved NBIBD for a biological experiment on the effect of inoculating plants with virus. Further use of this design was reported in 1965 by Kassanis and Kleczkowski [31, p. 211]. This experimental background led to Preece's 1967 statistical paper [42] where NBIBDs were defined for the first time and an incomplete table of them was given, $r \leqslant 15$.

The subsequent literature of NBIBDs has been small, and mostly statistical or relating to whist tournaments. Apart from the literature of tournaments, relevant papers, with years of publication, are as follows, but not all of them specifically mention NBIBDs:

1975: Homel and Robinson [26]; Preece and Cameron [44];
1981: Street [58];
1982: Agrawal and Prasad [3];
1983: Agrawal and Prasad [4]; Jimbo and Kuriki [29];
1984: Bailey et al. [12];
1986: Cheng [17]; Dey et al. [19];
1989: Sreenath [56];
1990: Uddin and Morgan [60];
1991: Iqbal [27]; Uddin and Morgan [61];
1992: Uddin [59];
1993: Jimbo [28]; Yin and Miao [64];
1994: Gupta and Kageyama [23];
1996: Morgan [40]; Sinha et al. [54]; Srivastav and Morgan [57];
1998: Das et al. [18]; Kageyama and Miao [30]; Saha et al. [50];
1999: Bailey [11]; Sinha and Mitra [53].

In particular, Morgan [40] gave a table listing references for NBIBDs for almost all possible sets of parameters with $v \leqslant 14$ and $r \leqslant 30$. Readers should however take particular care to note that Morgan [40], unlike Preece [42] and the present paper, used b_2 to denote k_1/k_2, not b_1k_1/k_2. We now prefer the parameters of $\mathscr{D}_1$ and $\mathscr{D}_2$ to be, respectively, (v, b_1, r, k_1) and (v, b_2, r, k_2), thereby facilitating reference to the important table of BIBDs in [37], where the parameters are taken in the order (v, b, r, k).

Gupta and Kageyama [23], followed by Das et al. [18], proposed the use of NBIBDs with $k_2 = 2$ for diallel-cross experiments in plant-breeding investigations of v cultivars.

As Preece [42, p. 481] pointed out in 1967, and Morgan [40, pp. 945–946] in 1996, the concept of an NBIBD can be extended to that of a 'doubly nested BIBD' (DNBIBD) [45] with blocks and sub-blocks as before, but also with sub-sub-blocks nested within sub-blocks, where the sub-sub-blocks too constitute a BIBD. Obvious further extensions can be made to 'triply nested BIBDs' and, in general, 'multiply nested BIBDs' (MNBIBDs) with multiple nesting of blocks of smaller sizes within blocks of larger sizes. Our main emphasis in this paper is on NBIBDs (singly nested), but we give a general powerful result that enables us to construct not only NBIBDs but also multiply nested BIBDs.

Another extension of the concept of an NBIBD can be visualised by supposing that the k_1 elements in each block of an NBIBD are arranged in a rectangular array with one row per sub-block. Writing $k_3 = k_1/k_2$, the array will then have k_3 rows (each containing k_2 treatments) and k_2 columns (each containing k_3 treatments). The definition of an NBIBD requires the full set of $b_2 = b_1k_3$ rows to constitute a BIBD. If we additionally require the full set of $b_3 = b_1k_2$ columns to constitute a BIBD, the overall arrangement can [42, p. 481] be called a 'criss-cross nested BIBD' (CCNBIBD). Some CCNBIBDs can readily be obtained from NBIBDs given in the present paper, but we do not discuss them further. Generalizing the CCNBIBDs, Singh and Dey [52] introduced a class of designs that they referred to as 'balanced incomplete block designs with nested rows and columns' (BIBRCs). In the terminology of Morgan [40, p. 960], 'completely balanced BIBRCs' are identical to CCNBIBDs.

A t-design (see, e.g., [35]) with parameters (v,b,r,k) has v treatments disposed in b blocks, each of size k, where $k < v$, with

(i) each treatment appearing exactly $r = bk/v$ times overall,
(ii) each treatment occurring no more than once per block, and
(iii) each t-subset of (distinct) treatments occurring in exactly

$$r\binom{k-1}{t-1}\bigg/\binom{v-1}{t-1}$$

blocks.

If the blocks of a t-design $\mathscr{T}_1$ with parameters (v,b_1,r,k_1) are each partitioned into sub-blocks whose size k_2 (>1) is a submultiple of k_1, and the $b_2 = b_1k_1/k_2$ sub-blocks themselves constitute a t-design $\mathscr{T}_2$ with parameters (v,b_2,r,k_2), then the system of blocks, sub-blocks and treatments can be defined as a 'nested t-design'. A nested t-design must have $k_1 \geqslant 2t$. A t-design with $t = 2$ is a BIBD, and a nested t-design with $t = 2$ is an NBIBD. Clearly, concepts of resolvability can be defined for t-designs and nested t-designs as for BIBDs and NBIBDs. Clearly, too, definitions given above can be adapted to give us the concepts of a 'doubly nested t-design', etc., and of a 'criss-cross nested t-design'.

As an example of a nested 3-design, we offer the following specimen due to D. H. Rees; it is an NBIBD with parameters $(12, 165, 330, 110, 8, 4)$:

$$
\begin{array}{l}
(\,2\;\,3\;\,6\;\,8 \mid 9\;\,7\;\,5\;\,4\,)(\,8\;\,1\;\,2\;\,10 \mid 3\;\,6\;\,9\;\,5\,)\\
(\,10\;\,4\;\,8\;\,7 \mid 1\;\,2\;\,3\;\,9\,)(\,7\;\,5\;\,10\;\,6 \mid 4\;\,8\;\,1\;\,3\,)\\
(\,6\;\,9\;\,7\;\,2 \mid 5\;\,10\;\,4\;\,1\,)(\,5\;\,8\;\,2\;\,\infty \mid 10\;\,3\;\,9\;\,7\,)\\
(\,9\;\,10\;\,8\;\,\infty \mid 7\;\,1\;\,3\;\,6\,)(\,3\;\,7\;\,10\;\,\infty \mid 6\;\,4\;\,1\;\,2\,)\\
(\,1\;\,6\;\,7\;\,\infty \mid 2\;\,5\;\,4\;\,8\,)(\,4\;\,2\;\,6\;\,\infty \mid 8\;\,9\;\,5\;\,10\,)\\
(\,\infty\;\,6\;\,5\;\,0 \mid 8\;\,2\;\,10\;\,9\,)(\,\infty\;\,2\;\,9\;\,0 \mid 10\;\,8\;\,7\;\,3\,)\\
(\,\infty\;\,8\;\,3\;\,0 \mid 7\;\,10\;\,6\;\,1\,)(\,\infty\;\,10\;\,1\;\,0 \mid 6\;\,7\;\,2\;\,4\,)\\
(\,\infty\;\,7\;\,4\;\,0 \mid 2\;\,6\;\,8\;\,5\,) \bmod 11.
\end{array}
$$

NBIBDs are component designs of 'nested pergolas' [48].

2. Isomorphism of NBIBDs; automorphism groups

If two BIBDS $\mathscr{D}$ and $\mathscr{D}^*$ have the same parameters (v, b, r, k), then $\mathscr{D}^*$ is defined to be isomorphic to $\mathscr{D}$ if $\mathscr{D}$ can be obtained from $\mathscr{D}^*$ by a combination of

(i) a permutation of the blocks of $\mathscr{D}^*$, and
(ii) a relabelling of the treatments of $\mathscr{D}^*$.

The permutation in (i) may be the identity permutation, and the relabelling in (ii) may be the identity relabelling. Likewise, if two NBIBDs $\mathscr{N}$ and $\mathscr{N}^*$ have the same parameters $(v, b_1, b_2, r, k_1, k_2)$, we define $\mathscr{N}^*$ to be isomorphic to $\mathscr{N}$ if $\mathscr{N}$ can be obtained from $\mathscr{N}^*$ by a combination of

(i) a permutation of the blocks of $\mathscr{N}^*$,
(ii) a permutation of sub-blocks within blocks of $\mathscr{N}^*$, and
(iii) a relabelling of the treatments of $\mathscr{N}^*$.

Either or both of the permutations may be an identity permutation, and the relabelling may be the identity relabelling.

For two NBIBDs to be isomorphic to one another, it is necessary but not sufficient for both of the following conditions to hold:

(a) their component BIBDs with block size k_1 must be isomorphic to one another;
(b) their component BIBDs with block size k_2 must be isomorphic to one another.

With the above definitions of isomorphism in place, the concepts of automorphism of a BIBD or NBIBD, and of the automorphism group of a BIBD or NBIBD, follow so naturally that we omit the formal definitions.

Let $\mathscr{A}$ denote the automorphism group of an NBIBD $\mathscr{N}$, and let $\mathscr{A}_1$ and $\mathscr{A}_2$ denote, respectively, the automorphism groups of the component BIBDs $\mathscr{D}_1$ (with b_1 blocks) and $\mathscr{D}_2$ (with b_2 blocks) of $\mathscr{N}$. An automorphism of $\mathscr{N}$ is an automorphism of $\mathscr{D}_1$ and also of $\mathscr{D}_2$. However, the converses are not necessarily true. In a sense that we illustrate in the next two paragraphs, an automorphism of $\mathscr{D}_1$ need not respect the sub-blocks of $\mathscr{D}_2$, or the nesting of sub-blocks within blocks, so we must have $|\mathscr{A}|$ equal to, or a factor of, $|\mathscr{A}_1|$. Likewise an automorphism of $\mathscr{D}_2$ need not respect the blocks of $\mathscr{D}_1$, so we must have $|\mathscr{A}|$ equal to, or a factor of, $|\mathscr{A}_2|$.

For illustration, consider the following NBIBD with treatments $0, 1, 2, \ldots, 6$ and parameters $(7, 7, 14, 6, 6, 3)$:

Block	Sub-blocks									
A	a	|	b	(1	2	4	|	6	5	3)
B	c	|	d	(2	3	5	|	0	6	4)
C	e	|	f	(3	4	6	|	1	0	5)
D	g	|	h	(4	5	0	|	2	1	6)
E	i	|	j	(5	6	1	|	3	2	0)
F	k	|	l	(6	0	2	|	4	3	1)
G	m	|	n	(0	1	3	|	5	4	2)

For this NBIBD, $|\mathscr{A}| = |\mathscr{A}_2| = 42$ but $|\mathscr{A}_1| = 5040 = 120|\mathscr{A}|$. Clearly automorphisms of $\mathscr{D}_1$ include (A B)(0 1), but this converts sub-block a to 0 2 4 , which does not appear in the NBIBD; thus the automorphism (A B)(0 1) does not respect the sub-blocks of the NBIBD.

Now, consider again the following NBIBD with parameters $(5, 5, 10, 4, 4, 2)$:

Block	Sub-blocks							
A	a	|	b	(1	4	|	2	3)
B	c	|	d	(2	0	|	3	4)
C	e	|	f	(3	1	|	4	0)
D	g	|	h	(4	2	|	0	1)
E	i	|	j	(0	3	|	1	2)

For this NBIBD, $|\mathscr{A}| = 20$ and $|\mathscr{A}_1| = |\mathscr{A}_2| = 120$. The automorphisms of $\mathscr{D}_1$ include (3 4)(D E), which converts block A to (1 3 | 2 4), whose sub-blocks appear in the NBIBD but not within the same block; thus the automorphism (3 4)(D E) does not respect the nesting of sub-blocks within blocks. The automorphisms of $\mathscr{D}_2$ include (3 4)(a e)(b g)(f i), which does not respect blocks.

This last NBIBD must have $|\mathscr{A}_1| = |\mathscr{A}_2|$, as each of $\mathscr{D}_1$ and $\mathscr{D}_2$ is an unreduced BIBD, and the automorphism group of an unreduced BIBD has order $v!$.

If we have two non-isomorphic NBIBDs with the same parameters $(v, b_1, b_2, r, k_1, k_2)$, then any one of the following properties may hold:

(1) Their component BIBDs with block size k_1 are not isomorphic to one another, nor are their component BIBDs with block size k_2.

(2) Their component BIBDs with block size k_1 are isomorphic to one another, but their component BIBDs with block size k_2 are not.
(3) Their component BIBDs with block size k_2 are isomorphic to one another, but their component BIBDs with block size k_1 are not.
(4) Their component BIBDs with block size k_1 are isomorphic to one another, as are their component BIBDs with block size k_2.

Consider, for example, the following three NBIBDs with parameters $(13, 26, 52, 12, 6, 3)$; each of these has two initial blocks that are to be developed modulo 13:

$$\begin{array}{lllllllllllllllll}
\text{(A)} & (1 & 3 & 9 & | & 4 & 12 & 10) & (2 & 6 & 5 & | & 7 & 8 & 11) \bmod 13; \\
\text{(B)} & (3 & 4 & 10 & | & 1 & 9 & 12) & (5 & 6 & 8 & | & 2 & 7 & 11) \bmod 13; \\
\text{(C)} & (0 & 6 & 7 & | & 3 & 11 & 1) & (0 & 1 & 3 & | & 11 & 2 & 7) \bmod 13.
\end{array}$$

Property (1) holds for pair (A) and (C); property (2) holds for (A) and (B); and property (3) holds for (B) and (C).

By analogy with a situation encountered for perfect Graeco–Latin balanced incomplete block designs (pergolas) [49], we define an NBIBD as 'synchronous' if $|\mathscr{A}| = |\mathscr{A}_1| = |\mathscr{A}_2|$, these equations implying that the three groups are all isomorphic to one another. Examples of synchronous NBIBDs include (A) and (C) from the previous paragraph; (A) has $|\mathscr{A}| = |\mathscr{A}_1| = |\mathscr{A}_2| = 156$, whereas (C) has $|\mathscr{A}| = |\mathscr{A}_1| = |\mathscr{A}_2| = 13$.

If two resolved (or α-resolved) NBIBDs $\mathscr{N}$ and $\mathscr{N}^*$ have the same parameters $(v, b_1, b_2, r, k_1, k_2)$, we can define $\mathscr{N}^*$ to be isomorphic to $\mathscr{N}$ if $\mathscr{N}$ can be obtained from $\mathscr{N}^*$ by a combination of

(i) a permutation of the resolution classes (or α-resolution classes) of $\mathscr{N}^*$,
(ii) a permutation of blocks within resolution classes (or α-resolution classes) of $\mathscr{N}^*$,
(iii) a permutation of sub-blocks within blocks of $\mathscr{N}^*$, and
(iv) a relabelling of the treatments of $\mathscr{N}^*$.

Under this definition, the automorphism group of a resolved (or α-resolved) NBIBD might well be smaller than the automorphism group of the same design with its resolvability (or α-resolvability) ignored. Such a situation would not, however, be important for the present paper, and we restrict ourselves to automorphisms ignoring resolvability (or α-resolvability, or indeed the near-resolvability of almost resolved NBIBDs).

3. Two special classes of NBIBDs

3.1. NBIBDs with $k_1 + k_2 = v$

A very few NBIBDs with parameters $(v, b_1, b_2, r, k_1, k_2)$ have $k_1 + k_2 = v$; we define such NBIBDs to be 'conformal'. If the component BIBDs of a conformal NBIBD are $\mathscr{D}_1$ (with b_1 blocks, as before) and $\mathscr{D}_2$ (with $b_2 = mb_1$, $m > 1$), then the complement of $\mathscr{D}_1$ is a BIBD with parameters

$$(v, b_1, b_1k_2/v, k_2) = (v, b_2/m, b_2k_2/mv, k_2).$$

We define a conformal NBIBD to be 'regular' if it has $\mathscr{D}_2$ isomorphic to m identical copies of the complement of $\mathscr{D}_1$, no relabelling of the treatments being permitted when the copies are made. The following example of a regular conformal NBIBD has parameters $(6, 15, 30, 10, 4, 2)$ and $m = 2$:

$$(1\quad 2\quad|\quad 3\quad 4)\quad (1\quad 3\quad|\quad 2\quad 5)\quad (1\quad 5\quad|\quad 2\quad 4) \bmod 6,\ \text{last block PC}(3),$$

where the notation PC(3) relating to the third initial block indicates that only a Partial Cycle, of length 3, is to be used in developing this particular block cyclically modulo 6. In this NBIBD, $\mathscr{D}_1$ is an unreduced BIBD, so $|\mathscr{A}_1| = v! = 720$, whereas $\mathscr{D}_2$ is m $(=2)$ copies of an unreduced BIBD with b_1 blocks, so

$$|\mathscr{A}_2| = v!(m!)^{b_1} = 720 \cdot 2^{15} = 23{,}592{,}960.$$

Alternatively, the example just given of a regular conformal NBIBD may be presented in 2-resolved form as follows, where the blocks within each set of double square brackets constitute a 2-resolution class:

$$[[(1\quad 3\quad|\quad 2\quad 5)\quad (4\quad 0\quad|\quad 5\quad 2)\quad (3\quad 1\quad|\quad 4\quad 0)]]\quad \text{PC}(3),$$

$$[[(1\quad 2\quad|\quad 3\quad 4)\quad (3\quad 4\quad|\quad 5\quad 0)\quad (5\quad 0\quad|\quad 1\quad 2)]]\quad \text{PC}(2) \bmod 6.$$

This and other regular and non-regular conformal NBIBDs are included in the table of NBIBDs that is given later in this paper. A non-regular example is the following, which has parameters (12, 33, 66, 22, 8, 4) and treatments $0, 1, 2, \ldots, 10, \infty$:

$$(0\quad 1\quad 2\quad 3\quad|\quad 4\quad 7\quad 8\quad 10)\quad (0\quad 1\quad 4\quad 7\quad|\quad 2\quad 3\quad 9\quad \infty)$$
$$(0\quad 2\quad 6\quad 8\quad|\quad 3\quad 7\quad 9\quad \infty) \bmod 11.$$

A possibility for a non-regular conformal NBIBD is for $\mathscr{D}_2$ to be partitionable into m BIBDs which are each isomorphic to the complement of $\mathscr{D}_1$, and so are isomorphic to one another, but are not all identical copies (without treatment relabelling) of $\mathscr{D}_1$. We define a non-regular conformal NBIBD with this weaker property than regularity to be 'semi-regular'. An example of a semi-regular conformal NBIBD for the parameters $(9, 12, 24, 8, 6, 3)$ is

$$[[(1\quad 3\quad 4\quad|\quad 2\quad 6\quad \infty)\,(5\quad 7\quad 0\quad|\quad 6\quad 2\quad \infty)\,(0\quad 1\quad 3\quad|\quad 4\quad 5\quad 7)]]$$
$$\text{PC}(4),\ \bmod 8.$$

The semi-regularity is seen by noting that the complement of the component $\mathscr{D}_1$ in this NBIBD is

$$[(5\quad 7\quad 0)\,(1\quad 3\quad 4)\,(2\quad 6\quad \infty)]\,\text{PC}(4),\ \bmod 8,$$

which is isomorphic to

$$[(3\quad 1\quad 0)\,(7\quad 5\quad 4)\,(6\quad 2\quad \infty)]\,\text{PC}(4),\ \bmod 8.$$

We have attempted no systematic study of regular or non-regular conformal NBIBDs.

