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Boundary Value Problems in Rectilinearly Anisotropic Thermoelastic Solids

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BOUNDARY VALUE PROBLEMS IN RECTILINEARLY ANISOTROPIC THERMOELASTIC SOLIDS

by

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ABSTRACT

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The boundary value problems which are considered are the type that arise due to the presence of a Griffith crack (or cracks) in an anisotropic thermoelastic solid. The thermoelastic field, in such materials, when the infinitesimal theory is employed, is governed by a set of elliptic partial differential equations. The general solution of these equations is expressed in terms of arbitrary analytic functions whose real parts, in turn, are expressed in terms of Fourier type integrals or Fourier series. Integral transform techniques are then used to determine the stress intensity factors (and other pertinent information) for various crack geometries. In certain cases the possibility of partial contact, of the crack faces, is also investigated.

DEDICATION

To my parents and my brother

and

to the Kerr, McDougall, Bradley and Armstrong families

ACKNOWLEDGEMENT

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CONTENTS

Chapter 1

Introduction

The subject matter of this dissertation was motivated by the fact that the majority of the thermoelastic studies that have been conducted over the past thirty years have been devoted to isotropic materials. In recent years, however, the option's of the structural engineer have grown, resulting in a substantial increase in the use of anisotropic materials. As a consequence, a need for corresponding thermoelastic studies, involving anisotropic materials, has emerged.

An anisotropic solid is one in which the elastic and thermal properties vary, in different directions. Perhaps the most familiar anisotropic material is wood; it is well known that the elastic modulus of wood in tension parallel to the grain is considerably greater than the corresponding modulus in tension perpendicular to the grain and, in addition, that it's elastic constants depend on

the direction in relation to the wood fibres. Other anisotropic materials include fibre-reinforced composites and certain metals, such as zinc, aluminium and 300-series stainless steel. To facilitate the ensuing, theoretical, discussion it is assumed that these materials may be considered to be homogeneous, anisotropic, thermoelastic solids [1],[2].

The main purpose of this dissertation is to determine the stress fields that are generated within these solids when either (or both) of the following situations arise.

1. The temperature field within the solid is non-uniform

2. Prescribed tractions are applied to the boundary surfaces of the solid

In particular, attention will be focused upon solids which are flawed by the presence of various line crack configurations. The typical line crack in a solid which extends infinitely in the x_3-direction will be denoted, in an appropriately orientated rectangular co-ordinate system (x_1, x_2, x_3), by

$$a < x_1 < b \quad : \quad x_2 = 0 \quad : \quad -\infty < x_3 < \infty$$

Cracks of this type are also referred to as ribbon cracks, by some authors. In all of the problems considered it is assumed, initially, that the crack is

1. traction free:–the stresses on the crack surfaces are zero, and

2. thermally insulated:–it acts as a thermal barrier, preventing the flow of heat.

As is the case with isotropic thermoelastic solids (Davidson [3]), a concentration of the stress field develops at each of the crack tips and, consequently, emphasis will be placed upon evaluating the stress intensity factors. The stress intensity factors, which measure the strength of the stress field singularities at each of the crack tips, can be used to predict crack growth, stability and closure: they are broken down into three different types (modes)

Mode I (opening or closing):–predicting how the crack faces deform relative to the x_2-direction

Mode II (shearing):–predicting how the crack faces deform relative to the x_1-direction

Mode III (tearing):–predicting how the crack faces deform relative to the x_3-direction

It should be noted that if either of the mode I stress intensity factors turn out to be negative it is an indication that the adjacent faces of the crack, at that particular tip, may be coming into contact. If this is indeed the case then it is an obvious violation of assumptions (1) and (2) and it will be necessary to reformulate the problem, if more meaningful results are to be obtained. In the

case of a single crack (Chapter 4) in an infinite solid, subjected to a uniform heat flow and an applied tensile stress, it is possible to assume a contact zone, at the offending tip, and to obtain an expression for the extent of this zone in terms of these thermomechanical loads. In the case of two cracks and an infinite array of cracks (Chapters 5 and 6), subjected to the same thermomechanical load, a condition guaranteeing the validity of stress intensity factors can be obtained. However, as of yet, the problem of establishing the extent of the respective contact zones remains unresolved.

Imposing the assumptions of small displacements and temperature gradients, together with a linear stress-strain-temperature law gives rise to the linear, quasi-static theory, used throughout this dissertation (A more detailed description of these assumptions is presented in Chapter 2.1). It is also assumed that the themomechanical loading and the resulting thermoelastic field are independent of the coordinate x_3; material deformations of this type are referred to as generalized plane-strain deformations. In Chapter 2.2 the basic relations and equations of this theory are combined to show that the temperature and displacement fields are solutions of the generalized Duhamel-Neumann equations. A study of these equations, attributable to Clements[4], in which the general solution is presented in terms of arbitrary analytic functions, is highlighted in Chapter 2.3. These equations and, consequently, their solution's can be further

simplified for various classes of materials, in which certain symmetries are inherent (e.g. transversely isotropic and orthotropic materials). In such cases, with an appropriately chosen set of axes, the equations can be separated into an "in-plane" and an "anti-plane" problem. The various symmetry classes are discussed in Chapter 2.5.

This linear theory permits the use of superposition, in solving the boundary value problems that appear in chapters 4-7. The strategy for solving these problems is to follow two consecutive steps:

1. Determine the undisturbed thermoelastic field which would be generated within the solid if it were unflawed and

2. obtain the perturbation of this solution due to the presence of the, thermally insulated and traction free, crack(s).

Using superposition the second step can be separated into appropriate thermal and isothermal problems. Employing Fourier type integrals (or Fourier series) to represent the real parts of the arbitrary analytic functions appearing in the general solution allows us to solve these component problems. In both, it is shown that the field components in the vicinity of the crack(s) are dependent upon the solution's of resulting sets of singular integral equations. In some cases these equations are solved analytically; in the others we have to be satisfied with established numerical techniques. A brief outline of these techniques

is presented in the Appendix. The appendix also features many of the results and details which, for the sake of continuity, are excluded in Chapter 4-7.

In chapter 7 an isothermal problem is addressed, in which the material is assumed to occupy the half space $x_1 > 0$. This restriction in the geometry requires us to augment Clements's solution, which is geared to "simplify" on the x_1-axis, with a second solution that simplifies on the x_2-axis. The necessary form of this second solution is disscussed in Chapter 2.4.

Chapter 2

The Governing Equations

2.1 Assumptions

In general, the equations that govern the thermoelastic behaviour of a homogeneous anisotropic solid are very complicated. Fortunately, however, they can be simplified, and yet still remain valid, for many cases of practical importance.

In order to justify the use of the linear, uncoupled, quasi-static theory of thermoelastisity, used throughout this dissertation, the following assumptions have to be made.

1. The solid, from the macroscopic viewpoint, is homogeneous.

2. The displacements are small in comparison with the solid's overall size and, therefore, no distinction need be made between the reference and

deformed states.

3. The temperature variations within the solid are small in comparison with the reference temperature.

As a consequence of (1) - (3), the strain may be considered to vary linearly with stress and temperature, the equation of heat conduction simplifies to the generalized Fourier Law and all of the coefficients occurring in these equations are constant.

4. Thermoelastic dissipation is negligible: i.e. the heat created within the solid due to the application of the mechanical loads is very small, and may be neglected.

5. The temperature changes occur very slowly with respect to time, and, as a result, inertia terms may also be neglected.

As a consequence of (4) and (5), the basic equations decouple: the equation of heat conduction separates from the remaining relations and the temperature field may be determined independently of the stress and strain

The interested reader is directed toward the works of [1] and [2] where a more thorough investigation of these assumptions and their consequences, is presented.

Further simplifications in the governing equations are also produced when the following assumptions are included:

6. The solid extends infinitely in the x_3-direction i.e. only cylinders with generators parallel to the x_3-axis will be considered.

7. The boundary conditions are independent of the variable x_3.

   As a result of (6) and (7) the quantities which constitute the thermoelastic field are functions of x_1 and x_2, only, in which case the solid is said to be in a state of generalized plane strain.

2.2 The Governing Equations

In the study of anisotropic thermoelastic solids the quantities of interest are the stresses σ_{ij}, strains e_{ij}, displacements u_i and heat fluxes q_i induced by a temperature field $T(x_i) = T_0 + \theta(x_i)$ $(i, j = 1, 2, 3)$ when the surfaces of the solid are subjected to prescribed thermal and mechanical loads. Hereafter, for brevity, we summarize this set of quantities by the letter $\mathbf{F}$. According to the linear, uncoupled, quasi-static thermoelastic theory of rectilinearly anisotropic solids these quantities are governed by the following equations:

Strain-displacement equations:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2.2.1}$$

Compatibility equations:

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0 \tag{2.2.2}$$

Equilibrium equations:

$$\sigma_{ij,j} = 0 \quad , \quad \sigma_{ij} = \sigma_{ji} \tag{2.2.3}$$

Heat flow equations:

$$q_{i,i} = 0 \tag{2.2.4}$$

Thermal constitutive equations:

$$q_i = -\kappa_{ij}\theta_{,j} \tag{2.2.5}$$

Mechanical constitutive equations:

$$e_{ij} = a_{ijkl}\sigma_{kl} + \alpha_{ij}\theta \tag{2.2.6}$$

or equivalently:

$$\sigma_{ij} = c_{ijkl}e_{kl} - \beta_{ij}\theta \tag{2.2.7}$$

The x_i in these equations are rectangular cartesian coordinates, T_0 is the reference temperature, comma denotes partial differentiation and repeated

indices are summed over the range 1 through 3. The quantities a_{ijkl}, c_{ijkl}, κ_{ij} and α_{ij} are respecively the elastic compliance tensor, the elastic stiffness tensor, the thermal conductivity tensor and the linear thermal expansion tensor, while $\beta_{ij} = c_{ijkl}\alpha_{kl}$. The compliance and stiffness tensors, due to the symmetry of σ_{ij} and e_{ij}, each exhibit symmetries of the type:

$$c_{ijkl} = c_{jikl} = c_{ijlk} \tag{2.2.8a}$$

which reduces their number of independent elements from 81 to 36. The existence of a potential energy function of the form

$$W = W_0(\theta) + \frac{1}{2}c_{ijkl}e_{ij}e_{kl} - c_{ijkl}e_{ij}\alpha_{kl}\theta \tag{2.2.9}$$

implies the additional symmetry

$$c_{ijkl} = c_{klij} \tag{2.2.8b}$$

which reduces the number of independent components from 36 to 21.

Material symmetries, if they exist, produce further simplifications in the five material tensors. A discussion of these simplifications if presented in section 5.

Now, in the case of generalized plane strain the temperature and displacement fields take the form

$$T = T_0 + \theta(x_1, x_2) \tag{2.2.10}$$

$$u_i = u_i(x_1, x_2) \qquad i = 1, 2, 3 \tag{2.2.11}$$

Observe here, that unlike plane strain, u_3 and σ_{i3} are, in general, non-zero, accounting for the possible warpings that the anisotropy may cause. Observe also, that

$$e_{33} = 0$$

and that the equations of equilibrium (2.2.3) are independent of σ_{33}, enabling us to maintain the state of generalized plane strain provided that

$$\begin{aligned} \sigma_{33} &= -\{a_{3311}\sigma_{11} + a_{3322}\sigma_{22} + 2a_{3312}\sigma_{12} \\ &\quad + 2a_{3313}\sigma_{13} + 2a_{3323}\sigma_{23} + \alpha_{33}\theta\}/a_{3333} \end{aligned} \tag{2.2.12}$$

Therefore σ_{33} is determined automatically when θ and the remaining components of stress are known.

Substituting from (2.2.5) and (2.2.7) into (2.2.4) and (2.2.3) respectively, reveals that the temperature and displacement fields are given by the generalized Duhamel-Neumann equations

$$\kappa_{jl}\theta_{,jl} = 0 \tag{2.2.13}$$

$$c_{ijkl}u_{k,jl} = \beta_{ij}\theta_{,j} \tag{2.2.14}$$

Before presenting the general solution of these equations it is worth noting that the Second Law of Thermodynamics dictates that the tensor κ_{jl} must be positive-definite (Carslaw and Jaeger [6]). Similarly, requiring the potential energy function to be everywhere positive implies that c_{ijkl} is positive-definite. Thus, the governing equations (2.2.13) and (2.2.14) are partial differential equations of elliptic type.

2.3 The General Solution of the Governing Equations

By seeking a solution of the form $\theta = f(x_1 + \tau x_2)$ it can be readily shown that the general solution of the temperature equation (2.2.13) may be written in the form

$$\theta = 2\mathrm{Re}[\omega'(z_0)] \tag{2.3.1}$$

where ω is an arbitrary analytic function of the complex variable

$z_0 = x_1 + \tau_0 x_2$ and $\tau_0 = r_0 + is_0$ is the root of the quadratic equation

$$\kappa_{11} + 2\kappa_{12}\tau + \kappa_{22}\tau^2 = 0 \tag{2.3.2}$$

with positive imaginary part. i.e.

$$r_0 = \frac{-\kappa_{12}}{\kappa_{22}} \quad , \quad s_0 = \frac{\sqrt{\kappa_{11}\kappa_{22} - \kappa_{12}^2}}{\kappa_{22}}$$

Hence, the expression (2.2.5) for the heat flux may be denoted by

$$q_i(x_1, x_2) = -2\mathrm{Re}\{(\kappa_{i1} + \tau_0\kappa_{i2})\omega''(z_0)\}. \tag{2.3.3}$$

The general solution of equation (2.2.14) is obtained by superimposing a particular solution with the general solution of the corresponding homogeneous equation. By inspection, it is clear that the functions

$$u_k(x_1, x_2) = 2\mathrm{Re}\{H_k\omega(z_0)\} \tag{2.3.4}$$

are particular solutions of the equations (2.2.14) if the constants H_k satisfy the linear algebraic system

$$[c_{i1k1} + (c_{i1k2} + c_{i2k1})\tau_0 + c_{i2k2}\tau_0^2]H_k = (\beta_{i1} + \tau_0\beta_{i2}) \tag{2.3.5}$$

For the homogeneous equations

$$c_{ijkl}u_{k,jl} = 0 \tag{2.3.6}$$

seeking solutions of the form $u_k = A_k f(x_1 + \tau x_2)$ yields the linear algebraic system

$$[c_{i1k1} + (c_{i1k2} + c_{i2k1})\tau + c_{i2k2}\tau^2]A_k = 0 \tag{2.3.7}$$

For this system to have a non-trivial solution it is necessary that the determinant of coefficients be zero. Hence the τ's must satisfy

$$\det[c_{i1k1} + (c_{i1k2} + c_{i2k1})\tau + c_{i2k2}\tau^2] = 0 \tag{2.3.8}$$

This determinant is a polynomial of degree 6 and Clements[5] has shown that the 6 roots are complex (3 conjugate pairs). In addition it will be assumed that they are distinct, if they are not then (from a numerical standpoint) it is always possible to treat the equal roots as a limiting case of distinct roots. The roots with positive imaginary part may be denoted by $\tau_\alpha = r_\alpha + is_\alpha$ and their corresponding eigenvectors as $A_{k\alpha}$. Real solutions of (2.3.6) may therefore be written in the form

$$u_k(x_1, x_2) = 2\mathrm{Re}[\sum_{\alpha=1}^{3} A_{k\alpha} f_\alpha(z_\alpha)] \tag{2.3.9}$$

where the f_α's are arbitrary analytic functions of the complex variable $z_\alpha = x_1 + \tau_\alpha x_2$.

It follows at once that (2.2.14) has general solution

$$u_k(x_1, x_2) = 2\mathrm{Re}\left\{\sum_{\alpha=1}^{3} A_{k\alpha} f_\alpha(z_\alpha) + H_k \omega(z_0)\right\} \tag{2.3.10}$$

and, via (2.2.7), that the corresponding stress field is given by

$$\sigma_{ij}(x_1, x_2) = 2\mathrm{Re}\left\{\sum_{\alpha=1}^{3} L_{ij\alpha} f'_\alpha(z_\alpha) + (\Lambda_{ij} - \beta_{ij})\omega'(z_0)\right\} \tag{2.3.11}$$

where

$$L_{ij\alpha} = (c_{ijk1} + \tau_\alpha c_{ijk2}) A_{k\alpha} \qquad (\alpha = 1, 2, 3) \tag{2.3.12}$$

and

$$\Lambda_{ij} = (c_{ijk1} + \tau_0 c_{ijk2}) H_k \tag{2.3.13}$$

When the boundary conditions are given in terms of the stresses it is much more convenient to rewrite the solution by introducing the functions

$$S_i(z) = \sum_{\alpha=1}^{3} L_{i2\alpha} f_\alpha(z) + (\Lambda_{i2} - \beta_{i2})\omega(z) \tag{2.3.14}$$

The matrix $L_{i2\alpha}$ is non-singular (Clements [5]), therefore (2.3.14) may be written in the form

$$f_\alpha(z) = M_{\alpha k}[S_k(z) - (\Lambda_{k2} - \beta_{k2})\omega(z)] \tag{2.3.15}$$

where $\sum_{\alpha=1}^{3} L_{i2\alpha} M_{\alpha k} = \delta_{ik}$. Hence, the revised expressions for the displacements and the stresses are given by

$$u_k(x_1, x_2) = 2\mathrm{Re}\left\{\sum_{\alpha=1}^{3} A_{k\alpha} M_{\alpha j}[S_j(z_\alpha) - (\Lambda_{j2} - \beta_{j2})\omega(z_\alpha)] + H_k \omega(z_0)\right\} \tag{2.3.16}$$

$$\begin{aligned} \sigma_{ij}(x_1, x_2) &= 2\mathrm{Re}\left\{\sum_{\alpha=1}^{3} L_{ij\alpha} M_{\alpha k}[S_k'(z_\alpha) - (\Lambda_{k2} - \beta_{k2})\omega'(z_\alpha)]\right. \\ &\qquad \left. + (\Lambda_{ij} - \beta_{ij})\omega'(z_0)\right\} \end{aligned} \tag{2.3.17}$$

On the boundary $x_2 = 0$, these simplify to

$$u_k(x_1, 0) = 2\mathrm{Re}\{B_{kj} S_j(x_1) + F_k \omega(x_1)\} \tag{2.3.18}$$

$$\sigma_{i2}(x_1, 0) = 2\mathrm{Re}\{S_i'(x_1)\} \tag{2.3.19}$$

where

$$F_k = H_k - B_{kj}(\Lambda_{j2} - \beta_{j2}) \tag{2.3.20}$$

and

$$B_{kj} = \sum_{\alpha=1}^{3} A_{k\alpha} M_{\alpha j} \tag{2.3.21}$$

2.4 Isothermal Problems

The solution in section 2.3 can also be used to study isothermal problems, by simply letting $w(z) = 0$. For isothermal problems involving the half-space $x_1 \geq 0$ it is also necessary to augment Clements's solution with a second solution that serves to satisfy the bounday conditions on the surface $x_1 = 0$.

With $\omega = 0$, the basic solution may be written in the form

$$u_k(x_1, x_2) = 2\mathrm{Re}\left\{\sum_{\alpha=1}^{3} A_{k\alpha} f_\alpha(z_\alpha)\right\} \tag{2.4.1}$$

and

$$\sigma_{ij}(x_1, x_2) = 2\mathrm{Re}\left\{\sum_{\alpha=1}^{3} L_{ij\alpha} f'_\alpha(z_\alpha)\right\} \tag{2.4.2}$$

Now choosing

$$f_\alpha(z) = M_{\alpha j} g_j(iz/\tau_\alpha) \tag{2.4.3}$$

(2.4.1) and (2.4.2) may be re-written as

$$u_k(x_1, x_2) = 2\mathrm{Re}\left\{\sum_{\alpha=1}^{3} A_{k\alpha} M_{\alpha j} g_j(iz_\alpha/\tau_\alpha)\right\} \tag{2.4.4}$$

and

$$\sigma_{jk}(x_1, x_2) = 2\mathrm{Re}\left\{\sum_{\alpha=1}^{3} L_{jk\alpha} M_{\alpha k} \frac{i}{\tau_\alpha} g_j'(iz_\alpha/\tau_\alpha)\right\} \tag{2.4.5}$$

On combining (2.2.8), (2.3.12) and (2.3.7) it can be readily shown that

$$L_{i1\alpha} = -\tau_\alpha L_{i2\alpha} \tag{2.4.6}$$

Revealing that, on the boundary $x_1 = 0$ (2.4.4) and (2.4.5) assume the forms

$$u_k(0, x_2) = 2\mathrm{Re}\left\{B_{kj} g_j(ix_2)\right\} \tag{2.4.7}$$

and

$$\sigma_{j1}(0, x_2) = 2\mathrm{Re}\left\{-ig_j'(ix_2)\right\} \tag{2.4.8}$$

2.5 Symmetry and the Material Coefficients

The material coefficients (properties) consist of the the thermal conductivity and thermal expansion tensors along with the elastic compliance tensor, or it's inverse, the elastic stiffness tensor. As stated at the outset, these properties are direction-dependent, which means that they vary with respect to the orientation of the chosen set of reference axes. As a result, their tabulated values (which can be found in Boas and McKenzie [7], for example) are always given, relative to the most conveniently oriented set

of axes. Throughout this study, the location of the crack(s) determines the reference axes, so it is important, in guaging the effects of anisotropy, to be able to calculate the corresponding coefficients, for arbitrary orientations of the material. This is achieved via the standard transformation laws of tensors and since the materials involved are rectinilinearly anisotropic, we may assume that these axes are rectangular. Therefore, if the coefficients are known for the system x_i then the corresponding values, for the system $\bar{x}_i = \bar{x}_i(x_j)$ are obtained from

$$\bar{a}_{ij}^{kl} = \frac{\partial \bar{x}_i}{\partial x_p}\frac{\partial \bar{x}_j}{\partial x_q}\frac{\partial \bar{x}_k}{\partial x_r}\frac{\partial \bar{x}_\ell}{\partial x_s} a_{pq}^{rs} \tag{2.5.1}$$

for each of the rank 4 tensors and

$$\bar{\alpha}_{ij} = \frac{\partial \bar{x}_i}{\partial x_r}\frac{\partial \bar{x}_j}{\partial x_s}\alpha_{rs} \tag{2.5.2}$$

for those of rank 2.

Now, we may also use these transformation laws to study the various types of material symmetries, and their effects upon the material coefficients. Due to the previously mentioned symmetries (2.2.8) in each of the rank 4 tensors, the 21 independent coefficients may be conveniently represented

in the following matrix type form:

$$
\begin{array}{cccccc}
a_{11}^{11} & a_{22}^{11} & a_{33}^{11} & a_{23}^{11} & a_{13}^{11} & a_{12}^{11} \\
 & a_{22}^{22} & a_{33}^{22} & a_{23}^{22} & a_{13}^{22} & a_{12}^{22} \\
 & & a_{33}^{33} & a_{23}^{33} & a_{13}^{33} & a_{12}^{33} \\
 & & & a_{23}^{23} & a_{13}^{23} & a_{12}^{23} \\
 & & & & a_{13}^{13} & a_{12}^{13} \\
 & & & & & a_{12}^{12}
\end{array}
\tag{2.5.3}
$$

While the 6 independent coefficients occuring in the rank 2 tensors may be written as:

$$
\begin{array}{ccc}
\alpha_{11} & \alpha_{12} & \alpha_{13} \\
 & \alpha_{22} & \alpha_{23} \\
 & & \alpha_{33}
\end{array}
\tag{2.5.4}
$$

A detailed discussion of symmetry, from the crystallographic viewpoint, is presented in Nye[2]. Mathematically, we can define the following classes of symmetry (Green and Zerna, [8]).

(a) Symmetry With Respect To A Plane (13 coefficients).

A material is said to be symmetric with respect to the plane $x_3 = 0$ if the components of the material tensors are invariant under the transformation of coordinates

$$\bar{x}_1 = x_1 \quad , \quad \bar{x}_2 = x_2 \quad , \quad \bar{x}_3 = -x_3 \tag{2.5.5}$$

For this change of coordinates

$$\frac{\partial \bar{x}_i}{\partial x_r} = \delta_{ir} \qquad r = 1,2 \qquad \text{and} \qquad \frac{\partial \bar{x}_i}{\partial x_3} = -\delta_{i3} \tag{2.5.6}$$

Therefore $\bar{\alpha}_{ij} = \alpha_{ij}$ if and only if $\alpha_{13} = \alpha_{23} = 0$ while $\bar{a}_{ij}^{kl} = a_{ij}^{kl}$ if and only if each of the a_{ij}^{kl} with indices containing an odd number of 3's is zero. Thus, the rank 4 tensors assume the form

$$\begin{array}{cccccc} a_{11}^{11} & a_{22}^{11} & a_{33}^{11} & 0 & 0 & a_{12}^{11} \\ & a_{22}^{22} & a_{33}^{22} & 0 & 0 & a_{12}^{22} \\ & & a_{33}^{33} & 0 & 0 & a_{12}^{33} \\ & & & a_{23}^{23} & a_{13}^{23} & 0 \\ & & & & a_{13}^{13} & 0 \\ & & & & & a_{12}^{12} \end{array} \tag{2.5.7}$$

with the rank 2's simplifying to

$$\begin{array}{ccc} \alpha_{11} & \alpha_{12} & 0 \\ & \alpha_{22} & 0 \\ & & \alpha_{33} \end{array} \tag{2.5.8}$$

(b) Symmetry With Respect to Two Orthogonal Planes: Orthotropy (9 coefficients)

A material is said to be symmetric with respect to the planes $x_3 = 0$ and $x_2 = 0$ if in addition to (2.5.5), the material tensors are also invariant

under the transformation

$$\bar{x}_1 = x_1 \qquad \bar{x}_2 = -x_2 \qquad \bar{x}_3 = x_3$$

This additional requirement leads to the rank 4 tensors assuming the form

$$\begin{matrix} a^{11}_{11} & a^{11}_{22} & a^{11}_{22} & 0 & 0 & 0 \\ & a^{22}_{22} & a^{22}_{33} & 0 & 0 & 0 \\ & & a^{33}_{33} & 0 & 0 & 0 \\ & & & a^{23}_{23} & 0 & 0 \\ & & & & a^{13}_{13} & 0 \\ & & & & & a^{12}_{12} \end{matrix} \tag{2.5.9}$$

With those of rank 2 simplifying to

$$\begin{matrix} \alpha_{11} & 0 & 0 \\ & \alpha_{22} & 0 \\ & & \alpha_{33} \end{matrix} \tag{2.5.10}$$

An examination of (2.5.9) and (2.5.10) reveals that symmetry with respect to two orthogonal planes implies symmetry with respect to a third plane, which is orthogonal to both of the original planes. Hence the material actually has three orthogonal planes of symmetery at each point, and such a material is said to be orthotropic.

In certain situations it is more convenient to express the non-zero components of the compliance tensor in terms of the "engineering constants" (Lekhnitskii, [9]) E_i, G_{ij} and ν_{ij} so that

$$a_{ij}^{k\ell} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & 0 & 0 & 0 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{31} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{12} \end{bmatrix} \tag{2.5.11}$$

where the relationships $E_2\nu_{12} = E_1\nu_{21}$; $E_1\nu_{31} = E_3\nu_{13}$; $E_3\nu_{23} = E_2\nu_{32}$ preserve the required symmetry and the E_i's are the Youngs Moduli, G_{ij}'s are the shear moduli and ν_{ij}'s are the Poisson's Ratios.

(c) Transverse Isotropy (5 coefficients)

This is obtained from an orthotorpic material if the material coefficients remain invariant under the transformation

$$\begin{aligned} \overline{x}_1 &= x_1 \\ \overline{x}_2 &= x_2\cos\theta + x_3\sin\theta \\ \overline{x}_3 &= -x_2\sin\theta + x_3\cos\theta \end{aligned} \tag{2.5.12}$$

where θ is arbitrary. In particular,(2.5.12) defines a material with the

x_1-axis normal to the transverse planes.

