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Estimating the Parameters of Truncated Distributions

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TRUNCATED DISTRIBUTIONS

by

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ABSTRACT

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The problem considered here is the estimation of the parameters of some special truncated distributions. If the sample observations are restricted to the interval $[0, T]$ with T known, then it is well known in the literature that the method of maximum likelihood fails to provide a finite estimate, for the mean of an exponential distribution, whenever the sample mean is greater than $T/2$ (Deemer and Votaw, 1955, Ann. Math. Statist. 26, 498-504). Not so well known is the nonexistence of the maximum likelihood estimator (m.l.e.), under certain conditions, for the scale parameter of a gamma distribution from a truncated sample, when the shape parameter of the distribution is assumed known (Broeder, 1955, Ann. Math. Statist. 26, 659-663). The above-mentioned results do not hold when the sample observations are truncated to an infinite interval, say to $[T, \infty)$, in which case the m.l.e. exists with probability one.

This research deals with similar results pertaining to the estimation of mean and standard deviation of a normal distribution from a doubly truncated sample, such that the sample observations are within the interval $[A, B]$, $-\infty < A < B < \infty$, A and B known.

It is proved here that the m.l.e.'s (which are the same as the moment estimators) of these parameters are nonexistent with positive probability. The cases for the two-parameter gamma and Weibull distributions are also examined with the help of Broeder's technique of standardizing the truncation interval to [0, 1] through a simple transformation.

In the cases considered here, the m.l.e.'s even when they exist, exhibit a tendency of blowing up near the upper boundary of the interval of their existence. In order to correct this problem, as well as to find estimators that exist with probability one, the class of Bayes modal, or modified maximum likelihood estimators is considered. The Bayes modal estimators were introduced by Blumenthal and Marcus (1975, J. Amer. Statist. Assoc. 70, 913-922). A new estimation procedure combining the m.l.e. and the Bayes modal estimator, called the mixed estimator, is proposed here. Simulations provide the comparison of the aggregate behavior of the m.l.e.'s, the modal estimators, and the mixed estimators.

to

my parents

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1. INTRODUCTION

Estimating the parameters of an assumed distribution, based on the truncated samples, is a common problem in quality control, life testing, and a large number of other areas. In quality control, a range may be specified for the dimensions of a manufactured product, and the items falling outside the range are rejected. Given a sample of passed items, the interest will center on estimating the parameters of the original manufacturing process. In an example given by Cohen (1957), a sample of bushings is sorted through go, no-go gauges, with the result that items having diameters less than 0.5985 inch or more than 0.6015 inch are discarded. Under the normal distribution assumption, this is a sample from a doubly truncated normal distribution. Knowledge of the mean and standard deviation of the original (complete) normal process will lead to the determination of the distributional structure of the truncated normal model.

In life testing, a number of items are put on test for a fixed period of time. The type of items that are subject to failure are unknown. Failures occurring within the time interval only are observable and the total number of items put on test is unknown. An example of this type of situation is given in Blumenthal and Marcus (1975). It consists of putting M items on life test for T (fixed) hours where N (unknown) of the items

have a certain defect and items with this defect can be identified only after failure. The variable of interest is the lifetime of an item with this particular defect.

A number of other examples considered in the literature, including the special distributions used, can be found in the survey article by Blumenthal (1981). A number of distributions are considered, important among them being Poisson, negative binomial, normal, exponential, gamma, Weibull, and log normal. Sometimes, the interest is not restricted to the estimation of the parameters alone but may extend to the estimation of the functions of these parameters. For example, Holla (1967), Sathe and Varde (1969) and Nath (1975) deal with unbiased estimation of the reliability function when the underlying models are the truncated exponential and gamma. Density estimation for the truncated normal (Crain, 1979) and estimating the probability of zero class for the truncated Poisson (Dahiya and Gross, 1973) are other examples of this type.

In addition to the examples leading to the generation of truncated data cited by Blumenthal (1981), Deemer and Votaw (DV) (1955) apply the truncated exponential model to the radial error, or the distance from the aiming point to the point of impact, in bombing accuracy studies. Truncation arises since, for example, in gun camera missions, the area of observation is restricted by the view angle of the camera. Also, the truncated normal model is sometimes used for inverse regression. In some situations, it might be more appropriate to use a truncated distribution to

model the observations, where a complete distribution is used currently. A case in point is the data on grade point averages of college students which is restricted on the interval [2, 4] and the complete normal distribution is commonly used as the basis for analysis. Unless the standard deviation of the complete model is really very small, the truncated normal would fit the data much better than the complete normal. It will be more so since, unlike complete normal, the truncated normal does not need to be symmetric about the mean. The population of heights, weights, and scores in an examination are other examples of this type where the truncated distributions might be more appropriate.