The definition of a conformal NBIBD can be extended in an obvious way to that of a conformal nested t-design, $t \geqslant 2$. The nested 3-design given at the end of Section 1 above is a conformal nested t-design.

3.2. NBIBDs with $k_1 = v/2$

As pointed out by Preece [43] and others, a BIBD whose parameters (v, b, r, k) satisfy $v = 2k$ may or may not be self-complementary in the sense of being isomorphic to its complement. Thus NBIBDs whose parameters $(v, b_1, b_2, r, k_1, k_2)$ satisfy $k_1 = v/2$ include some for which the component BIBD $\mathscr{D}_1$ (with block size k_1) is self-complementary and some for which it is not.

If, for a particular parameter set with $k_1 = v/2$, $\mathscr{N}$ is an NBIBD whose component BIBD $\mathscr{D}_1$ is not self-complementary, and $\mathscr{N}^*$ is an NBIBD whose $\mathscr{D}_1$ is the complement of that in $\mathscr{N}$, then $\mathscr{N}$ and $\mathscr{N}^*$ may, nevertheless, have component BIBDs $\mathscr{D}_2$ (with block size k_2) that are isomorphic to one another. Two such NBIBDs are

$$(0 \quad 1 \quad | \quad 3 \quad 7 \quad | \quad 8 \quad 10)(1 \quad 7 \quad | \quad 2 \quad 10 \quad | \quad 8 \quad \infty) \bmod 11$$

and

$$(4 \quad 9 \quad | \quad 5 \quad 6 \quad | \quad 2 \quad \infty)\,(0 \quad 3 \quad | \quad 4 \quad 6 \quad | \quad 5 \quad 9) \bmod 11$$

4. Existence and enumeration of NBIBDs

Necessary but not sufficient conditions for the existence of an NBIBD with parameters $(v, b_1, b_2, r, k_1, k_2)$, where $vr = b_1k_1 = b_2k_2$ and $k_2 < k_1$, are

(a) the existence of a BIBD with parameters (v, b_1, r, k_1), and
(b) the existence of a BIBD with parameters (v, b_2, r, k_2).

That the conditions are not sufficient is illustrated by the fact that there are 3 non-isomorphic BIBDs with parameters $(10, 15, 9, 6)$ and 960 non-isomorphic BIBDs with parameters $(10, 30, 9, 3)$ but [25] there is no NBIBD with parameters $(10, 15, 30, 9, 6, 3)$.

Necessary but not sufficient conditions for the existence of a resolvable NBIBD with parameters $(v, b_1, b_2, r, k_1, k_2)$, where $vr = b_1k_1 = b_2k_2$ and $k_2 < k_1$, are

(a) the existence of a resolvable BIBD with parameters (v, b_1, r, k_1), and
(b) the existence of a resolvable BIBD with parameters (v, b_2, r, k_2).

If, for a particular pair of values v, r, there exist BIBDs with parameters (v, b_1, r, k_1) and (v, b_2, r, k_2), where $k_2 < k_1$, but there is no NBIBD with parameters $(v, b_1, b_2, r, k_1, k_2)$, there may nevertheless be an NBIBD with parameters $(v, mb_1, mb_2, mr, k_1, k_2)$ for some integers m greater than 1. (This situation is akin to that for BIBDs, where a 'multiple' design may exist even though a 'basic' design does not.) Thus, with $m = 2$ and 3, NBIBDs with parameters $(10, 30, 60, 18, 6, 3)$ and $(10, 45, 90, 27, 6, 3)$ exist even though, as mentioned above, one with parameters $(10, 15, 30, 9, 6, 3)$ does not [25]. We refer to the NBIBDs with $m = 2$ and 3 as 'double' and 'triple' NBIBDs.

If, for a particular pair of values v, r, there exist n_1 non-isomorphic BIBDs $\mathscr{D}_1$ with parameters (v, b_1, r, k_1) and n_2 non-isomorphic BIBDs $\mathscr{D}_2$ with parameters (v, b_2, r, k_2), where k_2 is a factor of k_1, the enumeration of corresponding non-isomorphic NBIBDs can be considered at two levels:

(a) For how many of the $n_1 n_2$ pairs $\mathscr{D}_1, \mathscr{D}_2$ does an NBIBD exist?
(b) For any particular pair $\mathscr{D}_1, \mathscr{D}_2$, how many non-isomorphic NBIBDs exist?

The example earlier in this section, with 2880 pairs $\mathscr{D}_1, \mathscr{D}_2$ but no NBIBD, suggests that, more generally, the number of pairs $\mathscr{D}_1, \mathscr{D}_2$ that are productive of NBIBDs is likely to be small. This is hardly surprising, as the requirement that the blocks of $\mathscr{D}_1$ should partition to give the blocks of $\mathscr{D}_2$ is a strong one.

If $k_1 = v/2$ and $\mathscr{D}$ is a possible non-self-complementary choice for $\mathscr{D}_1$, it is tempting to conjecture that

(a) an NBIBD exists for as many pairs $\mathscr{D}_1$, $\mathscr{D}_2$ with $\mathscr{D}_1 = \mathscr{D}$ as with $\mathscr{D}_1 = \mathscr{D}^*$, where $\mathscr{D}^*$ is the complement of $\mathscr{D}$, and
(b) there are as many non-isomorphic NBIBDs with $\mathscr{D}_1 = \mathscr{D}$ as with $\mathscr{D}_1 = \mathscr{D}^*$.

We can, however, see no way in which such conjectures could be proved in general. Indeed, the second conjecture suggests a one–one correspondence between the NBIBDs with $\mathscr{D}_1 = \mathscr{D}$ and those with $\mathscr{D}_1 = \mathscr{D}^*$, but we can see no way in which any such one–one correspondence could be set up.

5. NBIBDs with $k_1 = v - 1$, $k_2 = 2$ and $v = b_1$

If v is odd, a 'starter' [20,62,63] in an abelian group of order v is a partition of the non-zero elements of the group into pairs x_i, y_i $(i = 1, 2, \ldots, (v-1)/2)$ such that the $v - 1$ differences $(x_i - y_i)$ and $(y_i - x_i)$ are all different. The $v - 1$ differences are thus the $v - 1$ non-zero elements of the group. With only a slight notational change, a starter can thus be used to produce the initial block in the representation of an NBIBD with $b_1 = v$, $k_1 = v - 1$ and $k_2 = 2$; the initial block contains the non-zero elements of the group, and the pairs of elements in the sub-blocks are the pairs in the starter. For example, the sole starter in $\mathbb{Z}_5$ is

$$1,4 \quad 2,3$$

which gives the initial block $(1\ 4 \,|\, 2\ 3)$ of the NBIBD

$$(1\ 4 \,|\, 2\ 3) \bmod 5 \quad (\text{with} |\mathscr{A}| = 20)$$

discussed in Section 2 above.

The number of distinct starters in $\mathbb{Z}_v$ has been enumerated [20, p. 469, Table 45.18] for $v = 5, 7, \ldots, 27$, which is helpful for the enumeration of the corresponding NBIBDs.

However, for a fixed v, distinct starters do not necessarily produce non-isomorphic NBIBDs. For example, $\mathbb{Z}_7$ has three distinct starters:

$$\begin{array}{lll} 1,6 & 2,5 & 3,4; \\ 1,3 & 2,6 & 4,5; \\ 1,5 & 2,3 & 4,6; \end{array}$$

but the third of these can be obtained by multiplying the elements of the second by 3 and reducing the products modulo 7. Thus the second and third starters are 'equivalent' [20, p. 469] in a sense that implies that the corresponding NBIBDs are isomorphic. The first starter is, however, not equivalent to either of the other two. So, for $\mathbb{Z}_7$, there are just two 'equivalence classes' of starters; these can be shown to correspond to 2 non-isomorphic NBIBDs, namely

$$\begin{array}{l} (1 \quad 6 \quad | \quad 2 \quad 5 \quad | \quad 3 \quad 4) \bmod 7 \ \ (\text{with}|\mathscr{A}| = 42), \\ (1 \quad 3 \quad | \quad 2 \quad 6 \quad | \quad 4 \quad 5) \bmod 7 \ \ (\text{with}|\mathscr{A}| = 168). \end{array}$$

The group $\mathbb{Z}_9$ has nine distinct starters, falling into just three equivalence classes, represented by the following three starters:

$$\begin{array}{llll} 1,8 & 2,7 & 3,6 & 4,5; \\ 1,2 & 3,6 & 4,8 & 5,7; \\ 1,6 & 2,8 & 3,4 & 5,7. \end{array}$$

These give the respective NBIBDs

$$\begin{array}{ll} \text{(a)} & (1 \quad 8 \quad | \quad 2 \quad 7 \quad | \quad 3 \quad 6 \quad | \quad 4 \quad 5) \bmod 9 \ \ (\text{with } |\mathscr{A}| = 54); \\ \text{(b)} & (1 \quad 2 \quad | \quad 3 \quad 6 \quad | \quad 4 \quad 8 \quad | \quad 5 \quad 7) \bmod 9 \ \ (\text{with } |\mathscr{A}| = 54); \\ \text{(c)} & (1 \quad 6 \quad | \quad 2 \quad 8 \quad | \quad 3 \quad 4 \quad | \quad 5 \quad 7) \bmod 9 \ \ (\text{with } |\mathscr{A}| = 9). \end{array}$$

However, applying the permutation (2 5 8) throughout *all* blocks of (a), and then doing some re-ordering of blocks and of sub-blocks within blocks, gives (b). So (a) is isomorphic to (b), even though (a) and (b) come from different equivalence classes of starters. As (a) and (b) are not isomorphic to (c), the final result for $\mathbb{Z}_9$ is that the nine distinct starters give just two non-isomorphic NBIBDs.

The group $\mathbb{Z}_{11}$ has 25 distinct starters. These fall into five equivalence classes containing, respectively, $1, 2, 2, 10$ and 10 starters. These five equivalence classes yield five non-isomorphic NBIBDs, whose respective values of $|\mathscr{A}|$ are 110, 55, $55, 11$ and 11.

Similarly $\mathbb{Z}_{13}$ has 133 distinct starters, falling into 14 equivalence classes. The numbers of distinct starters per equivalence class are 1 (for just one class), 4 (for each of three classes), and 12 (for each of ten classes). The 14 equivalence classes yield 14 non-isomorphic NBIBDs, whose values of $|\mathscr{A}|$ are 156 (for the single class containing just 1 starter), 39 (for each of the three classes each containing 4 distinct starters), and 13 (for each of the ten classes each containing 12 distinct starters).

Thus, for $v = 5$, 11 and 13, but not for $v = 7$ and 9, the number of non-isomorphic NBIBDs is the same as the number of equivalence classes, and all the NBIBDs have

$$|\mathscr{A}| = \frac{v(v-1)}{\text{size of equivalence class}}.$$

6. Our new table of NBIBDs

Table 1 lists over 200 NBIBDs. Subject to the restrictions $v \leqslant 16$ and $r \leqslant 30$, the table covers all parameter sets for which at least one NBIBD might be expected to exist, except that, if an NBIBD is known to exist for a particular parameter set, then no multiple of that parameter set is included in the table. The table includes only NBIBDs that can be generated easily or fairly easily from initial blocks. The results of some exhaustive searches for selected NBIBDs, together with information on some relationships to other designs, are reported on the third author's website: `http://www.davidhywel.freeserve.co.uk` .

For parameter sets nos. 29 and 30 in Table 1, the component design $\mathscr{D}_1$ would have parameters $(v,b,r,k)=(15,21,14,10)$ and so would be the complement of a BIBD with parameters $(15,21,7,5)$; but no such BIBD exists. Thus, there is no NBIBD for either of the parameter sets nos. 29 and 30, but the table gives NBIBDs for the corresponding double parameter sets, namely nos. 60 and 61. As pointed out earlier in the paper, no NBIBD exists for parameter set no. 11, namely $(v,b_1,b_2,r,k_1,k_2)=(10,15,30,9,6,3)$, but the table gives NBIBDs for the double and triple parameter sets, namely nos. 47 and 58. No NBIBD has been found for parameter set no. 39, but the table gives an NBIBD for the corresponding double parameter set, namely no. 68.

For the most part, we have made no attempt to provide a complete list of NBIBDs for an individual parameter set. We have, however, provided a wide selection of NBIBDs, to illustrate the wide diversity of types that exist. Thus, for example, a parameter set in the table may have NBIBDs with several different values of $|\mathscr{A}|$, or it may have some NBIBDs that are generated modulo v and others that are generated modulo $v-1$.

Where Table 1 gives more than one NBIBD for a particular parameter set, $k \neq v/2$, each NBIBD has a composite label, e.g. Cd2, which includes a capital letter followed by a lower-case letter. Throughout the parameter set, two NBIBDs have the same capital letter if and only if their component designs $\mathscr{D}_1$ are isomorphic to one another, and have the same lower-case letter if and only if their component designs $\mathscr{D}_2$ are isomorphic to one another. If two NBIBDs have isomorphic $\mathscr{D}_1$ designs and isomorphic $\mathscr{D}_2$ designs, they are distinguished by the integer following the lower-case letter. The same scheme of labelling is used for a parameter set with $k=v/2$, save that a refinement is used if $\mathscr{D}_1$ is not self-complementary; this happens for parameter set nos. 17 and 36, where the label $\bar{\mathrm{C}}$ refers to the complement of the non-self-complementary BIBD whose label is C.

7. Some methods of construction for NBIBDs

Several general methods for constructing NBIBDs are given here. Each is formulated as an MNBIBD construction, from which NBIBDs in Table 1 can be found as special cases. The first is a recursive technique.

Table 1
Nested balanced incomplete designs[a], $v \leqslant 16$, $r \leqslant 30$

[a] A letter w, p or c in the first column signifies a parameter set for, respectively, a whist tournament, a pitch tournament design, or a conformal NBIBD. Round brackets are used for blocks of size k_1. In resolved designs, square brackets [ ] are used for resolution classes. In 2-resolved designs, bracketting [[ ]] is used for 2-resolution classes. And so on. In almost resolved designs, angled brackets $\langle\,\rangle$ are used for the incomplete resolution classes that contain each of $v-1$ treatments exactly once. The letter u in either of the last 2 columns indicates that the component BIBD is unreduced, whereas $2u$ and $3u$ denote, respectively, 2 and 3 copies of the unreduced design. If a design is known to exist for a particular set of parameters, no multiple of that parameter set is included in the table.