Equating a_{ij}^{kl} and $\bar{a}_{ij}^{kl}$ in this case reveals that the rank 4 tensors assume the form

$$\begin{matrix} a_{11}^{11} & a_{22}^{11} & a_{22}^{11} & 0 & 0 & 0 \\ & a_{22}^{22} & a_{33}^{22} & 0 & 0 & 0 \\ & & a_{22}^{22} & 0 & 0 & 0 \\ & & & \frac{1}{2}(a_{22}^{22} - a_{33}^{22}) & 0 & 0 \\ & & & & a_{12}^{12} & 0 \\ & & & & & a_{12}^{12} \end{matrix} \tag{2.5.13}$$

While those of rank 2 simplify to

$$\begin{matrix} \alpha_{11} & 0 & 0 \\ & \alpha_{22} & 0 \\ & & \alpha_{22} \end{matrix} \tag{2.5.14}$$

(d) Fiber-Reinforced Composites

Undirectional full fiber-reinforced composites such as that shown in Figure 1 are macroscopically transversely isotropic with axis of transverse isotropy parallel to the fiber direction.

Given the elastic properties of the matrix and also those of the (possibly anisotropic) fiber we can calculate the elastic constants of the fiber-composite via Hahn's equations (see Vinson and Sierakowski, [20])

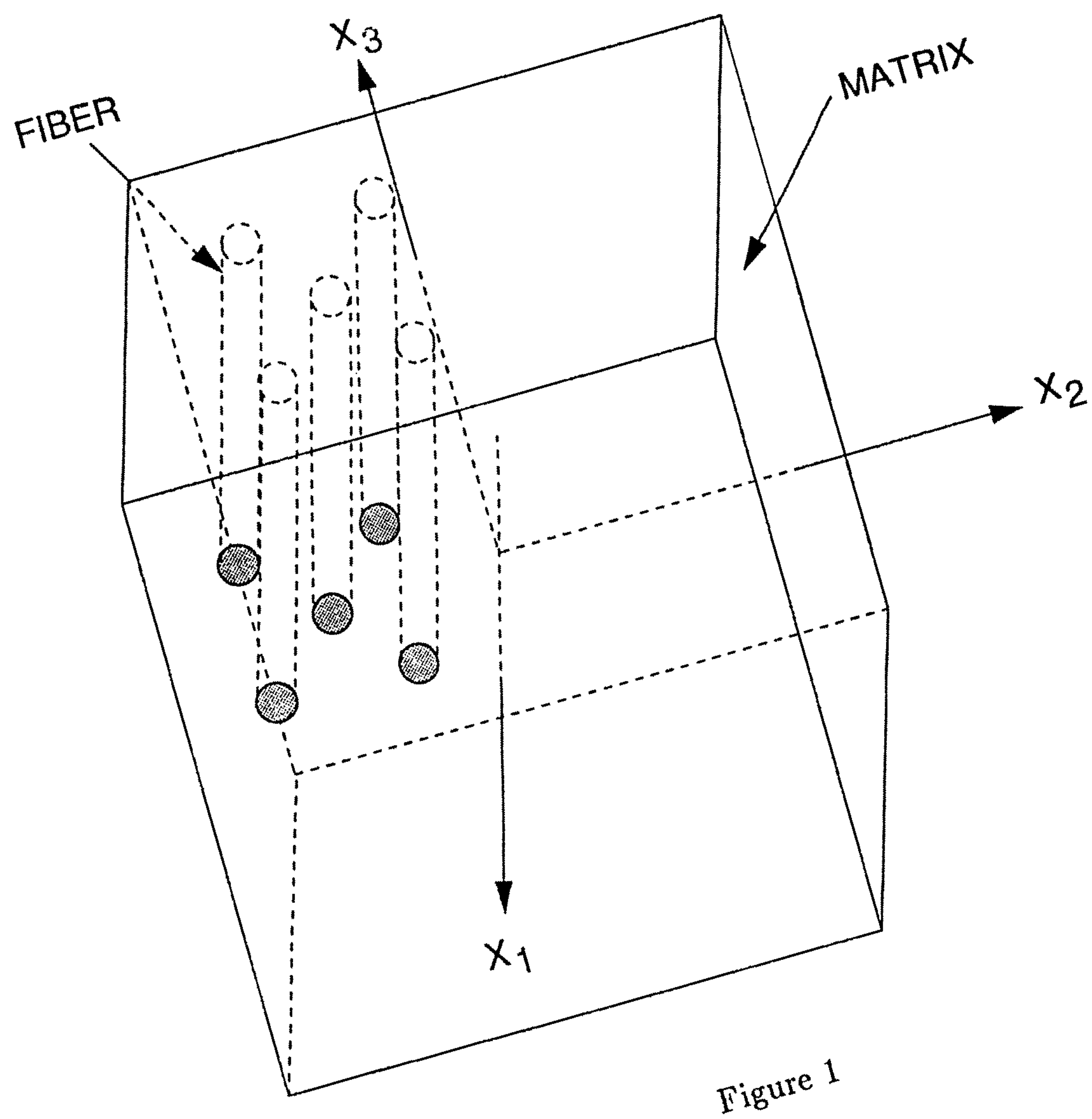

Figure 1

	Composite	Fiber	Matrix
Volume fraction $(V_f + V_m = 1)$	1	V_f	V_m
Young's Modulus: E_{11}	E_1	E_{11f}	E_m
Poisson's Ratio: ν_{12}	ν_{12}	ν_{12f}	ν_m
Shear Modulus: G_{12}	G_{12}	G_{12f}	G_m
Shear Modulus: G_{23}	G_{23}	G_{23f}	G_m
Plane Strain Bulk Modulus: K	K	K_f	K_m

Note:

$$\begin{aligned} K_m &= E_m/2(1-\nu_m). \\ K_f &= E_f/2(1-\nu_{12}f) \end{aligned} \tag{2.5.15a}$$

Hahn's Formulae

$$\begin{aligned} E_1 &= E_{11f}V_f + E_m V_m \\ \nu_{12} &= \nu_{12f}V_f + \nu_m V_m \end{aligned} \tag{2.5.15b}$$

$$\frac{1}{G_{12}} = \frac{\left(\frac{V_f}{G_{12f}} + \eta_6 \frac{V_m}{G_m}\right)}{V_f + \eta_6 V_m} \ , \ \eta_6 = \frac{1}{2}(1 + G_m/G_{12f})$$

$$\frac{1}{G_{23}} = \frac{\left(\frac{V_f}{G_{23f}} + \eta_4 \frac{V_m}{G_m}\right)}{V_f + \eta_4 V_m} \ , \ \eta_4 = \frac{3 - 4\nu_m + G_m/G_{23f}}{4(1 - \nu_m)}$$

$$\frac{1}{K} = \frac{\left(\frac{V_f}{K_f} + \eta_k \frac{V_m}{K_m}\right)}{V_f + \eta_k V_m} \ , \ \eta_k = \frac{1 + G_m/K_f}{2(1 - \nu_m)} \tag{2.5.15c}$$

$$E_2 = \frac{4KG_{23}}{K + mG_{23}} \ , \ m = 1 + \frac{4K\nu_{12}^2}{E_1}$$

$$\nu_{23} = \frac{E_2}{2G_{23}} - 1$$

It should be noted that if the direction of the cylindrical generators is parallel to the normal axis defining either of the aforementioned material symmetries, then the governing equations (and their solution) are simplified. Specific details can be found in Chapter 2 of Clements [5]. In each of the cases (a), (b) and (c) the equations decouple into an in-plane and an anti- plane problem. In case (c), however, the simplifications are somewhat unwelcome; it turns out that the polynomial (2.3.8) has roots $\pm i$, each of multiplicity 3, which violates the assumption of the roots being distinct. Problems of this type can be approximated by introducing a small

deviation between the respective x_3-axes but it would be more prudent to, instead, reformulate the problem using the classical theory of isotropic thermoelasticity (Parkus, [10]).

Chapter 3

Thermoelastic Fields in Unflawed Linearly Anisotropic Solids, Subjected to Prescribed Thermomechanical Loads

3.1 Introduction

The method used for determining the thermoelastic field in a flawed solid requires us, first, to determine the field in the equivalent unflawed solid; this field may be symbolized by a superscript 0. Two cases, which will play important roles in Chapters 4-7, are considered and in both the thermomechanical loading consists of a uniform heat flux

$$\mathbf{q}^0 = (q_1^0, q_2^0, q_3^0) \tag{3.1.1}$$

and a uniaxial tension in the x_2-direction

$$\sigma_{22}^0 = \sigma \tag{3.1.2}$$

3.2 Case 1

Determine the resulting thermoelastic field in the plane $x_3 = c$ if the solid is constrained in such a manner that the field is a function of x_1 and x_2 only (the solid, otherwise, is allowed to expand freely). This case coincides with that of generalized plane strain.

That is

$$\mathbf{F}^0 = \mathbf{F}^0(x_1, x_2) \tag{3.2.1}$$

$$e_{33}^0 = 0 \tag{3.2.2}$$

and the equilibrium equations (2.2.3) are independent of σ_{33}^0. Therefore, combining (3.2.2) with the relevant constitutive equation (2.2.6) reveals that a state of generalized plane strain can be maintained provided

$$\sigma_{33}^0 = -\{a_{3311}\sigma_{11}^0 + a_{3322}\sigma_{22}^0 + 2a_{3312}\sigma_{12}^0 + 2a_{3313}\sigma_{13}^0 + 2a_{3323}\sigma_{23}^0 + \alpha_{33}\theta^0\}/a_{3333} \tag{3.2.3}$$

Since the solid is, otherwise, stress free the stress field may be summarized

$$\sigma_{ij}^0(x_1, x_2) = \delta_{i2}\delta_{j2}\sigma - \delta_{i3}\delta_{j3}(\alpha_{33}\theta^0 + a_{3322}\sigma)/a_{3333} \tag{3.2.4}$$

which enables us to rewrite the remaining constitutive equations in the revised form

$$e^0_{ij}(x_1, x_2) = \lambda_{ij}\theta^0 + \mu_{ij}\sigma \tag{3.2.5}$$

where

$$\lambda_{ij} = \alpha_{ij} - \frac{a_{ij33}}{a_{3333}}\,\alpha_{33} \tag{3.2.6}$$

$$\mu_{ij} = a_{ij22} - \frac{a_{ij33}}{a_{3333}}\,a_{3322}, \tag{3.2.7}$$

$$(i, j = 1, 2, 3)$$

The equations (2.2.4) and (2.2.5) clearly indicate that a uniform heat flux can only be produced by a linear temperature field i.e.

$$\theta^0 = \theta_0 + \theta_1 x_1 + \theta_2 x_2 \tag{3.2.8}$$

Upon combining (3.2.5) and (3.2.8), it can be readily shown, to within a rigid body motion, that the displacement field is given by

$$\begin{aligned} u^0_1(x_1, x_2) &= \lambda_{1r}x_r\theta^0 - \tfrac{1}{2}\theta_1\lambda_{rs}x_r x_s + \mu_{1r}x_r\sigma \\ u^0_2(x_1, x_2) &= \lambda_{2r}x_r\theta^0 - \tfrac{1}{2}\theta_2\lambda_{rs}x_r x_s + \mu_{2r}x_r\sigma \\ u^0_3(x_1, x_2) &= \lambda_{3r}x_r(\theta^0 + \theta_0) + 2\mu_{3r}x_r\sigma \end{aligned} \tag{3.2.9}$$

where the subscripts r,s are summed over the range 1, 2 and where the additional requirement

$$\lambda_{31}\theta_2 = \lambda_{32}\theta_1 \tag{3.2.10}$$

is necessary to guarantee a unique expression for u_3.

Therefore, (3.2.10), (3.2.8), and (2.2.5) show that, in general, only one component of the heat flux may be specified in an arbitrary fashion. In the case where $x_3 = 0$ is a plane of symmetry however, $\lambda_{13} = \lambda_{23} = 0$ so that (3.2.10) is satisfied trivially and we may then prescribe two components arbitrarily. It should also be noted that, in the latter case $u_3(x_1, x_2) = 0$ so that the resulting solution is one of genuine plane strain.

3.3 Case 2

Determine the resulting thermoelastic field in the plane $x_3 = c$, $-\pi < x_1 < \pi$, $-\infty < x_2 < \infty$ if the solid is constrained in such a manner that the field is a function of x_2 only (the solid, otherwise, is allowed to expand freely). That is

$$\mathbf{F}^0 = \mathbf{F}^0(x_2) \tag{3.3.1}$$

$$e_{33}^0 = e_{11}^0 = e_{13}^0 = 0 \tag{3.3.2}$$

and the equilibrium equations (2.2.3) are independent of $\sigma_{33}^0, \sigma_{11}^0$ and σ_{13}^0. Therefore, combining (3.3.2) with the relevant constitutive equations (2.2.6)

reveals that the desired field can be maintained provided

$$\begin{pmatrix} \sigma_{11}^0 & + & \beta_{11}\theta^0 \\ \sigma_{33}^0 & + & \beta_{33}\theta^0 \\ \sigma_{13}^0 & + & \beta_{13}\theta^0 \end{pmatrix} = - \begin{bmatrix} A_{11}^{22} & 2A_{11}^{23} & 2A_{11}^{12} \\ A_{33}^{22} & 2A_{33}^{23} & 2A_{33}^{12} \\ A_{13}^{22} & 2A_{13}^{23} & 2A_{13}^{12} \end{bmatrix} \begin{pmatrix} \sigma_{22}^0 & + & \beta_{22}\theta^0 \\ \sigma_{23}^0 & + & \beta_{23}\theta^0 \\ \sigma_{12}^0 & + & \beta_{12}\theta^0 \end{pmatrix} \tag{3.3.3}$$

where

$$\begin{bmatrix} A_{11}^{22} & 2A_{11}^{23} & 2A_{11}^{12} \\ A_{33}^{22} & 2A_{33}^{23} & 2A_{33}^{12} \\ A_{13}^{22} & 2A_{13}^{23} & 2A_{13}^{12} \end{bmatrix} = \begin{bmatrix} a_{11}^{11} & a_{11}^{33} & 2a_{11}^{13} \\ a_{33}^{11} & a_{33}^{33} & 2a_{33}^{13} \\ a_{13}^{11} & a_{13}^{33} & 2a_{13}^{13} \end{bmatrix}^{-1} \begin{bmatrix} a_{11}^{22} & 2a_{11}^{23} & 2a_{11}^{12} \\ a_{33}^{22} & 2a_{33}^{23} & 2a_{33}^{12} \\ a_{13}^{22} & 2a_{13}^{23} & 2a_{13}^{12} \end{bmatrix} \tag{3.3.4}$$

Since the solid is, otherwise, stress-free the stress field may be summarized:

$$\begin{aligned} \sigma_{ij}^0 &= -A_{ij}^{22}\sigma - \{A_{ij}^{22}\beta_{22} + 2A_{ij}^{23}\beta_{23} + 2A_{ij}^{12}\beta_{12} + \beta_{ij}\}\theta^0 \\ &\quad (i,j = 1,3) \\ \sigma_{2j}^0 &= \delta_{2j}\sigma \end{aligned} \tag{3.3.5}$$

The remaining constitutive equations may now be written in revised form:

$$\begin{pmatrix} e_{22}^0 \\ e_{23}^0 \\ e_{21}^0 \end{pmatrix} = \begin{pmatrix} d_{22}^{22} & 2d_{22}^{23} & 2d_{22}^{12} \\ d_{23}^{22} & 2d_{23}^{23} & 2d_{23}^{12} \\ d_{21}^{22} & 2d_{21}^{23} & 2d_{21}^{12} \end{pmatrix} \begin{pmatrix} \sigma \ +\beta_{22}\theta^0 \\ \beta_{23}\theta^0 \\ \beta_{12}\theta^0 \end{pmatrix} \tag{3.3.6}$$

where

$$d_{ij}^{kl} = a_{ij}^{kl} - a_{ij}^{11}A_{11}^{kl} - a_{ij}^{33}A_{33}^{kl} - 2a_{ij}^{13}A_{13}^{kl} \tag{3.3.7}$$

The equations (2.2.4) and (2.2.5) clearly indicate that a uniform heat flux can only be produced by a linear temperature field, hence

$$\theta^0 = \theta_0 + \theta_2 x_2 \tag{3.3.8}$$

Combining (3.3.6) and (3.3.8), it can be readily shown, to within a rigid body motion, that the displacement field is given by:

$$
\begin{aligned}
u_1^0 &= 2d_{21}^{22}\sigma x_2 + 2D_{21}(\theta_0 x_2 + \tfrac{1}{2}\theta_2 x_2^2) \\
u_2^0 &= d_{22}^{22}\sigma x_2 + D_{22}(\theta_0 x_2 + \tfrac{1}{2}\theta_2 x_2^2) \\
u_3^0 &= 2d_{23}^{22}\sigma x_2 + 2D_{23}(\theta_0 x_2 + \tfrac{1}{2}\theta_2 x_2^2)
\end{aligned}
\tag{3.3.9}
$$

where

$$
D_{2j} = d_{2j}^{22}\beta_{22} + 2d_{2j}^{23}\beta_{23} + 2d_{2j}^{12}\beta_{12} \quad (j = 1, 2, 3) \tag{3.3.10}
$$

Chapter 4

The Disturbance of a Uniform Heat Flow by a Line Crack in an Infinite Anisotropic Thermoelastic Solid

4.1 Introduction

The first crack configuration to be considered is that of a single line crack in an infinite anisotropic solid. The reference axes are set up so that the crack, which is thermally insulated and traction free, is given by:

$$a < x_1 < b \quad , \quad x_2 = 0 \quad , \quad -\infty < x_3 < \infty$$

Under the conditions of generalized plane strain, the thermoelastic field in the vicinity of the crack is determined for the case in which the thermo-

mechanical loading consists of a uniform heat flux

$$\mathbf{q}^0 = (q_1^0, q_2^0, q_3^0)$$

and a uniaxial tension in the x_2-direction:

$$\sigma_{22} = \sigma$$

Choosing appropriate Fourier type integrals to represent the arbitrary analytic functions which appear in the general solution results in a system of singular integral equations. These equations are solved analytically and expressions for the stress intensity factors are obtained, along with a condition that guarantees their validity, by insuring that the crack remains open. The problem is then reformulated for the case in which the crack faces come into contact. An expression for the length of the contact zone is given, along with corresponding expressions for the revised stress intensity factors.

4.2 Resolution into Problems 1 and 2

In the absence of the crack the thermoelastic field is identical to that of Chapter 3.2. The presence of the crack disturbs this field thereby creating a new field which can be written:

$$\mathbf{F}^n(x_1, x_2) = \mathbf{F}^0(x_1, x_2) + \mathbf{F}^p(x_1, x_2)$$

The stress free movement of the crack requires that

$$\sigma_{i2}^{n}(x_1,0)=0 \qquad x_1\in(a,b)$$

while the prevention of heat flow across the crack requires that

$$q_2^{n}(x_1,0)=0 \qquad x_1\in(a,b)$$

We also require the thermoelastic field, far from the crack, to be the same as the field in the unflawed solid. Hence, recalling the form of $\mathbf{F}^0(x_1,x_2)$, together with appropriate continuity requirements, we seek a particular solution of the equations of generalized plane strain which satisfies the following boundary conditions:

$$\begin{aligned}
&(1)\quad \theta^p(x_1,x_2),\sigma_{ij}^p(x_1,x_2)\sim O(r^{-1}) \ \text{ and } \ u_k^p(x_1,x_2)\sim O(\ln r)\\
&\qquad \text{as} \quad r=\sqrt{x_1^2+x_2^2}\to\infty.\\
&(2)\quad [\![\sigma_{i2}^p(x_1,0)]\!]=0 && x_1\in\mathbb{R}\\
&(3)\quad [\![u_k^p(x_1,0)]\!]=0 && x_1\in(a,b)^c\\
&\qquad \sigma_{i2}^p(x_1,0+)=-\delta_{i2}\sigma && x_1\in(a,b)\\
&(4)\quad [\![q_2^p(x_1,0)]\!]=0 && x_1\in\mathbb{R}\\
&(5)\quad [\![\theta^p(x_1,0)]\!]=0 && x_1\in(a,b)^c\\
&\qquad q_2^p(x_1,0+)=-q_2^0 && x_1\in(a,b)
\end{aligned} \tag{4.2.1}$$

where the notation

$$[\![f(x,0)]\!]=f(x,0+)-f(x,0-)$$

is used to denote the jump in the relevant function on the x_1-axis.

This problem may be solved by superimposing the solution's of two component problems. In the first the thermal field is completely detemined by replacing the crack with a continuous distribution of heat dipoles. This leads, however, to an unwanted stress developing on the crack. In the second problem, which is isothermal, this stress is counter balanced by an equal and opposite stress, leaving the crack stress free, as desired. Problems 1 and 2 may thus be stated as follows:

Problem 1. Find a generalized plane strain solution $\theta^1(x_1,x_2), u_k^1(x_1,x_2)$ of the equations of anisotropic linear thermoelasticity in $\mathbb{R}^3$ subject to the boundary conditions:

$$(1)\quad \theta^1(x_1,x_2), \sigma_{ij}^1(x_1,x_2) \sim O(r^{-1}) \quad , \quad u_k^1(x_1,x_2) \sim O(\ln r)$$

$$\text{as} \qquad r = \sqrt{x_1^2+x_2^2} \to \infty.$$

$$(2)\quad [\![\sigma_{i2}^1(x_1,0)]\!] = 0 \qquad\qquad x_1 \in \mathbb{R}$$

$$(3)\quad [\![u_k^1(x_1,0)]\!] = 0 \qquad\qquad x_1 \in \mathbb{R}$$

$$(4)\quad [\![q_2^1(x_1,0)]\!] = 0 \qquad\qquad x_1 \in \mathbb{R}$$

$$(5)\quad [\![\theta^1(x_1,0)]\!] = 0 \qquad\qquad x_1 \in (a,b)^c$$

$$q_2^1(x_1,0+) = -q_2^0 \qquad\qquad x_1 \in (a,b)$$

Calculate: $\sigma_{i2}^1(x_1,0\pm) = p_i(x_1)$.

Problem 2. Find a generalized plane strain solution $u_k^2(x_1, x_2)$ of the equations of isothermal anisotropic linear elasticity in $\mathbb{R}^3$ cut along $x_2 = 0$, $a < x_1 < b$, $-\infty < x_3 < \infty$ subject to the boundary conditions:

(1) $\sigma_{ij}^2(x_1,x_2) \sim O(r^{-2})$, $\quad u_k^2(x_1,x_2) \sim O(r^{-1})$ as $r = \sqrt{x_1^2 + x_2^2} \to \infty$.

(2) $[\![\sigma_{i2}^2(x_1,0]\!] = 0 \qquad x_1 \in \mathbb{R}$

(3) $[\![u_k^2(x_1,0]\!] = 0 \qquad x_1 \in (a,b)^c$

$\sigma_{i2}^2(x_1,0+) = -\delta_{i2}\sigma - p_i(x_1) \qquad x_1 \in (a,b)$

4.3 The Solution of Problem 1

In the general solution (2.3.1), (2.3.3), (2.3.16), (2.3.17) take

$$\omega(z) = \kappa_0^{-1}\{\phi(x,y) + i\psi(x,y)\} \tag{4.3.1}$$

and

$$S_j(z) = \Omega_j(x,y) + i\Psi_j(x,y) \tag{4.3.2}$$

where

$$\phi(x,y) = \frac{1}{\sqrt{8\pi}} \int_{-\infty}^{\infty} \frac{A_0(\xi)}{|\xi|}(e^{-|\xi y|}e^{-i\xi x} - e^{-i\xi\frac{a+b}{2}})d\xi \tag{4.3.3}$$

$$\Omega_j(x,y) = \frac{1}{\sqrt{8\pi}} \int_{-\infty}^{\infty} \frac{A_j(\xi)}{|\xi|}(e^{-|\xi y|}e^{-i\xi x} - e^{-i\xi\frac{a+b}{2}})d\xi \tag{4.3.4}$$

and $\kappa_0 = is_0\kappa_{22}$

Then

$$\theta^1(x_1,x_2) = -\frac{2}{s_0\kappa_{22}}\phi_{,y}(x_1 + r_0x_2, s_0x_2) \tag{4.3.5}$$

$$q_2^1(x_1, x_2) = -2\phi_{,zz}(x_1 + r_0 x_2, s_0 x_2) \tag{4.3.6}$$

and boundary condition 4 is satisfied automatically. Now condition 5*i* is equivalent to

$$\frac{\partial}{\partial x_1}[\![\theta^1(x_1, 0)]\!] = G_0(x_1) = \begin{cases} g_0(x_1) & x_1 \in (a,b) \\ 0 & x_1 \in (a,b)^c \end{cases}$$

where $g_0(x_1)$ is an, as yet, undetermined function satisfying the condition

$$\int_a^b g_0(t)dt = 0 \tag{4.3.7}$$

Hence condition 5*i* is satisfied if

$$\frac{-2i}{s_0\kappa_{22}}\mathcal{F}^{-1}[\xi A_0(\xi)\,;\,x_1] = G_0(x_1) \qquad x_1 \in \mathbb{R}$$

or equivalently, if

$$A_0(\xi) = \frac{s_0\kappa_{22}}{2}\frac{i}{\xi}\frac{1}{\sqrt{2\pi}}\int_a^b g_0(t)e^{i\xi t}dt \tag{4.3.8}$$

where the inverse Fourier transform $\mathcal{F}^{-1}$ (Sneddon [11]) is defined by

$$\mathcal{F}^{-1}[f(\xi)\,;\,x] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)e^{-i\xi x}d\xi \tag{4.3.9}$$

Substituting (4.3.8) into (4.3.6) and making use of (8.1.1) reveals that

$$q_2^1(x_1, 0) = \frac{-s_0\kappa_{22}}{2}\frac{1}{\pi}\int_a^b \frac{g_0(t)}{t - x_1}dt \tag{4.3.10}$$

Therefore condition 5*ii* is satisfied if $g_0(t)$ is given by the singular integral equation

$$\frac{1}{\pi}\int_a^b \frac{g_0(t)}{t - x_1}dt = \frac{2}{s_0\kappa_{22}}q_2^0 \qquad x_1 \in (a,b) \tag{4.3.11}$$

with subsidiary condition (4.3.7). It now follows (8.2.1) that

$$g_0(t) = \frac{-q_2^0}{s_0\kappa_{22}} \frac{(a+b-2t)}{\Delta(t)} \tag{4.3.12}$$

where

$$\Delta(t) = \sqrt{(b-t)(t-a)} \tag{4.3.13}$$

Substituting (4.3.12) into (4.3.8) we see that

$$A_0(\xi) = -q_2^0 \frac{1}{\sqrt{2\pi}} \int_a^b \Delta(t) e^{i\xi t} dt \tag{4.3.14}$$

Hence, the temperature field is completely determined and, from (4.3.5), (4.3.14) and (8.1.3), is given by

$$\theta^1(x,y) = \frac{-q_2^0}{s_0\kappa_{22}} y \frac{1}{\pi} \int_a^b \frac{\Delta(t)}{y^2+(t-x)^2} dt \tag{4.3.15}$$

where, recall $\quad x = x_1 + r_0 x_2 \;\; ; \;\; y = s_0 x_2$

Before addressing the remaining boundary conditions it is worth noting that the corresponding imaginary parts of $\omega(z)$ and $S_j(z)$ are given by

$$\psi(x,y) = c_0 + i\,\mathrm{sgn}(y)\frac{1}{2}\mathcal{F}^{-1}\left[\frac{A_0(\xi)}{\xi}e^{-|\xi y|}\,;\,x\right] \tag{4.3.16}$$

$$\Psi_j(x,y) = c_j + i\mathrm{sgn}(y)\frac{1}{2}\mathcal{F}^{-1}\left[\frac{A_j(\xi)}{\xi}e^{-|\xi y|}\,;\,x\right] \tag{4.3.17}$$

where c_0, c_j $(j = 1,2,3)$ are arbitrary constants. Now, from (2.3.18)

$$\begin{aligned} u_k^1(x_1,0) &= (B_{kj}+\overline{B_{kj}})\Omega_j(x_1,0) + i(B_{kj}-\overline{B_{kj}})\Psi_j(x_1,0) \\ &+ \left(\frac{F_k}{\kappa_0}+\frac{\overline{F_k}}{\overline{\kappa}_0}\right)\phi(x_1,0) + i\left(\frac{F_k}{\kappa_0}-\frac{\overline{F_k}}{\overline{\kappa}_0}\right)\psi(x_1,0) \end{aligned} \tag{4.3.18}$$

Therefore, recalling the forms of the relevant functions (4.3.3,4,16, and 17), we see that

$$[\![u_k^1(x_1,0)]\!] = -\mathcal{F}^{-1}\left[\left\{(B_{kj}-\overline{B_{kj}})A_j(\xi)+\left(\frac{F_k}{\kappa_0}-\frac{\overline{F_k}}{\overline{\kappa_0}}\right)A_0(\xi)\right\}\xi^{-1};x_1\right]$$

Hence, boundary condition 3 is satisfied if

$$A_j(\xi) = \frac{1}{s_0\kappa_{22}} d_{jk}\mathrm{Re}(F_k)A_0(\xi) \tag{4.3.19}$$

where d_{jk} is the inverse of the non-singular matrix $b_{kj} = \mathrm{Im}[B_{kj}]$ (Clements, [5]).