The notion of truncation has to be distinguished from that of censoring wherein the total sample size (N), and hence the number of missing observations, is known. Literature on censoring to life testing, reliability problems, survival distributions and biomedical applications is collected and presented in a systematic manner in the books by Mann, Schafer and Singapurwalla (1975), Gross and Clark (1975), and Lawless (1982).

Truncated sampling problems can be classified into two main types. One type calls for the estimation of the unknown number of missing observations or the total sample size. Estimation of the other parameters is only incidental in this type of problem. A number of papers have dealt with this situation such as Sanathanan (1972, 1977), Dahiya and Gross (1973), Blumenthal and Marcus (1975), Blumenthal (1977), Blumenthal, Dahiya and Gross (1978), Dahiya

(1980), Watson and Blumenthal (1980), and Blumenthal (1981). The second type of truncated sampling problem focuses on the estimation of the distributional parameters although the number of missing observations is unknown. Estimation on the basis of truncated samples in the literature has largely been done using the conditional approach in which the unknown total sample size is eliminated from consideration by assuming the number of observations to be fixed and then examining the conditional distribution of the available observations, namely, the truncated distribution.

With this conditional approach, Deemer and Votaw (1955) (DV) prove that the maximum likelihood estimator (m.l.e.) of the mean of an exponential distribution from a sample truncated on (0, T) does not exist (becomes infinite) for certain samples. It is suggested in loc. cit. that if the parameter involved in the density is the reciprocal of the mean, there is no problem in estimating the parameter as the m.l.e. is now zero in the situation mentioned above. This, however, only masks the problem since estimating the mean of a distribution is of importance. Broeder (1955) has proved a similar result regarding the nonexistence of the m.l.e. for the scale parameter of the gamma distribution (shape parameter assumed known). Interestingly, no such problem of nonexistence occurs if the truncation is to an infinite interval, say to (T, ∞).

In this dissertation, we prove that the m.l.e.'s of the parameters of the doubly truncated normal distribution do not exist with positive probability. The case of the truncated Weibull distribution with both parameters (shape and scale) unknown is

also considered, wherein we have attempted to prove that the m.l.e.'s of the parameters are again nonexistent with positive probability. The truncated gamma distribution with both parameters (shape and scale) unknown proves rather intractable mathematically, but we include it for completeness sake. It is our belief that whenever a scale-parameter-dependent continuous density with infinite support is truncated to a finite interval, the m.l.e.'s would be nonexistent with positive probability. This being the case, we restrict our attention to some special distributions truncated to a finite interval in this dissertation and the conditional approach is used exclusively.

The common cure in the literature to the problem of the blowing up of the m.l.e.'s has been to reparametrize by considering the reciprocal of the original parameter with the result that the m.l.e. of the new parameter assumes the value zero. This strategy only skirts the issue since both zero and infinity are values which do not belong to the parameter space of the distribution. (The strategy has one advantage though: the estimator has finite expectation now.) A more positive approach to the taking care of this problem is to be found in the modified maximum likelihood estimators or the Bayes modal estimators, first considered in Blumenthal and Marcus (1975) and developed further subsequently by Blumenthal (1977), Blumenthal, Dahiya and Gross (1978), and Watson and Blumenthal (1980). The approach consists of multiplying the likelihood of the sample by a weight function (prior density) involving the parameter and then maxi-

mizing the resulting modified likelihood (posterior density).

The first two sections of chapter two review some literature pertaining to the problem of finding the m.l.e.'s from the normal, exponential, gamma and Weibull distributions. Section 2.3 deals with a review of the methods of modified maximum likelihood estimation, including a discussion on the stochastic expansions for these estimators, in order to study their asymptotic properties. Chapter three is devoted to the doubly truncated normal distribution for which we consider the proof of nonexistence of the m.l.e.'s of the parameters, derivation of the Bayes modal estimators, asymptotic variances of the m.l.e.'s, and simulation results comparing the aggregate behavior of the m.l.e.'s with that of the Bayes modal estimators. A new estimator, called the mixed estimator, which combines the Bayes modal estimator and the m.l.e. (when it exists and behaves rather well), is also proposed there. Much of the same is considered for the gamma and Weibull distributions in chapter four. The truncated Weibull, when the scale and the shape parameter are both unknown is discussed in detail - the existing literature does not deal with the nonexistence of the m.l.e.'s in this case.