* Denotes $\mathscr{D}_1$ is a 3-design.

	v	b_1	b_2	r	k_1	k_2		Blocks	$\lvert\mathscr{A}\rvert$	$\lvert\mathscr{A}_1\rvert$	$\lvert\mathscr{A}_2\rvert$
2.	7	7	21	6	6	2	Aa1	$(1\ 6 \mid 2\ 5 \mid 4\ 3) \bmod 7$	42	u	u
							Aa2	$(1\ 3 \mid 2\ 6 \mid 4\ 5) \bmod 7$	168	u	u
3.	7	7	14	6	6	3		$(1\ 2\ 4 \mid 6\ 5\ 3) \bmod 7$	42	u	42
4.w	8	14	28	7	4	2	Aa	$[(0\ 1 \mid 4\ 2)\ (3\ 6 \mid 5\ \infty)] \bmod 7$	56	1344*	u
							Ba	$(0\ 1 \mid 2\ 4)\ (1\ 4 \mid 2\ \infty) \bmod 7$	7	21	u
5.w	9	18	36	8	4	2		$\langle(01\ 02 \mid 10\ 20)\ (11\ 22 \mid 12\ 21)\rangle \bmod (3,3)$	144	144	u
6.	9	12	36	8	6	2	Aa1	$[[(1\ 2 \mid 3\ 6 \mid 4\ \infty)\ (5\ 6 \mid 7\ 2 \mid 0\ \infty)\ (0\ 4 \mid 1\ 7 \mid 3\ 5)]]$ PC(4), mod 8	144	432	u
							Aa2	$[[(0\ 7 \mid 2\ 6 \mid 3\ 4)\ (4\ 1 \mid 7\ 5 \mid 6\ \infty)\ (0\ 5 \mid 3\ 1 \mid 2\ \infty)]]$ PC(4), mod 8	72	432	u
							Aa3	$[[(2\ 3 \mid 1\ 4 \mid 6\ \infty)\ (6\ 7 \mid 5\ 0 \mid 2\ \infty)\ (0\ 4 \mid 1\ 7 \mid 3\ 5)]]$ PC(4), mod 8	8	432	u
7.c	9	12	24	8	6	3	Aa	$[[(1\ 3\ 4 \mid 2\ 6\ \infty)\ (5\ 7\ 0 \mid 2\ 6\ \infty)\ (1\ 3\ 4 \mid 5\ 7\ 0)]]$ PC(4), mod 8	432	432	$> 10^6$
							Ab	$[[(1\ 3\ 4 \mid 2\ 6\ \infty)\ (5\ 7\ 0 \mid 6\ 2\ \infty)\ (0\ 1\ 3 \mid 4\ 5\ 7)]]$ PC(4), mod 8	16	432	512
8.	9	9	36	8	8	2	Aa1	$(1\ 8 \mid 2\ 7 \mid 3\ 6 \mid 4\ 5) \bmod 9$	54	u	u
							Aa2	$(1\ 6 \mid 2\ 8 \mid 3\ 4 \mid 5\ 7) \bmod 9$	9	u	u
							Aa3	$(01\ 02 \mid 10\ 20 \mid 11\ 22 \mid 12\ 21) \bmod (3,3)$	432	u	u
9.p	9	9	18	8	8	4		$(01\ 02\ 10\ 20 \mid 11\ 22\ 12\ 21) \bmod (3,3)$	144	u	144
10.	10	15	45	9	6	2		$(0_0\ 2_0 \mid 3_0\ 2_1 \mid 3_1\ 4_1)\ (2_0\ 3_0 \mid 0_0\ 3_1 \mid 4_0\ 0_1)\ (0_0\ 0_1 \mid 1_0\ 3_1 \mid 2_1\ 4_1)$ mod 5, suffixes fixed	5	720	u

11.	10	15	30	9	6	3		No NBIBD exists, but doubles (No 47) and triples (No 58) exist.			
12.	10	10	30	9	9	3		$(1_0\ 2_0\ 4_1 \mid 3_0\ 4_0\ 3_1 \mid 0_1\ 1_1\ 2_1)\ (2_0\ 3_1\ 0_0 \mid 1_0\ 2_1\ 3_0 \mid 1_1\ 4_0\ 4_1)$	10	u	60
								mod 5, suffixes fixed			
13.c	6	15	30	10	4	2	Aa1	$[[(1\ 3 \mid 2\ 5)\ (4\ 0 \mid 5\ 2)\ (3\ 1 \mid 4\ 0)]]$ PC(3)	120	u	$2u$
								$[[(1\ 2 \mid 3\ 4)(3\ 4 \mid 5\ 0)\ (5\ 0 \mid 1\ 2)]]$ PC(2)			
								both mod 6			
							Aa2	$[[(0\ 1 \mid 2\ 4)\ (3\ 4 \mid 5\ 1)(2\ 0 \mid 3\ 5)]]$ PC(3)	6	u	$2u$
								$[[(0\ 3 \mid 1\ 2)\ (2\ 5 \mid 3\ 4)\ (4\ 1 \mid 5\ 0)]]$ PC(2)			
								both mod 6			
							Aa3	$[[(0\ 2 \mid 1\ 3)\ (\infty\ 0 \mid 3\ 4)\ (\infty\ 4 \mid 1\ 2)]]$ mod 5	5	u	$2u$
							Aa4	$[[(\infty\ 2 \mid 1\ 4)\ (\infty\ 0 \mid 3\ 4)\ (0\ 3 \mid 1\ 2)]]$ mod 5	5	u	$2u$
14.	11	11	55	10	10	2	Aa1	$(1\ 10 \mid 2\ 9 \mid 3\ 8 \mid 4\ 7 \mid 5\ 6)$ mod 11	110	u	u
							Aa2	$(1\ 2 \mid 3\ 6 \mid 4\ 8 \mid 5\ 10 \mid 9\ 7)$ mod 11	55	u	u
							Aa3	$(1\ 7 \mid 2\ 5 \mid 3\ 10 \mid 4\ 6 \mid 8\ 9)$ mod 11	55	u	u
							Aa4	$(1\ 2 \mid 3\ 8 \mid 4\ 6 \mid 5\ 9 \mid 7\ 10)$ mod 11	11	u	u
							Aa5	$(1\ 3 \mid 2\ 5 \mid 4\ 9 \mid 6\ 10 \mid 7\ 8)$ mod 11	11	u	u
15.	11	11	22	10	10	5		$(1\ 3\ 4\ 5\ 9 \mid 2\ 6\ 8\ 10\ 7)$ mod 11	110	u	110
16.w	12	33	66	11	4	2	Aa1	$[(0\ 1 \mid 3\ 7)(10\ 2 \mid 9\ 4)\ (8\ 6 \mid 5\ \infty)]$ mod 11	11	11	u
							Aa2	$[(0\ 7 \mid 1\ 3)\ (10\ 2 \mid 9\ 4)\ (5\ 6 \mid 8\ \infty)]$ mod 11	11	11	u
							Ba1	$(0\ 1 \mid 3\ 7)\ (10\ 2 \mid 9\ 4)\ (2\ 4 \mid 5\ \infty)$ mod 11	11	11	u
							Ba2	$(0\ 7 \mid 1\ 3)\ (10\ 2 \mid 9\ 4)\ (5\ 4 \mid 2\ \infty)$ mod 11	11	11	u
17.	12	22	66	11	6	2	Aa	$[(0\ 3 \mid 1\ 5 \mid 4\ 9)\ (8\ 10 \mid 7\ 6 \mid 2\ \infty)]$ mod 11	11	7920*	u
							Ba	$(0\ 3 \mid 1\ 5 \mid 4\ 9)\ (1\ 3 \mid 4\ 5 \mid 9\ \infty)$ mod 11	11	55	u
							Ca1	$(0\ 1 \mid 3\ 7 \mid 8\ 10)\ (1\ 7 \mid 2\ 10 \mid 8\ \infty)$ mod 11	11	11	u
							Ca2	$(1\ 5 \mid 7\ 9 \mid 2\ \infty)\ (1\ 7 \mid 2\ 10 \mid 8\ 9)$ mod 11	11	11	u
							$\bar{\text{C}}$a1	$(4\ 9 \mid 5\ 6 \mid 2\ \infty)\ (0\ 3 \mid 4\ 6 \mid 5\ 9)$ mod 11	11	11	u
							$\bar{\text{C}}$a2	$(0\ 3 \mid 4\ 10 \mid 6\ 8)\ (0\ 4 \mid 5\ 6 \mid 3\ \infty)$ mod 11	11	11	u

Table 1. (*continued*)

	v	b_1	b_2	r	k_1	k_2		Blocks	$\lvert\mathscr{A}\rvert$	$\lvert\mathscr{A}_1\rvert$	$\lvert\mathscr{A}_2\rvert$
18.	12	22	44	11	6	3	Aa	$[(0\ 1\ 3 \mid 4\ 5\ 9)\ (10\ 7\ \infty \mid 6\ 8\ 2)]$ mod 11	11	7920*	11
							Bb	$(0\ 1\ 3 \mid 4\ 5\ 9)\ (1\ 4\ \infty \mid 5\ 3\ 9)$ mod 11	11	55	11
19.	7	21	42	12	4	2	Aa	$(0\ 1 \mid 4\ 2)(0\ 2 \mid 1\ 4)(0\ 4 \mid 2\ 1)$ mod 7	168	$> 10^6$	$2u$
							Ba	$(0\ 1 \mid 4\ 2)(0\ 2 \mid 1\ 4)(0\ 3 \mid 5\ 6)$ mod 7	7	2688	$2u$
							Ca1	$(0\ 3 \mid 1\ 2)(0\ 6 \mid 2\ 4)(0\ 2 \mid 3\ 6)$ mod 7	42	42	$2u$
							Ca2	$(0\ 1 \mid 3\ \infty)(0\ 2 \mid 1\ \infty)(0\ 2 \mid 1\ 4)$ $(0\ 1 \mid 3\ 4)$ mod 6, last block PC(3)	6	42	$2u$
20.w	13	39	78	12	4	2	Aa	$\langle(1\ 12 \mid 5\ 8)(2\ 11 \mid 3\ 10)(4\ 9 \mid 6\ 7)\rangle$ mod 13	156	156	u
							Ba	$\langle(1\ 4 \mid 2\ 7)(3\ 12 \mid 6\ 8)(9\ 10 \mid 5\ 11)\rangle$ mod 13	39	39	u
							Ca	$(0\ 1 \mid 3\ 9)(0\ 3 \mid 1\ 9)(0\ 9 \mid 1\ 3)$ mod 13	5616	$> 10^6$	u
							Da	$(1\ 4 \mid 2\ 7)(3\ 12 \mid 6\ 8)(3\ 2 \mid 7\ 1)$ mod 13	13	13	u
							Ea	$\langle(\infty\ 11 \mid 4\ 7)(2\ 10 \mid 9\ 8)(6\ 1 \mid 5\ 3)\rangle$ mod 12 $\langle(0\ \ 6 \mid 3\ \ 9)(1\ 7 \mid 4\ 10)(2\ 8 \mid 5\ 11)\rangle$	12	12	u
21.	13	26	78	12	6	2	Aa1	$\langle(3\ 10 \mid 4\ 9 \mid 1\ 12)(5\ 8 \mid 11\ 2 \mid 6\ 7)\rangle$ mod 13	156	156	u
							Aa2	$\langle(1\ 4 \mid 3\ 12 \mid 9\ 10)(2\ 7 \mid 6\ 8 \mid 5\ 11)\rangle$ mod 13	39	156	u
							Aa3	$\langle(\infty\ 3 \mid 2\ 5 \mid 11\ 10)(6\ 1 \mid 7\ 9 \mid 8\ 4)\rangle$ mod 12 $\langle(0\ \ 6 \mid 2\ 8 \mid 4\ 10)(1\ 7 \mid 3\ 9 \mid 5\ 11)\rangle$	12	156	u
							Aa4	$\langle(\infty\ 3 \mid 2\ 11 \mid 5\ 10)(4\ 6 \mid 9\ 1 \mid 7\ 8)\rangle$ mod 12 $\langle(0\ \ 6 \mid 2\ 8 \mid 4\ 10)(1\ 7 \mid 3\ 9 \mid 5\ 11)\rangle$	12	156	u
							Ba	$(2\ 9 \mid 4\ 12 \mid 8\ 10)(2\ 11 \mid 5\ 8 \mid 6\ 7)$ mod 13	39	39	u

							Ca	(2 4 | 8 12 | 9 10)(4 9 | 5 8 | 6 12) mod 13	13	39	u
							Da	(0 3 | 6 11 | 7 1)(0 11 | 1 2 | 3 7) mod 13	13	13	u
							Ea	(∞ 11 | 3 2 | 5 10)(5 8 | 4 6 | 3 11) (0 6 | 2 8 | 4 10) mod 12, last block PC(2)	12	12	u
22.	13	26	52	12	6	3	Aa	⟨(1 3 9 | 4 12 10)(2 6 5 | 7 8 11)⟩ mod 13	156	156	156
							Ab	⟨(3 4 10 | 1 9 12)(5 6 8 | 2 7 11)⟩ mod 13	13	156	13
							Bb	(0 6 7 | 3 11 1)(0 1 3 | 11 2 7) mod 13	13	13	13
							Cc	(∞ 3 5 | 11 2 10)(8 6 3 | 5 4 11) (0 4 8 | 2 6 10) mod 12, last block PC(2)	12	12	12
23.	13	13	78	12	12	2	Aa1	(1 12 | 2 11 | 3 10 | 4 9 | 5 8 | 6 7) mod 13	156	u	u
							Aa2	(1 4 | 2 7 | 3 12 | 5 11 | 6 8 | 9 10) mod 13	39	u	u
							Aa3	(1 2 | 3 6 | 4 11 | 5 9 | 7 12 |8 10) mod 13	39	u	u
							Aa4	(1 5 | 2 3 | 4 11 | 6 9 | 7 12 | 8 10) mod 13	39	u	u
							Aa5	(1 2 | 3 7 | 4 11 | 5 10 | 6 8 | 9 12) mod 13	13	u	u
							Aa6	(1 2 | 3 6 | 4 9 | 5 12 | 7 11 | 8 10) mod 13	13	u	u
							Aa7	(1 2 | 3 6 | 4 11 | 5 10 | 7 9 | 8 12) mod 13	13	u	u
							Aa8	(1 2 | 3 11 | 4 10 | 5 7 | 6 9 | 8 12) mod 13	13	u	u
							Aa9	(1 2 | 3 9 | 4 12 | 5 7 | 6 10 | 8 11) mod 13	13	u	u
							Aa10	(1 2 | 3 7 | 4 12 | 5 11 | 6 9 | 8 10) mod 13	13	u	u

Table 1. (*continued*)

	v	b_1	b_2	r	k_1	k_2		Blocks	$\lvert\mathscr{A}\rvert$	$\lvert\mathscr{A}_1\rvert$	$\lvert\mathscr{A}_2\rvert$
							Aa11	$(1\ 2 \mid 3\ 10 \mid 4\ 8 \mid 5\ 7 \mid 6\ 11 \mid 9\ 12) \bmod 13$	13	u	u
							Aa12	$(1\ 2 \mid 3\ 12 \mid 4\ 10 \mid 5\ 8 \mid 6\ 11 \mid 7\ 9) \bmod 13$	13	u	u
							Aa13	$(1\ 3 \mid 2\ 6 \mid 4\ 11 \mid 5\ 8 \mid 7\ 12 \mid 9\ 10) \bmod 13$	13	u	u
							Aa14	$(1\ 3 \mid 2\ 5 \mid 4\ 11 \mid 6\ 10 \mid 7\ 12 \mid 8\ 9) \bmod 13$	13	u	u
24.	13	13	52	12	12	3	Aa	$(1\ 3\ 9 \mid 4\ 12\ 10 \mid 2\ 6\ 5 \mid 7\ 8\ 11) \bmod 13$	156	u	156
							Ab	$(3\ 4\ 10 \mid 1\ 9\ 12 \mid 5\ 6\ 8 \mid 2\ 7\ 11) \bmod 13$	13	u	13
25.	13	13	39	12	12	4	Aa	$(1\ 12\ 5\ 8 \mid 2\ 11\ 3\ 10 \mid 4\ 9\ 6\ 7) \bmod 13$	156	u	156
							Ab	$(1\ 4\ 2\ 7 \mid 3\ 12\ 6\ 8 \mid 9\ 10\ 5\ 11) \bmod 13$	39	u	39
26.	13	13	26	12	12	6		$(1\ 3\ 9\ 4\ 12\ 10 \mid 2\ 6\ 5\ 7\ 8\ 11) \bmod 13$	156	u	156
27.	15	35	105	14	6	2		$(1_1\ 0_0 \mid 2_1\ 0_1 \mid 4_1\ \infty)(0_0\ 3_0 \mid 0_1\ 5_0 \mid \infty\ 6_0)(2_0\ 1_0 \mid 4_0\ 3_1 \mid 1_1\ 0_1)$ $(2_0\ 0_1 \mid 5_0\ 1_1 \mid 3_1\ 3_0)(4_0\ 1_1 \mid 5_0\ 0_0 \mid 0_1\ 3_1)$ mod 7, suffixes fixed	7	21	u
28.	15	35	70	14	6	3		$(1_1\ 2_1\ 4_1 \mid 0_0\ 0_1\ \infty)(0_0\ 0_1\ \infty \mid 3_0\ 5_0\ 6_0)(2_0\ 4_0\ 1_1 \mid 1_0\ 3_1\ 0_1)$ $(2_0\ 5_0\ 3_1 \mid 0_1\ 1_1\ 3_0)(4_0\ 5_0\ 0_1 \mid 1_1\ 0_0\ 3_1)$ mod 7, suffixes fixed	21	21	2688
29.	15	21	105	14	10	2		No $\mathscr{D}_1$ exists, so no NBIBD exists, but a double exists (No 60).			
30.c	15	21	42	14	10	5		No $\mathscr{D}_1$ exists, so no NBIBD exists, but a double exists (No 61).			