Now, from (2.3.19)

$$\sigma_{j2}(x_1,0) = -i\,\mathcal{F}^{-1}[\mathrm{sgn}(\xi)A_j(\xi);x_1] \tag{4.3.20}$$

therefore boundary condition 2 is satisfied automatically. Also, from (4.3.20), (4.3.19), (4.3.14) and (8.1.1)

$$\sigma_{j2}(x_1,0) = -q_2^0 d_{jk}\frac{\mathrm{Re}(F_k)}{s_0\kappa_{22}}\frac{1}{\pi}\int_a^b \frac{\Delta(t)}{t-x_1}dt \tag{4.3.21}$$

therefore (appendix 8.2.3)

$$p_j(x_1) = Q_j\left\{\begin{array}{ll} a+b-2x_1-2\Delta_1(x_1) & (-\infty,a) \\ a+b-2x_1 & (a,b) \\ a+b-2x_1+2\Delta_1(x_1) & (b,\infty) \end{array}\right\} \tag{4.3.22}$$

where

$$\Delta_1(x) = \sqrt{(x-a)(x-b)} \tag{4.3.23}$$

and

$$Q_j = -q_2^0 d_{jk} \frac{\mathrm{Re}(F_k)}{2 s_0 \kappa_{22}} \tag{4.3.24}$$

Finally, we insure that condition 1 is satisfied.

Since $A_0(\xi)$ is a linear combination of the $A_j(\xi)$'s we need only study the behaviour of $\phi(x,y)$, for x,y "large".

From (4.3.5) and (4.3.15) we see that

$$\phi_{,y}(x,y) = \frac{q_2^0}{2} y \frac{1}{\pi} \int_a^b \frac{\Delta(t)}{y^2 + (t-x)^2} dt \tag{4.3.25}$$

therefore

$$\phi(x,y) = \frac{q_2^0}{4} \frac{1}{\pi} \int_a^b \Delta(t) \log \left| \frac{y^2 + (t-x)^2}{(t-x)^2} \right| dt + \phi(x,0) \tag{4.3.26}$$

Also, from (4.3.3),(4.3.14) and (8.1.1)

$$\phi_{,x}(x,0) = \frac{q_2^0}{2} \frac{1}{\pi} \int_a^b \frac{\Delta(t)}{x-t} dt$$

and $\quad \phi\left(\frac{a+b}{2}, 0\right) = 0$

therefore

$$\phi(x,0) = \frac{q_2^0}{2} \frac{1}{\pi} \int_a^b \Delta(t) \log \left| \frac{x-t}{\frac{a+b}{2} - t} \right| dt \tag{4.3.27}$$

and, combining (4.3.26) and (4.3.27)

$$\phi(x,y) = \frac{q_2^0}{4} \frac{1}{\pi} \int_a^b \Delta(t) \log \left| \frac{y^2 + (t-x)^2}{(\frac{a+b}{2} - t)^2} \right| dt \tag{4.3.28}$$

Clearly $\phi(x,y) \sim O(\log(x^2+y^2))$ as $\sqrt{x^2+y^2} \to \infty$ and condition 1 is satisfied.

4.4 The Solution of Problem 2

In the general solution (2.3.1), (2.3.3), (2.3.16), (2.3.17) take $\omega(z) = 0$ and

$$S_j(z) = \Omega_j(x, y) + i\Psi_j(x, y) \tag{4.4.1}$$

where

$$\Omega_j(x, y) = -\frac{1}{4}\mathcal{F}^{-1}[|\xi|^{-1}A_j(\xi)e^{-|\xi y|}; x] \tag{4.4.2}$$

then

$$\sigma_{j2}^2(x_1, 0) = \frac{1}{2}\mathcal{F}^{-1}[i\,\mathrm{sgn}(\xi)A_j(\xi); x_1] \tag{4.4.3}$$

and boundary condition 2 is satisfied automatically. Now condition 3i is equivalent to

$$\frac{\partial}{\partial x_1}[\![u_k^2(x_1, 0)]\!] = G_k(x_1) = \left\{ \begin{array}{ll} g_k(x_1) & x_1 \in (a, b) \\ 0 & x_1 \in (a, b)^c \end{array} \right\} \tag{4.4.4}$$

where the $g_k(x_1)$'s are, as yet, undetermined functions, each satisfying the condition

$$\int_a^b g_k(t)dt = 0 \tag{4.4.5}$$

Now, via the Cauchy Riemann equations, we have that

$$u_{k,x_1}^2(x_1, 0) = (B_{kj} + \overline{B}_{kj})\Omega_{j,x}(x_1, 0) - i(B_{kj} - \overline{B}_{kj})\Omega_{j,y}(x_1, 0)$$

Therefore boundary condition 3i is satisifed if

$$b_{kj}\mathcal{F}^{-1}[A_j(\xi); x_1] = G_k(x_1) \qquad x_1 \in \mathbb{R}$$

or equivalently, if

$$A_j(\xi) = d_{jk}\frac{1}{\sqrt{2\pi}}\int_a^b g_k(t)e^{i\xi t}dt \tag{4.4.6}$$

Condition 1 may now be verified as follows: observe, from (4.4.4), that the indefinite integral

$$\int g_k(t)dt = \int_a^t g_k(x)dx = [\![u_k^2(t,0)]\!]$$

Therefore

$$A_j(\xi) = -id_{jk}\xi\frac{1}{\sqrt{2\pi}}\int_a^b [\![u_k^2(t,0)]\!]e^{i\xi t}dt \tag{4.4.7}$$

and it now follows, by virtue of (8.1.5), that

$$\Omega_j(x,y) = \frac{d_{jk}}{4}\frac{1}{\pi}\int_a^b [\![u_k^2(t,0)]\!]\frac{x-t}{(x-t)^2+y^2}dt \tag{4.4.8}$$

these functions clearly exhibit the desired behaviour. Now combining (4.4.6), (4.4.3) and making use of (8.1.1) we see that

$$\sigma_{j2}^2(x_1,0) = -d_{jk}\frac{1}{2\pi}\int_a^b \frac{g_k(t)}{t-x_1}dt \tag{4.4.9}$$

Therefore condition 3ii is satisfied if $g_k(t)$ is given by the singular integral equation

$$\frac{d_{jk}}{2}\frac{1}{\pi}\int_a^b \frac{g_k(t)}{t-x_1}dt = \delta_{j2}\sigma + p_j(x_1) \qquad x_1 \in (a,b) \tag{4.4.10}$$

with subsidiary condition (4.4.5). It now follows (appendix 8.2.1) that

$$g_k(t) = -\frac{2b_{kj}}{\Delta(t)}\frac{1}{\pi}\int_a^b \frac{\Delta(x)}{x-t}\{\delta_{j2}\sigma + p_j(x)\}dx \tag{4.4.11}$$

Therefore, substituting (4.4.11) into (4.4.9), switching the order of integration and using (8.2.2) reveals that for $x_1 \in (a,b)^c$

$$\sigma_{j2}^2(x_1,0) = \frac{\operatorname{sgn}(a-x_1)}{\Delta_1(x_1)}\frac{1}{\pi}\int_a^b \frac{\Delta(t)}{t-x_1}\{\delta_{j2}\sigma + p_j(t)\}dt \tag{4.4.12}$$

where

$$\Delta_1(x) = \sqrt{(x-a)(x-b)} \tag{4.4.13}$$

Before evaluating the stress intensity factors and crack surface discontinuities it is worth noting from (4.3.22), (8.2.3), (8.2.7), that (4.4.11) and (4.4.12) may be alternately expressed as

$$\begin{aligned} g_k(t) &= 2b_{kj}\frac{1}{\Delta(t)}\left\{Q_j\left[\left(\frac{b-a}{2}\right)^2 - 2\left(\frac{a+b}{2}-t\right)^2\right]\right. \\ &\left. - \delta_{j2}\sigma\left[\frac{a+b}{2}-t\right]\right\} \qquad t \in (a,b) \end{aligned} \tag{4.4.14}$$

$$\begin{aligned} \sigma_{j2}^2(x_1,0) &= \frac{\operatorname{sgn}(a-x_1)}{\Delta_1(x_1)}\left\{Q_j\left[2\left(\frac{a+b}{2}-x_1\right)^2 - \left(\frac{b-a}{2}\right)^2\right]\right. \\ &\left. + \delta_{j2}\sigma\left[\frac{a+b}{2}-x_1\right]\right\} - \{Q_j(a+b-2x_1) + \delta_{j2}\sigma\} \qquad x_1 \in (a,b)^c \end{aligned} \tag{4.4.15}$$

4.5 The Stress Intensity Factors and Crack Surface Discontinuities

The mode I, II, and III stress intensity factors at the crack tips are defined by

$$k_I(a) = \Sigma_2(a), \qquad k_{II}(a) = \Sigma_1(a), \qquad k_{III}(a) = \Sigma_3(a), \tag{4.5.1}$$

$$k_I(b) = \Sigma_2(b), \qquad k_{II}(b) = \Sigma_1(b), \qquad k_{III}(b) = \Sigma_3(b), \tag{4.5.2}$$

where

$$\Sigma_j(a) = \lim_{x_1 \to a-} \sqrt{2(a - x_1)}\sigma_{j2}^2(x_1, 0) \qquad (j = 1, 2, 3) \tag{4.5.3}$$

and

$$\Sigma_j(b) = \lim_{x_1 \to b-} \sqrt{2(x_1 - b)}\sigma_{j2}^2(x_1, 0) \qquad (j = 1, 2, 3) \tag{4.5.4}$$

Therefore, by virtue of (4.4.15)

$$\Sigma_j(a) = \delta_{j2}\sigma\sqrt{\frac{b-a}{2}} + Q_j\left(\frac{b-a}{2}\right)^{3/2} \tag{4.5.5}$$

$$\Sigma_j(b) = \delta_{j2}\sigma\sqrt{\frac{b-a}{2}} - Q_j\left(\frac{b-a}{2}\right)^{3/2} \tag{4.5.6}$$

which yields

$$\begin{array}{lllllll}
k_I(a) & = & \sigma\sqrt{\frac{b-a}{2}} + Q_2\left(\frac{b-a}{2}\right)^{3/2} & \quad & k_I(b) & = & \sigma\sqrt{\frac{b-a}{2}} - Q_2\left(\frac{b-a}{2}\right)^{3/2} \\
k_{II}(a) & = & Q_1\left(\frac{b-a}{2}\right)^{3/2} & & k_{II}(b) & = & -Q_1\left(\frac{b-a}{2}\right)^{3/2} \\
k_{III}(a) & = & Q_3\left(\frac{b-a}{2}\right)^{3/2} & & k_{III}(b) & = & -Q_3\left(\frac{b-a}{2}\right)^{3/2}
\end{array} \tag{4.5.7}$$

Now, from (4.4.4)

$$[\![u_k^2(x_1,0)]\!] = \int_a^{x_1} g_k(t)dt \qquad x_1 \in (a,b) \tag{4.5.8}$$

or, substituting (4.4.14) into (4.5.8) and using (8.2.7) and (8.2.8)

$$[\![u_k^2(x_1,0)]\!] = -b_{kj}\{2\delta_{j2}\sigma + (a+b-2x_1)Q_j\}\Delta(x_1) \tag{4.5.9}$$

In contrast to the isotropic case [12], observe that the heat flux $\mathbf{q}^0$ produces opening mode stress intensity factors $k_I(a), k_I(b)$, and tearing mode stress intensity factors $k_{III}(a)$, $k_{III}(b)$ as well as sliding mode stress intensity facotrs $k_{II}(a)$ and $k_{II}(b)$. It has the effect of decreasing these quantities at one tip and increasing them at the other. If σ is small enough, and in particular if it is zero, the mode I stress intensity factors change sign from one tip to the other thereby indicating the possiblility of crack closure. In order to ensure the validity of the above solution therefore, we require that $[\![u_2^2(x_1,0)]\!] > 0$ for all $x_1 \in (a,b)$, Since $b_{22} < 0$ (Clements [5]) this leads to the condition

$$\sigma > -\frac{|q_2^0 \mathrm{ReF}_2|}{2 \mathrm{s}_0 \kappa_{22} \mathrm{b}_{22}} \left(\frac{b-a}{2}\right) \tag{4.5.10}$$

It is interesting to observe that this condition is identical to the condition that would result from requiring $k_I(a)$, $k_I(b) > 0$. It should be noted, however, that this situation does not, in general, occur.

If, for example, we let $q_2^0 = 0$ and apply additional shear stresses σ_{21}, σ_{23} then

$$k_I(a) = k_I(b) = \sigma\sqrt{\frac{b-a}{2}}$$

$$[\![u_2^2(x_1,0)]\!] = -2b_{22}\sigma - 2(b_{21}\sigma_{21} + b_{23}\sigma_{23})$$

Clearly, for any material, orientated in such a way that b_{21} and/or b_{23} are non-zero, it is possible to apply a loading which results in

$$k_1 > 0 \qquad \text{and} \qquad [\![u_2(x_1,0)]\!] < 0$$

or vice-versa.

4.6 Partial Closure (On The Right)

Whenever condition (4.5.10) fails to hold, the normal displacement discontinuity becomes negative at one of the crack tips and the preceeding solution ceases to be valid. This behaviour occurs at the tip $x_1 = a$, if $q_2^0\text{Re}(\text{F}_2) < 0$ and at the tip $x_1 = b$, if $q_2^0\text{Re}(\text{F}_2) > 0$. In such circumstances the problem has to be reformulated, if more meaningful results are to be obtained. This may be achieved by introducing a frictionless contact zone which is bounded by the offending tip and an interior point c of the crack (a, b). It is assumed that the heat flows freely across this

interval and that the crack surfaces separate "smoothly" at $x = c$ Here we may agree to deal with the case for which $q_2^0\text{Re}(\text{F}_2) > 0$ so that the contact zone is an interval of the from $a < c \le x_1 \le b$. As before, the uniform field of Chapter 3.2 is disturbed by the presence of the crack. In this case however, the perturbed field must satisfy the following boundary conditions:

(1) $\theta^p(x_1, x_2), \sigma_{ij}^p(x_1, x_2) \sim O(r^{-1})$ and $u_k^p(x_1, x_2) \sim O(\ln r)$

as $r = \sqrt{x_1^2 + x_2^2} \to \infty$.

(2) $[\![\sigma_{i2}^p(x_1, 0)]\!] = 0 \qquad x_1 \in \mathbb{R}$

(3) $[\![u_k^p(x_1, 0)]\!] = 0, \quad k = 1, 2, 3 \qquad x_1 \in (a, b)^c$

$[\![u_2^p(x_1, 0)]\!] = 0 \qquad x_1 \in (c, b)$

$\sigma_{i2}^p(x_1, 0+) = 0, \quad i = 1, 3 \qquad x_1 \in (a, b)$

$\sigma_{22}^p(x_1, 0+) = -\sigma \qquad x_1 \in (a, c)$

$\lim_{x_1 \to c+} \sigma_{22}^p(x_1, 0) < \infty$

(4) $[\![q_2^p(x_1, 0)]\!] = 0 \qquad x_1 \in \mathbb{R}$

(5) $[\![\theta^p(x_1, 0)]\!] = 0 \qquad x_1 \in (a, c)^c$

$q_2^p(x_1, 0+) = -q_2^0 \qquad x_1 \in (a, c)$

As before, this problem may be solved by superimposing the solutions of two other problems; 3 and 4, say. Problem 3 is identical to Problem 1 of section 3 except that the thermal barrier, in this case, is restricted to the

interval (a, c). Therefore, by analogy with problem 1 we discover that

$$\sigma^3_{j2}(x_1, 0+) = p_j(x_1) = \left\{ \begin{array}{ll} Q_j(a+c-2x_1) - 2Q_j\Delta_1(x_1) & (-\infty, a) \\ Q_j(a+c-2x_1) & (a, c) \\ Q_j(a+c-2x_1) + 2Q_j\Delta_1(x_1) & (c, \infty) \end{array} \right\} \tag{4.6.1}$$

where Q_j is given by (4.3.24) and $\Delta_1(x) = \sqrt{(x-c)(x-a)}$.

Problem 4 may be stated as follows:

Problem 4. Find a generalized plane strain solution $u^4_k(x_1, x_2)$ of the equations of isothermal anisotropic linear elasicity in $\mathbb{R}^3$ cut along $x_2 = 0, a < x_1 < b, -\infty < x_3 < \infty$ subject to the boundary conditions:

(1) $\sigma^4_{ij}(x_1, x_2) \sim O(r^{-2}), u^4_k(x_1, x_2) \sim O(r^{-1})$ as $r = \sqrt{x_1^2 + x_2^2} \to \infty$.

(2) $[\![\sigma^4_{i2}(x_1, 0]\!] = 0 \qquad x_1 \in \mathbb{R}$

(3) $[\![u^4_k(x_1, 0]\!] = 0 \qquad k = 1, 2, 3 \qquad x_1 \in (a, b)^c$

$[\![u^4_2(x_1, 0]\!] = 0 \qquad x_1 \in (c, b)$

$\sigma^4_{i2}(x_1, 0+) = -p_i(x_1), \qquad i = 1, 3 \qquad x_1 \in (a, b)$

$\sigma^4_{22}(x_1, 0+) = -\sigma - p_2(x_1) \qquad x_1 \in (a, c)$

$\lim_{x_1 \to c+} \sigma^4_{22}(x_1, 0) < \infty$

The solution of Problem 4 is obtained in the same manner as the solution of Problem 2: the same representations (4.4.1) and (4.4.2) will insure

conditions 1 and 2 are satisfied and that

$$[\![u^4_{k,x_1}(x_1,0)]\!] = b_{kj}\mathcal{F}^{-1}[A_j(\xi);x_1] \tag{4.6.2}$$

so that the first two parts of condition 3 will also be satisfied if

$$A_j(\xi) = \frac{d_{jk}}{\sqrt{2\pi}}\int_a^b g_k(t)e^{i\xi t}dt \tag{4.6.3}$$

where

$$\int_a^b g_k(t)dt = 0 \qquad (k = 1,2,3) \tag{4.6.4}$$

and, accomodating for the contact interval

$$g_2(t) = 0 \qquad t \in (c,b) \tag{4.6.5}$$

Again, we may write

$$\sigma^4_{j2}(x_1,0) = -d_{jk}\frac{1}{2\pi}\int_a^b \frac{g_k(t)}{t-x_1}dt$$

or, defining

$$P_j(t) = d_{jk}g_k(t) \qquad (j = 1,2,3) \tag{4.6.6}$$

$$\sigma^4_{j2}(x_1,0) = \frac{-1}{2\pi}\int_a^b \frac{P_j(t)}{t-x_1}dt \tag{4.6.7}$$

Therefore, recalling (4.6.1) and combining (4.6.4) and (4.6.6), boundary condition 3iii is satisifed if $P_j(t)$ $(j = 1,3)$ is given by the singular integral equation

$$\frac{1}{\pi}\int_a^b \frac{P_j(t)}{t-x}dt = 2Q_j\{a+c-2x+2H(x-c)\sqrt{(x-a)(x-c)}\} \qquad x \in (a,b) \tag{4.6.8}$$

with subsidiary condition

$$\int_a^b P_j(t)dt = 0 \tag{4.6.9}$$

and where $H(a-b) = \left\{ \begin{array}{ll} 1 & a > b \\ & \\ 0 & a < b \end{array} \right\}$ is the unit Heaviside function.

It now follows (appendix 8.2.1) that

$$P_j(t) = -2Q_j \frac{1}{\Delta(t)} \left\{ \frac{1}{\pi} \int_a^b \frac{(a+c-2x)\Delta(x)}{x-t} dx + \frac{2}{\pi} \int_c^b \frac{\Delta(x)\sqrt{(x-a)(x-c)}}{x-t} dx \right\} \tag{4.6.10}$$

or, from (8.2.3), (8.2.7)

$$P_j(t) = \frac{Q_j}{2} \left\{ 8H(c-t)\sqrt{(c-t)(t-a)} - \frac{(c-a)^2}{\Delta(t)} \right\} \tag{4.6.11}$$

Now, substituting (4.6.11) into (4.6.7) and making use of (8.2.2), (8.2.3) reveals that

$$\begin{aligned} \sigma_{j2}^4(x_1, 0) &= H[(x_1-b)(x_1-a)]\mathrm{sgn}(a-x_1) \frac{Q_j(c-a)^2}{4\sqrt{(x_1-b)(x_1-a)}} \\ &+ H[(x_1-c)(x_1-a)]\mathrm{sgn}(a-x_1) 2Q_j \sqrt{(x_1-c)(x_1-a)} \\ &- Q_j(a+c-2x_1) \qquad (j = 1, 3) \end{aligned} \tag{4.6.12}$$

Now, from condition (4.6.5), (4.6.6)

$$g_2(t) = b_{2j}P_j(t) = 0 \qquad t \in (c, b) \tag{4.6.13}$$

or equivalently, recalling (4.6.11)

$$P_2(t) = \frac{b_{21}Q_1 + b_{23}Q_3}{2b_{22}} \cdot \frac{(c-a)^2}{\Delta_1(t)} \qquad t \in (c, b) \tag{4.6.14}$$

Therefore boundary condition 3iv is satisfied if $P_2(t), t \in (a, c)$, is given by the singular integral equation

$$-\frac{1}{2\pi}\int_a^b \frac{P_2(t)}{t-x}dt = -\sigma - p_2(x) \qquad x \in (a, c)$$

with subsidiary condition $\frac{1}{\pi}\int_a^b P_2(t)dt = 0$. However, since $P_2(t)$ is known on the interval (c, b) this equation may be re-written as

$$\frac{1}{\pi}\int_a^c \frac{P_2(t)}{t-x}dt = 2(\sigma + p_2(x)) - \frac{1}{\pi}\int_c^b \frac{P_2(t)}{t-x}dt \qquad x \in (a, c) \tag{4.6.15}$$

with subsidiary condition

$$\frac{1}{\pi}\int_a^c P_2(t)dt = \frac{-1}{\pi}\int_c^b P_2(t)dt \tag{4.6.16}$$

It now follows (appendix 8.2.1) that

$$P_2(t) = \frac{1}{\sqrt{(t-a)(c-t)}}\left\{\frac{C_0}{\pi} - \frac{1}{\pi}\int_a^c \frac{\sqrt{(x-a)(c-x)}}{x-t}r(x)dx\right\} \tag{4.6.17}$$

where

$$C_0 = -\int_c^b P_2(t)dt \tag{4.6.18}$$

and

$$r(x) = 2(\sigma + p_2(x)) - \frac{1}{\pi}\int_c^b \frac{P_2(s)}{s-x}ds \tag{4.6.19}$$

Now

$$\frac{1}{\pi}\int_a^c \frac{\sqrt{(x-a)(c-x)}}{x-t} r(x)dx$$

$$\begin{aligned}
&= \frac{2}{\pi}\int_a^c \frac{\sqrt{(x-a)(c-x)}}{x-t}(\sigma + p_j(x))dx \\
&- \frac{1}{\pi}\int_a^c \frac{\sqrt{(x-a)(c-x)}}{x-t}\frac{1}{\pi}\int_c^b \frac{P_2(s)}{s-x} ds dx \\
&= I_1(t) - I_2(t) \qquad , \qquad \text{say}
\end{aligned}$$

and

$$\begin{aligned}
I_2(t) &= \frac{1}{\pi}\int_c^b \frac{P_2(s)}{s-t}\frac{1}{\pi}\int_a^c \frac{\sqrt{(x-a)(c-x)}}{x-t} dx ds \\
&- \frac{1}{\pi}\int_c^b \frac{P_2(s)}{s-t}\frac{1}{\pi}\int_a^c \frac{\sqrt{(x-a)(c-x)}}{x-s} dx\, ds
\end{aligned}$$

or, by virtue of (8.2.3)

$$I_2(t) = -\frac{C_0}{\pi} + \frac{1}{\pi}\int_c^b \frac{\sqrt{(s-c)(s-a)}}{s-t} P_2(s) ds$$

then, from (4.6.15) and (8.2.4)

$$I_2(t) = \frac{-C_0}{\pi} + \frac{b_{21}Q_1 + b_{23}Q_3}{2b_{22}}(c-a)^2\left(1 - \sqrt{\frac{c-t}{b-t}}\right) \tag{4.6.20}$$

Also, from (4.6.1) and (8.2.3)

$$I_1(t) = \sigma(a+c-2t) + Q_2((a+c-2t)^2 - \frac{1}{2}(c-a)^2) \tag{4.6.21}$$

Thus, combining (4.6.20), (4.6.21) and (4.6.15) we see that

$$
\begin{aligned}
P_2(t) &= -\frac{2\sigma(a+c-2t)+2Q_2(a+c-2t)^2-Q_2(c-a)^2}{2\sqrt{(c-t)(t-a)}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.6.22) \\
&- \frac{b_{21}Q_1+b_{23}Q_3}{2b_{22}}\cdot\frac{(c-a)^2}{\sqrt{(c-t)(t-a)}}\left\{1-\sqrt{\frac{c-t}{b-t}}\right\}, t\in(a,c)
\end{aligned}
$$

Now (4.6.22) and (4.6.11) may be combined and summarized by

$$
\begin{aligned}
P_2(t) &= \frac{H(c-t)}{\sqrt{(c-t)(t-a)}}\left\{4Q_2(c-t)(t-a)-\sigma(a+c-2t)-\frac{b_{2j}Q_j(c-a)^2}{2b_{22}}\right\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.6.23) \\
&+ \frac{[b_{21}Q_1+b_{23}Q_3](c-a)^2}{2b_{22}\sqrt{(b-t)(t-a)}}, \qquad t\in(a,b)
\end{aligned}
$$

Therefore, substituting (4.6.23) into (4.6.7) and using (8.2.2), (8.2.3) we find that

$$
\begin{aligned}
\sigma_{22}^4(x_1,0) &= -\sigma-Q_2(a+c-2x_1) \\
&- H[(x_1-b)(x_1-a)]\mathrm{sgn}(a-x_1)\frac{[b_{21}Q_1+b_{23}Q_3](c-a)^2}{4b_{22}\sqrt{(x_1-b)(x_1-a)}} \\
&+ H[(x_1-c)(x_1-a)]\mathrm{sgn}(a-x_1)\left\{2Q_2\sqrt{(x_1-c)(x_1-a)}\right. \\
&+ \left.\frac{b_{2j}Q_j(c-a)^2+2\sigma b_{22}(a+c-2x_1)}{4b_{22}\sqrt{(x_1-c)(x_1-a)}}\right\} \qquad (4.6.24)
\end{aligned}
$$

which together with the last part of condition 3 yields the expression

$$
c = a+\frac{2\sigma b_{22}}{b_{2j}Q_j} \qquad (4.6.25)
$$

for the determination of c.