2. REVIEW

In this chapter, we review some literature dealing with the maximum likelihood estimation for the parameters of the doubly truncated normal distribution (section 2.1) and also of the truncated exponential, gamma and Weibull distributions (section 2.2). Methods of Bayes modal estimation, especially with reference to the large sample properties based on the stochastic expansions of the estimator, are discussed in section 2.3.

2.1 The Doubly Truncated Normal Distribution

We consider here the problem of estimating the parameters of a doubly truncated normal distribution defined below.

Definition. A random variable has a doubly truncated normal distribution if its probability density function is

$$f(x;\mu,\sigma) = \frac{\exp[-(x-\mu)^2/2\sigma^2]}{\int_A^B \exp[-(t-\mu)^2/2\sigma^2]dt}, \qquad (2.1)$$

$A \le x \le B$, $-\infty < \mu < \infty$, $0 < \sigma < \infty$. The lower and upper truncation points are A and B respectively. The probability for the complete normal distribution below A and above B are the degrees of truncation. The distribution is singly truncated from

above or below, if A is replaced by $-\infty$ or B by ∞ respectively. As noted earlier, we shall deal with the double truncation exclusively, since the single truncation does not pose any problem regarding the existence of the m.l.e.'s of the parameters. Also we assume that A and B are known, which is commonly the case.

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with density (2.1). It is common knowledge that the maximum likelihood estimation of μ and σ from this sample is equivalent to estimation by equating the first and second sample and population moments (i.e., by the method of moments) (Johnson and Kotz, 1970 (JK)). The objective then is to try to solve two rather complicated non-linear equations in two unknowns. Some earlier attempts for this in the literature are by Cohen (1950a) and Cohen (1957). An iterative procedure for similar equations for the case of double censoring is given by Harter and Moore (1966).

The maximum likelihood (m.l.) equations are (cf. Cohen, 1950a)

$$\begin{aligned} \sigma(h_0 - z_1) - \eta_1 &= 0, \\ \sigma^2\left[1 - z_1(h_0 - z_1) - \frac{(B-A)\exp(-z_2^2/2)}{\Phi(z_2) - \Phi(z_1)}\right] - \eta_2 &= 0, \end{aligned} \tag{2.2}$$

where $z_1 = (A-\mu)/\sigma$, $z_2 = (B-\mu)/\sigma$, $h_0 = \dfrac{\exp(-z_1^2/2) - \exp(-z_2^2/2)}{\Phi(z_2) - \Phi(z_1)}$,

$\eta_k = \Sigma(x_i - A)^k/n$, $k=1,2$, and $\Phi(.)$ is the cumulative function (distribution function) of the complete normal distribution.

Using a modified Newton-Raphson method for solving two equations in two unknowns, iterative solution for (σ, z_1) is found,

which in turn provides the estimate for μ.

A chart simplifying the process of estimation is presented by Cohen (1957). The chart can be used to read rough estimates of μ and σ, for given sample moments. If more accurate values are needed, the rough estimates can be used as convenient starting values for an iterative solution. Slight modification in (2.2) is carried out, (a) on replacing σ by $(B-A)/(z_2-z_1)$ where z_1 and z_2 are the standardized ordinates for the truncation points as defined earlier, and (b) by writing the second equation in terms of the central moments in place of the earlier version containing the moments about the point A. The new equations are

$$\begin{aligned} [(h_0-z_1)/(z_2-z_1)] - [\eta_1/(B-A)] &= 0, \\ [(1+h_1-h_0^2)/(z_2-z_1)^2] - [s^2/(B-A)^2] &= 0, \end{aligned} \tag{2.3}$$

where $h_1 = \dfrac{z_1\exp(-z_1^2/2)-z_2\exp(-z_2^2/2)}{\Phi(z_2)-\Phi(z_1)}$ and $s^2 = \Sigma(x_i-\bar{x})^2/n$.

The chart consists of sets of two curves, one for the values of $\eta_1/(B-A) = (\bar{x}-A)/(B-A)$ (from 0.175 to 0.825), another for $s^2/(B-A)^2$ (from 0.015 to 0.0833). The values for z_1 and z_2 can be read on the two vertical axes for given values of $(\bar{x}-A)/(B-A)$ and $s^2/(B-A)^2$. Estimation of μ and σ follows from: $\hat{\sigma} = (B-A)/(\hat{z}_2-\hat{z}_1)$, $\hat{\mu} = A - \hat{\sigma}\hat{z}_1$. The tables used to prepare the chart were taken from a report by Thomson, Friedman and Garelis (1954), which was not available to us. (An apparently smaller table incorporating the first two moments of the truncated normal distribution, as functions of μ and σ, for different repre-

sentative degrees of truncation is available in JK (p.84).)