31.	15	15	105	14	14	2	Aa1	$(1\ 14 \mid 2\ 13 \mid 3\ 12 \mid 4\ 11 \mid 5\ 10 \mid 6\ 9 \mid 7\ 8)$ mod 15	120	*u*	*u*
							Aa2	$(\infty\ 0_1 \mid 4_0\ 6_1 \mid 1_0\ 5_1 \mid 1_1\ 5_0 \mid 2_0\ 3_1 \mid 4_1\ 6_0 \mid 2_1\ 3_0)$ $(0_0\ \infty \mid 2_0\ 6_0 \mid 4_0\ 5_0 \mid 5_1\ 4_1 \mid 1_0\ 3_0 \mid 6_1\ 2_1 \mid 3_1\ 1_1)$ mod 7, suffixes fixed, with $(0_1\ 0_0 \mid 1_1\ 1_0 \mid 2_1\ 2_0 \mid 3_1\ 3_0 \mid 4_1\ 4_0 \mid 5_1\ 5_0 \mid 6_1\ 6_0)$	21	*u*	*u*
32.	15	15	30	14	14	7		$(\infty\ 4_0 1_0 1_1\ 2_0\ 4_1\ 2_1 \mid 0_1\ 6_1\ 5_1\ 5_0\ 3_1 6_0\ 3_0)$ $(0_0\ 2_0\ 4_0\ 5_1\ 1_0\ 6_1\ 3_1 \mid \infty\ 6_0\ 5_0\ 4_1\ 3_0\ 2_1\ 1_1)$ mod 7, suffixes fixed, with $(0_1\ 1_1\ 2_1\ 3_1\ 4_1\ 5_1\ 6_1 \mid 0_0\ 1_0\ 2_0\ 3_0\ 4_0\ 5_0\ 6_0)$	21	*u*	21
33.w	16	60	120	15	4	2	Aa	$[(\infty\ 0 \mid 5\ 10)\ (1\ 2 \mid 4\ 8)\ (6\ 9 \mid 7\ 13)\ (11\ 3 \mid 12\ 14)]$ mod 15	5760	$> 10^6$	*u*
							Ba	$[(\infty\ 0 \mid 3\ 14)\ (1\ 4 \mid 9\ 7)\ (2\ 8 \mid 6\ 13)\ (5\ 10 \mid 11\ 12)]$ mod 15	240	480	*u*
							Ca	$(\infty\ 0 \mid 5\ 10)\ (1\ 2 \mid 4\ 8)\ (6\ 9 \mid 7\ 13)\ (3\ 11 \mid 2\ 0)$ mod 15	30	$> 10^6$	*u*
							Da	$(\infty\ 0 \mid 3\ 14)\ (1\ 4 \mid 9\ 7)\ (2\ 8 \mid 6\ 13)\ (9\ 4 \mid 3\ 2)$ mod 15	15	15	*u*
34.	16	40	120	15	6	2	Aa1	$(0\ 1 \mid 9\ 3 \mid 5\ 12)\ (0\ 3 \mid 1\ 12 \mid 6\ 2)\ (11\ 9 \mid 1\ 3 \mid 0\ 8)$ mod 16, last block PC(8)	16	16	*u*
							Aa2	$(0\ 9 \mid 5\ 3 \mid 12\ 1)\ (6\ 3 \mid 12\ 0 \mid 1\ 2)\ (1\ 11 \mid 8\ 0 \mid 9\ 3)$ mod 16, last block PC(8)	16	16	*u*
							Ba1	$(0\ 1 \mid 9\ 3 \mid 5\ 12)\ (0\ 13 \mid 15\ 4 \mid 10\ 14)\ (11\ 9 \mid 1\ 3 \mid 0\ 8)$ mod 16, last block PC(8)	16	16	*u*
							Ba2	$(0\ 9 \mid 5\ 3 \mid 12\ 1)\ (10\ 13 \mid 4\ 0 \mid 15\ 14)\ (1\ 11 \mid 8\ 0 \mid 9\ 3)$ mod 16, last block PC(8)	16	16	*u*
							Cb	$[[[(0\ 5 \mid 1\ 4 \mid 7\ 13)\ (2\ 6 \mid 8\ 10 \mid 9\ \infty)\ (5\ 10 \mid 6\ 9 \mid 12\ 3)\ (7\ 11 \mid 13\ 0 \mid 14\ \infty)$ $(10\ 0 \mid 11\ 14 \mid 2\ 8)\ (12\ 1 \mid 3\ 5 \mid 4\ \infty)\ (1\ 9 \mid 6\ 14 \mid 11\ 4)\ (8\ 7 \mid 13\ 12 \mid 3\ 2)]]]$ PC(5), mod 15	15	60	*u*
35.	16	40	80	15	6	3		$(0_0\ 0_1\ 0_2 \mid 1_1\ 2_1\ 3_1)\ (1_0\ 3_0\ 0_2 \mid 1_2\ 0_1\ 2_1)$ $(1_0\ 3_0\ 0_1 \mid 0_0\ 3_2\ 4_1)\ (2_0\ 3_0\ 4_1 \mid 4_0\ 0_2\ 1_2)$ $(0_0\ 0_1\ 0_2 \mid 1_2\ 2_2\ 4_2)\ (\infty\ 3_0\ 4_0 \mid 0_0\ 3_1\ 4_2)$ $(\infty\ 2_1\ 4_2 \mid 0_0\ 3_1\ 1_2)\ (\infty\ 3_1\ 1_2 \mid 0_0\ 1_1\ 3_2)$ mod 5, suffixes fixed	5	5	160

Table 1. (*continued*)

	v	b_1	b_2	r	k_1	k_2		Blocks	$\lvert\mathscr{A}\rvert$	$\lvert\mathscr{A}_1\rvert$	$\lvert\mathscr{A}_2\rvert$
36.	16	30	120	15	8	2	Aa	$[(\infty\ 0 \mid 3\ 14 \mid 1\ 4 \mid 9\ 7)\ (2\ 8 \mid 6\ 13 \mid 5\ 10 \mid 11\ 12)]$ mod 15	240	322560	u
							Ba	$(\infty\ 0 \mid 11\ 14 \mid 1\ 12 \mid 8\ 6)\ (2\ 8 \mid 6\ 13 \mid 5\ 10 \mid 11\ 12)$ mod 15	15	60	u
							Ca	$(\infty\ 7 \mid 2\ 4 \mid 9\ 14 \mid 10\ 13)\ (0\ 6 \mid 2\ 10 \mid 7\ 11 \mid 8\ 9)$ mod 15	15	15	u
							$\bar{\text{C}}$a	$(\infty\ 14 \mid 1\ 4 \mid 3\ 12 \mid 5\ 13)\ (0\ 1 \mid 3\ 5 \mid 6\ 11 \mid 8\ 12)$ mod 15	15	15	u
37.p	16	30	60	15	8	4	Aa	$[(0\ 1\ 3\ 7 \mid 4\ 9\ 14\ \infty)\ (2\ 10\ 11\ 13 \mid 5\ 6\ 8\ 12)]$ mod 15	5760	322560	$> 10^6$
							Ab1	$[(1\ 4\ 7\ 9 \mid 0\ 3\ 14\ \infty)\ (2\ 6\ 8\ 13 \mid 5\ 10\ 11\ 12)]$ mod 15	480	322560	480
							Ab2	$[(3\ 7\ 9\ 14 \mid 0\ 1\ 4\ \infty)\ (2\ 5\ 8\ 10 \mid 6\ 11\ 12\ 13)]$ mod 15	240	322560	480
							Ac	$[(0\ 3\ 4\ 14 \mid 1\ 7\ 9\ \infty)\ (2\ 5\ 11\ 12 \mid 6\ 8\ 10\ 13)]$ mod 15	15	322560	15
							Bd	$[(1\ 4\ 7\ 9 \mid 0\ 3\ 14\ \infty)\ (12\ 8\ 6\ 1 \mid 9\ 4\ 3\ 2)]$ mod 15	30	60	30
38.	16	24	120	15	10	2		$(\infty_1\ 0_1 \mid \infty_2\ 0_2 \mid 1_2\ 1_3 \mid 2_2\ 2_4 \mid 2_3\ 1_4)$ $(\infty_2\ 0_3 \mid \infty_3\ 0_2 \mid 1_1\ 2_4 \mid 2_1\ 2_3 \mid 1_3\ 1_4)$ $(\infty_1\ 0_3 \mid \infty_3\ 0_1 \mid 1_1\ 2_2 \mid 2_1\ 1_4 \mid 1_2\ 2_4)$ $(\infty_2\ 2_4 \mid \infty_3\ 2_3 \mid \infty_4\ 0_2 \mid 0_1\ 2_2 \mid 1_1\ 1_2)$ $(\infty_1\ 0_2 \mid \infty_3\ 2_4 \mid \infty_4\ 0_3 \mid 2_1\ 1_3 \mid 1_2\ 2_3)$ $(\infty_1\ 2_4 \mid \infty_2\ 1_1 \mid \infty_4\ 0_1 \mid 2_1\ 0_3 \mid 2_2\ 1_3)$ $(\infty_4\ 0_4 \mid 1_1\ 2_1 \mid 1_2\ 2_2 \mid 1_3\ 2_3 \mid 1_4\ 2_4)$ mod 3, suffixes fixed, with $(\infty_1\ \infty_2 \mid \infty_3\ \infty_4 \mid 2_1\ 2_4 \mid 2_2\ 1_4 \mid 2_3\ 0_4)$ $(\infty_1\ \infty_3 \mid \infty_2\ \infty_4 \mid 0_1\ 0_4 \mid 0_2\ 2_4 \mid 0_3\ 1_4)$ $(\infty_1\ \infty_4 \mid \infty_2\ \infty_3 \mid 1_1\ 1_4 \mid 1_2\ 0_4 \mid 1_3\ 2_4)$	1	72	u
39.	16	24	48	15	10	5		No NBIBD has been found, but a double exists (No 68).			

<table>
<tr><td>40.</td><td>16</td><td>20</td><td>120</td><td>15</td><td>12</td><td>2</td><td>Aa1</td><td>[[[(1 2 | 4 8 | 6 13 | 7 9 | 0 5 | 10 ∞) (6 7 | 9 13 | 11 3 | 12 14 | 5 10 | 0 ∞)
(11 12 | 14 3 | 1 8 | 2 4 | 10 0 | 5 ∞) (1 4 | 6 9 | 11 14 | 2 8 | 7 13 | 12 3)]]]
PC(5), mod 15</td><td>960</td><td>5760</td><td>u</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>Aa2</td><td>[[[(0 5 | 1 4 | 7 13 | 2 6 | 8 10 | 9 ∞) (5 10 | 6 9 | 12 3 | 7 11 | 13 0 | 14 ∞)
(10 0 | 11 14 | 2 8 | 12 1 | 3 5 | 4 ∞) (1 9 | 6 14 | 11 4 | 8 7 | 13 12 | 3 2)]]]
PC(5), mod 15</td><td>15</td><td>5760</td><td>u</td></tr>
<tr><td>41.</td><td>16</td><td>20</td><td>80</td><td>15</td><td>12</td><td>3</td><td></td><td>[[[(0 1 5 | 2 8 10 | 6 7 9 | 13 4 ∞) (5 6 10 | 7 13 0 | 11 12 14 | 3 9 ∞)
(10 11 0 | 12 3 5 | 1 2 4 | 8 14 ∞) (2 7 12 | 1 4 8 | 6 9 13 | 3 11 14)]]]
PC(5), mod 15</td><td>15</td><td>5760</td><td>15</td></tr>
<tr><td>42.c</td><td>16</td><td>20</td><td>60</td><td>15</td><td>12</td><td>4</td><td></td><td>[[[(1 2 4 8 | 6 7 9 13 | 0 5 10 ∞) (6 7 9 13 | 11 12 14 3 | 5 10 0 ∞)
(11 12 14 3 | 1 2 4 8 | 10 0 5 ∞) (1 2 4 8 | 6 7 9 13 | 11 12 14 3)]]]
PC(5), mod 15</td><td>5760</td><td>5760</td><td>$> 10^6$</td></tr>
<tr><td>43.</td><td>16</td><td>20</td><td>40</td><td>15</td><td>12</td><td>6</td><td></td><td>[[[(0 5 1 4 7 13 | 2 6 8 10 9 ∞) (5 10 6 9 12 3 | 7 11 13 0 14 ∞)
(10 0 11 14 2 8 | 12 1 3 5 4 ∞) (1 9 6 14 11 4 | 8 7 13 12 3 2)]]]
PC(5),mod 15</td><td>60</td><td>5760</td><td>60</td></tr>
<tr><td>44.</td><td>16</td><td>16</td><td>80</td><td>15</td><td>15</td><td>3</td><td>Aa</td><td>(1 5 8 | 2 10 12 | 3 4 7 | 6 11 13 | 9 14 15) mod 16</td><td>16</td><td>u</td><td>16</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>Ab</td><td>(0001 0110 0111 | 0010 1100 1110 |
0100 1011 1111 | 1000 0101 1101 | 0011 1010 1001)
mod (2,2,2,2)</td><td>5760</td><td>u</td><td>5760</td></tr>
<tr><td>45.</td><td>16</td><td>16</td><td>48</td><td>15</td><td>15</td><td>5</td><td>Aa</td><td>(3 14 10 2 1 | 12 5 8 6 11 | 9 15 4 7 13) mod 16</td><td>16</td><td>u</td><td>16</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>Ab</td><td>(10 13 3 4 8 | 9 7 1 2 5 | 6 12 11 14 15) mod 16</td><td>16</td><td>u</td><td>16</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>Ac</td><td>(0001 1000 1100 1010 1111 |
0010 0011 1011 0111 1101 | 0100 0110 0101 1110 1001)
mod (2,2,2,2)</td><td>960</td><td>u</td><td>960</td></tr>
</table>

Table 1. (*continued*)

	v	b_1	b_2	r	k_1	k_2		Blocks	$\lvert\mathscr{A}\rvert$	$\lvert\mathscr{A}_1\rvert$	$\lvert\mathscr{A}_2\rvert$
46.	10	45	90	18	4	2	Aa	(1 2 | 3 5) (1 4 | 3 7) (0 2 | 4 5) (0 3 | 1 6) (1 7 | 2 6) mod 10, last block PC(5)	10	10	$2u$
							Ba	(1 3 | 2 5) (1 4 | 3 7) (0 8 | 3 4) (0 5 | 7 8) (1 7 | 2 6) mod 10, last block PC(5)	10	10	$2u$
							Ca1	(1 4 | 2 ∞) (5 8 | 1 ∞) (2 4 | 1 8) (1 6 | 2 3) (1 6 | 2 3) mod 9	4608	4608	$2u$
							Ca2	(2 4 | 1 ∞) (5 8 | 1 ∞) (1 2 | 4 8) (1 2 | 3 6) (1 3 | 2 6) mod 9	9	4608	$2u$
							Da1	(1 4 | 2 ∞) (5 8 | 1 ∞) (2 4 | 1 8) (1 6 | 2 3) (3 8 | 6 7) mod 9	9	9	$2u$
							Da2	(2 4 | 1 ∞) (5 8 | 1 ∞) (1 2 | 4 8) (1 2 | 3 6) (3 7 | 6 8) mod 9	9	9	$2u$
47.	10	30	60	18	6	3	Aa	(1 2 4 | 5 6 9)(1 2 7 | 3 5 8)(1 2 4 | 3 5 9) mod 10	10	10	10240
							Ab	(1 6 9 | 2 4 5)(1 5 8 | 2 3 7)(1 5 9 | 2 3 4) mod 10	10	10	10
							Bb	(1 6 9 | 2 4 5)(1 5 8 | 2 3 7)(1 5 7 | 2 3 4) mod 10	10	10	10
							Bc	(1 2 4 | 5 6 9)(1 2 7 | 3 5 8)(1 3 7 | 2 4 5) mod 10	10	10	10
							Cd1	(1 3 5 | 2 6 ∞)(1 2 3 | 5 7 ∞)(1 2 5 | 3 4 7) (1 4 7 | 2 5 8) mod 9, last block PC(3)	9	9	36864
							Cd2	(1 2 6 | 3 5 ∞)(1 5 7 | 2 3 ∞)(1 5 7 | 2 3 4) (1 4 7 | 2 5 8) mod 9, last block PC(3)	9	9	36864
							Ce1	(1 2 6 | 3 5 ∞)(1 2 7 | 3 5 ∞)(1 5 7 | 2 3 4)	9	9	36864
							Ce2	(1 5 6 | 2 3 ∞)(1 5 7 | 2 3 ∞) (1 2 4 | 3 5 7) (1 4 7 | 2 5 8) mod 9, last block PC(3) (1 4 7 | 2 5 8) mod 9, last block PC(3)	9	9	36864