By virtue of (4.6.12), (4.6.24) and the definitions (4.5.1)-(4.5.4) of the stress intensity factors we now discover that

$$\begin{aligned}
k_I(a) &= \frac{2b_{2j}Q_j}{b_{22}}\left(\frac{c-a}{2}\right)^{3/2} - \frac{b_{21}Q_1 + b_{23}Q_3}{b_{22}}\left(\frac{c-a}{2}\right)^2\sqrt{\frac{2}{b-a}} \\
k_I(b) &= \frac{b_{21}Q_1 + b_{23}Q_3}{b_{22}}\left(\frac{c-a}{2}\right)^2\sqrt{\frac{2}{b-a}} \\
k_{II}(a) &= Q_1\left(\frac{c-a}{2}\right)^2\sqrt{\frac{2}{b-a}}\ ,\quad k_{II}(b) = -Q_1\left(\frac{c-a}{2}\right)\sqrt{\frac{2}{b-a}} \\
k_{III}(a) &= Q_3\left(\frac{c-a}{2}\right)^2\sqrt{\frac{2}{b-a}}\ ,\quad k_{III}(b) = -Q_3\left(\frac{c-a}{2}\right)\sqrt{\frac{2}{b-a}}
\end{aligned} \tag{4.6.26}$$

Lastly we observe that

$$\begin{aligned}
[\![u^4_{2,x_1}(x_1,0)]\!] &= H[(b-x_1)(x_1-a)]b_{2j}P_j(x_1) \\
&= -\frac{H[(c-x_1)(x_1-a)]}{2\sqrt{(c-x_1)(x_1-a)}}\{2\sigma b_{22}(a+c-2x_1) \\
&+ b_{2j}Q_j[(c-a)^2 - 8(c-x_1)(x_1-a)]\}
\end{aligned} \tag{4.6.27}$$

and hence from (4.6.25), (8.2.7), (8.2.8) that the crack opening displacement is given by

$$[\![u^4_2(x_1,0]\!] = -4\sigma b_{22}H[(c-x_1)(x_1-a)]\frac{(c-x_1)}{(c-a)}\sqrt{(c-x_1)(x_1-a)} \tag{4.6.28}$$

4.7 Partial Closure (On The Left)

If condition (4.5.10) fails to hold and $q_2^0 \mathrm{ReF}_2 < 0$ the contact zone is an interval of the type $a \le x_1 \le c < b$. An analysis similar to that presented in section 6 shows that the edge of the contact zone is given by

$$c = b + \frac{2\sigma b_{22}}{b_{2j} Q_j} \tag{4.7.1}$$

the crack opening displacement by

$$[\![u_2^4(x_1, 0]\!] = -4\sigma b_{22} H[(b - x_1)(x_1 - c)] \frac{(x_1 - c)}{(b - c)} \sqrt{(b - x_1)(x_1 - c)} \tag{4.7.2}$$

and the stress intensity factors by

$$\begin{aligned}
k_I(a) &= -\frac{b_{21}Q_1 + b_{23}Q_3}{b_{22}} \left(\frac{b-c}{2}\right)^2 \sqrt{\frac{2}{b-a}} \\
k_I(b) &= \frac{b_{21}Q_1 + b_{23}Q_3}{b_{22}} \left(\frac{b-c}{2}\right)^2 \sqrt{\frac{2}{b-a}} - \frac{2b_{2j}Q_j}{b_{22}} \left(\frac{b-c}{2}\right)^{3/2} \\
k_{II} &= Q_1 \left(\frac{b-c}{2}\right)^2 \sqrt{\frac{2}{b-a}} \,, \quad k_{II}(b) = -Q_1 \left(\frac{b-c}{2}\right)^2 \sqrt{\frac{2}{b-a}} \\
k_{III} &= Q_3 \left(\frac{b-c}{2}\right)^2 \sqrt{\frac{2}{b-a}} \,, \quad k_{III}(b) = -Q_3 \left(\frac{b-c}{2}\right)^2 \sqrt{\frac{2}{b-a}}
\end{aligned} \tag{4.7.3}$$

Chapter 5

The Disturbance of a Uniform Heat Flow by Two Line Cracks in an Infinite Anisotropic Thermoelastic Solid

5.1 Introduction

The crack configuration to be considered in this chapter consists of two co-linear cracks of equal length in an infinite anisotropic solid. The reference axes are set up so that the cracks, which are thermally insulated and traction free, are given by:

$$0 < a < |x_1| < b \quad , \quad x_2 = 0 \quad , \quad -\infty < x_3 < \infty$$

Under the conditions of generalized plane strain, the thermoelastic field in the vicinity of the cracks is determined for the case in which the thermomechanical loading consists of a uniform heat flux:

$$\mathbf{q}^0 = (q_1^0, q_2^0, q_3^0)$$

and a uniaxial tension in the x_2-direction:

$$\sigma_{22} = \sigma$$

Choosing appropriate Fourier type integrals to represent the arbitrary analytic functions which appear in the general solution results in a system of singular integral equations. These equations are solved analytically and expressions for the stress intensity factors are obtained, along with a condition that guarantees their validity, by insuring that each crack remains open.

5.2 Resolution into Problems 1 and 2

In the absence of the cracks the thermoelastic field is identical to that of Chapter 3.2. The presence of the cracks disturbs this field thereby creating a new field which can be written:

$$\mathbf{F}^n(x_1, x_2) = \mathbf{F}^0(x_1, x_2) + \mathbf{F}^p(x_1, x_2)$$

The stress free movement of the cracks requires that

$$\sigma_{i2}^{n}(x_1,0)=0 \qquad |x_1|\in(a,b)$$

while the prevention of heat flow across the cracks requires that

$$q_2^{n}(x_1,0)=0 \qquad |x_1|\in(a,b)$$

We also require the thermoelastic field, far from the cracks, to be the same as the field in the unflawed solid. Hence, recalling the form of $\mathbf{F}^0(x_1,x_2)$ together with appropriate continuity requirements, we seek a particular solution of the equations of generalized plane strain which satisfies the following boundary conditions:

(1) $\theta^p(x_1,x_2), \sigma_{ij}^p(x_1,x_2) \sim O(r^{-1})$ and $u_k^p(x_1,x_2) \sim O(\ln r)$
as $r=\sqrt{x_1^2+x_2^2}\to\infty$.

(2) $[\![\sigma_{i2}^p(x_1,0)]\!]=0 \qquad x_1\in\mathbb{R}$

(3) $[\![u_k^p(x_1,0)]\!]=0 \qquad |x_1|\in(a,b)^c$

$\sigma_{i2}^p(x_1,0+)=-\delta_{i2}\sigma \qquad |x_1|\in(a,b)$

(4) $[\![q_2^p(x_1,0)]\!]=0 \qquad x_1\in\mathbb{R}$

(5) $[\![\theta^p(x_1,0)]\!]=0 \qquad |x_1|\in(a,b)^c$

$q_2^p(x_1,0+)=-q_2^0 \qquad |x_1|\in(a,b)$

This problem may be solved by superimposing the solution's of two component problems. In the first the thermal field is completely determined

by replacing the cracks with a continuous distribution of heat dipoles. This leads, however, to an unwanted stress developing upon the cracks. In the second problem, which is isothermal, this stress is counter balanced by an equal and opposite stress, leaving the cracks stress free, as desired. Problems 1 and 2 may thus be stated as follows:

<u>Problem 1.</u> Find a generalized plane strain solution $\theta^1(x_1,x_2), u_k^1(x_1,x_2)$ of the equations of anisotropic linear thermoelasticity in $\mathbb{R}^3$ subject to the bounday conditions:

(1) $\theta^1(x_1,x_2), \sigma_{ij}^1(x_1,x_2) \sim O(r^{-1}) \quad , \quad u_k^1(x_1,x_2) \sim O(\ln r)$

as $\quad r = \sqrt{x_1^2+x_2^2} \to \infty.$

(2) $[\![\sigma_{i2}^1(x_1,0)]\!] = 0 \qquad x_1 \in \mathbb{R}$

(3) $[\![u_k^1(x_1,0)]\!] = 0 \qquad x_1 \in \mathbb{R}$

(4) $[\![q_2^1(x_1,0)]\!] = 0 \qquad x_1 \in \mathbb{R}$

(5) $[\![\theta^1(x_1,0)]\!] = 0 \qquad |x_1| \in (a,b)^c$

$q_2^1(x_1,0+) = -q_2^0 \qquad |x_1| \in (a,b)$

<u>Calculate:</u> $\quad \sigma_{i2}^1(x_1,0\pm) = p_i(x_1).$

<u>Problem 2</u>. Find a generalized plane strain solution $u_k^2(x_1,x_2)$ of the equations of isothermal anisotropic linear elasticity in $\mathbb{R}^3$ cut along $x_2 = 0$, $a < |x_1| < b$, $-\infty < x_3 < \infty$ subject to the boundary conditions:

(1) $\sigma_{ij}^2(x_1,x_2) \sim O(r^{-2}), u_k^2(x_1,x_2) \sim O(r^{-1})$ as $r = \sqrt{x_1^2+x_2^2} \to \infty.$

(2) $[\![\sigma_{i2}^2(x_1,0)]\!] = 0 \qquad x_1 \in \mathbb{R}$

(3) $[\![u_k^2(x_1,0)]\!] = 0 \qquad |x_1| \in (a,b)^c$

$\sigma_{i2}^2(x_1,0+) = -\delta_{i2}\sigma - p_i(x_1) \qquad |x_1| \in (a,b)$

5.3 The Solution of Problem 1

In the general solution (2.3.1), (2.3.3), (2.3.16), (2.3.17) take

$$\omega(z) = \kappa_0^{-1}\{\phi(x,y) + i\psi(x,y)\} \tag{5.3.1}$$

$$S_j(z) = \Omega_j(x,y) + i\Psi_j(x,y) \tag{5.3.2}$$

where

$$\phi(x,y) = \frac{1}{\sqrt{8\pi}}\int_{-\infty}^{\infty}\frac{A_0(\xi)}{|\xi|}(e^{-|\xi y|}e^{-i\xi x} - 1)d\xi \tag{5.3.3}$$

$$\Omega_j(x,y) = \frac{1}{\sqrt{8\pi}}\int_{-\infty}^{\infty}\frac{A_j(\xi)}{|\xi|}(e^{-|\xi v|}e^{-i\xi x} - 1)d\xi \tag{5.3.4}$$

and $\kappa_0 = i\, s_0\kappa_{22}$

then

$$\theta^1(x_1,x_2) = \frac{-2}{s_0\kappa_{22}}\phi_{,y}(x_1 + r_0x_2, s_0x_2) \tag{5.3.5}$$

$$q_2^1(x_1,x_2) = -2\phi_{,xx}(x_1 + r_0x_2, s_0x_2) \tag{5.3.6}$$

and boundary condition 4 is satisfied automatically.

Now condition $5i$ is equivalent to

$$\frac{\partial}{\partial x_1}[\theta^1(x_1,0] = G_0(x_1) = \left\{ \begin{array}{ll} g_0(x_1) & x_1 \in L \\ 0 & x_1 \in L^c \end{array} \right\}$$

where $L_1 = (-b,-a)$, $L_2 = (a,b)$, $L = L_1 \cup L_2$ and $g_0(x_1)$ is an, as yet, undetermined function which may be characterized by

$$g_0(x_1) = \begin{cases} g_{01}(x_1) & x_1 \in L_1 \\ g_{02}(x_1) & x_1 \in L_2 \end{cases}$$

and which satisfies the condition

$$\int_L g_0(t)dt = \int_L \operatorname{sgn}(t) g_0(t)dt = 0 \qquad (5.3.7)$$

Hence, condition $5ii$ is satisfied if

$$\frac{-2i}{s_0\kappa_{22}}\mathcal{F}^{-1}[\xi A_0(\xi);x_1] = G_0(x_1) \qquad x_1 \in \mathbb{R}$$

or equivalently, if

$$A_0(\xi) = \frac{s_0\kappa_{22}}{2}\,\frac{i}{\xi}\,\frac{1}{\sqrt{2\pi}}\int_L g_0(t)e^{i\xi t}dt \qquad (5.3.8)$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform defined by (4.3.9). Substituting (5.3.8) into (5.3.6) and making use of (8.1.1) reveals that

$$q_2^1(x_1,0) = \frac{-s_0\kappa_{22}}{2}\,\frac{1}{\pi}\int_L \frac{g_0(t)}{t-x_1}dt \qquad (5.3.9)$$

Therefore, condition $5ii$ is satisfied if $g_0(t)$ is given by the singular integral equation

$$\frac{1}{\pi}\int_L \frac{g_0(t)}{t-x_1}dt = \frac{2}{s_0\kappa_{22}}q_2^0 \qquad x_1 \in L \tag{5.3.10}$$

with subsidiary condition (5.3.7). It now follows (appendix 8.3.1) that

$$g_0(t) = \frac{-2}{s_0\kappa_{22}}q_2^0\frac{\operatorname{sgn}(t)}{K}\frac{b^2E - t^2K}{\Delta(t)} \tag{5.3.11}$$

where $K = K(k_1)$ and $E = E(k_1)$ are complete elliptic integrals of the first and second kinds respectively with parameter $k_1 = \sqrt{1 - a^2/b^2}$ and

$$\Delta(t) = \sqrt{(t^2 - a^2)(b^2 - t^2)} \tag{5.3.12}$$

The temperature field is now completely determined and, from (5.3.5), (5.3.8) and (8.1.2), is given by

$$\theta^1(x,y) = \frac{1}{2\pi}\int_L g_0(t)\tan^{-1}\left(\frac{x-t}{y}\right)dt \tag{5.3.13}$$

where $\qquad x = x_1 + r_0x_2 \;\; ; \;\; y = s_0x_2.$

Repeating steps (3.16) through (3.20) of chapter 4.3, we note that boundary condition 3 is satisifed if

$$A_j(\xi) = \frac{1}{s_0\kappa_{22}}d_{jk}\mathrm{Re}(F_k)A_0(\xi) \tag{5.3.14}$$

boundary condition 2 is satisfied automatically and

$$\sigma_{j2}^1(x_1,0) = -i\mathcal{F}^{-1}[\operatorname{sgn}(\xi)A_j(\xi)\,;\,x_1] \tag{5.3.15}$$

The $p_i(x)$ are calculated, indirectly, via

$$\sigma^1_{j2,x_1}(x_1,0) = -\mathcal{F}^{-1}[|\xi|A_j(\xi)\,;\,x_1] \tag{5.3.16}$$

substituting (5.3.14), (5.3.8) into (5.3.16) and using (8.1.1), it can be shown that

$$\sigma^1_{j2,x_1}(x_1,0) = \frac{1}{2}d_{jk}\mathrm{Re}(F_k)\frac{1}{\pi}\int_L \frac{g_0(t)}{t-x_1}dt \tag{5.3.17}$$

Then, substituting (5.3.11) into (5.3.17) and using (8.3.3) and (8.3.5) reveals that

$$\sigma^1_{j2,x_1}(x_1,0) = \left\{ \begin{array}{ll} -2Q_j\left\{1 - \frac{b^2E - x_1^2K}{K\Delta_1(x_1)}\right\} & |x_1| < a \\ -2Q_j & a < |x_1| < b \\ -2Q_j\left\{1 + \frac{b^2E - x_1^2K}{K\Delta_1(x_1)}\right\} & |x_1| > b \end{array} \right\} \tag{5.3.18}$$

where

$$\Delta_1(x) = \sqrt{(x^2-a^2)(x^2-b^2)}. \tag{5.3.19}$$

Integrating (5.3.18), with $\sigma^1_{j2}(-\infty,0) = 0$, it follows (8.4.14) that

$$p_j(x_1) = \left\{ \begin{array}{ll} \frac{\pi b Q_j}{K}[1 - \Lambda_0(\alpha,k_1)]\mathrm{sgn}(x_1) - 2Q_jx_1[1 - \sqrt{\frac{a^2-x_1^2}{b^2-x_1^2}}] & |x_1| \in (0,a) \\ 2Q_j[\frac{\pi b}{2K} - |x_1|]\mathrm{sgn}(x_1) & |x_1| \in (a,b) \\ \frac{\pi b Q_j}{K}[1 - \Lambda_0(\beta,k_1)]\mathrm{sgn}(x_1) - 2Q_jx_1[1 - \sqrt{\frac{x_1^2-b^2}{x_1^2-a^2}}] & |x_1| \in (b,\infty) \end{array} \right\} \tag{5.3.20}$$

where Λ_0 is Heuman's Lambda Function (Byrd and Friedman, [13]) and $k_1 = \sqrt{1 - \frac{a^2}{b^2}}$,

$$\alpha = \sin^{-1}\left\{\sqrt{\frac{b^2(a^2-x_1^2)}{a^2(b^2-x_1^2)}}\right\}, \beta = \sin^{-1}\left\{\sqrt{\frac{x_1^2-b^2}{x_1^2-a^2}}\right\}$$

Finally, we insure that condition 1 is satisfied. By the same reasoning of chapter 4.3, consider, from (5.3.13)

$$\phi_{,y}(x,y) = \frac{-s_0\kappa_{22}}{4}\frac{1}{\pi}\int_L g_0(t)\tan^{-1}\left(\frac{x-t}{y}\right)dt \tag{5.3.21}$$

If $|x|,|y| >> |t|$ then

$$\tan^{-1}\left(\frac{x-t}{y}\right) \simeq \tan^{-1}\left(\frac{x}{y}\right) - \frac{yt}{x^2+y^2} \tag{5.3.22}$$

In which case, by virtue of (5.3.7)

$$\phi_{,y}(x,y) \simeq \frac{s_0\kappa_{22}}{4}\frac{1}{\pi}\int_L g_0(t)\frac{yt}{x^2+y^2}dt$$

therefore

$$\phi(x,y) \simeq \frac{s_0\kappa_{22}}{8}\frac{1}{\pi}\int_L t g_0(t)\log\left|\frac{x^2+y^2}{x^2}\right|dt + \phi(x,0) \tag{5.3.23}$$

Also, from (5.3.3), (5.3.8)

$$\phi_{,x}(x,0) = \frac{s_0\kappa_{22}}{4}\frac{1}{\sqrt{2\pi}}\int_L g_0(t)\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-i\xi(x-t)}}{|\xi|}d\xi\right\}dt \tag{5.3.24}$$

However, $g_0(t)$ is an odd function, therefore

$$\begin{aligned}\phi_{,x}(x,0) &= \frac{s_0\kappa_{22}}{4}\frac{i}{\sqrt{2\pi}}\int_L g_0(t)\mathcal{F}^{-1}[|\xi|^{-1}\sin(\xi t)\,;\,x]dt \\ &= \frac{s_0\kappa_{22}}{4}\frac{1}{2\pi}\int_L g_0(t)\log\left|\frac{x+t}{x-t}\right|dt \end{aligned} \tag{5.3.25}$$

by (8.1.4). Now, observe that $|x| >> |t|$

$$\log\left|\frac{x+t}{x-t}\right| \simeq \frac{2t}{x} \tag{5.3.26}$$

Combining (5.3.25), (5.3.26) and recalling $\phi(0,0) = 0$ reveals that

$$\phi(x,0) \simeq \frac{s_0\kappa_{22}}{4}\frac{1}{\pi}\int_L tg_0(t)\log|x|dt \tag{5.3.27}$$

finally, combining (5.3.27) and (5.3.23)

$$\phi(x,y) \simeq \frac{s_0\kappa_{22}}{8}\frac{1}{\pi}\int_L tg_0(t)\log|x^2+y^2|dt \tag{5.3.28}$$

Clearly, $\phi(x,y) \sim O(\log(x^2+y^2))$ as $x^2+y^2 \to \infty$ and condition 1 is satisfied.

5.4 The Solution of Problem 2

In the general solution (2.3.1), (2.3.3), (2.3.16), (2.3.17) take $\omega(z) = 0$ and

$$S_j(z) = \Omega_j(x,y) + i\Psi_j(x,y) \tag{5.4.1}$$

where

$$\Omega_j(x,y) = -\frac{1}{4}\mathcal{F}^{-1}[|\xi|^{-1}A_j(\xi)e^{-|\xi y|};x] \tag{5.4.2}$$

then

$$\sigma^2_{j2}(x_1,0) = \frac{1}{2}\mathcal{F}^{-1}[i\mathrm{sgn}(\xi)A_j(\xi);x_1] \tag{5.4.3}$$

and condition 2 is satisfied automatically. Now condition 3i is equivalent to

$$\frac{\partial}{\partial x_1}[\![u^2_k(x_1,,0)]\!] = G_k(x_1) = \left\{\begin{array}{ll} g_k(x_1) & x_1 \in L \\ 0 & x_1 \in L^c \end{array}\right\} \tag{5.4.4}$$

where the $g_k(x_1)$ are, as yet, undetermined functions, each satisfying the condition

$$\int_L g_k(t)dt = \int_L \text{sgn}(t) g_k(t)dt = 0 \tag{5.4.5}$$

Therefore condition $3i$ is satisfied if

$$A_j(\xi) = \frac{d_{jk}}{\sqrt{2\pi}} \int_L g_k(t) e^{i\xi t} dt \tag{5.4.6}$$

Repeating steps 4.6 through 4.8 of Chapter 4.4, reveals that

$$\Omega_j(x,y) = \frac{d_{jk}}{4} \frac{1}{\pi} \int_a^b [\![u_k^2(t,0)]\!] \frac{x-t}{(x-t)^2+y^2} dt \tag{5.4.7}$$

again, confirming that condition 1 is satisfied.

Now, combining (5.4.3), (5.4.6) and making use of (8.1.1) we see that

$$\sigma_{j2}^2(x_1,0) = \frac{-d_{jk}}{2\pi} \int_L \frac{g_k(t)}{t-x_1} dt \tag{5.4.8}$$

Therefore condition $3ii$ is satisfied is $g_k(t)$ is given by the singular integral equation

$$\frac{d_{jk}}{2\pi} \int_L \frac{g_k(t)}{t-x_1} dt = \delta_{j2}\sigma + p_j(x_1) \qquad x_1 \in L \tag{5.4.9}$$

with subsidiary condition (5.4.6). It now follows (8.3.1) that

$$g_k(t) = \frac{-2b_{kj}\text{sgn}(t)}{\Delta(t)} \left\{ c_j + \frac{1}{\pi} \int_L \frac{[\delta_{j2}\sigma + p_j(x)]\text{sgn}(x)\Delta(x))}{x-t} dx \right\} \tag{5.4.10}$$

where

$$\Delta(x) = \sqrt{(x^2-a^2)(b^2-x^2)} \tag{5.4.11}$$

and

$$c_j = \frac{-b}{2\pi K}\int_L \frac{1}{\Delta(t)}\int_L \frac{[\delta_{j2}\sigma + p_j(x)]\mathrm{sgn}(x)\Delta(x)}{x-t}dx\,dt \tag{5.4.12}$$

Substituting (5.4.10) into (5.4.8), reversing the order of integration and using (8.3.3) now reveals that for $x_1 \in L^c$

$$\sigma_{j2}^2(x_1,0) = \frac{\mathrm{sgn}(a^2-x_1^2)}{\Delta_1(x_1)}\left\{c_j + \frac{1}{\pi}\int_L \frac{[\delta_{j2}\sigma + p_j(t)]\mathrm{sgn}(t)\Delta(t)}{t-x_1}dt\right\} \tag{5.4.13}$$

where

$$\Delta_1(x) = \sqrt{(x^2-a^2)(x^2-b^2)} \tag{5.4.14}$$

Before considering the stress intensity factors and crack surface discontinuities it is worth spending some effort upon simplifying expressions (5.4.12), (5.4.10) and (5.4.13). In each case we begin by recalling the relevant form (5.3.20) of $p_j(x)$.

Separating the integrand into odd and even parts we see that

$$c_j = \frac{-b}{2\pi K}\delta_{j2}\sigma\int_L \frac{1}{\Delta(t)}\int_L \frac{\mathrm{sgn}(x)\Delta(x)}{x-t}dx\,dt$$

and then, by virtue of (8.3.4), (8.4.10) and (8.4.11)

$$c_j = \frac{\delta_{j2}\sigma}{2K}\{2b^2E - (a^2+b^2)K\} \tag{5.4.15}$$

Now consider

$$I(x) = \frac{1}{\pi}\int_L \frac{[\delta_{j2}\sigma + p_j(t)]\mathrm{sgn}(t)\Delta(t)}{t-x}dt \tag{5.4.16}$$

Using the results (8.3.4), (8.3.6), this may be rewritten as

$$\begin{aligned} I(x) &= (\delta_{j2}\sigma - 2Q_j x)\left[\frac{a^2+b^2}{2} - x^2 - \mathrm{sgn}(a^2-x^2)H((a^2-x^2)(b^2-x^2))\Delta_1(x)\right] \\ &+ \frac{2Q_j bx}{K}\int_a^b \frac{\Delta(t)}{t^2-x^2}dt \end{aligned} \tag{5.4.17}$$

Therefore, regarding the stress intensity factors, we note, via (8.4.12,13), that the appropriate limiting values are given by

$$\begin{aligned} I(\pm a) &= (\delta_{j2}\sigma \mp 2Q_j a)\left(\frac{b^2-a^2}{2}\right) \pm \frac{2Q_j ab^2}{K}(K-E) \\ I(\pm b) &= (\delta_{j2}\sigma \mp 2Q_j b)\left(\frac{a^2-b^2}{2}\right) \pm \frac{2Q_j b^2}{K}\left(\frac{a^2}{b}K - bE\right) \end{aligned} \tag{5.4.18}$$

Regarding the crack surface discontinuities, we note, via (8.4.15), that for $x \in L$

$$\begin{aligned} I(x) &= (\delta_{j2}\sigma - 2Q_j x)\left(\frac{a^2+b^2}{2} - x^2\right) \\ &+ \frac{2Q_j bx}{K}\left\{-bE + (a^2+b^2-x^2)\frac{K}{b} - \frac{\Delta(x)}{|x|}KZ(\beta,k)\right\} \end{aligned} \tag{5.4.19}$$

where $Z(\beta,k)$ is the Jacobian Zeta function with $k^2 = 1 - a^2/b^2$ and $\beta = \sin^{-1}\left(\sqrt{\frac{b^2-x^2}{b^2-a^2}}\right)$

5.5 The Stress Intensity Factors and Crack Surface Discontinuities

The mode I, II and III stress intensity factors at the crack tips are given respectively by

$$k_I(\pm a) = \Sigma_2(\pm a), \quad k_{II}(\pm a) = \Sigma_1(\pm a), \quad k_{III}(\pm a) = \Sigma_3(\pm a) \tag{5.5.1}$$

$$k_I(\pm b) = \Sigma_2(\pm b), \quad k_{II}(\pm b) = \Sigma_1(\pm b), \quad k_{III}(\pm b) = \Sigma_3(\pm b) \tag{5.5.2}$$

where

$$\Sigma_j(a) = \lim_{x_1 \to a-} \sqrt{2(a - x_1)}\sigma_{j2}^2(x_1, 0) \tag{5.5.3}$$

$$\Sigma_j(-a) = \lim_{x_1 \to -a+} \sqrt{2(x_1 + a)}\sigma_{j2}^2(x_1, 0) \tag{5.5.4}$$

$$\Sigma_j(b) = \lim_{x_1 \to b+} \sqrt{2(x_1 - b)}\sigma_{j2}^2(x_1, 0) \tag{5.5.5}$$

and

$$\Sigma_j(-b) = \lim_{x_1 \to -b-} \sqrt{2(-b - x_1)}\sigma_{j2}^2(x_1, 0) \tag{5.5.6}$$

$(j = 1, 2, 3)$.