A closer examination of the chart reveals that there is no suggestion of having a curve for values of $s^2/(B-A)^2$ greater than $0.0833 = 1/12$, while it is quite possible that $s^2/(B-A)^2$ be greater than 1/12. This seems to be the only (faint at best) indication to the presence of the fact that the m.l.e.'s of μ and σ do not exist for $s^2 > (B-A)^2/12$ (to be proved in chapter three).

A recent paper on the problem of the existence of the m.l.e.'s from the doubly truncated normal samples is that by Crain (1979). He transforms the interval [A, B] into [-1,1] by

$$y = u(x) = 2[(x-A)/(B-A)] - 1, \tag{2.4}$$

$A \le x \le B$, with the inverse transformation being

$$x = w(y) = (B-A)(y/2) + (A+B)/2, \tag{2.5}$$

$-1 \le y \le 1$. The corresponding reparametrization of the density function results in the parameter vector $\tau = (\tau_1, \tau_2)$ for the new density, where τ_1 and τ_2 are defined by:

$$\begin{aligned} -(1/2\sigma^2) &= (45/8)^{\frac{1}{2}}[2/(B-A)]^2\tau_2, \\ (\mu/\sigma^2) &= (3/2)^{\frac{1}{2}}[2/(B-A)]\tau_1 \\ &\quad - (45/8)^{\frac{1}{2}}[4(B+A)/(B-A)^2]\tau_2. \end{aligned} \tag{2.6}$$

Applying results from the theory of exponential families of distributions, it is proved in loc. cit. that the m.l.e. of τ exists with probability one if $n \ge 2$. Hence the m.l.e.'s of μ

and σ are also stated to exist with probability one if $n \geq 2$. Results on the consistency and asymptotic normality of the m.l.e.'s, on the estimation of the density function and the distribution function are also stated.

At this point, two facts deserve attention. First, the transformation of the truncation interval from $[A, B]$ to $[-1, 1]$ does not serve any useful purpose. On the other hand, it complicates the analysis and masks the problem by presenting the density in the form of the parameters without clear meaning, while the original set of parameters identifies the mean and standard deviation of the complete normal distribution. Secondly, other arguments being valid, the author has not examined the possible values of $\hat{\tau}_1$ and $\hat{\tau}_2$ carefully. For example, a value of $\hat{\tau}_2 = 0^+$ will result in $\hat{\sigma}^2 = -\infty$ (clearly untenable), and $\hat{\tau}_2 = 0^-$ will give $\hat{\sigma}^2 = +\infty$. Also the estimates of μ will blow up in this situation. These estimates of μ and σ^2 do not belong to the parameter space of the distribution and the m.l.e.'s are to be then called nonexistent. In the light of the results we prove in chapter three, it is obvious that the main result of Crain (1979) about the existence of the m.l.e.'s is in error.

Some results that have come to our attention recently regarding this problem are available in Barndorff-Nielsen (1978). In an example on the existence of the m.l.e.'s from the doubly truncated normal distribution, necessary and sufficient conditions for such existence are given. These conditions are rather difficult to check and are dependent on the whole previous development

of the subject based on the convex exponential family theory. In chapter three, we derive simple sufficient condition(s) for the nonexistence of the m.l.e.'s in this case, which require rather elementary analysis.

2.2 The Truncated Exponential, Gamma and Weibull Distributions

We next consider other truncated distributions defined below.

Definition. A random variable has a truncated exponential distribution with parameter θ if its probability density function is

$$f(x;\theta) = \exp(-x/\theta)/[\theta(1-\exp(-T/\theta))], \tag{2.7}$$

$0 \le x \le T,\ 0 < \theta < \infty.$

Definition. A random variable has a truncated gamma distribution with shape parameter α and scale parameter θ if its probability density function is

$$f(x;\alpha,\theta) = x^{\alpha-1}\exp(-x/\theta)/ \int_0^T t^{\alpha-1}\exp(-t/\theta)dt\ , \tag{2.8}$$

$0 \le x \le T,\ 0 < \alpha < \infty,\ 0 < \theta < \infty.$

Definition. A random variable has a truncated Weibull distribution with shape parameter α and scale parameter θ if its probability density function is

$$f(x;\alpha,\theta) = \alpha x^{\alpha-1}\exp(-x^{\alpha}/\theta)/[\theta(1-\exp(-T^{\alpha}/\theta))], \tag{2.9}$$

$0 \le x \le T,\ 0 < \alpha < \infty,\ 0 < \theta < \infty.$

The truncated exponential is an obvious special case of the

truncated gamma and the truncated Weibull distributions both. The truncation point T is assumed known throughout.