<table>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>Cf</td><td>(1 5 6 | 2 3 ∞)(1 2 7 | 3 5 ∞)(1 2 4 | 3 5 7)
(1 4 7 | 2 5 8) mod 9, last block PC(3)</td><td>9</td><td>9</td><td>72</td></tr>
<tr><td>48.</td><td>13</td><td>26</td><td>78</td><td>18</td><td>9</td><td>3</td><td>Aa</td><td>(2 6 5 | 4 12 10 | 8 11 7)(2 4 8 | 6 12 11 | 5 10 7) mod 13</td><td>39</td><td>$> 10^6$</td><td>39</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>Bb</td><td>(2 6 5 | 4 12 10 | 8 11 7)(1 3 9 | 4 12 10 | 8 11 7) mod 13</td><td>39</td><td>39</td><td>$> 10^6$</td></tr>
<tr><td>49.</td><td>11</td><td>55</td><td>110</td><td>20</td><td>4</td><td>2</td><td>Aa</td><td>(10 9 | 1 2)(9 7 | 2 4)(7 3 | 4 8)(3 6 | 8 5)(6 1 | 5 10) mod 11</td><td>110</td><td>110</td><td>$2u$</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>Ba1</td><td>(8 7 | 10 6)(10 6 | 7 2)(7 2 | 6 8)(6 8 | 2 10)(2 10 | 8 7) mod 11</td><td>55</td><td>660</td><td>$2u$</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>Ba2</td><td>(8 7 | 10 6)(10 7 | 6 2)(6 7 | 2 8)(6 8 | 2 10)(2 7 | 8 10) mod 11</td><td>11</td><td>660</td><td>$2u$</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>Ca</td><td>(∞ 1 | 0 3)(∞ 0 | 2 4)(0 1 | 3 4)(0 7 | 1 3)(0 6 | 1 5)
(0 5 | 2 7) mod 10, last block PC(5)</td><td>320</td><td>320</td><td>$2u$</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>Da</td><td>(∞ 1 | 0 3)(∞ 0 | 2 6)(0 1 | 3 4)(0 2 | 1 5)(0 8 | 2 5)
(0 5 | 1 6) mod 10, last block PC(5)</td><td>10</td><td>10</td><td>$2u$</td></tr>
<tr><td>50.c</td><td>8</td><td>28</td><td>84</td><td>21</td><td>6</td><td>2</td><td>Aa1</td><td>(2 3 | 4 5 | 6 7) (1 3 | 4 6 | 5 7)(4 7 | 1 6 | 2 5)
(1 5 | 2 6 | 3 7) mod 8, last block PC(4)</td><td>32</td><td>u</td><td>$3u$</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>Aa2</td><td>(2 3 | 4 7 | 5 6)(1 3 | 4 6 | 5 7)(1 4 | 6 7 | 2 5)
(1 5 | 2 6 | 3 7) mod 8, last block PC(4)</td><td>16</td><td>u</td><td>$3u$</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>Aa3</td><td>[[[(1 3 | 2 6 | 4 5)(∞ 0 | 4 5 | 2 6)(4 5 | ∞ 0 | 1 3)(2 6 | 1 3 | ∞ 0)]]] mod 7</td><td>1344</td><td>u</td><td>$3u$</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>Aa4</td><td>[[[(1 6 | 2 5 | 4 3)(∞ 0 | 2 5 | 4 3)(∞ 0 | 4 3 | 1 6)(∞ 0 | 1 6 | 2 5)]]] mod 7</td><td>42</td><td>u</td><td>$3u$</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>Aa5</td><td>[[[(1 6 | 2 5 | 4 3)(∞ 2 | 0 5 | 4 3)(∞ 4 | 0 3 | 1 6)(∞ 1 | 0 6 | 2 5)]]] mod 7</td><td>21</td><td>u</td><td>$3u$</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>Aa6</td><td>[[[(1 6 | 2 5 | 4 3)(∞ 2 | 0 4 | 3 5)(∞ 4 | 0 1 | 6 3)(∞ 1 | 0 2 | 5 6)]]] mod 7</td><td>21</td><td>u</td><td>$3u$</td></tr>
<tr><td>51.</td><td>8</td><td>28</td><td>56</td><td>21</td><td>6</td><td>3</td><td>Aa</td><td>(0 1 3 | 2 6 7)(0 5 7 | 2 4 6)(0 1 3 | 2 5 6)
(0 1 6 | 4 5 2) mod8, last block PC(4)</td><td>16</td><td>u</td><td>$> 10^6$</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>Ab1</td><td>[[[(1 2 4 | 3 6 5)(∞ 4 2 | 0 5 6)(4 ∞ 1 | 5 0 3)(2 1 ∞ | 6 3 0)]]] mod 7</td><td>168</td><td>u</td><td>40320</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>Ab2</td><td>[[[(1 2 4 | 6 5 3)(∞ 2 4 | 0 5 3)(∞ 4 1 | 0 3 6)(∞ 1 2 | 0 6 5)]]] mod 7</td><td>21</td><td>u</td><td>40320</td></tr>
</table>

Table 1. (*continued*)

	v	b_1	b_2	r	k_1	k_2		Blocks	$\lvert\mathscr{A}\rvert$	$\lvert\mathscr{A}_1\rvert$	$\lvert\mathscr{A}_2\rvert$
52.	15	35	105	21	9	3		$(0_1\ 4_1\ 5_2 \mid 1_1\ 3_1\ 4_2 \mid 2_1\ 5_1\ 2_2)(0_1\ 1_1\ 3_1 \mid 0_2\ 1_2\ 3_2 \mid 2_2\ 4_2\ 5_2)(0_1\ 2_2\ 3_2 \mid 2_1\ 3_1\ 1_2 \mid 4_1\ 6_2\ \infty)$ $(1_1\ 1_2\ 4_2 \mid 2_1\ 3_1\ 0_2 \mid 6_1\ 6_2\ \infty)(0_1\ 2_1\ 1_2 \mid 1_1\ 3_2\ 5_2 \mid 5_1\ 4_2\ \infty)$ mod 7, suffixes fixed	7	21	7
53.	12	33	132	22	8	2	Aa1	[[(2 10 | 4 9 | 6 8 | 5 ∞)(0 1 | 3 7 | 6 8 | 5 ∞)(0 1 | 3 7 | 2 10 | 4 9)]] mod 11	11	11	$2u$
							Aa2	[[(2 4 | 9 10 | 6 8 | 5 ∞)(0 1 | 3 7 | 5 8 | 6 ∞)(0 3 | 1 7 | 2 9 | 4 10)]] mod 11	11	11	$2u$
							Aa3	[[(2 4 | 9 8 | 6 5 | 10 ∞)(0 3 | 1 5 | 6 8 | 7 ∞)(0 3 | 1 7 | 2 9 | 4 10)]] mod 11	11	11	$2u$
							Ba	(0 1 | 3 8 | 4 7 | 5 6)(0 2 | 3 6 | 5 9 | 4 ∞)(0 4 | 1 6 | 7 9 | 5 ∞) mod 11	11	11	$2u$
54.*c*	12	33	66	22	8	4	Aa	[[(2 10 4 9 | 6 8 5 ∞)(0 1 3 7 | 6 8 5 ∞)(0 1 3 7 | 2 10 4 9)]] mod 11	11	11	$> 10^6$
							Bb	(0 1 2 3 | 4 7 8 10)(0 1 4 7 | 2 3 9 ∞)(0 2 6 8 | 3 7 9 ∞) mod 11	11	11	11
55.	13	39	156	24	8	2	Aa	(1 12 | 5 8 | 2 11 | 3 10)(4 9 | 6 7 | 1 12 | 5 8)(2 11 | 3 10 | 4 9 | 6 7) mod 13	156	156	$2u$
							Ba	(1 5 | 8 10 | 9 6 | 7 12)(3 2 | 11 4 | 1 5 | 8 10)(9 6 | 7 12 | 3 2 | 11 4) mod 13	39	39	$2u$
							Ca	(1 4 | 2 8 | 3 12 | 6 11)(9 10 | 5 7 | 1 4 | 2 8)(3 12 | 6 11 | 9 10 | 5 7) mod 13	39	39	$2u$
							Da	(0 6 | 4 7 | 8 12 | 10 11)(3 8 | 5 6 | 9 11 | 10 12)(0 10 | 2 8 | 4 12 | 5 9) mod 13	13	13	$2u$
							Ea1	(0 6 | 4 8 | 7 10 | 11 ∞)(4 11 | 6 7 | 9 10 | 0 ∞)(0 5 | 3 6 | 8 10 | 7 11) (0 2 | 3 5 | 6 8 | 9 11) mod 12, last block PC(3)	12	12	$2u$
							Ea2	(0 11 | 4 8 | 7 10 | 6 ∞)(4 11 | 6 7 | 0 10 | 9 ∞)(0 6 | 5 10 |8 11 | 3 7) (0 2 | 3 5 | 6 8 | 9 11) mod 12, last block PC(3)	12	12	$2u$
56.	13	39	78	24	8	4	Aa	(1 12 5 8 | 2 11 3 10)(4 9 6 7 | 1 12 5 8)(2 11 3 10 | 4 9 6 7) mod 13	156	156	$> 10^6$
							Bb	(1 5 8 10 | 9 6 7 12)(3 2 11 4 | 1 5 8 10)(9 6 7 12 | 3 2 11 4) mod 13	39	39	$> 10^6$
							Cb	(1 4 2 8 | 3 12 6 11)(9 10 5 7 | 1 4 2 8)(3 12 6 11 | 9 10 5 7) mod 13	39	39	$> 10^6$

							Ad	$(6\ 3\ 8\ 7 \mid \infty\ 9\ 2\ 5)(10\ \infty\ 0\ 7 \mid 5\ 1\ 11\ 3)(9\ 1\ 10\ 2 \mid 7\ 6\ 0\ 5)$ $(1\ 7\ 10\ 4 \mid 2\ 8\ 5\ 11)$ mod 12, last block PC(3)	12	156	96
							Ae	$(8\ 2\ 3\ \infty \mid 9\ 7\ 5\ 6)(11\ 10\ 1\ \infty \mid 0\ 5\ 3\ 7)(9\ 10\ 2\ 6 \mid 5\ 0\ 7\ 1)$ $(1\ 7\ 10\ 4 \mid 2\ 8\ 5\ 11)$ mod 12, last block PC(3)	12	156	96
57.	14	91	182	26	4	2	Aa	$(0\ 1 \mid 9\ 8)(0\ 2 \mid 5\ 3)(0\ 8 \mid 1\ 9)(0\ 3 \mid 2\ 5)(0\ 4 \mid 7\ 11)(\infty\ 0 \mid 2\ 9)(\infty\ 0 \mid 3\ 9)$ mod 13	13	$> 10^6$	$2u$
							Ba	$(1\ 2 \mid 4\ 9)(2\ 4 \mid 8\ 5)(4\ 8 \mid 3\ 10)(8\ 3 \mid 6\ 7)(3\ 6 \mid 12\ 1)(\infty\ 0 \mid 1\ 5)(\infty\ 0 \mid 4\ 10)$ mod 13	13	13	$2u$
							Ca1	$(0_1\ 0_2 \mid 1_1\ 1_2)(0_1\ 2_2 \mid 2_1\ 0_2)(0_1\ 4_1 \mid 0_2\ 4_2)(2_1\ 3_1 \mid 0_2\ 1_2)(6_1\ 0_2 \mid 4_1\ 2_2)$ $(1_1\ 0_2 \mid 5_1\ 4_2)(4_1\ 0_2 \mid 5_1\ 1_2)(1_1\ 3_1 \mid 0_2\ 2_2)(6_1\ 0_2 \mid 2_1\ 4_2)$ $(0_2\ 1_2 \mid 6_1\ 3_2)(2_1\ 3_1 \mid 0_1\ 4_2)(0_2\ 3_2 \mid 2_2\ 4_2)(0_1\ 2_1 \mid 1_1\ 4_1)$ mod 7, suffixes fixed	14	42*	$2u$
							Ca2	$(0_1\ 0_2 \mid 1_1\ 1_2)(0_1\ 2_1 \mid 0_2\ 2_2)(0_1\ 4_1 \mid 0_2\ 4_2)(2_1\ 3_1 \mid 0_2\ 1_2)(4_1\ 0_2 \mid 6_1\ 2_2)$ $(1_1\ 0_2 \mid 5_1\ 4_2)(5_1\ 0_2 \mid 4_1\ 1_2)(1_1\ 2_2 \mid 3_1\ 0_2)(2_1\ 0_2 \mid 6_1\ 4_2)$ $(1_2\ 3_2 \mid 6_1\ 0_2)(0_1\ 3_1 \mid 2_1\ 4_2)(2_2\ 3_2 \mid 0_2\ 4_2)(0_1\ 1_1 \mid 2_1\ 4_1)$ mod 7, suffixes fixed	7	42*	$2u$
							Ca3	$(0_1\ 0_2 \mid 1_1\ 1_2)(0_1\ 2_1 \mid 0_2\ 2_2)(0_1\ 4_1 \mid 0_2\ 4_2)(2_1\ 0_2 \mid 3_1\ 1_2)(4_1\ 0_2 \mid 6_1\ 2_2)$ $(1_1\ 0_2 \mid 5_1\ 4_2)(4_1\ 5_1 \mid 0_2\ 1_2)(1_1\ 2_2 \mid 3_1\ 0_2)(6_1\ 0_2 \mid 2_1\ 4_2)$ $(0_2\ 1_2 \mid 6_1\ 3_2)(0_1\ 3_1 \mid 2_1\ 4_2)(0_2\ 3_2 \mid 2_2\ 4_2)(0_1\ 1_1 \mid 2_1\ 4_1)$ mod 7, suffixes fixed	7	42*	$2u$
58.	10	45	90	27	6	3	Aa	$(2\ 3\ 4 \mid 1\ 6\ \infty)(1\ 2\ 5 \mid 3\ 6\ \infty)(1\ 3\ 5 \mid 2\ 8\ \infty)(1\ 3\ 5 \mid 2\ 8\ 7)(1\ 2\ 7 \mid 3\ 4\ 6)$ mod 9	9	9	$> 10^6$
							Ab	$(1\ 3\ \infty \mid 6\ 4\ 2)(1\ \infty\ 3 \mid 6\ 2\ 5)(5\ 2\ 8 \mid 1\ \infty\ 3)(7\ 3\ 8 \mid 1\ 5\ 2)(4\ 3\ 2 \mid 6\ 7\ 1)$ mod 9	9	9	$> 10^6$
							Ac	$(6\ 4\ 3 \mid \infty\ 2\ 1)(\infty\ 5\ 3 \mid 1\ 6\ 2)(5\ 8\ \infty \mid 3\ 2\ 1)(3\ 8\ 2 \mid 7\ 1\ 5)(6\ 1\ 3 \mid 4\ 2\ 7)$ mod 9	9	9	$> 10^6$
59.	15	105	210	28	4	2	Aa1	$(1\ 6 \mid 3\ 10)(1\ 13 \mid 2\ 8)(1\ 2 \mid 6\ 8)(1\ 2 \mid 7\ 14)(1\ 4 \mid 2\ 8)(1\ 12 \mid 2\ 4)(1\ 5 \mid 2\ 7)$ mod 15	15	15	$2u$
							Aa2	$(1\ 3 \mid 6\ 10)(1\ 8 \mid 2\ 13)(1\ 6 \mid 2\ 8)(1\ 2 \mid 7\ 14)(1\ 4 \mid 2\ 8)(1\ 4 \mid 2\ 12)(1\ 2 \mid 5\ 7)$ mod 15	15	15	$2u$
							Ba	$(1\ 3 \mid 6\ 10)(1\ 8 \mid 2\ 13)(1\ 11 \mid 3\ 9)(1\ 2 \mid 7\ 14)(1\ 4 \mid 2\ 8)(1\ 6 \mid 2\ 5)(1\ 2 \mid 5\ 7)$ mod 15	15	15	$2u$
							Ca	$(0\ 1 \mid 2\ 6)(0\ 2 \mid 5\ 3)(0\ 3 \mid 7\ 14)(0\ 4 \mid 9\ 10)(0\ 5 \mid 12\ 7)(0\ 6 \mid 4\ 13)(0\ 7 \mid 1\ 4)$ mod 15	15	15	$2u$