Therefore, by virtue of (5.4.13, 15, 18), we discover that

$$\Sigma_j(\pm a) = \frac{\sigma\delta_{j2}[b^2E - a^2K] \mp aQ_j[2b^2E - (a^2 + b^2)K]}{K\sqrt{a(b^2 - a^2)}} \tag{5.5.7}$$

and

$$\Sigma_j(\pm b) = \frac{\sigma\delta_{j2}[b^2K - b^2E] \pm bQ_j[2b^2E - (a^2 + b^2)K]}{K\sqrt{b(b^2 - a^2)}} \tag{5.5.8}$$

which yields

$$
\begin{aligned}
k_I(\pm a) &= \frac{\sigma[b^2E - a^2K] \mp aQ_2[2b^2E - (a^2+b^2)K]}{K\sqrt{a(b^2-a^2)}} \\
k_I(\pm b) &= \frac{\sigma[b^2K - b^2E] \pm bQ_2[2b^2E - (a^2+b^2)K]}{K\sqrt{b(b^2-a^2)}} \\
k_{II}(\pm a) &= \mp\frac{aQ_1[2b^2E - (a^2+b^2)K]}{K\sqrt{a(b^2-a^2)}} \\
k_{II}(\pm b) &= \pm\frac{bQ_1[2b^2E - (a^2+b^2)K]}{K\sqrt{b(b^2-a^2)}} \\
k_{III}(\pm a) &= \mp\frac{aQ_3[2b^2E - (a^2+b^2)K]}{K\sqrt{a(b^2-a^2)}} \\
k_{III}(\pm b) &= \pm\frac{bQ_3[2b^2E - (a^2+b^2)K]}{K\sqrt{b(b^2-a^2)}} \qquad (5.5.9)
\end{aligned}
$$

Now, from (5.4.4), we have that

$$
[\![u_k^2(x_1,0)]\!] = \int_{-b}^{x_1} g_k(t)dt \qquad x_1 \in L
$$

Therefore, from (5.4.10,15,19) we see that

$$
[\![u_k^2(x_1,0)]\!] = -2b_{kj}[\delta_{j2}\sigma I_1(x_1) - 2Q_j(I_2(x_1) - I_3(x_1))] \qquad (5.5.10)
$$

where

$$
\begin{aligned}
I_1(x) &= \operatorname{sgn}(x)\int_{-b}^{x}\frac{b^2E - Kt^2}{K\Delta(t)}dt \qquad (5.5.11) \\
I_2(x) &= \frac{1}{2}\int_{-b}^{x}\frac{|t|(a^2+b^2-2t^2)}{\Delta(t)}dt = \frac{1}{2}\operatorname{sgn}(x)\Delta(x) \qquad (5.5.12) \\
I_3(x) &= \int_{-b}^{x}\frac{|t|\left[-b^2\frac{E}{K} + a^2 + b^2 - t^2 - b\frac{\Delta(t)}{|t|}Z(\beta,k)\right]}{\Delta(t)}dt \qquad (5.5.13)
\end{aligned}
$$

Integrals $I_1(x)$ and $I_3(x)$ appear in the appendix; see (8.4.16) and (8.4.17), respectively.

Therefore

$$\begin{aligned} [\![u_k^2(x_1,0)]\!] &= 2b_{kj}Q_j\{2bx_1Z(\chi,k) - \Delta(x_1)\mathrm{sgn}(x_1\} \\ &- 2\sigma bb_{k2}Z(\chi,k) \end{aligned} \tag{5.5.14}$$

where

$$\chi = \sin^{-1}\sqrt{\frac{b^2-x^2}{b^2-a^2}}$$

In contrast to the isotropic case [14], we observe that the heat flux $\mathbf{q}^0$ produces opening mode stress intensity factors $k_I(\pm a)$, $k_I(\pm b)$, and tearing mode stress intensity factors $k_{III}(\pm a)$, $k_{III}(\pm b)$ as well as sliding mode stress intensity factors $k_{II}(\pm a)$ and $k_{II}(\pm b)$. It has the effect of decreasing these quantities at one tip of each crack and increasing them at the other. If σ is small enough, and in particular if it is zero, the mode I stress intensity factors change sign from one tip to the other thereby indicating the possibility of crack closure. In order to ensure the validity of the above solution therefore, we require that $[\![u_2^2(x_1,0)]\!] > 0$ for all $x_1 \in L$. Since $b_{22} < 0$ (Clements [5]) this leads to the condition

$$\sigma > \max_{x \in L}\left\{\frac{b_{2j}Q_j}{b_{22}}v(x)\right\} \tag{5.5.15}$$

where

$$v(x) = 2x - \frac{\sqrt{(b^2 - x^2)(x^2 - a^2)}}{bZ[\chi, k_1]}\operatorname{sgn}(x)$$

A numerical investigation reveals that $v(x)$ is odd and monotone increasing on L. In addition, we discover that $|v(a)| < |v(b)|$, which enables us to rewrite (5.5.15) in the simpler form.

$$\sigma > \frac{|q_2^0 \mathrm{Re}(\mathrm{F}_2)|}{2\mathrm{s}_0 \mathrm{b}_{22} \kappa_{22}} \left\{ \frac{(a^2 + b^2)K - 2b^2 E}{b(K - E)} \right\}$$

Finally, it is noted once again that the above condition is identical to the one which would result from requiring all four mode I stress intensity factors to be positive.

Chapter 6

The Disturbance of a Uniform Heat Flow by an Infinite Array of Line Cracks in an Infinite Anisotropic Thermoelastic Solid

6.1 Introduction

The crack configuration to be considered in this chapter consists of an infinite array of equally spaced co-linear cracks in an infinite anisotropic solid. The reference axes are set up so that the cracks, which are thermally insulated and traction free, are given by:

$$|x_1 - 2n\pi| < a : n \in I \ , \ \ x_2 = 0 \ , \ \ -\infty < x_3 < \infty$$

Under the conditions of generalized plane strain, the thermoelastic field in the vicinity of the cracks is determined for the case in which the thermoemachanical loading consists of a uniform heat flux:

$$\mathbf{q}^0 = (q_1^0, q_2^0, q_3^0)$$

and a uniaxial tension in the x_2-direction:

$$\sigma_{22} = \sigma$$

Choosing appropriate Fourier series to represent the arbitrary analytic functions which appear in the general solution results in a system of singular integral equations. These equations are solved analytically and expressions for the stress intensity factors are obtained, along with a condition that guarantees their validity, by insuring that each crack remains open.

6.2 Resolutions into Problems 1 and 2

The geometry of the crack configuration dictates that the field must be periodic, with period 2π. That is

$$\mathbf{F}(x_1, x_2) = \mathbf{F}(x_1 + 2n\pi, x_2) \quad : \quad n \in I$$

Therefore, in the absence of the cracks the thermoelastic field should also exhibit this periodic behavior. A uniform heat flux, however, can only

be produced by a linear temperature field, indicating that $\theta^0 = \theta^0(x_2)$. Similarly, the equations (2.2.7) indicate that the displacement field must also be a function of x_2, only. Hence the thermoelastic field in the unflawed solid is identical to that of Chapter 3.3. The presence of the cracks disturbs this field thereby creating a new field which can be written:

$$\mathbf{F}^n(x_1, x_2) = \mathbf{F}^0(x_2) + \mathbf{F}^p(x_1, x_2)$$

Due to the periodic nature of the field, we may limit our study to the single crack in the region $-\pi < x_1 < \pi$. The stress free movement of the crack requires that

$$\sigma^n_{i2}(x_1, 0) = 0 \qquad |x_1| < a$$

while the prevention of heat flow across the crack requires that

$$q^n_2(x_1, 0) = 0 \qquad |x_1| < a$$

We also require the thermoelastic field, far from the crack, to be the same as the field in the unflawed solid. Unlike the preceeding problems, we find that the infinite array of heat dipole distributions causes the temperature (in problem 1) to be constant, for x_2 large. Note, however, that this will not adversely alter the field in the unflawed solid because θ_0 in (3.3.8) is arbitrary. Hence, recalling the form of $\mathbf{F}^0(x_1)$, together with appropriate

continuity requirements, we seek a particular solution of the equations of generalized plane strain which satisfies the following boundary conditions:

(1) $\theta^p(x_1,x_2), \sigma^p_{ij}(x_1,x_2) \sim O(1)$ and $u^p_k(x_1,x_2) \sim O(x_2)$

as $x_2 \to \infty$

(2) $\theta^p(-\pi^+,x_2) = \theta^p(\pi^-,x_2) \qquad x_2 \in \mathbb{R}$

$u^p_k(-\pi^+,x_2) = u^p_k(\pi^-,x_2) \qquad x_2 \in \mathbb{R}$

(3) $[\![\sigma^p_{j2}(x_1,0]\!] = 0 \qquad |x_1| < \pi$

(4) $[\![u^p_k(x_1,0)]\!] = 0 \qquad a < |x_1| < \pi$

$\sigma^p_{j2}(x_1,0^+) = -\delta_{j2}\sigma \qquad |x_1| < a$

(5) $[\![q^p_2(x_1,0)]\!] = 0 \qquad |x_1| < \pi$

(6) $[\![\theta^p(x_1,0)]\!] = 0 \qquad a < |x_1| < \pi$

$q^p_2(x_1,0^+) = -q^0_2 \qquad |x_1| < a$

This problem may be solved by superimposing the solution's of two component problems. In the first the thermal field is completely determined by replacing the crack with a continuous distribution of heat dipoles. This leads, however, to an unwanted stress developing upon the crack. In the second problem, which is isothermal, this stress is counter balanced by an equal and opposite stress, leaving the crack stress free, as desired. Problems 1 and 2 may thus be stated as follows:

Problem 1. Find a generalized plane strain solution $\theta^1(x_1,x_2)$, $u_k^1(x_1,x_2)$ of the equations of anisotropic linear thermoelasticity in the region $|x_1| < \pi$ subject to the boundary conditions:

(1) $\theta^1(x_1,x_2), \sigma_{ij}^1(x_1,x_2) \sim O(1)$, $u_k^1(x_1,x_2) \sim O(x_2)$

as $x_2 \to \infty$

(2) $u_k^1(-\pi^+,x_2) = u_k^1(\pi^-,x_2)$ $\quad x_2 \in \mathbb{R}$

$\theta^1(-\pi^+,x_2) = \theta^1(\pi^-,x_2)$ $\quad x_2 \in \mathbb{R}$

(3) $[\![\sigma_{j2}^1(x_1,0)]\!] = 0$ $\quad |x_1| < \pi$

(4) $[\![u_k^1(x_1,0)]\!] = 0$ $\quad |x_1| < \pi$

(5) $[\![q_2^1(x_1,0)]\!] = 0$ $\quad |x_1| < \pi$

(6) $[\![\theta^1(x_1,0]\!] = 0$ $\quad a < |x_1| < \pi$

$q_2^1(x_1,0^+) = -q_2^0 |x_1| < a$

Calculate: $\sigma_{i2}^1(x_1,0^+) = p_i(x_1)$

Problem 2. Find a generalized plane strain solution $u_k^2(x_1,x_2)$ of the equations of isothermal anisotropic linear elasticity in $|x_1| < \pi$ cut along $x_2 = 0, |x_1| < a$, $-\infty < x_3 < \infty$ subject to the boundary conditions:

(1) $\sigma^2_{ij}(x_1,x_2) \sim O(x_2^{-1})$, $u^2_k(x_1,x_2) \sim O(1)$

as $x_2 \to \infty$

(2) $u^2_k(-\pi^+,x_2) = u^2_k(\pi^-,x_2)$ $\qquad x_2 \in \mathbb{R}$

(3) $[\![\sigma^2_{j2}(x_1,0)]\!] = 0$ $\qquad |x_1| < \pi$

(4) $[\![u^2_k(x_1,0)]\!] = 0$ $\qquad a < |x_1| < \pi$

$\sigma^2_{j2}(x_1,0^+) = -\delta_{j2}\sigma - p_j(x_1)$ $\qquad |x_1| < a$

6.3 The Solution of Problem 1

In the general solution (2.3.1), (2.3.3), (2.3.16), (2.3.17) take

$$\omega(z) = \kappa_0^{-1}\{\phi(x,y) + i\psi(x,y)\} \tag{6.3.1}$$

and

$$S_j(z) = \Omega_j(x,y) + i\Psi(x,y) \tag{6.3.2}$$

where

$$\phi(x,y) = -\frac{s_0\kappa_{22}}{2}E^0 y + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |n|^{-1} E^0_n e^{-|ny|} e^{inx} \tag{6.3.3}$$

$$\Omega_j(x,y) = E^0_j y + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |n|^{-1} E_{nj} e^{-|ny|} e^{inx} \tag{6.3.4}$$

and $\kappa_0 = i\, s_0\kappa_{22}$ then

$$\theta^1(x_1,x_2) = E^0 + \operatorname{sgn}(y)\frac{2}{s_0\kappa_{22}} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} E^0_n e^{-|ny|} e^{inx} \tag{6.3.5}$$

$$q_2^1(x_1,x_2) = 2 \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |n| E_n^0 e^{-|ny|} e^{inx} \tag{6.3.6}$$

where $\quad x = x_1 + \lambda_0 x_2 \ , \ y = s_0 x_2$

Therefore boundary conditions 1, 2ii and 5 are satisfied automatically. Now, condition 6i is equivalent to

$$\frac{\partial}{\partial x_1}[\![\theta^1(x_1,0)]\!] = G_0(x_1) = \begin{cases} g_0(x_1) & |x_1| < a \\ 0 & a < |x_1| < \pi \end{cases}$$

where $g_0(x_1)$ is an, as yet, undetermined function satisfying the condition

$$\int_{-a}^{a} g_0(t)dt = 0 \tag{6.3.7}$$

Hence, condition 6i is satisfied if

$$\frac{4i}{s_0\kappa_{22}} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} nE_n^0 e^{inx_1} = G_0(x_1) \qquad |x_1| < \pi$$

or equivalently, if

$$E_n^0 = \frac{s_0\kappa_{22}}{8i\,n}\frac{1}{\pi}\int_{-a}^{a} g_0(t)e^{-int}dt \tag{6.3.8}$$

Substituting (6.3.8) into (6.3.6) reveals that

$$q_2^1(x_1,0) = -\frac{s_0\kappa_{22}}{2\pi}\int_{-a}^{a} g_0(t)\frac{\partial}{\partial x_1}\left\{\sum_{n=1}^{\infty}\frac{\cos(n(x_1-t))}{n}\right\}dt$$

therefore, via (8.1.7)

$$q_2^1(x_1,0) = \frac{s_0\kappa_{22}}{4\pi}\int_{-a}^{a} g_0(t)\cot\left(\frac{x_1-t}{2}\right)dt \tag{6.3.9}$$

Now, the trigonometric identity

$$\cot\left(\frac{x-t}{2}\right) = -\tan\left(\frac{x}{2}\right) + \frac{\sec^2(x/2)}{\tan(x/2)-\tan(t/2)} \tag{6.3.10}$$

and condition (6.3.7) can be combined, revealing that

$$q_2^1(x_1,0) = \frac{-s_0\kappa_{22}}{4\pi}\sec^2\left(\frac{x_1}{2}\right)\int_{-a}^{a}\frac{g_0(t)}{\tan(t/2)-\tan(x_1/2)}dt \tag{6.3.11}$$

Therefore condition 6ii is satisfied if $g_0(t)$ is given by the singular integral equation

$$\frac{1}{\pi}\int_{-a}^{a}\frac{g_0(t)}{\tan(t/2)-\tan(x/2)}dt = \frac{4}{s_0\kappa_{22}}q_2^0\cos^2(x/2) \qquad |x|<a \tag{6.3.12}$$

Introducing the change of variables $\tau = \tan(t/2)$; $\xi = \tan(x/2)$ enables us to rewrite (6.3.12) and (6.3.7) in the form

$$\int_{-\alpha}^{\alpha}\frac{h_0(\tau)}{\tau-\xi}d\tau = \frac{4}{s_0\kappa_{22}}q_2^0\frac{1}{\xi^2+1} \qquad |\xi|<\alpha \tag{6.3.13}$$

and

$$\int_{-\alpha}^{\alpha}h_0(\tau)d\tau = 0 \tag{6.3.14}$$

where

$$h_0(\tau) = \frac{2g_0(2\tan^{-1}(\tau))}{\tau^2+1} \quad ; \quad \alpha = \tan(a/2)$$

From (8.2.1), (8.2.6) we now see that

$$h_0(\tau) = \frac{4q_2^0}{s_0\kappa_{22}}\frac{\tau}{\Delta(\tau)}\frac{\sqrt{\alpha^2+1}}{\tau^2+1}$$

where

$$\Delta(\tau) = \sqrt{\alpha^2 - \tau^2}$$

or, in terms of the original variables

$$g_0(t) = \frac{q_2^0}{s_0\kappa_{22}}\sec\left(\frac{a}{2}\right)\frac{\sin(t)\sec^2(t/2)}{\Delta(t)} \tag{6.3.15}$$

where

$$\Delta(t) = \sqrt{\tan^2(a/2) - \tan^2(t/2)} \tag{6.3.16}$$

Note that $g_0(t)$ is an odd function and, from (6.3.8)

$$E_n^0 = E_{-n}^0 = \frac{-s_0\kappa_{22}}{4n}\frac{1}{\pi}\int_0^a g_0(t)\sin(nt)dt \tag{6.3.17}$$

Hence, the temperature field is completely determined and from (6.3.5), (6.3.17) is given by

$$\theta^1(x,y) = E^0 - \mathrm{sgn}(y)\frac{1}{\pi}\int_0^a g_0(t)[J(x+t,y) + J(t-x,y)]dt \tag{6.3.18}$$

where

$$J(x,y) = \frac{1}{2}\tan^{-1}\left[\frac{\sin(x)}{e^{|y|} - \cos(x)}\right] \tag{6.3.19}$$

comes from using the appendix result (8.1.9).

Repeating steps (4.3.16) through (4.3.20) of chapter 4.3, we note that condition 4 is satisfied if

$$E_{nj} = \frac{1}{s_0\kappa_{22}}d_{jk}\mathrm{Re}(F_k)E_n^0 \tag{6.3.20}$$

boundary conditions 2i and 3 are satisfied automatically and

$$\sigma_{j2}^{1}(x_1,0) = -4\sum_{n=1}^{\infty} E_{nj}\sin(n\,x_1) \tag{6.3.21}$$

Then, combining (6.3.21), (6.3.17) and (6.3.20) we find that

$$\sigma_{j2}^{1}(x_1,0) = d_{jk}\mathrm{Re}(F_k)\frac{1}{\pi}\int_0^a g_0(t)\sum_{n=1}^{\infty}\frac{\sin(nx_1)\sin(nt)}{n}dt \tag{6.3.22}$$

or, via (8.1.8)

$$\sigma_{j2}^{1}(x_1,0) = d_{jk}\mathrm{Re}(F_k)\frac{1}{2\pi}\int_0^a \log\left|\frac{\sin\left(\frac{x_1+t}{2}\right)}{\sin\left(\frac{x_1-t}{2}\right)}\right| g_0(t)dt \tag{6.3.23}$$

The $p_i(x)$ are calculated, indirectly, via

$$\frac{\partial}{\partial x_1}[\sigma_{j2}^{1}(x_1,0)] = d_{jk}\mathrm{Re}(F_k)\frac{1}{4\pi}\int_0^a \left[\cot\left(\frac{x_1+t}{2}\right) - \cot\left(\frac{x_1-t}{2}\right)\right] g_0(t)dt \tag{6.3.24}$$

which, recalling (6.3.15), may be written as

$$\frac{\partial}{\partial x_1}[\sigma_{j2}^{1}(x_1,0)] = Q_j\sec(\frac{a}{2})\frac{1}{\pi}\int_0^a \frac{\sec^2(t/2)\sin^2(t)}{\Delta(t)(\cos t - \cos x_1)}dt \tag{6.3.25}$$

where Q_j is given by (4.3.24).

Introducing the change of variables $T = \cos(t)$; $\xi = \cos(x_1)$ enables us to rewrite (6.3.25) as

$$\frac{\partial}{\partial x_1}[\sigma_{j2}^{1}(x_1,0)] = Q_j\frac{2}{\pi}\int_\alpha^1 \sqrt{\frac{1-T}{T-\alpha}}\frac{dT}{T-\xi} \tag{6.3.26}$$

where $\alpha = \cos(a)$

Now, using appendix result (8.2.5) and reverting back to the original variables, we see that

$$\frac{\partial}{\partial x_1}[\sigma_{j2}^1(x_1,0)] = 2Q_j \begin{cases} -1 & |x_1| < a \\ -1 + \sqrt{\dfrac{1-\cos x_1}{\cos(a) - \cos x_1}} & a < |x_1| < \pi \end{cases} \tag{6.3.27}$$

Therefore, since from (6.3.21) $\sigma_{j2}^1(0,0) = 0$, we find that

$$p_i(x) = -2Q_i x \qquad |x| < a \tag{6.3.28}$$

6.4 The Solution of Problem 2

In the general solution (2.3.1), (2.3.3), (2.3.16), (2.3.17) take $\omega(z) = 0$ and

$$S_j(z) = \Omega_j(x,y) + i\,\Psi_j(x,y) \tag{6.4.1}$$

where

$$\Omega_j(x,y) = E_j^0 + \frac{1}{2}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |n|^{-1} E_{nj} e^{-|ny|} e^{inx} \tag{6.4.2}$$

then

$$\sigma_{j2}^2(x_1,0) = i \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \operatorname{sgn}(n) E_{nj} e^{inx_1} \tag{6.4.3}$$

and boundary conditions 1, 2 and 3 are satisfied automatically. Now condition 4i is equivalent to

$$\frac{\partial}{\partial x_1}[\![u_k^2(x_1,0)]\!] = G_k(x_1) = \left\{ \begin{array}{ll} g_k(x_1) & |x_1| < a \\ 0 & a < |x_1| < \pi \end{array} \right\} \tag{6.4.4}$$

where the $g_k(x_1)$'s are, as yet, undetermined functions, each satisfying the condition

$$\int_{-a}^{a} g_k(t)dt = 0 \tag{6.4.5}$$

Therefore boundary condition 4i is satisfied if

$$-b_{kj} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} E_{nj} e^{inx} = G_k(x) \qquad |x| < \pi$$

or equivalently, if

$$E_{nj} = \frac{-d_{jk}}{2\pi} \int_{-a}^{a} g_k(t) e^{-int} dt \tag{6.4.6}$$

Now, combining (6.4.3) and (6.4.6) and making use of (8.1.7) we see that

$$\sigma_{j2}^2(x_1, 0) = \frac{d_{jk}}{2\pi} \int_{-a}^{a} g_k(t) \cot\left(\frac{x_1 - t}{2}\right) dt \tag{6.4.7}$$

Therefore boundary condition 4ii is satisfied if $g_k(t)$ is given by the singular integral equation

$$\frac{d_{jk}}{2\pi} \int_{-a}^{a} g_k(t) \cot\left(\frac{x_1 - t}{2}\right) dt = -\delta_{j2}\sigma - p_j(x_1) \qquad |x_1| < a \tag{6.4.8}$$

with subsidiary condition (6.4.5).

The same process that took us from (3.9) to (3.16) in the previous section may be followed again, revealing that the solution of equation (6.4.8) is given by

$$g_k(t) = -\frac{b_{kj}\sec^2(t/2)}{2\Delta(t)} \frac{1}{\pi} \int_{-a}^{a} \frac{\Delta(y)[\delta_{j2}\sigma + p_j(y)]dy}{\tan(y/2) - \tan(t/2)} \tag{6.4.9}$$

where

$$\Delta(t) = \sqrt{\tan^2(a/2) - \tan^2(t/2)} \tag{6.4.10}$$

Therefore, substituting (6.4.9) into (6.4.7), switching the order of integration and making use of (8.2.2) we see that for $a < |x_1| < \pi$

$$\sigma_{j2}^2(x_1, 0) = \frac{-\text{sgn}(x_1)\sec^2(\frac{x_1}{2})}{2\Delta_1(x_1)} \frac{1}{\pi} \int_{-a}^{a} \frac{\Delta(y)[\delta_{j2}\sigma + p_j(y)]}{\tan(y/2) - \tan(x_1/2)} dy \tag{6.4.11}$$

where

$$\Delta_1(x) = \sqrt{\tan^2(x/2) - \tan^2(a/2)} \tag{6.4.12}$$

6.5 The Stress Intensity Factors and Crack Surface Discontinuities

The mode I, II and III stress intensity factors are again defined by (4.5.1.) - (4.5.4), with a replaced by $-a$, b replaced by a.

Thus, from (6.3.28) and (6.4.11), we find that

$$\begin{aligned} \Sigma_j(-a) &= \lim_{x_1 \to -a^-} \sqrt{2(-a - x_1)}\sigma_{j2}^2(x_1, 0) \\ &= \frac{1}{\sqrt{\sin a}} \frac{1}{\pi} \int_{-a}^{a} K(a, y)[\delta_{j2}\sigma + 2Q_j y] dy \end{aligned} \tag{6.5.1}$$

$$\begin{aligned} \Sigma_j(a) &= \lim_{x_1 \to a^+} \sqrt{2(x_1 - a)}\sigma_{j2}^2(x_1, 0) \\ &= \frac{1}{\sqrt{\sin a}} \frac{1}{\pi} \int_{-a}^{a} K(a, y)[\delta_{j2}\sigma - 2Q_j y] dy \end{aligned} \tag{6.5.2}$$

where

$$K(a,y) = \sqrt{\frac{\tan(\frac{a}{2}) + \tan(\frac{y}{2})}{\tan(\frac{a}{2}) - \tan(\frac{y}{2})}} \tag{6.5.3}$$

which yields

$$k_I(-a) = \frac{1}{\sqrt{\sin a}}\frac{1}{\pi}\int_{-a}^{a} K(a,y)[\sigma + 2Q_2 y]dy$$

$$k_I(a) = \frac{1}{\sqrt{\sin a}}\frac{1}{\pi}\int_{-a}^{a} K(a,y)[\sigma - 2Q_2 y]dy$$

$$k_{II}(-a) = \frac{2Q_1}{\sqrt{\sin a}}\frac{1}{\pi}\int_{-a}^{a} K(a,y)y\,dy \qquad k_{II}(a) = \frac{-2Q_1}{\sqrt{\sin a}}\frac{1}{\pi}\int_{-a}^{a} K(a,y)y\,dy$$

$$k_{III}(-a) = \frac{2Q_3}{\sqrt{\sin a}}\frac{1}{\pi}\int_{-a}^{a} K(a,y)y\,dy \qquad k_{III}(a) = \frac{-2Q_3}{\sqrt{\sin a}}\frac{1}{\pi}\int_{-a}^{a} K(a,y)y\,dy \tag{6.5.4}$$

We now focus attention upon obtaining a condition that will guarantee the validity of the above expressions.