Like the normal case, the moment estimator and the m.l.e. of θ are the same for the truncated exponential. Deemer and Votaw (1955) (DV) show that the m.l.e. of θ, say $\hat{\theta}$, is the solution of the equation

$$(\overline{x}/T) = (\hat{\theta}/T) - 1/[\exp(T/\hat{\theta})-1], \tag{2.10}$$

if $\overline{x} < T/2$, where $\overline{x}$ is the mean of the sample. If $\overline{x} > T/2$, the likelihood of the truncated sample attains its maximum for $\hat{\theta} = \infty$. JK comment on this situation that $\hat{\theta} = \infty$ "may be taken to mean that a truncated exponential distribution is inappropriate" and that a uniform (rectangular) distribution over the interval $[0, T]$ should be used instead. But one can easily visualize a situation where the sample has been specifically drawn from a truncated exponential distribution with moderate value of θ and still the probability of $\hat{\theta} = \infty$ can be substantial. [For a table of the estimates of these probabilities, see Table 3, Blumenthal and Marcus (1975).]

Some other results pertaining to the truncated exponential are available in the literature, namely, a table for solving the equation (2.10) or giving the right hand side of (2.10) as a function of $\hat{\theta}/T$, the exact distribution of Σx_i and an approximation to the distribution of $\overline{x}/T$ by a beta density. These are available in DV, Bain and Weeks (1964), Bain et al. (1977), and JK.

Immediately following the DV paper (in the same issue of the Annals of Mathematical Statistics) appeared the paper by Broeder (1955) in which the m.l. estimation of θ for the truncated gamma distribution, for known α, is discussed. The interval of the truncated sample values is first transformed from $[0, T]$ to $[0, 1]$. This can be done without loss of generality by transforming X to $Y = X/T$ (since T is known) and the density of Y remains a truncated gamma density (same shape parameter, θ/T the new scale parameter). (The same technique is later applied with questionable success by Crain (1979) for truncated normal. In this case, the density changes a great deal.) For the discussion that follows we assume that the sample values are truncated within the interval $[0, 1]$.

The m.l. estimator and the moment estimator of θ are again the same and $\hat{\theta}$ is the solution of the equation

$$\overline{x} = \frac{\int_0^1 t^{\alpha}\exp(-t/\hat{\theta})dt}{\int_0^1 t^{\alpha-1}\exp(-t/\hat{\theta})dt}$$

$$= \alpha\hat{\theta} - \frac{\exp(1/\hat{\theta})}{\int_0^1 t^{\alpha-1}\exp(-t/\hat{\theta})dt}, \tag{2.11}$$

if $\overline{x} < \alpha/(\alpha+1)$. If $\overline{x} > \alpha/(\alpha+1)$, the likelihood of the truncated sample attains a maximum for $\hat{\theta} = \infty$. The right side of the equation (2.11) is the mean of the truncated gamma $[E(X)]$ and is an increasing function of θ since $(\partial/\partial\theta)E(X) = \mathrm{Var}(x) > 0$. Parallel to JK's comment about the unsuitability of the truncated

exponential model, one can, in this case suggest that for $\bar{x}$ greater than $\alpha/(\alpha + 1)$, the density given by

$$f(x;\alpha) = \alpha x^{\alpha-1}, \qquad 0 \le x \le 1, \qquad \alpha > 0, \tag{2.12}$$

(which is the limit of $f(x;\alpha,\theta)$ of (2.8) as $\theta \to \infty$) would be more appropriate instead of the truncated gamma. Again we could be given a truncated sample specifically from the gamma distribution and $\bar{x}$ could still be greater than $\alpha/(\alpha + 1)$ with positive probability. Broeder's density involves $1/\theta$ in place of θ (which was also the case with DV's density for the truncated exponential), and the m.l.e. of $1/\theta$ takes value zero when $\bar{x}$ exceeds $\alpha/(\alpha + 1)$, hence seemingly no problem arises, but the parameter space of $1/\theta$ is still restricted away from zero and the m.l.e. of $1/\theta$ is nonexistent.

Inspite of Broeder's paper being in the same genre as that of DV, surprisingly little note has been taken of it in the later literature. JK do not mention his contribution in chapter 17, section 8.1 on the truncated gamma distribution (but do list the paper in the bibliography). Gross (1971), while proving monotonicity properties of the moments of the truncated gamma (and truncated Weibull), does not cite Broeder's work, even though Broeder also discusses monotonicity of the first moment of the truncated gamma as a function of θ. And subsequently, Gross and Clark (1975) do not refer to Broeder's work in their book which contains a separate section on the truncated gamma distribution.