Table 1. (*continued*)

	v	b_1	b_2	r	k_1	k_2		Blocks	$\lvert\mathscr{A}\rvert$	$\lvert\mathscr{A}_1\rvert$	$\lvert\mathscr{A}_2\rvert$
							Da	$(0\ 1 \mid 2\ 10)(0\ 2 \mid 5\ 6)(0\ 3 \mid 7\ 1)(0\ 4 \mid 9\ 11)(0\ 5 \mid 12\ 2)(0\ 6 \mid 4\ 7)(0\ 7 \mid 1\ 12)$ mod 15	15	15	$2u$
							Ea	$(0\ 1 \mid 6\ 10)(0\ 2 \mid 3\ 6)(0\ 3 \mid 14\ 1)(0\ 4 \mid 10\ 11)(0\ 5 \mid 7\ 2)(0\ 6 \mid 13\ 7)(0\ 7 \mid 4\ 12)$ mod 15	15	15	$2u$
							Fa	$(0\ 2 \mid 6\ 10)(0\ 5 \mid 3\ 6)(0\ 7 \mid 14\ 1)(0\ 9 \mid 10\ 11)(0\ 12 \mid 7\ 2)(0\ 4 \mid 13\ 7)(0\ 1 \mid 4\ 12)$ mod 15	15	15	$2u$
							Ga	$(1\ 2 \mid 6\ 10)(2\ 5 \mid 3\ 6)(3\ 7 \mid 14\ 1)(4\ 9 \mid 10\ 11)(5\ 12 \mid 7\ 2)(6\ 4 \mid 13\ 7)(7\ 1 \mid 4\ 12)$ mod 15	15	15	$2u$
							Ha1	$(0\ 3 \mid 1\ \infty)(3\ 9 \mid 0\ \infty)(0\ 3 \mid 1\ 7)(0\ 1 \mid 5\ 7)(0\ 4 \mid 3\ 8)(0\ 1 \mid 4\ 6)(0\ 4 \mid 1\ 6)(0\ 7 \mid 2\ 9)$ mod 14, last block PC(7)	14	229376	$2u$
							Ha2	$(0\ 3 \mid 1\ \infty)(0\ 9 \mid 3\ \infty)(0\ 3 \mid 1\ 7)(0\ 7 \mid 1\ 5)(0\ 8 \mid 3\ 4)(0\ 4 \mid 1\ 6)(0\ 1 \mid 4\ 6)(0\ 2 \mid 7\ 9)$ mod 14, last block PC(7)	14	229376	$2u$
							Ia1	$(0\ 3 \mid 1\ \infty)(3\ 9 \mid 0\ \infty)(0\ 3 \mid 1\ 7)(0\ 1 \mid 5\ 7)(0\ 4 \mid 3\ 8)(0\ 1 \mid 4\ 6)(0\ 10 \mid 13\ 8)(0\ 7 \mid 2\ 9)$ mod 14, last block PC(7)	14	14	$2u$
							Ia2	$(0\ 3 \mid 1\ \infty)(0\ 9 \mid 3\ \infty)(0\ 3 \mid 1\ 7)(0\ 7 \mid 1\ 5)(0\ 8 \mid 3\ 4)(0\ 4 \mid 1\ 6)(0\ 13 \mid 10\ 8)(0\ 2 \mid 7\ 9)$ mod 14, last block PC(7)	14	14	$2u$
							Ja	$[[[[(1_2\ 0_2 \mid 6_2\ 5_2)(3_2\ 3_1 \mid 4_2\ \infty)(0_1\ 5_1 \mid 2_1\ 4_1)(5_2\ 0_2 \mid 4_1\ 1_1)(6_1\ 2_2 \mid 1_2\ 0_1)(6_2\ \infty \mid 3_1\ 3_2)$ $(1_1\ \infty \mid 6_1\ 4_2)(2_1\ 5_2 \mid 0_2\ 3_2)(6_2\ 5_1 \mid 2_2\ 0_1)(0_1\ 1_1 \mid 0_2\ 5_1)(2_1\ \infty \mid 4_2\ 6_1)(1_2\ 4_1 \mid 2_2\ 3_1)$ $(5_2\ 3_2 \mid 3_1\ 6_1)(1_2\ 4_1 \mid 4_2\ 5_1)(2_2\ 6_2 \mid 1_1\ 2_1)]]]]$ mod 7, suffixes fixed	7	7	$2u$
							Ka	$[[[[(6_2\ 3_1 \mid 5_2\ 1_1)(4_2\ 6_1 \mid 0_2\ 0_1)(3_2\ 2_1 \mid 1_2\ \infty)(3_2\ 4_1 \mid 0_2\ 2_1)(1_2\ 6_1 \mid 6_2\ 0_1)(2_2\ 5_1 \mid 5_2\ \infty)$ $(3_2\ 0_1 \mid 4_2\ 4_1)(2_2\ 1_1 \mid 0_2\ 5_1)(6_2\ 5_2 \mid 3_1\ \infty)(2_2\ 0_2 \mid 0_1\ 4_1)(1_2\ 4_2 \mid 6_1\ 5_1)(1_1\ 3_1 \mid 2_1\ \infty)$ $(3_2\ 2_2 \mid 5_1\ 2_1)(1_2\ 5_2 \mid 6_2\ 4_2)(3_1\ 4_1 \mid 1_1\ 6_1)]]]]$ mod 7, suffixes fixed	7	7	$2u$
							La	$[[[[(4_1\ \infty \mid 6_2\ 5_2)(3_2\ 0_2 \mid 5_1\ 1_2)(4_2\ 2_1 \mid 0_1\ 6_1)(5_1\ 2_2 \mid 6_2\ 3_1)(1_1\ 0_1 \mid 3_2\ \infty)(1_2\ 4_1 \mid 5_2\ 0_2)$ $(2_2\ 1_1 \mid 3_2\ 3_1)(6_1\ \infty \mid 5_2\ 6_2)(2_1\ 4_2 \mid 0_1\ 0_2)(0_1\ 3_1 \mid 4_1\ 6_1)(4_2\ 2_2 \mid 1_1\ 0_2)(5_1\ 2_1 \mid \infty\ 1_2)$ $(5_2\ 2_2 \mid 3_2\ 5_1)(4_2\ 3_1 \mid 4_1\ 6_1)(1_2\ 2_1 \mid 6_2\ 1_1)]]]]$ mod 7, suffixes fixed	7	7	$2u$
60.	15	42	210	28	10	2		$(0\ 2 \mid 3\ 11 \mid 4\ 13 \mid 5\ 12 \mid 6\ 9)(0\ 1 \mid 2\ 5 \mid 3\ 13 \mid 4\ 10 \mid 9\ \infty)(0\ 1 \mid 3\ 5 \mid 7\ 11 \mid 8\ 13 \mid 10\ \infty)$ mod 14	14	14	$2u$

61.*c*	15	42	84	28	10	5	Aa	$(0\ 7\ 8\ 9\ 11 \mid 2\ 3\ 4\ 5\ 10)(0\ 5\ 6\ 9\ 10 \mid 1\ 4\ 7\ 8\ \infty)(0\ 2\ 5\ 7\ 11 \mid 4\ 8\ 10\ 12\ \infty)$ mod 14	14	14	14
							Bb	$(0\ 7\ 6\ 5\ 3 \mid 12\ 11\ 10\ 9\ 4)(0\ 5\ 6\ 9\ 10 \mid 1\ 4\ 7\ 8\ \infty)(0\ 2\ 5\ 7\ 11 \mid 4\ 8\ 10\ 12\ \infty)$ mod 14	14	14	14
62.	15	35	210	28	12	2	Aa1	$[[[[(1_2\ 0_2 \mid 6_2\ 5_2 \mid 3_2\ 3_1 \mid 4_2\ \infty \mid 0_1\ 5_1 \mid 2_1\ 4_1)(5_2\ 0_2 \mid 4_1\ 1_1 \mid 6_1\ 2_2 \mid 1_2\ 0_1 \mid 6_2\ \infty \mid 3_1\ 3_2)$ $(1_1\ \infty \mid 6_1\ 4_2 \mid 2_1\ 5_2 \mid 0_2\ 3_2 \mid 6_2\ 5_1 \mid 2_2\ 0_1)(0_1\ 1_1 \mid 0_2\ 5_1 \mid 2_1\ \infty \mid 4_2\ 6_1 \mid 1_2\ 4_1 \mid 2_2\ 3_1)$ $(5_2\ 3_2 \mid 3_1\ 6_1 \mid 1_2\ 4_1 \mid 4_2\ 5_1 \mid 2_2\ 6_2 \mid 1_1\ 2_1)]]]]$ mod 7, suffixes fixed	7	20160	$2u$
							Aa2	$[[[[(6_2\ 3_1 \mid 5_2\ 1_1 \mid 4_2\ 6_1 \mid 0_2\ 0_1 \mid 3_2\ 2_1 \mid 1_2\ \infty)(3_2\ 4_1 \mid 0_2\ 2_1 \mid 1_2\ 6_1 \mid 6_2\ 0_1 \mid 2_2\ 5_1 \mid 5_2\ \infty)$ $(3_2\ 0_1 \mid 4_2\ 4_1 \mid 2_2\ 1_1 \mid 0_2\ 5_1 \mid 6_2\ 5_2 \mid 3_1\ \infty)(2_2\ 0_2 \mid 0_1\ 4_1 \mid 1_2\ 4_2 \mid 6_1\ 5_1 \mid 1_1\ 3_1 \mid 2_1\ \infty)$ $(3_2\ 2_2 \mid 5_1\ 2_1 \mid 1_2\ 5_2 \mid 6_2\ 4_2 \mid 3_1\ 4_1 \mid 1_1\ 6_1)]]]]$ mod 7, suffixes fixed	7	20160	$2u$
							Ba	$[[[[(4_1\ \infty \mid 6_2\ 5_2 \mid 3_2\ 0_2 \mid 5_1\ 1_2 \mid 4_2\ 2_1 \mid 0_1\ 6_1)(5_1\ 2_2 \mid 6_2\ 3_1 \mid 1_1\ 0_1 \mid 3_2\ \infty \mid 1_2\ 4_1 \mid 5_2\ 0_2)$ $(2_2\ 1_1 \mid 3_2\ 3_1 \mid 6_1\ \infty \mid 5_2\ 6_2 \mid 2_1\ 4_2 \mid 0_1\ 0_2)(0_1\ 3_1 \mid 4_1\ 6_1 \mid 4_2\ 2_2 \mid 1_1\ 0_2 \mid 5_1\ 2_1 \mid \infty\ 1_2)$ $(5_2\ 2_2 \mid 3_2\ 5_1 \mid 4_2\ 3_1 \mid 4_1\ 6_1 \mid 1_2\ 2_1 \mid 6_2\ 1_1)]]]]$ mod 7, suffixes fixed	7	21	$2u$
63.*c*	15	35	140	28	12	3	Aa	$(0\ 3\ 14 \mid 7\ 8\ 11 \mid 9\ 10\ 13 \mid 4\ 6\ 12)(0\ 2\ 8 \mid 4\ 11\ 13 \mid 5\ 12\ 14 \mid 6\ 7\ 10)$ $(1\ 6\ 11 \mid 2\ 7\ 12 \mid 3\ 8\ 13 \mid 4\ 9\ 14)$ mod 15, last block PC(5)	15	20160	$> 10^6$
							Ab1	$[[[[(0_1\ 0_2 \infty \mid 3_2\ 5_2\ 6_2 \mid 4_1\ 3_1\ 1_2 \mid 2_1\ 5_1\ 4_2)(1_1\ 6_1\ 2_2 \mid 0_1\ 0_2\ \infty \mid 3_2\ 5_2\ 6_2 \mid 4_1\ 3_1\ 1_2)$ $(2_1\ 5_1\ 4_2 \mid 1_1\ 6_1\ 2_2 \mid 0_1\ 0_2\ \infty \mid 3_2\ 5_2\ 6_2)(4_1\ 3_1\ 1_2 \mid 2_1\ 5_1\ 4_2 \mid 1_1\ 6_1\ 2_2 \mid 0_1\ 0_2\ \infty)$ $(3_2\ 5_2\ 6_2 \mid 4_1\ 3_1\ 1_2 \mid 2_1\ 5_1\ 4_2 \mid 1_1\ 6_1\ 2_2)]]]]$ mod 7, suffixes fixed	168	20160	$> 10^6$
							Ab2	$[[[[(0_1\ 0_2\ \infty \mid 3_2\ 5_2\ 6_2 \mid 2_1\ 6_1\ 1_2 \mid 1_1\ 3_1\ 4_2)(4_1\ 5_1\ 2_2 \mid 0_1\ 0_2\ \infty \mid 3_2\ 5_2\ 6_2 \mid 2_1\ 6_1\ 1_2)$ $(1_1\ 3_1\ 4_2 \mid 4_1\ 5_1\ 2_2 \mid 0_1\ 0_2\ \infty \mid 3_2\ 5_2\ 6_2)(2_1\ 6_1\ 1_2 \mid 1_1\ 3_1\ 4_2 \mid 4_1\ 5_1\ 2_2 \mid 0_1\ 0_2\ \infty)$ $(3_2\ 5_2\ 6_2 \mid 2_1\ 6_1\ 1_2 \mid 1_1\ 3_1\ 4_2 \mid 4_1\ 5_1\ 2_2)]]]]$ mod 7, suffixes fixed	168	20160	$> 10^6$
							Bc	$[[[[(0_1\ 0_2\ \infty \mid 3_2\ 5_2\ 6_2 \mid 4_1\ 5_1\ 1_2 \mid 2_1\ 6_1\ 4_2)(1_1\ 3_1\ 2_2 \mid 0_1\ 0_2\ \infty \mid 3_2\ 5_2\ 6_2 \mid 4_1\ 5_1\ 1_2)$ $(2_1\ 6_1\ 4_2 \mid 1_1\ 3_1\ 2_2 \mid 0_1\ 0_2\ \infty \mid 3_2\ 5_2\ 6_2)(4_1\ 5_1\ 1_2 \mid 2_1\ 6_1\ 4_2 \mid 1_1\ 3_1\ 2_2 \mid 0_1\ 0_2\ \infty)$ $(3_2\ 5_2\ 6_2 \mid 4_1\ 5_1\ 1_2 \mid 2_1\ 6_1\ 4_2 \mid 1_1\ 3_1\ 2_2)]]]]$ mod 7, suffixes fixed	21	21	$> 10^6$
64.	15	35	105	28	12	4	Aa	$[[[[(1_2\ 0_2\ 6_2\ 5_2 \mid 3_2\ 3_1\ 4_2\ \infty \mid 0_1\ 5_1\ 2_1\ 4_1)(5_2\ 0_2\ 4_1\ 1_1 \mid 6_1\ 2_2\ 1_2\ 0_1 \mid 6_2\ \infty\ 3_1\ 3_2)$ $(1_1\ \infty\ 6_1\ 4_2 \mid 2_1\ 5_2\ 0_2\ 3_2 \mid 6_2\ 5_1\ 2_2\ 0_1)(0_1\ 1_1\ 0_2\ 5_1 \mid 2_1\ \infty\ 4_2\ 6_1 \mid 1_2\ 4_1\ 2_2\ 3_1)$ $(5_2\ 3_2\ 3_1\ 6_1 \mid 1_2\ 4_1\ 4_2\ 5_1 \mid 2_2\ 6_2\ 1_1\ 2_1)]]]]$ mod 7, suffixes fixed	7	20160	7

Table 1. (*continued*)