In both of the previous chapters we were able to obtain closed form expressions for $[\![u_2^2(x_1,0)]\!]$. In this chapter a slightly different approach will be taken. Due to the nature of the applied stress it is reasonable to assume that the entire crack will remain fully open, provided it is open at both ends. Consequently, we need only consider the following quantities:

$$S_\ell \equiv \operatorname{sgn}\left\{\lim_{x_1\to -a^+} [\![u_2^2(x_1,0)]\!]\right\} \tag{6.5.5}$$

$$S_r \equiv \operatorname{sgn}\left\{\lim_{x_1\to a^-} [\![u_2^2(x_1,0)]\!]\right\} \tag{6.5.6}$$

First, from (6.4.4), we recall that

$$[\![u_2^2(x_1,0)]\!] = \int_{-a}^{x_1} g_2(t)dt \tag{6.5.7}$$

Then substituting (6.4.9) into the above expression and introducing the change of variables $T = \tan(t/2)$, $Y = \tan(y/2)$, $\alpha = \tan(a/2)$ we find, upon taking the appropriate limits, that

$$S_\ell = \operatorname{sgn}\left\{\int_{-\alpha}^{\alpha}\sqrt{\frac{\alpha+Y}{\alpha-Y}}\frac{1}{Y^2+1}P_2[2\tan^{-1}(-Y)]dY\right\} \tag{6.5.8}$$

$$S_r = \operatorname{sgn}\left\{\int_{-\alpha}^{\alpha}\sqrt{\frac{\alpha+Y}{\alpha-Y}}\frac{1}{Y^2+1}P_2[2\tan^{-1}(Y)]dY\right\} \tag{6.5.9}$$

where

$$P_2(y) = -2b_{22}\sigma + 4b_{2j}Q_j\, y \tag{6.5.10}$$

which, by virtue of (4.3.24), may be written as

$$P_2(y) = -2b_{22}\sigma\left[1 + \frac{q_2^0\mathrm{Re}(F_2)}{s_0\kappa_{22}b_{22}\sigma}y\right] \tag{6.5.11}$$

Noting that $b_{22} < 0$ (Clements[5]) we now discover that

$$S_\ell = \operatorname{sgn}\{I_1 - \beta I_2\} \tag{6.5.12}$$

$$S_r = \operatorname{sgn}\{I_1 + \beta I_2\} \tag{6.5.13}$$

where

$$I_1 = \int_{-\alpha}^{\alpha}\sqrt{\frac{\alpha+Y}{\alpha-Y}}\frac{1}{Y^2+1}dY \tag{6.5.14}$$

$$I_2 = \int_{-\alpha}^{\alpha}\sqrt{\frac{\alpha+Y}{\alpha-Y}}\frac{\tan^{-1}(Y)}{Y^2+1}dY \tag{6.5.15}$$

and

$$\beta = \frac{2q_2^0\mathrm{Re}(F_2)}{s_0\kappa_{22}b_{22}\sigma} \tag{6.5.16}$$

Therefore, observing I_1, $I_2 > 0$, the crack will remain fully open provided that

$$\frac{1}{|\beta|} = \left|\frac{s_0 \kappa_{22} b_{22} \sigma}{2q_2^0 \mathrm{Re}(F_2)}\right| > \frac{I_2}{I_1}$$

or equivalently, if

$$\sigma > \frac{-2|q_2^0 \mathrm{Re}(F_2)|}{s_0 \kappa_{22} b_{22}} \frac{I_2}{I_1} \tag{6.5.17}$$

I_1 may be evaluated exactly; we find that

$$I_1 = \frac{\pi \alpha}{\sqrt{\alpha^2 + 1}} \tag{6.5.18}$$

I_2 may be accurately approximated using the software package QUADPACK [21]. A plot of how I_2/I_1 varies with the crack tip location, a, is shown in Figure 2. Note also that condition (6.5.17) is identical to the one that would result from requiring $k_I(-a), k_I(a) > 0$.

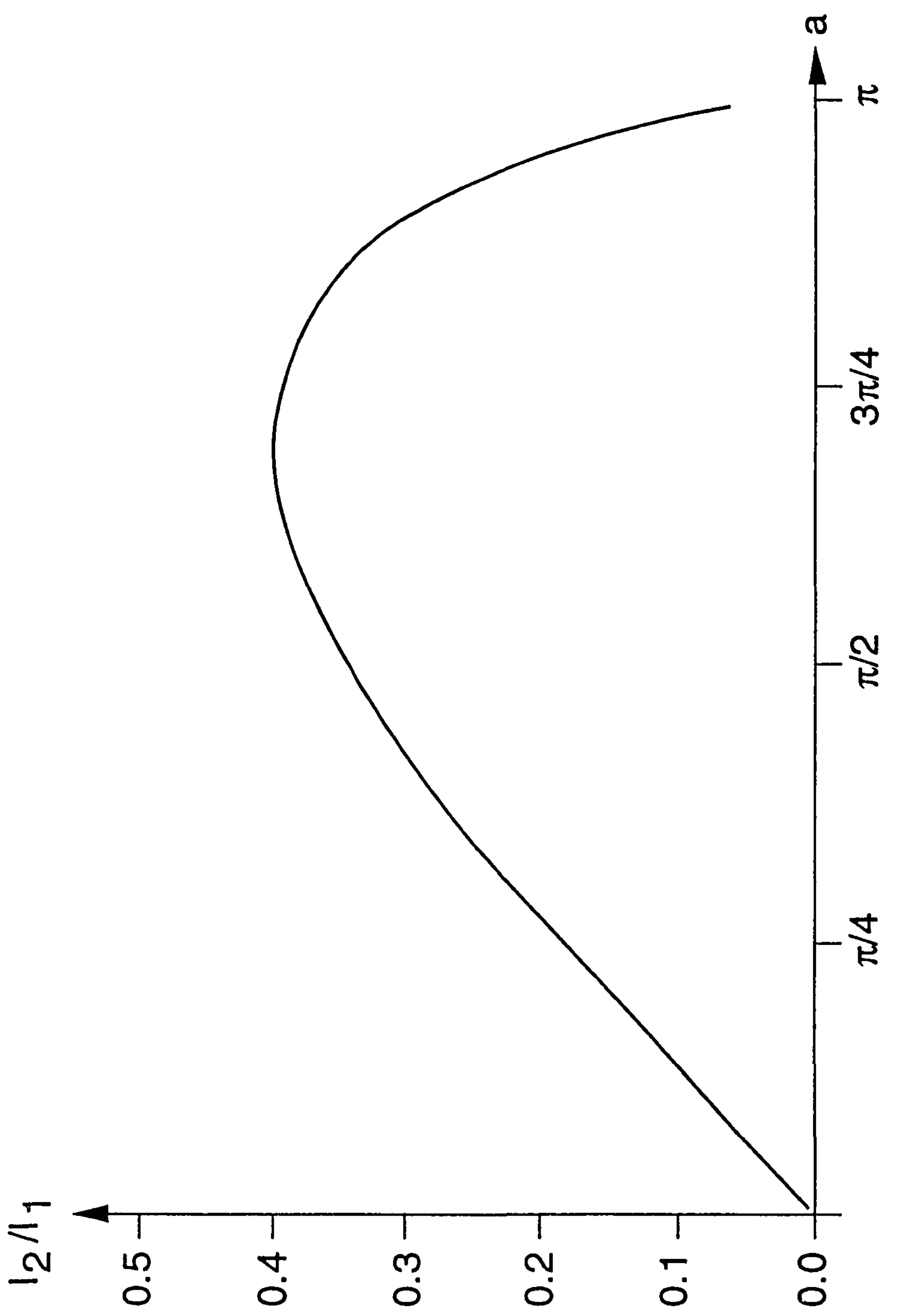

Figure 2: I_2/I_1 v a

Chapter 7

A Single Line Crack in a Semi-infinite Anisotropic Elastic Solid

7.1 Introduction

The final crack configuration to be considered is that of a single crack in a semi-infinite anisotropic solid. The reference axes are set up so that the crack, which is subjected to a prescribed mechanical loading, is given by:

$$0 \leq a < x_1 < b \quad , \quad x_2 = 0 \quad , \quad -\infty < x_3 < \infty$$

while the solid occupies the region $x_1 \geq 0$. Note that the case when the crack occurs at the edge will also be considered.

Under the (isothermal) conditions of generalized plane strain, the elastic field in the vicinity of the crack is determined, numerically, for certain

selected pressures:

$$\sigma_{2j}(x_1, 0) = -p_j(x_1, 0)$$

Choosinig appropriate Fourier type integrals to represent the arbitrary analytic functions which appear in the general solutions of Chapters 2.3 and 2.4, results in a set of triple integral equations, in the case of an internal crack and a double set for the edge crack case. Both these sets of equations may be solved numerically and approximations for the stress intensity factors may be obtained.

7.2 Statement of the Problem

Find a generalized plane strain solution $u_k(x_1, x_2)$ of the equations of isothermal anisotropic linear elasticity in the region $x_1 \geq 0$ cut along $x_2 = 0, -\infty < x_3 < \infty, 0 \leq a < x_1 < b$ subject to the following boundary conditions:

$$\begin{array}{llll}
(1) & u_k(x_1, x_2) \to 0 & \text{as} & \sqrt{x_1^2 + x_2^2} \to \infty \\
(2) & \sigma_{1k}(0, x_2) = 0 & & x_2 \in \mathbb{R} \\
(3) & [\![\sigma_{2j}(x_1, 0)]\!] = 0 & & x_1 \geq 0 \\
(4) & [\![u_k(x_1, 0)]\!] = 0 & & x_1 > b \\
 & \sigma_{2j}(x_1, 0) = -p_j(x_1) & & x_1 \in (a, b)
\end{array}$$

The case of an edge crack may be labeled case(i), in which case $a = 0$, and we impose the additional condition

$$\lim_{x_1 \to 0+} \left[\!\left[\frac{\partial u_k(x_1, 0)}{\partial x_1} \right]\!\right] = 0$$

The case of an internal crack may be labeled case(ii), in which case $a > 0$, and we impose the additional condition

$$[\![u_k(x_1, 0)]\!] = 0 \qquad 0 \leq x_1 < a$$

7.3 The Solution of the Problem

In the general solution (2.3.16), (2.3.17) take $w(z) = 0$ and

$$S_j(z) = \pm i \frac{1}{\sqrt{2\pi}} \int_0^\infty \xi^{-1} \Phi_j(\xi) e^{\pm i \xi z} d\xi \tag{7.3.1}$$

where we choose $+$ for $x_2 > 0$ and $-$ for $x_2 < 0$.

In the general solution (2.4.4) and (2.4.5) take

$$g_j(z) = i \frac{1}{\sqrt{2\pi}} \int_0^\infty \eta^{-1} \Psi_j(\eta) e^{-\eta z} d\eta \tag{7.3.2}$$

then

$$S_j'(z) = -\frac{1}{\sqrt{2\pi}} \int_0^\infty \Phi_j(\xi) e^{\pm i \xi z} d\xi \tag{7.3.3}$$

and

$$g_j'(z) = -i \frac{1}{\sqrt{2\pi}} \int_0^\infty \Psi_j(\eta) e^{-\eta z} d\eta \tag{7.3.4}$$

Therefore, from (2.3.17), (2.4.6) and (2.4.8)

$$\begin{aligned}
\sigma_{j1}(0,x_2) &= -\mathrm{Re}\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty \Psi_j(\eta)e^{-i\eta x_2}d\eta\right. \\
&- \left.\sum_{\alpha=1}^{3}\tau_\alpha L_{j2\alpha}M_{\alpha k}\sqrt{\frac{2}{\pi}}\int_0^\infty \Phi_k(\xi)e^{i\xi\tau_\alpha|x_2|}d\xi\right\} \qquad (7.3.5) \\
&= -\sqrt{\frac{2}{\pi}}\int_0^\infty \Psi_j(\eta)\cos(\eta x_2)d\eta \\
&+ \sqrt{\frac{2}{\pi}}\int_0^\infty \mathrm{Re}\left\{\sum_{\alpha=1}^{3}\tau_\alpha L_{j2\alpha}M_{\alpha k}e^{i\xi\tau_\alpha|x_2|}\right\}\Phi_k(\xi)d\xi \qquad (7.3.6)
\end{aligned}$$

Noting that $\mathrm{Re}\{-i\xi\tau_\alpha\} = \xi s_\alpha > 0$ permits us to make use of (8.1.3), revealing that boundary condition 2 is satisfied if

$$\Psi_j(\eta) = \frac{1}{\pi}\int_0^\infty \sum_{\alpha=1}^{3}\left[\overline{L}_{j2\alpha}\overline{M}_{\alpha k}\frac{i\xi\overline{\tau}_\alpha^2}{\eta^2-\xi^2\overline{\tau}_\alpha^2} - L_{j2\alpha}M_{\alpha k}\frac{i\xi\tau_\alpha^2}{\eta^2-\xi^2\tau_\alpha^2}\right]\Phi_k(\xi)d\xi \qquad (7.3.7)$$

Now, from (2.3.18) and (2.4.4)

$$u_k(x_1,0^\pm) = 2\mathrm{Re}\left\{B_{kj}S_j^\pm(x_1) + \sum_{\alpha=1}^{3}A_{k\alpha}M_{\alpha j}g_j^\pm\left(\frac{ix_1}{\tau_\alpha}\right)\right\} \qquad (7.3.8)$$

Therefore

$$[\![u_k(x_1,0]\!] = -2b_{kj}\sqrt{\frac{2}{\pi}}\int_0^\infty \xi^{-1}\Phi_j(\xi)\cos(\xi x_1)d\xi \qquad (7.3.9)$$

where, again, $b_{kj} = \mathrm{Im}[B_{kj}]$ and $d_{jk} = [b_{jk}]^{-1}$. Now, from (2.3.19) and (2.4.5)

$$\sigma_{2j}(x_1,0^\pm) = 2\mathrm{Re}\left\{S_j'^\pm(x_1) + i\sum_{\alpha=1}^{3}\tau_\alpha^{-1}L_{j2\alpha}M_{\alpha k}g_k'^\pm\left(\frac{ix_1}{\tau_\alpha}\right)\right\} \tag{7.3.10}$$

$$= -\sqrt{\frac{2}{\pi}}\int_0^\infty \Phi_j(\xi)\cos(\xi x_1)d\xi$$
$$+ \frac{1}{\sqrt{2\pi}}\int_0^\infty\sum_{\alpha=1}^{3}\left[L_{j2\alpha}M_{\alpha k}\frac{e^{-i\eta x_1/\tau_\alpha}}{\tau_\alpha} + \overline{L}_{j2\alpha}\overline{M}_{\alpha k}\frac{e^{i\eta x_1/\overline{\tau}_\alpha}}{\overline{\tau}_\alpha}\right]\Psi_k(\eta)d\eta \tag{7.3.11}$$

Therefore boundary condition 3 is satisfied automatically while boundary condition 4 is satisfied.

In case i if

$$-2b_{kj}\sqrt{\frac{2}{\pi}}\int_0^\infty \xi^{-1}\Phi_j(\xi)\cos(\xi x_1)d\xi = 0 \qquad x_1 > b \tag{7.3.12}$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \Phi_j(\xi)\cos(\xi x_1)d\xi$$
$$- \sqrt{\frac{2}{\pi}}\int_0^\infty \mathrm{Re}\left[\sum_{\alpha=1}^{3}L_{j2\alpha}M_{\alpha k}\tau_\alpha^{-1}e^{-i\eta x_1/\tau_\alpha}\right]\Psi_k(\eta)d\eta = p_j(x_1) \qquad x_1\in(0,b) \tag{7.3.13}$$

and

$$\lim_{x_1\to 0} 2b_{kj}\sqrt{\frac{2}{\pi}}\int_0^\infty \Phi_j(\xi)\sin(\xi x_1)d\xi = 0 \tag{7.3.14}$$

while

In case ii if

$$-2b_{kj}\sqrt{\frac{2}{\pi}}\int_0^\infty \xi^{-1}\Phi_j(\xi)\cos(\xi x_1)d\xi = 0 \qquad x_1\in(a,b)^c \tag{7.3.15}$$

and

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \Phi_j(\xi)\cos(\xi x_1)d\xi$$
$$-\sqrt{\frac{2}{\pi}}\int_0^\infty \mathrm{Re}\left[\sum_{\alpha=1}^{3} L_{j2\alpha}M_{\alpha k}\tau_\alpha^{-1}e^{-i\eta x_1/\tau_\alpha}\right]\Psi_k(\eta)d\eta = p_j(x_1) \quad x_1\in(a,b) \tag{7.3.16}$$

Now let

$$\Phi_j(\xi) = d_{jk}\frac{1}{\sqrt{2\pi}}\int_a^b \frac{g_k(t)}{\Delta(t)}\sin(\xi t)dt \tag{7.3.17}$$

where

$$\Delta(t) = \sqrt{(t-a)(b-t)} \tag{7.3.18}$$

and $g_k(t)$'s are, as yet, undetermined functions.

Then substituting (7.3.17) into (7.3.9) and making use of (8.1.6) reveals that

$$\begin{aligned}[\![u_k(x_1,0)]\!] &= -\int_a^b \frac{g_k(t)}{\Delta(t)}H(t-x_1)dt \\ &= \left\{\begin{array}{cc} -\int_a^b \frac{g_k(t)}{\Delta(t)}dt & x_1<a \\ -\int_{x_1}^b \frac{g_k(t)}{\Delta(t)}dt & a<x_1<b \\ 0 & x_1>b \end{array}\right\}\end{aligned} \tag{7.3.19}$$

Therefore, in case i, conditions (7.3.12) and (7.3.14) are satisfied if

$$g_k(0) = 0 \tag{7.3.20}$$

While, in case ii, condition (7.3.15) is satisfied if

$$\int_a^b \frac{g_k(t)}{\Delta(t)} dt = 0 \tag{7.3.21}$$

We now focus our attention on the various components that appear in expression (7.3.11).

First, recalling (7.3.17), we see that

$$\begin{aligned} I_{1j}(x_1) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \Phi_j(\xi) \cos(\xi x_1) d\xi \\ &= d_{jk} \frac{1}{\pi} \int_a^b \frac{g_k(t)}{\Delta(t)} \frac{\partial}{\partial x_1} \int_0^\infty \frac{\sin(\xi x_1)\sin(\xi t)}{\xi} d\xi\, dt \end{aligned}$$

or, via (8.1.4)

$$I_{1j}(x) = d_{jk} \frac{1}{\pi} \int_a^b \frac{g_k(t)}{\Delta(t)} \frac{t}{t^2 - x_1^2} dt \tag{7.3.22}$$

Secondly, substituting (7.3.17) into (7.3.7), reveals that

$$\Psi_j(\eta) = \frac{1}{\pi} \int_a^b \mathrm{Re}\left[\sum_{\alpha=1}^{3} L_{j2\alpha} M_{\alpha k} \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{i\,\xi}{\xi^2 + \left(\frac{-\eta}{i\tau_\alpha}\right)^2} \sin(\xi t) d\xi \right] d_{k\ell} \frac{g_\ell(t)}{\Delta(t)} dt \tag{7.3.23}$$

Therefore, noting that $\mathrm{Re}\left(\frac{-\eta}{i\tau_\alpha}\right) = \frac{\eta\, s_\alpha}{|\tau_\alpha|^2} > 0$ permits us to make use of (8.1.5) and we discover that

$$\Psi_j(\eta) = \frac{1}{\sqrt{2\pi}} \int_a^b \mathrm{Re}\left[\sum_{\alpha=1}^{3} L_{j2\alpha} M_{\alpha k} i e^{\frac{-i\eta t}{\tau_\alpha}} \right] d_{kl} \frac{g_\ell(t)}{\Delta(t)} dt \tag{7.3.24}$$

Finally, substituting (7.3.24) into the second integral of expression (7.3.11), we see that

$$
\begin{aligned}
I_{2j}(x_1) &= \sqrt{\frac{2}{\pi}}\int_0^\infty \mathrm{Re}\left[\sum_{\alpha=1}^{3} L_{j2\alpha}M_{\alpha k}\tau_\alpha^{-1}e^{\frac{-i\eta x_1}{\tau_\alpha}}\right]\Psi_k(\eta)d\eta \\
&= \frac{1}{\pi}\int_a^b\int_0^\infty F_{jr}(\eta,x_1,t)d\eta\, d_{r\ell}\frac{g_\ell(t)}{\Delta(t)}dt \qquad (7.3.25)
\end{aligned}
$$

where

$$
\begin{aligned}
F_{jr}(\eta,x_1,t) &= \mathrm{Re}\left[\sum_{\alpha=1}^{3} L_{j2\alpha}M_{\alpha k}\tau_\alpha^{-1}e^{-i\eta x_1/\tau_\alpha}\right]\times \\
&\quad \mathrm{Re}\left[\sum_{\beta=1}^{3} L_{k2\beta}M_{\beta r}i\, e^{\frac{-i\eta t}{\tau_\beta}}\right] \qquad (7.3.26)
\end{aligned}
$$

Some algebra, coupled with the fact that

$$
L_{i2j}M_{jk} = \delta_{ik} \qquad (7.3.27)
$$

enables us to rewrite the above expression as

$$
\begin{aligned}
F_{jr}(\eta,x_1,t) &= \frac{1}{4}\left\{\sum_{\alpha=1}^{3} L_{j2\alpha}M_{\alpha r}i\,\tau_\alpha^{-1}e^{-i\eta(t+x_1)/\tau_\alpha}\right. \\
&- \sum_{\alpha=1}^{3}\overline{L}_{j2\alpha}\overline{M}_{\alpha r}i\overline{\tau}_\alpha^{-1}e^{i\eta(t+x_1)/\overline{\tau}_\alpha} \\
&+ \sum_{\alpha,\beta=1}^{3}\overline{L}_{j2\alpha}\overline{M}_{\alpha k}L_{k2\beta}M_{\beta r}i\overline{\tau}_\alpha^{-1}e^{i\eta\left(\frac{x_1}{\overline{\tau}_\alpha}-\frac{t}{\tau_\beta}\right)} \qquad (7.3.28) \\
&- \left.\sum_{\alpha,\beta=1}^{3} L_{j2\alpha}M_{\alpha k}\overline{L}_{k2\beta}\overline{M}_{\beta r}i\tau_\alpha^{-1}e^{-i\eta\left(\frac{x_1}{\tau_\alpha}-\frac{t}{\overline{\tau}_\beta}\right)}\right\}
\end{aligned}
$$

Noting that the real parts of each of the four exponential terms are negative, we find that

$$\int_0^\infty F_{jr}(\eta, x_1, t)d\eta = \frac{1}{4}\left[\sum_{\alpha=1}^{3} L_{j2\alpha}M_{\alpha r}\frac{1}{t+x_1}\right.$$
$$+ \sum_{\alpha=1}^{3} \overline{L}_{j2\alpha}\overline{M}_{\alpha r}\frac{1}{t+x_1} + \sum_{\alpha,\beta=1}^{3} \overline{L}_{j2\alpha}\overline{M}_{\alpha k}L_{k2\beta}M_{\beta r}\frac{\tau_\beta}{\overline{\tau}_\alpha t - \tau_\beta x_1}$$
$$\left.+ \sum_{\alpha,\beta=1}^{3} L_{j2\alpha}M_{\alpha k}\overline{L}_{k2\beta}\overline{M}_{\beta r}\frac{\overline{\tau}_\beta}{\tau_\alpha t - \overline{\tau}_\beta x_1}\right] \tag{7.3.29}$$

which, again using (7.3.27), may be written as

$$\int_0^\infty F_{jr}(\eta, x_1, t)d\eta = \frac{1}{4}\left[\frac{2\delta_{jr}}{t+x_1} + 2\mathrm{Re}\left[\sum_{\alpha,\beta=1}^{3} L_{j2\alpha}M_{\alpha k}\overline{L}_{k2\beta}\overline{M}_{\beta r}\frac{\overline{\tau}_\beta}{\tau_\alpha t - \overline{\tau}_\beta x_1}\right]\right] \tag{7.3.30}$$

Substituting (7.3.30) into (7.3.25) it now follows that

$$I_{2j}(x_1) = \frac{1}{4\pi}\int_a^b \left[\frac{2\delta_{jr}}{t+x_1} + 2K_{jr}(t, x_1)\right] d_{r\ell}\frac{g_\ell(t)}{\Delta(t)}dt \tag{7.3.31}$$

where

$$K_{jr}(t, x_1) = \mathrm{Re}\left[\sum_{\alpha,\beta=1}^{3} L_{j2\alpha}M_{\alpha k}\overline{L}_{k2\beta}\overline{M}_{\beta r}\frac{\overline{\tau}_\beta}{\tau_\alpha t - \overline{\tau}_\beta x_1}\right] \tag{7.3.32}$$

Therefore, combining (7.3.22) and (7.3.31), condition (7.3.16) may be rewritten in the form

$$\frac{1}{\pi}\int_a^b \left[\frac{\delta_{jk}}{t-x_1} - K_{jk}(t, x_1)\right]\frac{G_k(t)}{\Delta(t)}dt = p_j(x_1) \qquad x_1 \in (a, b) \tag{7.3.33}$$

where

$$G_k(t) = \frac{1}{2}d_{k\ell}g_\ell(t)$$

Now, since $d_{k\ell}$ is non-singular, the corresponding subsidiary conditions may be written in the form

$$G_k(0) = 0 \qquad k = 1,2,3 \qquad \text{(case i)} \tag{7.3.34}$$

$$\int_a^b \frac{G_k(t)}{\Delta(t)} dt = 0 \qquad k = 1,2,3 \qquad \text{(case ii)} \tag{7.3.35}$$

7.4 The Stress Intensity Factors

The mode I, II, and III stress intensity factors are again defined by (4.5.1)-(4.5.4). Now, utilizing the alternative method of determining the stress intensity factors (Appendix 8.5) it follows that

$$\Sigma_j(a) = -\frac{1}{2}\sqrt{\frac{2}{b-a}}\, d_{jk} g_k(a) = -\sqrt{\frac{2}{b-a}}\, G_j(a) \tag{7.4.1}$$

$$\Sigma_j(b) = \frac{1}{2}\sqrt{\frac{2}{b-a}}\, d_{jk} g_k(b) = \sqrt{\frac{2}{b-a}}\, G_j(b) \tag{7.4.2}$$

The quantities $G_j(a)$ and $G_j(b)$ may be approximated by numerically solving equation (7.3.33), along with the appropriate subsidiary condition. For demonstrative purposes, we may consider a crack in

(a) a transversely isotropic crystal of zinc and

(b) a unidirectional fiber-reinforced carbon/epoxy composite

with, for each material, an applied tensile loading $p_j(x) = \delta_{j2}\sigma^0$. Before progressing with a numerical investigation it is worth spending some effort upon reducing the number of parameters.