For α unknown, the m.l. estimation leads to the comparison

of the first population and sample moments of logX and X. Because the integral of probability density cannot be expressed in close form, it is particularly difficult to come up with a proof of the nonexistence of the m.l.e.'s of α and θ. Still an attempt is made in chapter four.

Following the traditional nomenclature of calling gamma type distributions as the Pearson Type III distributions, some authors have discussed the estimation of α and θ. They are Cohen (1950b, 1951) (the moment estimators), Des Raj (1953) (the m.l. and moment estimators), Das (1955), Chapman (1956) and Iyer and Singh (1963) (the censored sample). Most of these papers express the density function in terms of the moments of the complete distribution (Des Raj assumes the third non-central moment of the distribution known), commonly use iterative methods to solve equations in order to estimate the parameters, and do not examine the question of the existence or nonexistence of the solution of equations.

As already noted, Gross (1971) proves monotonicity properties for the moments of the truncated gamma and Weibull distributions. Employing the total positivity technique, he proves for the truncated gamma distribution that

(i) $\mu_r'(\theta)$ is an increasing function of θ;

(ii) $\mu_r'(\alpha)$ is an increasing function of α;

(iii) $[\mu_{r+1}'(\theta)/\mu_r'(\theta)]$ is an increasing function of θ;

(iv) $0 \le \mu(\theta) \le \alpha T/(\alpha + 1)$ for all $\theta \ge 0$; and

(v) $0 \le \mu_{r+s}'(\theta)/\mu_r'(\theta) \le (\alpha + r)T^s/(\alpha + r + s)$ for all $\theta \ge 0$,

where μ_r' is the r-th moment about the origin, $r \ge 0$, s any

positive integer and $\mu = \mu_1'$. Except for the result (ii), rest of the four results hold as it is, if the underlying distribution is the truncated Weibull. Harter and Moore (1965, 1967) consider the m.l. estimation and asymptotic variances of the m.l.e.'s for the parameters of the gamma and Weibull distributions based on the censored samples.

A result like (iv) above for the truncated gamma cannot be used to prove the nonexistence of the m.l.e. of θ for the truncated Weibull, given α known, since the m.l.e. and the moment estimator are different here. But it does follow that, if in place of the m.l. estimation, we tried the method of moments, the moment estimator of θ will still have a problem of blowing up whenever $\overline{x}$ exceeds $\alpha T/(\alpha + 1)$.

The nonexistence of the m.l.e. of θ (for α known) for the truncated Weibull is still not difficult to prove, given the property that $X^{\alpha} = Y$ is distributed as exponential, when X follows the Weibull distribution. It can be seen that the m.l.e. of θ, say $\hat{\theta}$, is the solution of

$$(\Sigma x_i^{\alpha}/nT^{\alpha}) = (\hat{\theta}/T^{\alpha}) - 1/[\exp(T^{\alpha}/\hat{\theta})-1], \tag{2.13}$$

when $\Sigma x_i^{\alpha} < nT^{\alpha}/2$, and $\hat{\theta} = \infty$ for $\Sigma x_i^{\alpha} > nT^{\alpha}/2$. The limiting density as $\theta \to \infty$ is still (2.12), same as that for the truncated gamma, which would seem to be appropriate if $\Sigma x_i^{\alpha} > nT^{\alpha}/2$.

2.3 The Bayes Modal Estimation

Working with the truncated exponential distribution, and estimating θ and the total population size N, Blumenthal and Marcus (1975) note, quoting DV, that for small samples and moderate probabilities of truncation, the estimate of N is infinite with surprisingly large probability and correspondingly the m.l.e. of θ (conditional m.l.e. in their terminology) also fails to exist with this probability.

In order to avoid this problem they use the Bayes approach in which the likelihood of the truncated sample is multiplied by a prior density (conjugate prior) $p(\theta)$ for θ (and uniform prior for N) and is then maximized. The resulting estimators are then called the Bayes modal estimators or the modified maximum likelihood estimators. Expansions for the estimators of N and θ allow the comparison between different types of estimators by concentrating on the first and second order properties. Stochastic expansions can also be used to find the optimum values of the parameters of the prior density.

In subsequent papers by Blumenthal (1977), Blumenthal, Dahiya and Gross (1978), and Watson and Blumenthal (1980), this method is applied to the general case under some mild regularity conditions and to other special distributions (e.g., Poisson). Optimum values for the parameters of the prior density are also obtained using the criteria of minimax bias and minimum mean square error. These and other developments are summarized in Blumenthal (1981).

Since it will be useful later, a summary of the results, based on Blumenthal (1982), is presented here. The estimation of N not being our concern, we restrict our attention to the estimation of θ. Sufficient regularity conditions for $f(x;\theta)$ and $p(\theta)$ are assumed to hold.