	v	b_1	b_2	r	k_1	k_2		Blocks	$\vert\mathscr{A}\vert$	$\vert\mathscr{A}_1\vert$	$\vert\mathscr{A}_2\vert$
							Ab	$[[[[(6_2\ 3_1\ 5_2\ 1_1 \vert 4_2\ 6_1\ 0_2\ 0_1 \vert 3_2\ 2_1\ 1_2\ \infty)(3_2\ 4_1\ 0_2\ 2_1 \vert 1_2\ 6_1\ 6_2\ 0_1 \vert 2_2\ 5_1\ 5_2\ \infty)$ $(3_2\ 0_1\ 4_2\ 4_1 \vert 2_2\ 1_1\ 0_2\ 5_1 \vert 6_2\ 5_2\ 3_1\ \infty)(2_2\ 0_2\ 0_1\ 4_1 \vert 1_2\ 4_2\ 6_1\ 5_1 \vert 1_1\ 3_1\ 2_1\ \infty)$ $(3_2\ 2_2\ 5_1\ 2_1 \vert 1_2\ 5_2\ 6_2\ 4_2 \vert 3_1\ 4_1\ 1_1\ 6_1)]]]]$ mod 7, suffixes fixed	7	20160	7
							Bc	$[[[[(4_1\ \infty\ 6_2\ 5_2 \vert 3_2\ 0_2\ 5_1\ 1_2 \vert 4_2\ 2_1\ 0_1\ 6_1)(5_1\ 2_2\ 6_2\ 3_1 \vert 1_1\ 0_1\ 3_2\ \infty \vert 1_2\ 4_1\ 5_2\ 0_2)$ $(2_2\ 1_1\ 3_2\ 3_1 \vert 6_1\ \infty\ 5_2\ 6_2 \vert 2_1\ 4_2\ 0_1\ 0_2)(0_1\ 3_1\ 4_1\ 6_1 \vert 4_2\ 2_2\ 1_1\ 0_2 \vert 5_1\ 2_1\ \infty\ 1_2)$ $(5_2\ 2_2\ 3_2\ 5_1 \vert 4_2\ 3_1\ 4_1\ 6_1 \vert 1_2\ 2_1\ 6_2 1_1)]]]]$ mod 7, suffixes fixed	7	21	7
65.	15	35	70	28	12	6	Aa	$[[[[(0_1\ 0_2\ \infty\ 2_1\ 5_1\ 4_2 \vert 3_2\ 5_2\ 6_2\ 4_1\ 3_1\ 1_2)(1_1\ 6_1\ 2_2\ 4_1\ 3_1\ 1_2 \vert 0_1\ 0_2\ \infty\ 3_2\ 5_2\ 6_2)$ $(2_1\ 5_1\ 4_2\ 3_2\ 5_2\ 6_2 \vert 1_1\ 6_1\ 2_2\ 0_1\ 0_2\ \infty)(4_1\ 3_1\ 1_2\ 0_1\ 0_2\ \infty \vert 2_1\ 5_1\ 4_2\ 1_1\ 6_1\ 2_2)$ $(3_2\ 5_2\ 6_2\ 1_1\ 6_1\ 2_2 \vert 4_1\ 3_1\ 1_2\ 2_1\ 5_1\ 4_2)]]]]$ mod 7, suffixes fixed	7	20160	168
							Ab	$[[[[(0_1\ 0_2\ \infty\ 1_1\ 3_1\ 4_2 \vert 3_2\ 5_2\ 6_2\ 2_1\ 6_1\ 1_2)(4_1\ 5_1\ 2_2\ 2_1\ 6_1\ 1_2 \vert 0_1\ 0_2\ \infty\ 3_2\ 5_2\ 6_2)$ $(1_1\ 3_1\ 4_2\ 3_2\ 5_2\ 6_2 \vert 4_1\ 5_1\ 2_2\ 0_1\ 0_2\ \infty)(2_1\ 6_1\ 1_2\ 0_1\ 0_2\ \infty \vert 1_1\ 3_1\ 4_2\ 4_1\ 5_1\ 2_2)$ $(3_2\ 5_2\ 6_2\ 4_1\ 5_1\ 2_2 \vert 2_1\ 6_1\ 1_2\ 1_1\ 3_1\ 4_2)]]]]$ mod 7, suffixes fixed	7	20160	168
							Bc	$[[[[(0_1\ 0_2\ \infty\ 2_1\ 6_1\ 4_2 \vert 3_2\ 5_2\ 6_2\ 4_1\ 5_1\ 1_2)(1_1\ 3_1\ 2_2\ 4_1\ 5_1\ 1_2 \vert 0_1\ 0_2\ \infty\ 3_2\ 5_2\ 6_2)$ $(2_1\ 6_1\ 4_2\ 3_2\ 5_2\ 6_2 \vert 1_1\ 3_1\ 2_2\ 0_1\ 0_2\ \infty)(4_1\ 5_1\ 1_2\ 0_1\ 0_2\ \infty \vert 2_1\ 6_1\ 4_2\ 1_1\ 3_1\ 2_2)$ $(3_2\ 5_2\ 6_2\ 1_1\ 3_1\ 2_2 \vert 4_1\ 5_1\ 1_2\ 2_1\ 6_1\ 4_2)]]]]$ mod 7, suffixes fixed	7	21	21
66.	11	55	165	30	6	2	Aa	$(0\ 2 \vert 8\ 7 \vert 10\ 6)(0\ 8 \vert 10\ 6 \vert 7\ 2)(0\ 10 \vert 7\ 2 \vert 6\ 8)(0\ 7 \vert 6\ 8 \vert 2\ 10)(0\ 6 \vert 2\ 10 \vert 8\ 7)$ mod 11	55	$> 10^6$	$3u$
							Ba	$(0\ 9 \vert 3\ 4 \vert 1\ 5)(0\ 8 \vert 10\ 6 \vert 7\ 2)(0\ 10 \vert 7\ 2 \vert 6\ 8)(0\ 7 \vert 6\ 8 \vert 2\ 10)(0\ 6 \vert 2\ 10 \vert 8\ 7)$ mod 11	11	$> 10^6$	$3u$
							Ca	$(0\ 9 \vert 3\ 4 \vert 1\ 5)(0\ 3 \vert 1\ 5 \vert 4\ 9)(0\ 10 \vert 7\ 2 \vert 6\ 8)(0\ 7 \vert 6\ 8 \vert 2\ 10)(0\ 6 \vert 2\ 10 \vert 8\ 7)$ mod 11	11	$> 10^6$	$3u$
							Da1	$(4\ 5 \vert 8\ 9 \vert 6\ \infty)(2\ 6 \vert 4\ 8 \vert 9\ \infty)(2\ 8 \vert 4\ 5 \vert 9\ \infty)(4\ 7 \vert 5\ 8 \vert 6\ 9)(1\ 9 \vert 3\ 5 \vert 6\ 8)(2\ 7 \vert 3\ 8 \vert 4\ 9)$ mod 10, last block PC(5)	10	10	$3u$
							Da2	$(4\ 5 \vert 8\ 9 \vert 6\ \infty)(2\ 6 \vert 4\ 8 \vert 9\ \infty)(2\ 4 \vert 5\ 8 \vert 9\ \infty)(4\ 8 \vert 5\ 6 \vert 7\ 9)(1\ 3 \vert 5\ 8 \vert 6\ 9)(2\ 7 \vert 3\ 8 \vert 4\ 9)$ mod 10, last block PC(5)	10	10	$3u$
							Da3	$(4\ 5 \vert 6\ 8 \vert 9\ \infty)(2\ 9 \vert 4\ 8 \vert 6\ \infty)(2\ 8 \vert 4\ 5 \vert 9\ \infty)(4\ 6 \vert 7\ 9 \vert 5\ 8)(1\ 5 \vert 3\ 6 \vert 8\ 9)(2\ 7 \vert 3\ 8 \vert 4\ 9)$ mod 10, last block PC(5)	10	10	$3u$

67.	11	55	110	30	6	3	Aa	(0 2 8 | 7 10 6)(0 6 2 | 10 8 7)(0 7 6 | 8 2 10)(0 10 7 | 2 6 8)(0 8 10 | 6 7 2) mod 11	55	$> 10^6$	110
							Ba	(10 4 6 | 1 7 5)(4 6 9 | 7 5 2)(6 9 8 | 5 2 3)(9 8 1 | 2 3 10)(8 1 7 | 3 10 4) mod 11	110	110	110
							Cb	(0 9 3 | 4 1 5)(0 6 2 | 10 8 7)(0 7 6 | 8 2 10)(0 10 7 | 2 6 8)(0 8 10 | 6 7 2) mod 11	11	$> 10^6$	$> 10^6$
							Dc	(0 9 3 | 4 1 5)(0 5 9 | 1 3 4)(0 7 6 | 8 2 10)(0 10 7 | 2 6 8)(0 8 10 | 6 7 2) mod 11	11	$> 10^6$	$> 10^6$
							Ed	(4 5 8 | 6 9 ∞)(2 8 9 | 4 6 ∞)(4 5 9 | 2 8 ∞)(4 7 9 | 5 6 8)(1 5 9 | 3 6 8)(2 3 4 | 7 8 9) mod 10, last block PC(5)	10	10	$> 10^6$
							Ee	(4 5 8 | 6 9 ∞)(2 8 9 | 4 6 ∞)(4 5 9 | 2 8 ∞)(4 6 9 | 5 7 8)(1 5 9 | 3 6 8)(2 3 4 | 7 8 9) mod 10, last block PC(5)	10	10	10240
							Ff	(4 5 8 | 6 9 ∞)(8 2 1 | 6 4 ∞)(4 5 9 | 2 8 ∞)(4 6 9 | 5 7 8)(1 5 9 | 3 6 8)(2 3 4 | 7 8 9) mod 10, last block PC(5)	10	10	10
68.	16	48	96	30	10	5		(1101 0011 0111 0010 1011 |0110 1110 0100 0101 1001) (0110 1110 0100 0101 1001 | 1111 1000 1010 0001 1100) (1111 1000 1010 0001 1100 | 0011 0111 0010 1011 1101) mod $(2, 2, 2, 2)$	960	960	$> 10^6$

7.1. A recursive construction

Let $\mathscr{M}_1$ be an MNBIBD $(\bar{v}, \bar{b}_1, \bar{b}_2, \ldots, \bar{b}_s, \bar{r}, \bar{k}_1, \bar{k}_2, \ldots, \bar{k}_s)$ with $s \geqslant 1$ component designs (if $s = 1$ then $\mathscr{M}_1$ is a BIBD; if $s = 2$ then an NBIBD; and if $s > 2$ then an MNBIBD). Let $\mathscr{M}_2$ be an MNBIBD $(\hat{v}, \hat{b}_1, \hat{b}_2, \ldots, \hat{b}_t, \hat{r}, \hat{k}_1, \hat{k}_2, \ldots, \hat{k}_t)$ with $t \geqslant 2$ component designs, and with $\hat{k}_1/\hat{k}_q = \bar{v}$ for some $2 \leqslant q \leqslant t$.

From $\mathscr{M}_1$ and $\mathscr{M}_2$, a new design is constructed as follows. Select one block of size $\hat{k}_1$ from $\mathscr{M}_2$ and label its sub-blocks of size $\hat{k}_q$ with the symbols $1, 2, \ldots, \bar{v}$, which are the treatment symbols of $\mathscr{M}_1$. Now replace each symbol in $\mathscr{M}_1$ by the correspondingly labeled sub-block of the selected block from $\mathscr{M}_2$. Each large block of the so modified $\mathscr{M}_1$ is now of size $k_1 = \bar{k}_1\hat{k}_q$ and contains successively nested blocks of sizes $k_2, k_3, \ldots, k_{s+t-q+1}$ where

$$k_2 = \bar{k}_2\hat{k}_q, \ldots, k_s = \bar{k}_s\hat{k}_q$$

and

$$k_{s+1} = \hat{k}_q,\ k_{s+2} = \hat{k}_{q+1}, \ldots, k_{s+t-q+1} = \hat{k}_t.$$

Repeat this process $\hat{b}_1$ times, using a new copy of $\mathscr{M}_1$ for each of the $\hat{b}_1$ blocks of $\mathscr{M}_2$. The resulting design $\mathscr{M}$ is an MNBIBD $(v, b_1, b_2, \ldots, b_{s+t-q+1}, r, k_1, k_2, \ldots, k_{s+t-q+1})$ with $v = \hat{v}$, $r = \bar{r}\hat{r}$, block sizes k_j $(j = 1, \ldots, s+t-q+1)$ as specified above, and numbers of blocks

$$b_1 = \bar{b}_1\hat{b}_1,\ b_2 = \bar{b}_2\hat{b}_1, \ldots, b_s = \bar{b}_s\hat{b}_1$$

and

$$b_{s+1} = \bar{k}_s\bar{b}_s\hat{b}_1,\ b_{s+2} = \bar{k}_s\bar{b}_s\hat{b}_1\hat{k}_q/\hat{k}_{q+1}, \ldots, b_{s+t-q+1} = \bar{k}_s\bar{b}_s\hat{b}_1\hat{k}_q/\hat{k}_t.$$

To see that $\mathscr{M}$ is indeed an MNBIBD, we must show that each of its $s + t - q + 1$ component designs is a BIBD. Let the concurrence parameters of the designs $\mathscr{M}_1$, $\mathscr{M}_2$, and $\mathscr{M}$ be respectively $\bar{\lambda}_j$ for $j=1, \ldots, s$; $\hat{\lambda}_j$ for $j=1, \ldots, t$; and λ_j for $j=1, \ldots, s+t-q+1$.

For $1 \leqslant j \leqslant s$, two treatments appear together in a block of $\mathscr{M}$ of size k_j exactly $\bar{r}$ times for each time they occur together in a block of size $\hat{k}_q$ of $\mathscr{M}_2$, and exactly $\bar{\lambda}_j$ times for each time they occur together in a block of size $\hat{k}_1$ of $\mathscr{M}_2$ without being together in a block of size $\hat{k}_q$. Hence

$$\lambda_j = \bar{r}\hat{\lambda}_q + \bar{\lambda}_j(\hat{\lambda}_1 - \hat{\lambda}_q).$$

For $s + 1 \leqslant j \leqslant s + t - q + 1$, two treatments appear together in a block of $\mathscr{M}$ of size k_j exactly $\bar{r}$ times for each time they occur together in a block of size $\hat{k}_{q+j-s-1}$ of $\mathscr{M}_2$, and so

$$\lambda_j = \bar{r}\hat{\lambda}_{q+j-s-1}.$$

There are several important special cases of our construction of $\mathscr{M}$, some of which have appeared previously in the literature. To tie these all together we broaden our definition of MNBIBD to include certain limiting cases (this is done for the context of this discussion only). We allow $\bar{k}_1$ to be equal to $\bar{v}$, so that $\mathscr{M}_1$ is then a resolved (and, if $s>2$, nested) BIBD. Similarly, $\mathscr{M}_2$ is allowed to be resolved. For $\mathscr{M}_2$ we allow $\hat{k}_q=\hat{k}_t=1$, in which case we effectively have only $t-1$ component designs and thus $\mathscr{M}$ will have $s+t-q$ component designs; if this is done with $t=2$ then $\mathscr{M}_2$ is a BIBD with no nesting, $\bar{v}=\hat{k}_1$, and $\mathscr{M}$ has s (not $s+1$) component designs.

Special case 1. Let $s=1$, $t=2$, $\hat{k}_2=1$, and $\hat{v}>\hat{k}_1=\bar{v}$. Then $\mathscr{M}$ is the composition of the two BIBDs $\mathscr{M}_1$ and $\mathscr{M}_2$, i.e. it is the BIBD($v=\hat{v},\ b=\bar{b}_1\hat{b}_1,\ r=\bar{r}\hat{r},\ k=\bar{k}_1$) consisting of $\hat{b}_1$ copies of $\mathscr{M}_1$, where the $\bar{v}$ treatments in the ith copy ($i=1,\dots,\hat{b}_1$) are the $\bar{v}$ treatments from block i of $\mathscr{M}_2$.

Special case 2. Let $s=1$ so that $\mathscr{M}_1$ is a BIBD, and $t=2$ with $\hat{v}>\hat{k}_1>\hat{k}_2>1$. Then this is the NBIBD construction of Theorem 4.2 of Morgan [40], which appears again as Theorem 3.1 of Sinha and Mitra [53]. If $\hat{v}=\hat{k}_1$ then $\mathscr{M}_2$ is an RBIBD and this is construction (ii) of Dey et al. [19, p. 163].

Special case 3. Let $s=t=2$, $\bar{v}=\bar{k}_1$, and $\hat{k}_2=1$. Then the NBIBD(v,b_1,b_2,r,k_1,k_2) $\mathscr{M}$ is found by constructing the RBIBD $\mathscr{M}_1$ for the treatments in each block of the BIBD $\mathscr{M}_2$. This method was employed, though not elucidated as a general technique, in construction (i) of Dey et al. [19, p. 162].

Special case 4. Let $s=t=2$, $\bar{v}>\bar{k}_1$, and $\hat{k}_2=1$. Then the NBIBD(v,b_1,b_2,r,k_1,k_2) $\mathscr{M}$ is found by constructing the NBIBD $\mathscr{M}_1$ for the treatments in each block of the BIBD $\mathscr{M}_2$. This is Theorem 1 of Jimbo and Kuriki [29].

This recursive technique is most effective for constructing MNBIBDs, tending to produce relatively large r for NBIBDs. However, designs for four parameter sets within the range of Table 1 can be produced. The numbers of these sets, followed by parameter specifications of $\mathscr{M}_1$ and $\mathscr{M}_2$, are

49: NBIBD(5,5,10,4,4,2):BIBD(11,11,5,5),
63: BIBD(5,5,4,4):RBIBD(15,7,35,7,15,3),
66: RBIBD(6,5,15,5,6,2):BIBD(11,11,6,6), and
67: RBIBD(6,5,10,5,6,3):BIBD(11,11,6,6).

7.2. A difference construction

Our second method of constructing MNBIBDs is a difference construction, using finite fields GF_v where v is a prime power. We use x to denote a primitive element of GF_v, and we use a Kronecker product notation for initial blocks of size k_1. Thus, for example, an initial block of an MNBIBD with $k_1=12$, $k_2=6$, $k_3=3$ might be

$$\begin{aligned}&(x^0\,x^4\,x^8 : x^2\,x^6\,x^{10}\,|\,x^1\,x^5\,x^9 : x^3\,x^7\,x^{11})\\ &\quad=(x^0,x^1)\otimes(x^0\,x^4\,x^8\,|\,x^2\,x^6\,x^{10})\\ &\quad=(x^0,x^1)\otimes(x^0,x^2)\otimes(x^0,x^4,x^8).\end{aligned}$$

Theorem 1. *Let v be a prime power of the form $v=a_0a_1a_2\cdots a_n+1$ ($a_0\geqslant 1$, $a_n\geqslant 1$ and $a_i\geqslant 2$ for $1\leqslant i\leqslant n-1$ are integers). Then there is an MNBIBD with n component* *designs having*

$$k_1 = ua_1a_2\cdots a_n,\ k_2 = ua_2a_3\cdots a_n,\ \ldots, k_n = ua_n$$

and with a_0v blocks of size k_1, for any integer u with $1\leqslant u\leqslant a_0$ and $u>1$ if $a_n=1$. If integer $t\geqslant 2$ is chosen so that $2\leqslant tu\leqslant a_0$, then there is an MNBIBD with $n+1$ component designs, with the same number of big blocks but of size $k_0=tk_1$, and with its n other block sizes being $k_1,\ldots,k_n$ as given above.