Introducing the change of variables

$$\xi = \frac{x_1}{b} \quad , \quad \tau = \frac{t}{b} \tag{7.4.3}$$

in equation (7.3.33) results in the revised equation

$$\frac{1}{\pi}\int_{\alpha}^{1}\left[\frac{\delta_{jk}}{\xi-\tau} - K_{jk}(\xi,\tau)\right]\frac{H_k(\xi)}{\overline{\Delta}(\xi)}d\xi = 1 \quad , \quad \tau \in (\alpha,1) \quad , \quad j = 1,2,3 \tag{7.4.4}$$

where

$$\overline{\Delta}(\xi) = \sqrt{(\xi-\alpha)(1-\xi)} \quad , \quad \alpha = \frac{a}{b} \tag{7.4.5}$$

and

$$H_k(\xi) = \frac{1}{b\sigma^0}G_k(b\xi) \tag{7.4.6}$$

In case i $\alpha = 0$ and the subsidiary conditions assume the revised form:

$$H_k(0) = 0 \quad , \quad k = 1,2,3 \tag{7.4.7}$$

Therefore the stress intensity factors may be non-dimensionalized via:

$$\frac{\Sigma_j(b)}{k_0} = \sqrt{2}H_j(1) \quad , \quad k_0 = \sqrt{b}\sigma^0 \tag{7.4.8}$$

While, in case ii the subsidiary conditions assume the form:

$$\int_{\alpha}^{1}\frac{H_k(\xi)}{\overline{\Delta}(\xi)}d\xi = 0 \quad , \quad k = 1,2,3 \tag{7.4.9}$$

and the stress intensity factors may be non-dimensionalized via:

$$\frac{\Sigma_j(a)}{k_0} = -\frac{2}{1-\alpha}H_j(\alpha) \quad , \quad \frac{\Sigma_j(b)}{k_0} = \frac{2}{1-\alpha}H_j(1) \tag{7.4.10}$$

$$k_0 = \sqrt{\frac{b-a}{2}}\sigma^0$$

In order to guage the effects of the anisotropy the stress intensity factors may be computed for different orientations of the grain/fibers. If the fibers run parallel to the x_1-axis, the non-zero components of the elastic stiffness tensor assume the form:

$$\begin{aligned} &c_{1111} = C \quad , \quad c_{2222} = c_{3333} = A \quad , \quad c_{1122} = c_{1133} = F \\ &c_{2233} = N \quad , \quad c_{1212} = c_{1313} = L \quad , \quad c_{2323} = \frac{1}{2}(A - N) \end{aligned} \tag{7.4.11}$$

Hence, if the material is (positively) rotated through an angle of θ_2 degrees about the x_2-axis, followed by a similar rotation, of θ_3, about the x_3-axis then the constants will be transformed via:

$$\bar{c}_{ij}^{kl} = a_{ip}a_{jq}a_{kr}a_{ls}c_{pq}^{rs} \tag{7.4.12}$$

where

$$a_{ij} = \begin{bmatrix} \cos(\theta_3)\cos(\theta_2) & -\sin(\theta_3) & \cos(\theta_3)\sin(\theta_2) \\ \sin(\theta_3)\cos(\theta_2) & \cos(\theta_3) & \sin(\theta_3)\sin(\theta_2) \\ -\sin(\theta_2) & 0 & \cos(\theta_2) \end{bmatrix} \tag{7.4.13}$$

Once the material properties are known, (7.4.4.) may be solved by the numerical method described in Chapter 8.6. For the case of zinc the material constants may be borrowed from Atkinson and Clements [22]. That is:

$$C = 6.2 \ , \ A = 16.5 \ , \ F = 5.0 \ , \ N = 3.1 \ \text{and} \ L = 3.92$$

where, upon multiplication by 10^{11}, the units for the constants are dynes/cm^2.

For the case of a carbon/epoxy composite the material constants may be calculated via Hahn's formulae (2.5.15). The necessary data may be borrowed from Ashton, Halpin and Petit [23]. That is, for carbon:

$$E_{11} = 40 \times 10^6\text{psi} \ , \ E_{22} = 1.5 \times 10^6\text{psi} \ , \ G_{12} = 4 \times 10^6\text{psi} \ \text{and} \ \nu_{12} = .2$$

while, for epoxy:

$$E_{11} = E_{22} = .5 \times 10^6\text{psi} \ , \ G_{12} = .185 \times 10^6\text{psi} \ \text{and} \ \nu_{12} = .35$$

Assuming the carbon fibers run parallel to the x_1-axis and account for 80% of the material it may be shown that:

$$C = 3.26 \ , \ A = .324 \ , \ F = .101 \ , \ N = .115 \ \text{and} \ L = .118$$

where, upon multiplication by 10^6, the units for the constants are psi.

In case (a), where the effects of the anistropy are limited, the stress intensity factors are listed in Table 1. In case (b), where they are much

more pronounced, the stress intensity factors are illustrated graphically in Figures 3 through 9. In each case an edge crack was considered with incremental fiber rotations of the form:

$$(\theta_2, \theta_3) = \frac{1}{24}(m\pi, n\pi) \quad , \quad (m, n = 1, 2, 3, ..., 12)$$

θ_1	θ_2	k_I/k_0	k_{II}/k_0	k_{III}/k_0
0.000E+00	0.000E+00	0.115E+01	0.171E-16	0.000E+00
0.000E+00	0.131E+00	0.114E+01	-.226E-01	0.000E+00
0.000E+00	0.262E+00	0.114E+01	-.438E-01	0.000E+00
0.000E+00	0.393E+00	0.113E+01	-.628E-01	0.000E+00
0.000E+00	0.524E+00	0.112E+01	-.793E-01	0.000E+00
0.000E+00	0.654E+00	0.112E+01	-.934E-01	0.000E+00
0.000E+00	0.785E+00	0.113E+01	-.104E+00	0.000E+00
0.000E+00	0.916E+00	0.113E+01	-.110E+00	0.000E+00
0.000E+00	0.105E+01	0.114E+01	-.106E+00	0.000E+00
0.000E+00	0.118E+01	0.114E+01	-.928E-01	0.000E+00
0.000E+00	0.131E+01	0.115E+01	-.687E-01	0.000E+00
0.000E+00	0.144E+01	0.115E+01	-.366E-01	0.000E+00
0.000E+00	0.157E+01	0.115E+01	-.114E-15	0.000E+00
0.131E+00	0.000E+00	0.115E+01	-.304E-16	-.249E-16
0.131E+00	0.131E+00	0.114E+01	-.217E-01	0.300E-02
0.131E+00	0.262E+00	0.114E+01	-.419E-01	0.523E-02
0.131E+00	0.393E+00	0.113E+01	-.600E-01	0.623E-02
0.131E+00	0.524E+00	0.112E+01	-.756E-01	0.602E-02
0.131E+00	0.654E+00	0.112E+01	-.890E-01	0.513E-02
0.131E+00	0.785E+00	0.112E+01	-.992E-01	0.429E-02
0.131E+00	0.916E+00	0.113E+01	-.104E+00	0.407E-02
0.131E+00	0.105E+01	0.114E+01	-.101E+00	0.460E-02
0.131E+00	0.118E+01	0.114E+01	-.883E-01	0.564E-02
0.131E+00	0.131E+01	0.114E+01	-.655E-01	0.678E-02
0.131E+00	0.144E+01	0.115E+01	-.349E-01	0.763E-02
0.131E+00	0.157E+01	0.115E+01	0.224E-16	0.794E-02
0.262E+00	0.000E+00	0.114E+01	0.565E-16	0.692E-16
0.262E+00	0.131E+00	0.114E+01	-.191E-01	0.558E-02
0.262E+00	0.262E+00	0.113E+01	-.368E-01	0.974E-02
0.262E+00	0.393E+00	0.113E+01	-.524E-01	0.116E-01
0.262E+00	0.524E+00	0.112E+01	-.659E-01	0.113E-01
0.262E+00	0.654E+00	0.112E+01	-.772E-01	0.965E-02
0.262E+00	0.785E+00	0.112E+01	-.858E-01	0.802E-02
0.262E+00	0.916E+00	0.113E+01	-.900E-01	0.741E-02
0.262E+00	0.105E+01	0.113E+01	-.875E-01	0.813E-02
0.262E+00	0.118E+01	0.114E+01	-.765E-01	0.980E-02
0.262E+00	0.131E+01	0.114E+01	-.569E-01	0.117E-01
0.262E+00	0.144E+01	0.114E+01	-.303E-01	0.132E-01
0.262E+00	0.157E+01	0.114E+01	0.682E-16	0.138E-01
0.393E+00	0.000E+00	0.114E+01	0.744E-16	0.380E-17
0.393E+00	0.131E+00	0.114E+01	-.155E-01	0.730E-02
0.393E+00	0.262E+00	0.113E+01	-.298E-01	0.128E-01
0.393E+00	0.393E+00	0.112E+01	-.423E-01	0.154E-01
0.393E+00	0.524E+00	0.112E+01	-.529E-01	0.151E-01
0.393E+00	0.654E+00	0.112E+01	-.618E-01	0.132E-01

Table 1: Stress intensity factors for various fiber rotations

0.393E+00	0.785E+00	0.112E+01	-.684E-01	0.111E-01
0.393E+00	0.916E+00	0.112E+01	-.717E-01	0.100E-01
0.393E+00	0.105E+01	0.113E+01	-.697E-01	0.105E-01
0.393E+00	0.118E+01	0.114E+01	-.611E-01	0.123E-01
0.393E+00	0.131E+01	0.114E+01	-.456E-01	0.144E-01
0.393E+00	0.144E+01	0.114E+01	-.244E-01	0.161E-01
0.393E+00	0.157E+01	0.114E+01	-.151E-15	0.168E-01
0.524E+00	0.000E+00	0.114E+01	-.354E-15	-.242E-15
0.524E+00	0.131E+00	0.114E+01	-.118E-01	0.782E-02
0.524E+00	0.262E+00	0.113E+01	-.226E-01	0.138E-01
0.524E+00	0.393E+00	0.112E+01	-.321E-01	0.169E-01
0.524E+00	0.524E+00	0.112E+01	-.401E-01	0.172E-01
0.524E+00	0.654E+00	0.112E+01	-.466E-01	0.155E-01
0.524E+00	0.785E+00	0.112E+01	-.514E-01	0.135E-01
0.524E+00	0.916E+00	0.112E+01	-.536E-01	0.123E-01
0.524E+00	0.105E+01	0.113E+01	-.521E-01	0.125E-01
0.524E+00	0.118E+01	0.113E+01	-.457E-01	0.138E-01
0.524E+00	0.131E+01	0.114E+01	-.341E-01	0.157E-01
0.524E+00	0.144E+01	0.114E+01	-.183E-01	0.172E-01
0.524E+00	0.157E+01	0.114E+01	-.278E-15	0.178E-01
0.654E+00	0.000E+00	0.113E+01	0.132E-14	0.401E-15
0.654E+00	0.131E+00	0.113E+01	-.853E-02	0.718E-02
0.654E+00	0.262E+00	0.113E+01	-.165E-01	0.129E-01
0.654E+00	0.393E+00	0.112E+01	-.234E-01	0.163E-01
0.654E+00	0.524E+00	0.112E+01	-.292E-01	0.173E-01
0.654E+00	0.654E+00	0.112E+01	-.338E-01	0.166E-01
0.654E+00	0.785E+00	0.112E+01	-.370E-01	0.153E-01
0.654E+00	0.916E+00	0.112E+01	-.383E-01	0.144E-01
0.654E+00	0.105E+01	0.112E+01	-.370E-01	0.143E-01
0.654E+00	0.118E+01	0.113E+01	-.323E-01	0.152E-01
0.654E+00	0.131E+01	0.113E+01	-.240E-01	0.164E-01
0.654E+00	0.144E+01	0.113E+01	-.129E-01	0.175E-01
0.654E+00	0.157E+01	0.113E+01	0.562E-15	0.179E-01
0.785E+00	0.000E+00	0.113E+01	-.151E-15	0.594E-16
0.785E+00	0.131E+00	0.113E+01	-.599E-02	0.585E-02
0.785E+00	0.262E+00	0.113E+01	-.116E-01	0.107E-01
0.785E+00	0.393E+00	0.112E+01	-.165E-01	0.141E-01
0.785E+00	0.524E+00	0.112E+01	-.206E-01	0.157E-01
0.785E+00	0.654E+00	0.112E+01	-.238E-01	0.160E-01
0.785E+00	0.785E+00	0.112E+01	-.258E-01	0.157E-01
0.785E+00	0.916E+00	0.112E+01	-.264E-01	0.154E-01
0.785E+00	0.105E+01	0.112E+01	-.251E-01	0.154E-01
0.785E+00	0.118E+01	0.113E+01	-.217E-01	0.158E-01
0.785E+00	0.131E+01	0.113E+01	-.160E-01	0.165E-01
0.785E+00	0.144E+01	0.113E+01	-.851E-02	0.170E-01
0.785E+00	0.157E+01	0.113E+01	0.143E-14	0.172E-01

Table 1 (continued)