Let $f(x;\theta)$ be the truncated density with real θ and $p(\theta)$ an arbitrary weight function, then the modified likelihood is given by

$$L_m = [\Pi\ f(x_i;\theta)]\ p(\theta), \tag{2.14}$$

where the product extends over the range $1 \le i \le n$, as are all the summations later on. We consider the expansion of the type

$$\tilde{\theta} = \theta + (a/\sqrt{n}) + (b/n) + O(n^{-3/2}), \tag{2.15}$$

for the Bayes modal estimator $\tilde{\theta}$, which is the solution of the equation

$$0 = (\partial/\partial\theta)\log L_m = \Sigma S(x_i;\tilde{\theta}) + \xi(\tilde{\theta}), \tag{2.16}$$

where

$$\begin{aligned} S(x;\theta) &= (\partial/\partial\theta)\log f(x;\theta) = f'(x;\theta)/f(x;\theta), \\ \xi(\theta) &= (\partial/\partial\theta)\log p(\theta) = p'(\theta)/p(\theta), \end{aligned} \tag{2.17}$$

and the prime notation indicates differentiation w.r.t. θ.

Expanding $S(x;\tilde{\theta})$ and $\xi(\tilde{\theta})$ in Taylor series around $S(x;\theta)$ and $\xi(\theta)$ respectively in (2.16) and using the expansion (2.15), we obtain

$$0 = n^{\frac{1}{2}}(Z_1\sqrt{L_2} + aL_{01}) + [bL_{01} + aZ_{01}\sqrt{V_{01}} + (a^2L_{001}/2) + \xi] + O(n^{-\frac{1}{2}}), \qquad (2.18)$$

where $L_1 = E[S(X;\theta)] = 0$; $L_2 = E[S(X;\theta)]^2$; $L_{01} = E[S'(X;\theta)] = -L_2$; $L_{02} = E[S'(X;\theta)]^2$; $L_{001} = E[S''(X;\theta)]$; $V_1 = L_2 - L_1^2$; $V_{01} = L_{02} - L_{01}^2$; $Z_1 = [\Sigma S(x_i;\theta) - nL_1]/\sqrt{nV_1}$; $Z_{01} = [\Sigma S'(x_i;\theta) - nL_{01}]/\sqrt{nV_{01}}$, and all expectations are w.r.t. the truncated density, $f(x;\theta)$.

Equating to zero the coefficient of $n^{\frac{1}{2}}$ and the constant term in (2.18), we get a and b in terms of the Z's and L's as follows

$$a = (Z_1/\sqrt{L_2}),$$
$$b = L_2^{-2}[Z_1Z_{01}\sqrt{V_{01}L_2} + (Z_1^2L_{001}/2) + L_2\xi]. \qquad (2.19)$$

Note that the influence of the prior density appears only in b or in the bias term while the consistency property is free from it. From the form of a, $\tilde{\theta} - \theta = Z_1/\sqrt{nL_2}$, which converges stochastically to zero, hence consistency of $\tilde{\theta}$ follows at once, and we can write

$$\sqrt{n}(\tilde{\theta}-\theta) = (Z_1/\sqrt{L_2}) + (b/\sqrt{n}).$$

Since b has a legitimate limiting distribution, $(b/\sqrt{n}) \to 0$ and the central limit theorem applies to Z_1, giving the usual asymptotic normality of $\tilde{\theta}$. From (2.15) and (2.19), we find that

$$E(\tilde{\theta}) = \theta + [E(b)/n] + O(n^{-3/2}), \qquad (2.20)$$

where $E(b)$, being the $O(1/n)$ term of the bias, can be regarded as the "asymptotic bias" of $\tilde{\theta}$. Using $E(Z_1Z_{01}\sqrt{V_{01}L_2}) = L_{11} =$

$E[S(X;\theta)S'(X;\theta)]$ which is easy to see from the definitions of Z_1 and Z_{01}, we find that

$$E(b) = L_2^{-2}\,[L_{11} + (L_{001}/2) + L_2\xi],$$

since $E(Z_1^2) = 1$. Furthermore, the L's can be expressed in terms of μ's, the more fundamental quantities (if integrals and differentials can be interchanged), as follows

$$L_2 = \mu_2;\; L_{01} = -\mu_2;\; L_{11} = \mu_{11}-\mu_3;\; L_{001} = -3\mu_{11}+2\mu_3, \qquad (2.21)$$

where

$$\begin{aligned} \mu_2 &= E[f'(X;\theta)/f(X;\theta)]^2, \\ \mu_{11} &= E[\{f'(X;\theta)/f(X;\theta)\}\{f''(X;\theta)/f(X;\theta)\}], \qquad (2.22) \\ \mu_3 &= E[f'(X;\theta)/f(X;\theta)]^3, \end{aligned}$$

and so on.