Proof. The designs are cyclically constructed using the finite field GF_v with primitive element x. To specify the initial blocks, let the sets L_j for $j=1,\ldots,t$ be disjoint u-subsets of $\{x^0,x^1,\ldots,x^{a_0-1}\}$, where $t=1$ if the n-component design is desired and $t>1$ for $n+1$ components. The initial blocks of size k_1 are

$$\begin{aligned} x^s &\otimes (x^0,x^{a_0},x^{2a_0},\ldots,x^{(a_1-1)a_0})\\ &\otimes (x^0,x^{a_0a_1},x^{2a_0a_1},\ldots,x^{(a_2-1)a_0a_1})\\ &\otimes (x^0,x^{a_0a_1a_2},x^{2a_0a_1a_2},\ldots,x^{(a_3-1)a_0a_1a_2})\\ &\ \vdots\\ &\otimes (x^0,x^{a_0a_1\cdots a_{n-1}},x^{2a_0a_1\cdots a_{n-1}},\ldots,x^{(a_n-1)a_0a_1\cdots a_{n-1}})\otimes L_j \end{aligned}\tag{1}$$

for $s=0,1,\ldots,a_0-1$ and $j=1,\ldots,t$; if $t>1$ then for fixed s these are the t sub-blocks of size k_1 in the sth block of size k_0. For $i>1$, the $u(v-1)/a_0k_i$ consecutive, disjoint subsets of size k_i in each of these a_0t initial blocks of size $k_1=u(v-1)/a_0$ are the initial sub-blocks for the component BIBD with block size k_i.

The MNBIBD property is established if the differences from within the $ta_0\cdots a_{i-1}$ initial sub-blocks of size k_i can be shown to be symmetrically repeated. Expression (1) for an initial block of size k_1 is the Kronecker product of x^s, L_j, and n other terms, the ith of which is a vector of length a_i. Thus the general form of an initial sub-block of size k_i for any $i\geqslant 1$ is x^s times the Kronecker product of L_j and the last $n-i+1$ of these terms, multiplied by any single member of the Kronecker product of the first $i-1$ terms. As the product of the last $n-i+1$ terms yields all elements that can be written as x raised to a multiple of $a_0a_1a_2\cdots a_{i-1}=u(v-1)/k_i$, the general initial sub-block is

$$x^{s+l}(x^0,x^{\frac{u(v-1)}{k_i}},x^{\frac{2u(v-1)}{k_i}},\ldots,x^{\frac{(k_i-u)(v-1)}{k_i}})\otimes L_j.\tag{2}$$

The collection of all of these initial sub-blocks in a given block of size k_1 (that is, fixing s and j) is generated as l takes all of its values

$$l=0,a_0,2a_0,\ldots,a_0a_1a_2\cdots a_{i-1}-a_0=\frac{u(v-1)}{k_i}-a_0.\tag{3}$$

The differences within the displayed sub-block (2) may be written in two lists.

First, the differences among elements of the sub-block that are multiplied by the same element of L_j (x^e, say) are

$$x^{s+l+e}(x^0, x^{\frac{u(v-1)}{k_i}}, x^{\frac{2u(v-1)}{k_i}}, \ldots, x^{\frac{(k_i-u)(v-1)}{k_i}})$$
$$\otimes (1 - x^{\frac{u(v-1)}{k_i}}, 1 - x^{\frac{2u(v-1)}{k_i}}, \ldots, 1 - x^{\frac{(k_i-u)(v-1)}{k_i}}) \tag{4}$$

which as s and l vary gives every non-zero element of GF_v exactly $k_i/u - 1$ times. The differences between elements of (2) that are multiplied by two different elements of L_j (x^e and x^f, say) are

$$\pm x^{s+l}(x^0, x^{\frac{u(v-1)}{k_i}}, x^{\frac{2u(v-1)}{k_i}}, \ldots, x^{\frac{(k_i-u)(v-1)}{k_i}})$$
$$\otimes (x^e - x^f, x^e - x^{f+\frac{u(v-1)}{k_i}}, x^e - x^{f+\frac{2u(v-1)}{k_i}}, \ldots, x^e - x^{f+\frac{(k_i-u)(v-1)}{k_i}}) \tag{5}$$

which as s and l vary gives every non-zero element of GF_v exactly $2k_i/u$ times.

It remains to investigate the differences within the blocks of size k_0 for $t \geqslant 2$. The sth block of size k_0 is composed of the size k_1 sub-blocks (1) for $j = 1, \ldots, t$. Having already established that the differences within the size k_1 sub-blocks are balanced, it remains to investigate differences between these sub-blocks. Analogous to (5), these differences for fixed s are

$$\pm x^{s}(x^0, x^{\frac{u(v-1)}{k_1}}, x^{\frac{2u(v-1)}{k_1}}, \ldots, x^{\frac{(k_1-u)(v-1)}{k_1}})$$
$$\otimes (x^e - x^f, x^e - x^{f+\frac{u(v-1)}{k_1}}, x^e - x^{f+\frac{2u(v-1)}{k_1}}, \ldots, x^e - x^{f+\frac{(k_1-u)(v-1)}{k_1}}), \tag{6}$$

where now $x^e \in L_j$ and $x^f \in L_{j'}$ for $j \neq j'$. Since $u(v-1)/k_1 = a_0$, as s varies this list generates every non-zero element of GF_v exactly $2k_1/u$ times. This establishes the multiply nested BIBD property. □

Theorem 2. *With the conditions of Theorem 1, if a_0 is even and a_i is odd for $i \geqslant 1$, then MNBIBDs can be constructed with the same block sizes but with $a_0 v/2$ blocks of size k_1.*

Proof. The initial blocks are the same, except that now the range of s is restricted to $s = 0, 1, \ldots, a_0/2 - 1$. To show that the differences are still balanced, consider first the right-most vector in list (4). Since for any w,

$$1 - x^{(v-1)-w\frac{u(v-1)}{k_i}} = -x^{-w\frac{u(v-1)}{k_i}}(1 - x^{w\frac{u(v-1)}{k_i}})$$
$$= x^{\frac{k_i}{u}\frac{u(v-1)}{2k_i}} x^{-w\frac{u(v-1)}{k_i}}(1 - x^{w\frac{u(v-1)}{k_i}}),$$

then if k_i/u is odd (assured by odd $a_1a_2\cdots a_n$) and $u(v-1)/k_i$ is even (assured by even a_0) the list of differences (4) can be written

$$x^{s+l+e}(x^0, x^{\frac{u(v-1)}{2k_i}}, x^{\frac{u(v-1)}{k_i}}, \ldots, x^{\frac{(2k_i-u)(v-1)}{2k_i}})$$
$$\otimes(1-x^{\frac{u(v-1)}{k_i}}, 1-x^{\frac{2u(v-1)}{k_i}}, \ldots, 1-x^{\frac{(k_i-u)(v-1)}{2k_i}})$$

which as l varies through its range (3) and $s=0,1,\ldots,a_0/2-1$ gives every non-zero element of GF_v exactly $(k_i-u)/2u$ times. For (5), because -1 is an odd power of $x^{u(v-1)/2k_i}$, this list similarly gives every non-zero element of GF_v exactly k_i/u times as l and s vary. The same reasoning shows that (6) is balanced for the restricted range of s. $\square$

Theorems 1 and 2 generalize previously known results for construction of NBIBDs. Theorem 3 of Jimbo and Kuriki [29] results when $n=1$ and $t>1$. Theorem 4 of Jimbo and Kuriki [29] is the case $n=2$ and $t=1$. Setting $n=t=1$ gives the BIBD construction due to Sprott [55], while $u=t=1$ gives the result of Preece et al. [45].

Parameter sets in Table 1 for which NBIBDs can be directly constructed from Theorems 1 and 2, followed by values of the theorem variables (t,u,a_0,a_1,a_2), are

1: (1,1,1,2,2), 2: (1,1,1,3,2), 3: (1,1,1,2,3) or (2,1,2,3,.), 5: (1,1,2,2,2), 8: (1,1,1,4,2), 9: (1,1,1,2,4), 14: (1,1,1,5,2), 15: (1,1,1,2,5) or (2,1,2,5,.), 19: (2,1,3,2,.), 20: (1,1,3,2,2), 21: (1,1,2,3,2), 22: (1,1,2,2,3) or (2,1,4,3,.), 23: (1,1,1,6,2), 24: (1,1,1,4,3), 25: (1,1,1,3,4), 26: (1,1,1,2,6), 44: (1,1,1,5,3), 45: (1,1,1,3,5), 48: (3,1,4,3,.), 49: (2,1,5,2,.), 56: (1,2,3,2,2) or (2,1,3,4,.), 66: (3,1,5,2,.), 68: (2,1,3,5,.).

7.3. Construction from perpendicular arrays

A ‘perpendicular array’ $\mathrm{PA}_f(s,k,v)$ is a $k\times fv(v-1)/2$ array with v entries such that the columns of each $s\times fv(v-1)/2$ subarray comprise each s-subset of the v entries with equal frequency f. It is known that, if v is even and $s\geqslant 2$, then an array $\mathrm{PA}_f(s,k,v)$ must have f even.

Our interest is in perpendicular arrays of strength $s=2$. Let $k_1,k_2,\ldots,k_n$ be integers such that $k_1\geqslant 4$ and k_i is a subfactor of k_{i-1} for $i\geqslant 2$. Then the columns of $\mathrm{PA}_f(2,k_1,v)$ are the blocks of a MNBIBD with block sizes $k_1,k_2,\ldots,k_n$ and $b_1=fv(v-1)/2$.

Perpendicular arrays have received considerable attention in the combinatorial literature in the past 20 years; see [14] for references and a summary of existence results. Perpendicular arrays are known in the statistical literature as ‘semibalanced arrays’, so renamed in 1973 by Rao [47], who had originally introduced them in 1961 as ‘orthogonal arrays of type II’ [46]. Recent statistical interest in semibalanced arrays has focused on their use as ‘neighbour designs’; see [36] for a survey, or [41] which introduced the arrays in that context.

Within the bounds of Table 1, parameter sets for which a perpendicular array would be a solution are nos. 19, 49, 59, 66, and 67. Perpendicular arrays can be found for all of these, save no. 59, using Rao's 1961 prime power construction [46]. For parameter set no. 59, a 5-row and 15-symbols perpendicular array given by Schellenberg et al. [51] may be used.

8. Constructing NBIBDs by a modified Kramer–Mesner technique

For some parameter sets for which formal methods of construction such as those described above are not available, search techniques can be used to produce NBIBDs. Search techniques can also be used to produce further NBIBDs for parameter sets for which formal methods are known. The techniques can be used to produce nested t-designs too, but in this paper we restrict our description to NBIBDs.

One such technique is a much simplified version of the method developed by Kramer and Mesner [34] for finding t-designs, $t>2$. The method, as described by Kramer et al. [33], employed large groups. Mutually exclusive and exhaustive orbits (under a selected group) were derived from initial blocks, and for the t-sets of the treatments. A matrix was then built up, one column for each block orbit, one row for each t-set orbit, each entry in the matrix being the number of occurrences of the corresponding t-set orbit in the corresponding block orbit.

A t-design was then obtained as a sum of multiples of the columns, such that, over this sum of multiples, the total number of occurrences of each t-set was the required constant $\lambda(t)$. This is equivalent to finding integral solutions for the linear equations $\mathbf{Ax}=\mathbf{b}$, where $\mathbf{A}$ is the matrix described in the previous paragraph and $\mathbf{b}$ is $\lambda(t)$ times the unit vector. Because of the sizes of the group and of the design considered, many of these calculations were non-trivial. Solutions without repeated blocks were obtained by restricting the values of the entries in $\mathbf{x}$ to 0 and 1.

Now suppose that an NBIBD is required with parameters (v,b_1,b_2,r,k_1,k_2) and that its component BIBDs $\mathscr{D}_1$ and $\mathscr{D}_2$ have parameters (v,b_1,r,k_1,λ_1) and (v,b_2,r,k_2,λ_2), respectively. Suppose further that a BIBD for $\mathscr{D}_1$ is known and that it has a known, non-trivial automorphism group $\mathscr{G}$, and p initial blocks, say

$$(a\ b\ c\ d\ e\ f\ \ldots),\ldots,(u\ v\ w\ldots x\ y\ z).$$

Partition each of these p initial blocks in all possible ways into blocks of size k_2, giving say pq initial blocks of the form

$$(a\ b\ c\mid\ldots\mid d\ e\ f),\ldots,(u\ v\ w\mid\ldots\mid x\ y\ z).$$

For each of the pq initial blocks, calculate the frequency of occurrence of each 2-set orbit within all the sub-blocks of the initial block. Then set up a matrix with pq columns and with one row for each distinct 2-set orbit. An NBIBD will be obtained, as required, if one column can be selected from each of the p sets of q columns, such that the selected columns add to λ_2 times the unit vector. (The original $\mathscr{D}_1$ must, of course, be preserved.)

When $\mathscr{G}$ is cyclic or k_1-rotational, the problem of finding the orbits of the 2-sets is reduced to finding all the differences in a set of sub-blocks of a block. Cyclic and k_1-rotational groups are the most likely automorphism groups $\mathscr{G}$ to be used in looking for NBIBDs.

The size of the matrix **A** can be reduced by eliminating duplicated columns, cyclically equivalent columns, and columns that cannot possibly be part of an NBIBD. Elimination of duplicate or equivalent columns may influence the search for non-isomorphic designs.

The advantage of this technique is that, for the group selected, it gives all the NBIBDs in one go if there are any, and proves non-existence otherwise. The disadvantages include the rapid growth in the number of columns as parameter-values increase (in particular the ratio of k_1 to k_2), the problems of determining the various orbits, and the difficulty of solving the linear equations (specialized methods of solution being needed for all but the smallest NBIBDs). The matrix could be expanded to solve for both BIBD and NBIBD simultaneously, if the available BIBDs are not suitable, or alternative NBIBDs are sought. In practice, a two-stage investigation may be preferable.

For an example, suppose that we want all the NBIBDs that have the parameters (10, 45, 90, 18, 4, 2) and that are based on the $\mathscr{D}_1$ with initial blocks:

$$(1\ 2\ 4\ \infty)\ (1\ 5\ 8\ \infty)\ (1\ 2\ 4\ 8)\ (1\ 2\ 3\ 6)\ (3\ 6\ 7\ 8) \bmod 9.$$

Each of the above blocks can be split in 3 ways: for the first block this gives

$$(1\ 2\,|\,4\ \infty),\ (1\ 4\,|\,2\ \infty) \quad \text{and} \quad (1\ \infty\,|\,2\ 4).$$

From the first of these, there are 2 orbits based on a difference of 1 and of ∞ (and their negations); from the second, 3 and ∞; and from the third, 2 and ∞. Similar calculations can be made for the other blocks. Hence the 5×15 matrix below, where the rows comprise one for the ∞ difference, and one each for the non-zero residues 1 to 4, modulo 9, in that order, and the columns represent the 5 sets of 3 different possible ways of selecting sub-blocks from the 5 initial blocks. There are duplicate columns, but wherever this happens the 2 columns concerned belong to different sets, so it would not be appropriate to eliminate any columns in this instance:

$$\begin{bmatrix} 1&1&1&1&1&1&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&1&0&0&1&0&1&1&0&1 \\ 0&0&1&0&1&0&0&2&0&0&1&0&0&1&0 \\ 0&1&0&0&0&1&0&0&2&1&0&0&1&0&0 \\ 0&0&0&1&0&0&1&0&0&0&1&1&0&1&1 \end{bmatrix}.$$

An NBIBD is obtained for each selection of 5 columns of the above, one from each set of 3, such that the 5 columns add to twice the unit vector. One such is based on columns 3, 6, 7, 10, 14, given in the table as

$$(2\ 4\,|\,1\ \infty)\ (5\ 8\,|\,1\ \infty)\ (1\ 2\,|\,4\ 8)\ (1\ 2\,|\,3\ 6)\ (1\ 3\,|\,2\ 6) \bmod 9.$$

9. Constructing NBIBDs by a randomised search technique

A second search technique for constructing NBIBDs is a randomised hill-climbing (or, strictly speaking, hill-descending) search.

Suppose that a cyclic BIBD is given for $\mathscr{D}_1$. At random, partition each of the initial blocks of $\mathscr{D}_1$ into the appropriate number of sub-blocks. Calculate the value of an objective function measuring the discrepancy between (a) the observed number of occurrences of the differences between treatments within sub-blocks and (b) the required number of occurrences λ_2. If this value is not zero, exchange a pair of treatments chosen at random from a pair of randomly chosen sub-blocks from a randomly chosen initial block of $\mathscr{D}_1$. If the value of the objective function is thereby reduced, accept the change and repeat the procedure. Continue the exchanges until the value of the objective function is zero or until some arbitrary stopping limit (based on the number of iterations) is reached. In practice, the objective function may well have local minima, so acceptance of some changes that do not reduce the value of the objective function is desirable.

The advantage of this technique is its simplicity. Its disadvantage is that it does not guarantee to find any NBIBD for a particular parameter-set, let alone all of those that exist. Repeating the search many times is desirable, as different randomised starting points may vary from each other by distances greater than those that the randomised steps are likely to cover, or because the process may have difficulties emerging from some of the local minima, and to get a spread of solutions. The restriction to cyclic groups is not necessary.

Proceeding in the reverse direction, by combining the blocks of a known BIBD $\mathscr{D}_2$ to obtain the blocks of $\mathscr{D}_1$, is not so straightforward. This is because each of the sub-blocks within a block of $\mathscr{D}_1$ can be cyclically offset with respect to the others without destroying the properties of the nested design. This would very rapidly increase the number of combinations to be considered.

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