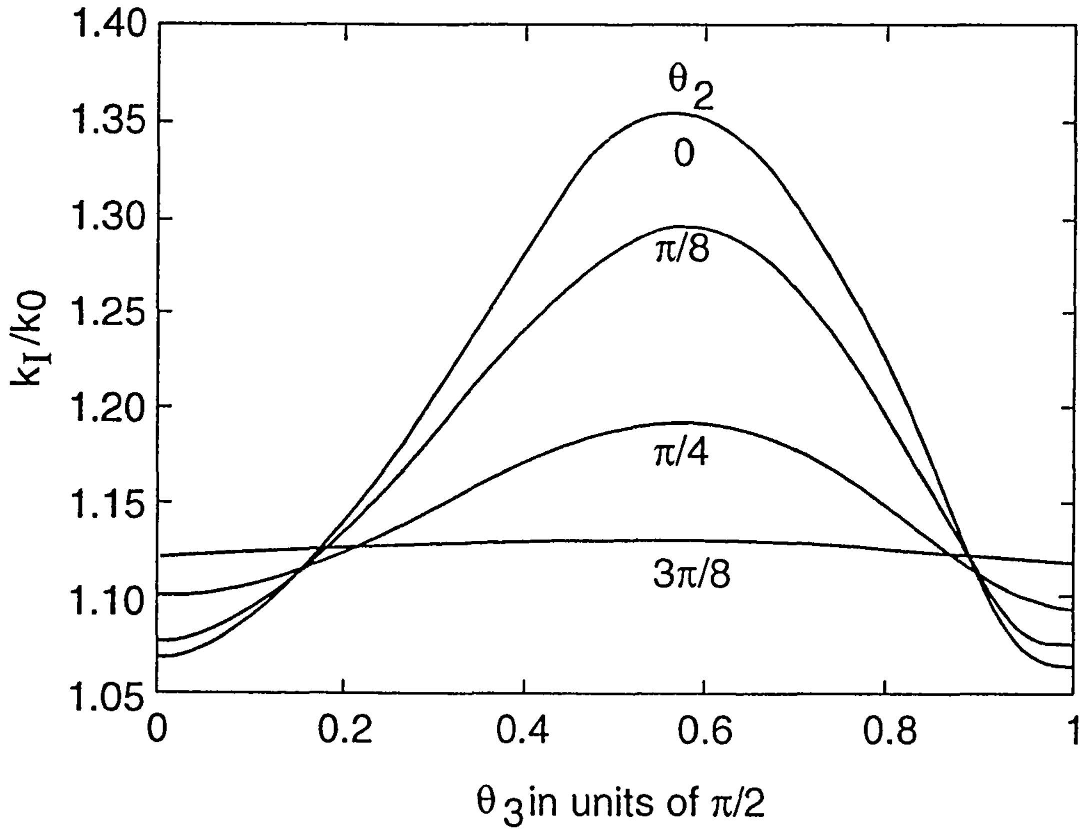

Figure 3: k_I/k_0 v θ_3 for several values of θ_2

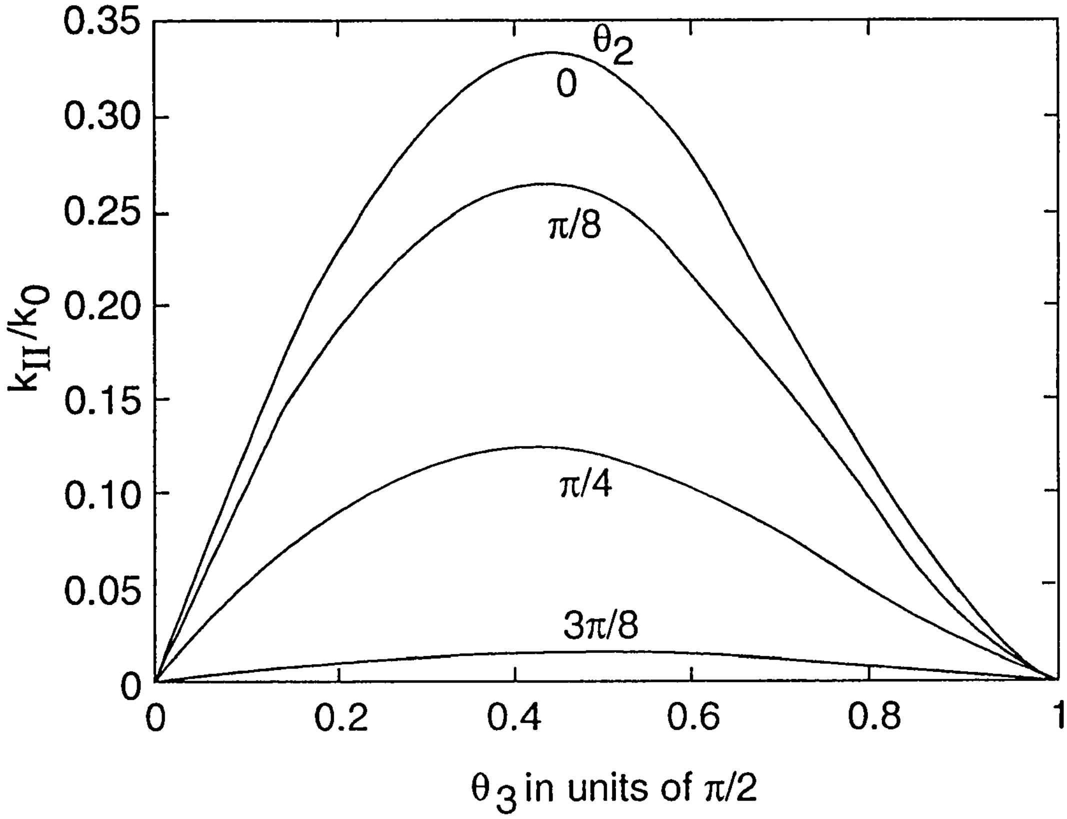

Figure 4: k_{II}/k_0 v θ_3 for several values of θ_2

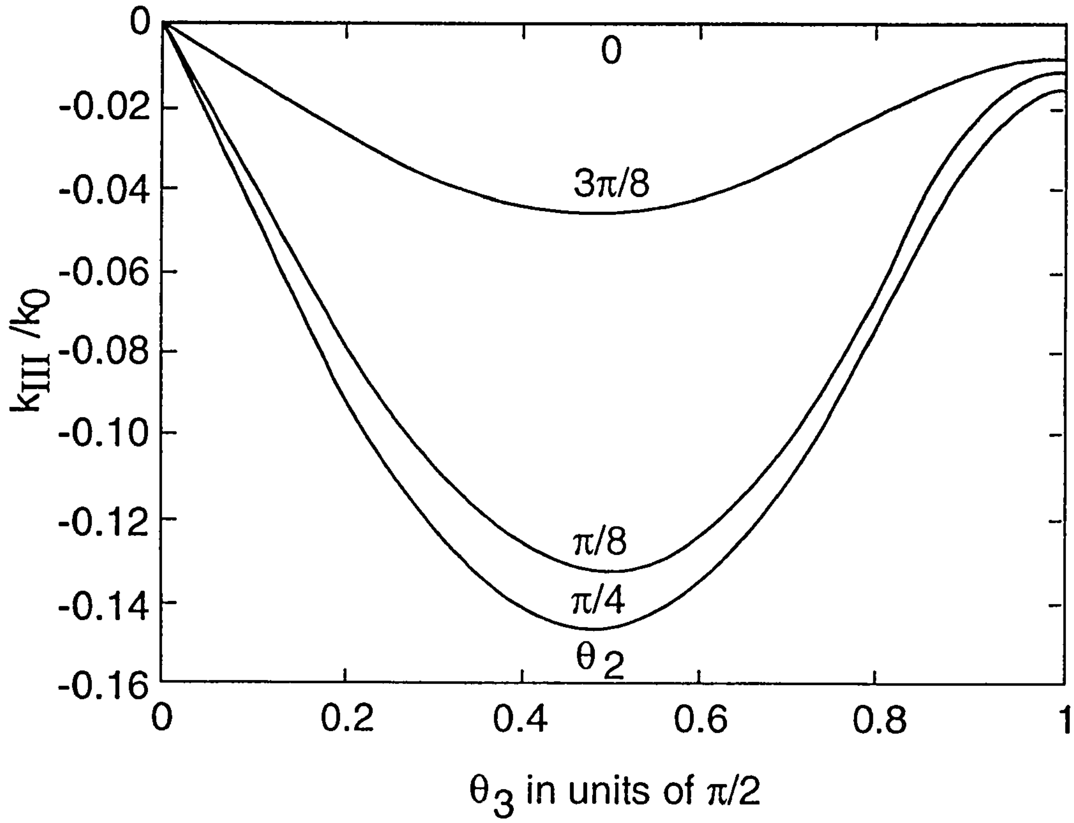

Figure 5: k_{III}/k_0 v θ_3 for several values of θ_2

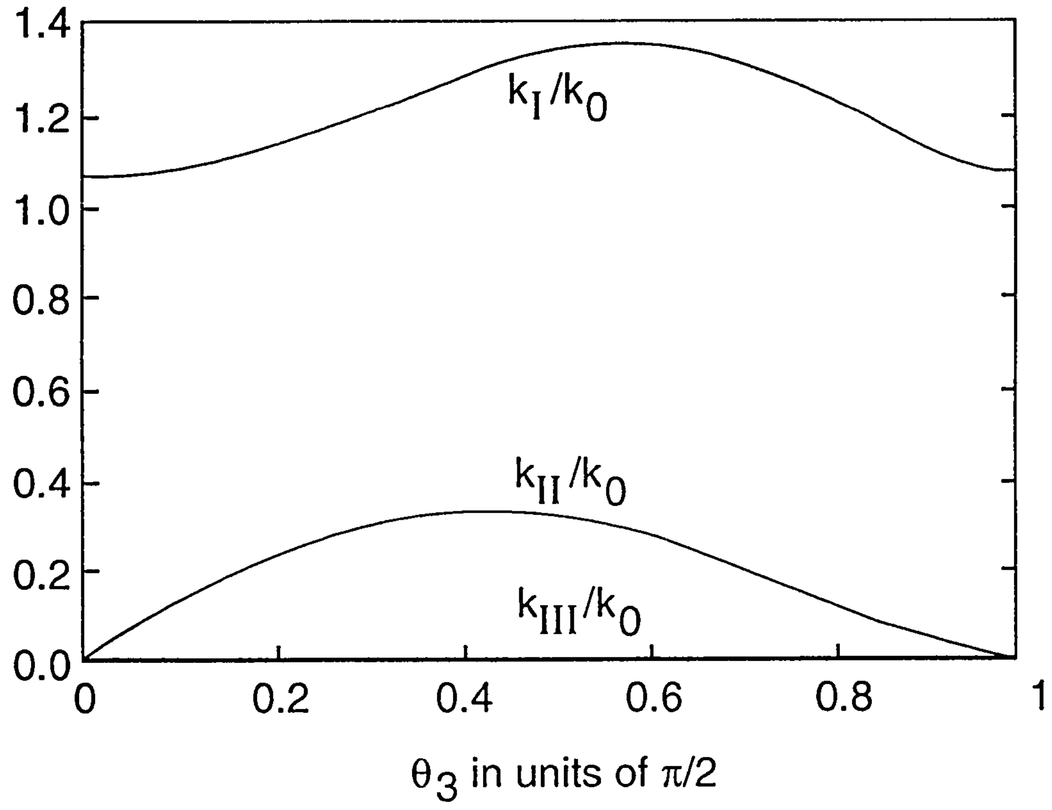

Figure 6: Stress intensity factors v θ_3 for $\theta_2 = 0$

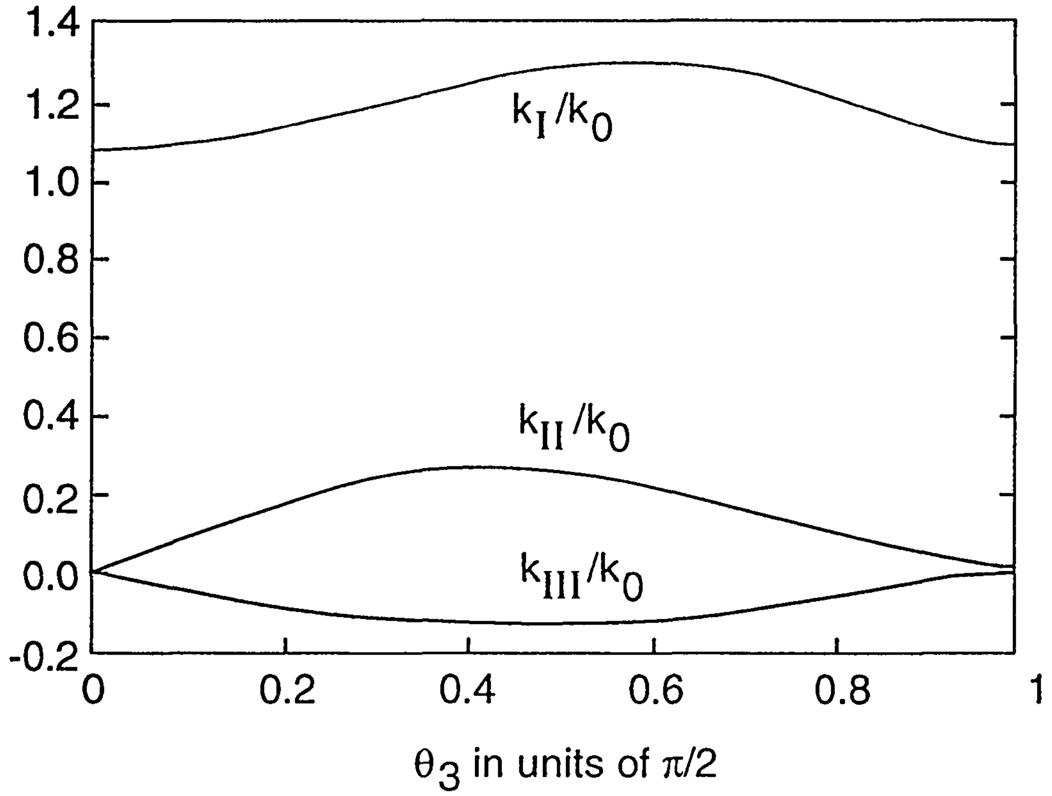

Figure 7: Stress intensity factors v θ_3 for $\theta_2 = \pi/8$

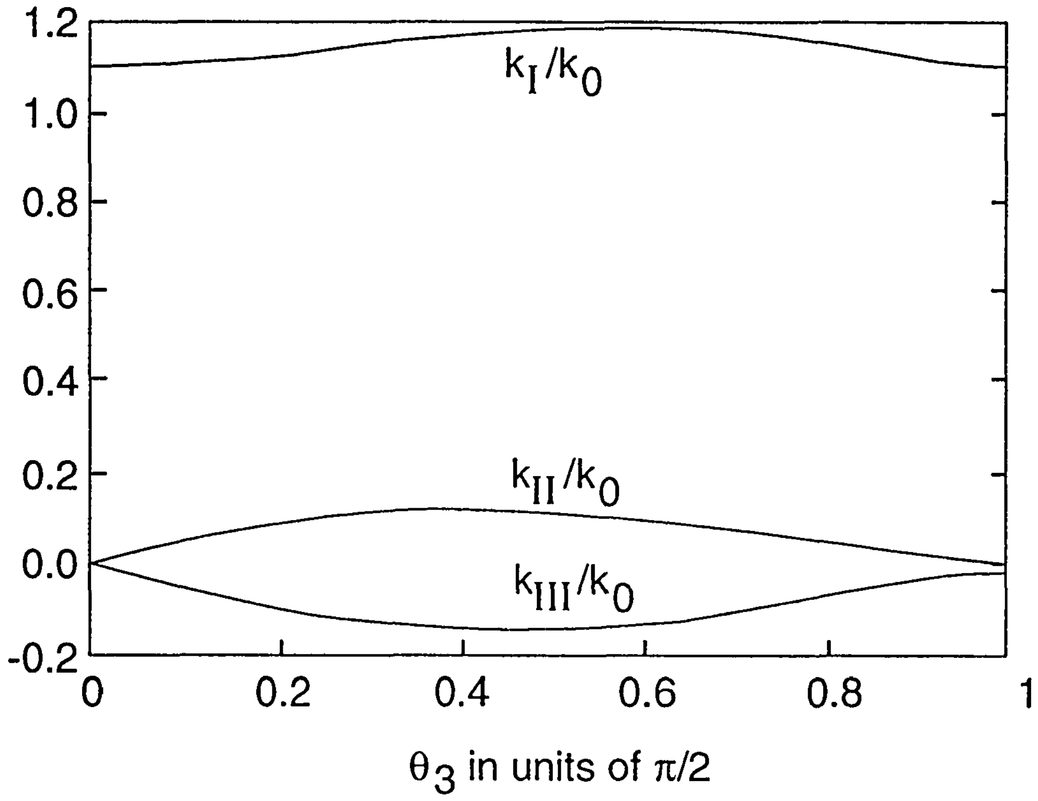

Figure 8: Stress intensity factors v θ_3 for $\theta_2 = \pi/4$

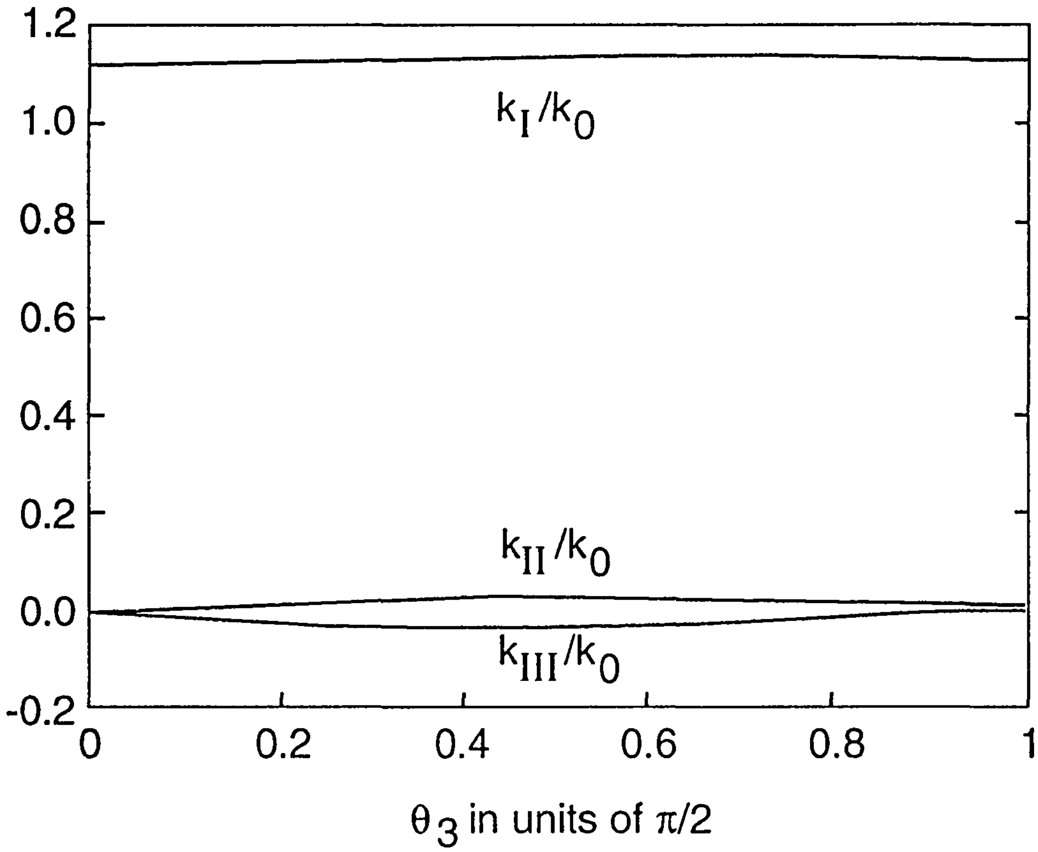

Figure 9: Stress intensity factors v θ_3 for $\theta_2 = 3\pi/8$

Chapter 8

Appendix

8.1 Fourier Transform and Fourier Series Results

(a) Fourier Transform Results

The following results, which may be found (or readily derived from others) on pages 1147 -1153 of Gradshteyn and Rhyzik [17], were referenced at least once in the preceeding chapters.

Defining the inverse Fourier transform (Sneddon [11]) by

$$\mathcal{F}^{-1}[f(\xi);x] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)e^{-i\xi x}d\xi$$

we have (in tabulated form)

$f(\xi)$	$\mathcal{F}^{-1}[f(\xi);x]$		
$\operatorname{sgn}(\xi)$	$-i\sqrt{\frac{2}{\pi}}\frac{1}{x}$		(8.1.1)
$\xi^{-1}e^{-a\lvert\xi\rvert}$	$-i\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{a}\right)$	, $\operatorname{Re}(a)>0$	(8.1.2)
$e^{-a\lvert\xi\rvert}$	$\sqrt{\frac{2}{\pi}}\frac{a}{a^2+x^2}$	, $\operatorname{Re}(a)>0$	(8.1.3)
$\lvert\xi\rvert^{-1}\sin(a\xi)$	$\frac{i}{\sqrt{2\pi}}\log\left\lvert\frac{x-a}{x+a}\right\rvert$		(8.1.4)
$\operatorname{sgn}(\xi)e^{-a\lvert\xi\rvert}$	$-i\sqrt{\frac{2}{\pi}}\frac{x}{a^2+x^2}$	, $\operatorname{Re}(a)>0$	(8.1.5)
$\xi^{-1}\sin(a\xi)$	$\begin{cases}\sqrt{\frac{\pi}{2}} & \lvert x\rvert<a \\ 0 & \lvert x\rvert>a\end{cases}$		(8.1.6)

(b) Fourier Series Results

The following results, which can be found (or readily derived from others) on pages 8 and 16 of Oberhettinger [16], were referenced at least once in preceeding chapters

$$\sum_{n=1}^{\infty}\frac{\cos(n(x-t))}{n}=-\log\left|2\sin\left(\frac{x-t}{2}\right)\right| \tag{8.1.7}$$

$$\sum_{n=1}^{\infty}\frac{\sin(nt)\sin(nx)}{n}=\frac{1}{2}\log\left|\frac{\sin\left(\frac{x+t}{2}\right)}{\sin\left(\frac{x-t}{2}\right)}\right| \tag{8.1.8}$$

$$\sum_{n=1}^{\infty}\frac{e^{-nt}}{n}\sin(nx)=\tan^{-1}\left[\frac{\sin x}{e^t-\cos x}\right]\quad,\quad t>0 \tag{8.1.9}$$

8.2 The Finite Hilbert Transform and Singular Integrals Over the Interval (a,b)

<u>Notation</u> for $a < b$ define $L = (a, b)$; $L^c = (-\infty, a) \cup (b, \infty)$

$\Delta(x) = \sqrt{(x-a)(b-x)}$

and

$\Delta_1(x) = \sqrt{(x-a)(x-b)}$

Then we have the following (Tricomi, [15])

The singular integral equation:

$$\frac{1}{\pi}\int_L \frac{f(x)}{x-t}dx = g(t) \qquad t \in L \tag{8.2.1a}$$

with subsidiary condition

$$\int_L f(x)dx = c_0 \tag{8.2.1b}$$

has solution

$$f(x) = \frac{1}{\pi}\frac{1}{\Delta(x)}\left[c_0 - \int_L \frac{\Delta(t)g(t)}{t-x}dt\right] \qquad x \in L \tag{8.2.1c}$$

Using the residue theorem of complex variables, along with an appropriate contour and a branch cut on L, the following results may be readily derived

$$\frac{1}{\pi}\int_L \frac{dt}{\Delta(t)(t-x)} = \begin{cases} 0 & x \in L \\ \dfrac{\operatorname{sgn}(b-x)}{\Delta_1(x)} & x \in L^c \end{cases} \tag{8.2.2}$$

$$\frac{1}{\pi}\int_L \frac{\Delta(t)dt}{t-x} = \begin{cases} \dfrac{b+a}{2} - x & x \in L \\ \dfrac{b+a}{2} - x - \operatorname{sgn}(b-x)\Delta_1(x) & x \in L^c \end{cases} \tag{8.2.3}$$

$$\frac{1}{\pi}\int_L \sqrt{\frac{t-a}{b-t}}\,\frac{dt}{t-x} = \begin{cases} 1 & x \in L \\ 1 - \sqrt{\frac{x-a}{x-b}} & x \in L^c \end{cases} \tag{8.2.4}$$

$$\frac{1}{\pi}\int_L \sqrt{\frac{b-t}{t-a}}\,\frac{dt}{t-x} = \begin{cases} -1 & x \in L \\ -1 + \sqrt{\dfrac{x-b}{x-a}} & x \in L^c \end{cases} \tag{8.2.5}$$

$$\frac{1}{\pi}\int_{-a}^{a} \frac{\sqrt{a^2-t^2}}{(t^2+1)(t-x)}dt = \frac{-x\sqrt{a^2+1}}{x^2+1} \qquad |x| < a \tag{8.2.6}$$

From elementary calculus we also note

$$\begin{aligned} \int_a^x \Delta(t)dt &= \frac{1}{2}\left(\frac{b-a}{2}\right)^2 \left[\sin^{-1}\left(\frac{2x-a-b}{b-a}\right) + \frac{\pi}{2}\right] \\ &\quad + \frac{1}{2}\left(x - \frac{a+b}{2}\right)\Delta(x) \quad , \quad x \in L \end{aligned} \tag{8.2.7}$$

with

$$\int_a^b \Delta(t)dt = \frac{\pi}{2}\left(\frac{b-a}{2}\right)^2$$

and

$$\int_a^x \frac{dt}{\Delta(t)} = \sin^{-1}\left(\frac{2x-a-b}{b-a}\right) + \frac{\pi}{2} \quad , \quad x \in L \tag{8.2.8}$$

with

$$\int_a^b \frac{dt}{\Delta(t)} = \pi$$

8.3 Singular Integral Equations and Singular Integrals Over the Set $(-b,-a) \cup (a,b)$

Notation for $0 < a < b$ define $L_1 = (-b,-a)$, $L_2 = (a,b)$, $L = L_1 \cup L_2$

$\Delta(x) = \sqrt{(x^2-a^2)(b^2-x^2)}$

and

$\Delta_1(x) = \sqrt{(x^2-a^2)(x^2-b^2)}$

Then we have the following (Lewin [18])

The singular integral equation

$$\frac{1}{\pi}\int_L \frac{f(x)}{x-t}dx = g(t) \qquad t \in L \tag{8.3.1a}$$

with subsidiary condition

$$\int_L f(x)dx = \int_L \operatorname{sgn}(x) f(x)dx = 0 \tag{8.3.1b}$$

has solution

$$f(x) = \frac{\operatorname{sgn}(x)}{\Delta(x)} \left\{ c_0 - \frac{1}{\pi} \int_L \frac{\operatorname{sgn}(t)\Delta(t) f(t) dt}{t - x} \right\} \qquad x \in L \tag{8.3.1c}$$

where

$$c_0 = \frac{b}{2K} \int_L \frac{1}{\Delta(t)} \int_L \frac{\operatorname{sgn}(x)\Delta(x) f(x)}{x - t} dx\, dt \tag{8.3.1d}$$

and K is the complete elliptic integral (see section 8.4) of the first kind with modulus $k^2 = 1 - a^2/b^2$.

Note that if $g(x) = c_1$ (a constant) then, by virtue of (8.3.4) and (8.4.10,11) we have that

$$f(x) = c_1 \frac{\operatorname{sgn}(x)}{\Delta(x)} [x^2 - b^2 E/K] \tag{8.3.2}$$

where E is the complete elliptic integral (see section 8.4) of the second kind, again with $k^2 = 1 - a^2/b^2$.

Using the residue theorem of complex variables, along with an appropriate contour and branch cuts on L_1 and L_2, the following results may be readily derived

$$\frac{1}{\pi} \int_L \frac{\operatorname{sgn}(t)}{\Delta(t)(t - x)} dt = \begin{cases} 0 & x \in L \\ \dfrac{\operatorname{sgn}(a^2 - x^2)}{\Delta_1(x)} & x \in L^c \end{cases} \tag{8.3.3}$$

$$\frac{1}{\pi} \int_L \frac{\Delta(t) \operatorname{sgn}(t) dt}{t - x} = \begin{cases} \dfrac{a^2 + b^2}{2} - x^2 & x \in L \\ \dfrac{a^2 + b^2}{2} - x^2 - \operatorname{sgn}(a^2 - x^2)\Delta_1(x) & x \in L^c \end{cases} \tag{8.3.4}$$

$$\frac{1}{\pi}\int_L \frac{\operatorname{sgn}(t)t^2}{\Delta(t)(t-x)}dt = \begin{cases} 1 & x \in L \\ 1+\operatorname{sgn}(a^2-x^2)\dfrac{x^2}{\Delta_1(x)} & x \in L^c \end{cases} \tag{8.3.5}$$

$$\frac{1}{\pi}\int_L \frac{|t|\Delta(t)}{t-x}dt = \begin{cases} x\left(\dfrac{a^2+b^2}{2}-x^2\right) & x \in L \\ x\left(\dfrac{a^2+b^2}{2}-x^2\right)-\operatorname{sgn}(a^2-x^2)x\Delta_1(x) & x \in L^c \end{cases} \tag{8.3.6}$$

8.4 Elliptic Integrals

The continuity of Chapter 5 was preserved by referencing the following results. Here, in the appendix, some of the intermediate steps are highlighted, along with the basic definitions of the functions involved. All of the results (and basic definitions) were obtained with the aid of Byrd and Friedman's "Handbook of Ellliptic Integrals for Engineers and Scientists", [13]. All page/reference numbers, accompanied by *, may be found there.

Definitions:

Elliptic Integral of the first kind:

$$F(\phi,k) = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \tag{8.4.1}$$

Elliptic Integral of the second kind:

$$E(\phi,k) = \int_0^\phi \sqrt{1-k^2\sin^2\theta}\,d\theta \tag{8.4.2}$$

If $\phi = \frac{\pi}{2}$ then both integrals are said to be complete, and

$$F\left(\frac{\pi}{2}, k\right) \equiv K(k) \equiv K \tag{8.4.3}$$

$$E\left(\frac{\pi}{2}, k\right) \equiv E(k) \equiv E \tag{8.4.4}$$

Heuman's Lambda function:

$$\Lambda_0(\phi, k) \equiv \frac{2}{\pi}[E(k)F(\phi, k') + K(k)E(\phi, k') - K(k)F(\phi, k')] \tag{8.4.5}$$

Jacobian Zeta function:

$$Z(\phi, k) \equiv E(\phi, k) - \frac{E}{K}F(\phi, k) \tag{8.4.6}$$

In each definition the parameter k is referred to as the *modulus*, while k' (the *complementary modulus*) is given by

$$k' = \sqrt{1 - k^2} \tag{8.4.7}$$

Note, each of the above definitions may also be expressed in terms of the Jacobian Elliptic functions (*p18-20).

Notation (same as Section 8.3)

For our purposes we may select the modulus to be constant throughout this entire section; in particular

$$k = \sqrt{1 - a^2/b^2} \tag{8.4.8}$$

with

$$k' = a/b \tag{8.4.9}$$

Then, firstly, (from *P56-57), we find that:

$$\int_a^b \frac{dt}{\Delta(t)} = \frac{1}{b}K \tag{8.4.10}$$

$$\int_a^b \frac{t^2 dt}{\Delta(t)} = bE \tag{8.4.11}$$

Now, from the above results, it may be readily shown that:

$$\int_a^b \sqrt{\frac{t^2 - a^2}{b^2 - t^2}}\, dt = bE - \frac{a^2}{b}K \tag{8.4.12}$$

$$\int_a^b \sqrt{\frac{b^2 - t^2}{t^2 - a^2}}\, dt = b(K - E) \tag{8.4.13}$$

<u>Other Results</u>:

<u>Result 1</u>

In deriving expression (5.3.20) from (5.3.18) the following result was used:

$$\begin{aligned} &\lim_{R\to\infty} \int_{-R}^{x} \left[1 + \frac{b^2E - t^2K}{K\Delta_1(t)}\right] dt = \lim_{R\to\infty} I_1(x) \\ &= \frac{\pi b}{2K}[1 - \Lambda_0(\psi_2, k)] + x\left(1 - \sqrt{\frac{x^2 - b^2}{x^2 - a^2}}\right) \quad , \quad x < -b \end{aligned} \tag{8.4.14}$$

where

$$\psi_2 = \sin^{-1}\sqrt{\frac{x^2 - b^2}{x^2 - a^2}}$$

<u>Details</u>:

Introduce the change of variables:

$$
\begin{aligned}
sn^2 u &= \frac{t^2-b^2}{t^2-a^2} &&, \quad k_1 = \tfrac{a}{b}^{\dagger} \\
\psi_1 = am(u_1) &= \sin^{-1}\sqrt{\frac{R^2-b^2}{R^2-a^2}} &&, \quad sn(u_1) = \sin(\psi_1) \\
\psi_2 = am(u_2) &= \sin^{-1}\sqrt{\frac{x^2-b^2}{x^2-a^2}} &&, \quad sn(u_2) = \sin(\psi_1)
\end{aligned}
$$

† Note, the k in the result is indeed the k given by (8.4.8). However, in the proof we select $\frac{a}{b}$ as the modulus; to (hopefully) avoid confusion, it is labelled k_1. With the change of variables we find that

$$t^2 = b^2 sn^2 u \quad , \quad \frac{dt}{\Delta_1(t)} = \frac{-1}{b} du$$

in which case

$$I_1(x) = x + R - \frac{b}{K}\int_{u_1}^{u_2} (E - K dc^2 u) du$$

then, from (*321.02)

$$
\begin{aligned}
I_1(x) &= x + R - \frac{b}{K}\left[(Eu - K[u - E(u) + dn\, u\, tn\, u])\Big|_{u_1}^{u_2}\right] \\
&= x + R - \frac{b}{K}\left[E\,F(\psi_2,k_1) - K\,F(\psi_2,k_1) + K\,E(\psi_2,k_1) + \frac{Kx}{b}\sqrt{\frac{x^2-b^2}{x^2-a^2}}\right] \\
&\quad + \frac{b}{K}\left[E\,F(\psi_1,k_1) - K\,F(\psi_1,k_1) + K\,E(\psi_1,k_1) - \frac{KR}{b}\sqrt{\frac{R^2-b^2}{R^2-a^2}}\right]
\end{aligned}
$$

which, via (8.4.5) and (*110.10), may be written as

$$
\begin{aligned}
I_1(x) &= x + R - \frac{b}{K}\left(\frac{\pi}{2}\Lambda_0(\psi_2, k) + \frac{K}{b}x\sqrt{\frac{x^2-b^2}{x^2-a^2}}\right) \\
&\quad + \frac{b}{K}\left(\frac{\pi}{2} - \frac{K}{b}R\sqrt{\frac{R^2-b^2}{R^2-a^2}}\right)
\end{aligned}
$$

Finally, letting $R \to \infty$, the required result is obtained. The other terms in expression (5.3.20) may be derived in a similar manner.

Result 2

$$
\begin{aligned}
&\int_a^b \frac{\Delta(t)}{t^2-x^2}dt = I_2(x) \\
&\quad = -bE + (a^2+b^2-x^2)\frac{K}{b} - \frac{\Delta(x)}{|x|}KZ(\beta, k) \qquad x \in L
\end{aligned}
\tag{8.4.15}
$$

where

$$
\beta = \sin^{-1}\sqrt{\frac{b^2-x^2}{b^2-a^2}}
$$

Details:

Rewrite

$$
\frac{\Delta(t)}{t^2-x^2} = \frac{1}{\Delta(t)}\left\{-t^2 + (a^2+b^2-x^2) + \frac{(x^2-a^2)(b^2-x^2)}{t^2-x^2}\right\}
$$

then, by virtue of (8.4.10,11)

$$
I_2(x) = -bE + (a^2+b^2-x^2)\frac{K}{b} + \Delta^2(x)\int_a^b \frac{dt}{\Delta(t)(t^2-x^2)}
$$

Also, from (*218.02), (*415.01), we find that

$$\int_a^b \frac{dt}{\Delta(t)(t^2 - x^2)} = \frac{-K\,Z(\beta, k)}{|\,x\,|\,\Delta(x)}$$

Hence, the result.

<u>Result 3</u>

$$\begin{aligned} I_3(x) &= \operatorname{sgn}(x)\int_{-b}^{x} \frac{b^2E - Kt^2}{K\Delta(t)}dt \\ &= b\,Z(\chi, k) \qquad x \in L \end{aligned}$$

where (8.4.16)

$$\chi = \sin^{-1}\sqrt{\frac{b^2 - x^2}{b^2 - a^2}}$$

<u>Details</u>:

Using the change of variables on *P56, the integral becomes

$$I_3(x) = b\int_0^{u_1} \left(\frac{E}{K} - dn^2u\right) du$$

where $sn(u_1) = \sin(\chi)$

then, from (* 314.02)

$$I_3(x) = b\frac{E}{K}F(\chi, k) - E(\chi, k) = bZ(\chi, k)$$

Result 4

$$I_4(x) = \int_{-b}^{x} \frac{|t|\,(-b^2\frac{E}{K} + a^2 + b^2 - t^2 - b\frac{\Delta(t)}{|t|}Z(\beta,k))}{\Delta(t)}dt$$
$$= \mathrm{sgn}(x)\Delta(x) - bxZ(\beta,k) \qquad x \in L$$

where (8.4.17)

$$\beta = \sin^{-1}\sqrt{\frac{b^2-x^2}{b^2-a^2}}$$

Details: Introduce the change of variables

$$\sin^2\phi = \frac{b^2-t^2}{b^2-a^2} \ , \ \chi = \sin^{-1}\sqrt{\frac{b^2-x^2}{b^2-a^2}}$$

then

$$d\phi = \frac{-tdt}{\Delta(t)} \ , \ |t| = b\sqrt{1-k^2\sin^2\phi} \ , \ \Delta(t) = (b^2-a^2)\sin\phi\cos\phi$$

therefore

$$I_4(x) = -\mathrm{sgn}(x)\int_0^{\chi}\left[-b\frac{E}{K} + a^2 + b^2 - b^2(1-k^2\sin^2\phi) - \right.$$
$$\left. - \frac{b^2k^2\sin\phi\cos\phi Z(\phi,k)}{\sqrt{1-k^2\sin^2\phi}}\right]d\phi$$

then, via (* 630.73), we find that

$I_4(x) = \mathrm{sgn}(x)\Delta(x) - bxZ(\chi,k)$, as required.

8.5 An Alternative Method for Determining the Stress Intensity Factors, for a Single Crack

This derivation is accomplished by retracing some of the solution steps from Chapter 4.

We begin by replacing condition 3ii of problem 2 with the more general condition

$\sigma^2_{j2}(x_1, 0^+) = P_j(x_1)$, where $P_j(x_1)$ is arbitrary.

In which case (4.4.11) and (4.4.12) will be replaced by

$$g_k(t) = \frac{-2b_{kj}}{\Delta(t)} \frac{1}{\pi} \int_a^b \frac{\Delta(x) P_j(x)}{x - t} dt \qquad t \in (a, b) \tag{8.5.1}$$

and

$$\sigma^2_{j2}(x_1, 0) = \frac{\mathrm{sgn}(a - x_1)}{\Delta_1(x_1)} \frac{1}{\pi} \int_a^b \frac{\Delta(t) P_j(t)}{t - x_1} dt \qquad x_1 \in (a, b)^c \tag{8.5.2}$$

Thus, from (4.5.3) and (4.5.4)

$$\Sigma_j(a) = \sqrt{\frac{2}{b-a}} \frac{1}{\pi} \int_a^b \sqrt{\frac{t-a}{b-t}} P_j(t) dt \tag{8.5.3}$$

$$\Sigma_j(b) = \sqrt{\frac{2}{b-a}} \frac{1}{\pi} \int_a^b \sqrt{\frac{b-t}{t-a}} P_j(t) dt \tag{8.5.4}$$

Now, define

$$
\begin{aligned}
h_k(t) &= \Delta(t)g_k(t) \\
&= -2b_{kj}\frac{1}{\pi}\int_a^b \frac{\Delta(x)}{x-t}P_j(x)dx
\end{aligned}
\tag{8.5.5}
$$

then

$$h_k(a) = -2b_{kj}\frac{1}{\pi}\int_a^b \sqrt{\frac{b-x}{x-a}}P_j(x)dx \tag{8.5.6}$$

$$h_k(b) = 2b_{kj}\frac{1}{\pi}\int_a^b \sqrt{\frac{x-a}{b-x}}P_j(x)dx \tag{8.5.7}$$

Finally, comparing (8.4.6) and (8.4.7) with (8.4.3) and (8.4.4), we discover that

$$\Sigma_j(a) = -\frac{1}{2}\sqrt{\frac{2}{b-a}}d_{jk}h_k(a) \tag{8.5.8}$$

$$\Sigma_j(b) = \frac{1}{2}\sqrt{\frac{2}{b-a}}d_{jk}h_k(b) \tag{8.5.9}$$

8.6 Numerical Procedure

The method used for solving the system of integral equations (7.4.4) is known as the Labatto-Chebyshev method. A detailed description of the method can be found in Theocaris and Ioakimidis, [19]. However, for completeness, a brief summary will also be presented here.

The integral itself is approximated by the quadrature formula:

$$\frac{1}{\pi}\int_\alpha^1 \left[\frac{\delta_{jk}}{\xi-\tau} - K_{jk}(\xi,\tau)\right]\frac{H_k(\xi)}{\Delta(\xi)}d\xi =$$

$$\frac{1}{\pi}\sum_{i=1}^{N} w_i \left[\frac{\delta_{jk}}{\xi_i-\tau} - K_{jk}(\xi_i,\tau)\right] H_k(\xi_i) \quad , \quad j=1,2,3 \tag{8.6.1}$$

where

$$w_i = \frac{\pi}{N-1}\begin{cases} \frac{1}{2} & i=1,N \\ 1 & i=2,3,...,N-1 \end{cases} \tag{8.6.2}$$

and

$$\xi_i = \left(\frac{1+\alpha}{2}\right)\cos\left(\frac{i-1}{N-1}\right)\pi + \left(\frac{1-\alpha}{2}\right) \quad , \quad i=1,2,...,N \tag{8.6.3}$$

A set of linear algebraic equations is then generated by requiring the quadrature formula to agree with the right hand side of (7.4.4) at the set of collocation points given by:

$$\tau_j = \left(\frac{1+\alpha}{2}\right)\cos\left(\frac{j-\frac{1}{2}}{N-1}\right)\pi + \left(\frac{1-\alpha}{2}\right) \quad , \quad j=1,2,...,N-1 \tag{8.6.4}$$

Note that the quadrature and collocation points satisfy the equations

$$U_{N-2}(\xi_i) = 0 \quad , \quad i=2,3,...,N-1 \tag{8.6.5}$$

$$T_{N-1}(\tau_j) = 0 \quad , \quad j=1,2,...,N-1 \tag{8.6.6}$$

where T and U are, respectively, the Chebyshev polynomials of the first and second kinds on the interval $(\alpha, 1)$.

Now, in case i the subsidiary conditions (7.4.7) translates to

$$H_k(\xi_N) = 0 \quad , \quad k = 1, 2, 3 \tag{8.6.7}$$

in which case, the number of algebraic equations, $3(N-1)$, coincides with the number of unknowns.

In case ii the above quadrature scheme/collocation points may also be used to take account of the subsidiary equations (7.4.9), resulting in a set of 3N linear alagebraic equations with 3N unknowns.

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