Thus,

$$E(b) = \mu_2^{-2}\,[-(\mu_{11}/2)+\mu_2\xi]. \qquad (2.23)$$

It is this expression which has been utilized in chapter three to minimize the asymptotic bias of the Bayes modal estimator. One can go further and write the expression for the mean square error of $\tilde{\theta}$, i.e., $E[\sqrt{n}(\tilde{\theta}-\theta)^2]$ in terms of similar μ's but the mathematics is relatively tedious. For more details, see Watson and Blumenthal (1980) and Blumenthal (1982).

3. THE DOUBLY TRUNCATED NORMAL DISTRIBUTION

In this chapter, we discuss the doubly truncated normal distribution divided into two main cases: (a) μ known, and (b) both μ and σ unknown. We discuss the conditions under which the m.l.e.'s of the parameters are nonexistent (sections 3.1 and 3.2), derivation of the Bayes modal estimators including an analysis for the optimum value of the prior density parameter (sections 3.3 and 3.4), the asymptotic variances of the m.l.e.'s (section 3.5), and some simulation results indicating the usefulness of the Bayes modal estimators (sections 3.6 and 3.7).

3.1 The M.L. Estimation. Case I - μ Known

For the sake of simplicity, we first consider the case when μ is known, and without loss of generality, is equal to zero. Let $\sigma^2 = \theta$. We now prove that the m.l.e. of θ is nonexistent for certain sample configurations. The truncated normal density is

$$f(x;\theta) = \exp(-x^2/2\theta)/[\int_A^B \exp(-t^2/2\theta)dt], \qquad (3.1)$$

$A \leq x \leq B$, $0 < \theta < \infty$. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables having the density $f(x;\theta)$, then the likelihood for the sample is

$$L(\underline{x};\theta) = \exp(-\Sigma x_i^2/2\theta)/[\int_A^B \exp(-t^2/2\theta)dt]^n, \tag{3.2}$$

and the m.l.e. of θ is the solution of the equation

$$(\Sigma x_i^2/n) = \frac{\int_A^B t^2\exp(-t^2/2\theta)dt}{\int_A^B \exp(-t^2/2\theta)dt} = g(\theta), \tag{3.3}$$

say. The ratio of the integrals on the right side of (3.3), $g(\theta)$, is the second moment about the origin of the random variable with the density (3.1). Thus the m.l.e. and the moment estimator are the same. To examine the existence of the solution of the equation (3.3) for θ, we need to examine the behavior of the function $g(\theta)$. Now

$$\begin{aligned} g'(\theta) &= (\partial/\partial\theta)\, g(\theta) \\ &= (1/2\theta^2)[E(X^4) - (E(X^2))^2] \\ &= (1/2\theta^2)\, \mathrm{Var}(X^2) > 0, \end{aligned} \tag{3.4}$$

where all the expectations are taken w.r.t. the density in (3.1). Thus $g(\theta)$ is an increasing function of θ assuring a unique solution for the equation (3.3), if it exists at all. Also

$$\begin{aligned} \lim_{\theta \to \infty} g(\theta) &= \frac{\int_A^B t^2 dt}{\int_A^B dt} = (B^3-A^3)/[3(B-A)] \\ &= (A^2+B^2+AB)/3 = M_1, \end{aligned} \tag{3.5}$$

say. The quantity M_1 is also the second moment about the origin of the uniform distribution over the range [A, B] (or U[A, B]). Further, if zero is contained in the interval [A, B],

$$\lim_{\theta \downarrow 0} g(\theta) = 0, \tag{3.6}$$

since the distribution, indicated by the density in (3.1), tends to a distribution degenerate at zero (as zero is the mean of the complete normal distribution). (If zero is to the left of the interval [A, B], the distribution tends to be degenerate at A and for zero to the right, it degenerates at B.)

Thus $g(\theta)$ takes values in the interval $[0, M_1]$ while $\Sigma x_i^2/n$, the quantity on the left in (3.3), can assume values in the interval $[0, (A^2+B^2)/2]$. It is obvious that $(A^2+B^2)/2 > M_1$ and, therefore, whenever $\Sigma x_i^2/n$ exceeds M_1, no solution to (3.3) exists. It can be directly seen that when this happens, the maximum of $L(\underset{\sim}{x};\theta)$ occurs at $\hat{\theta} = \infty$.

It is to be noted that the problem does not resolve itself, if, instead of the m.l.e., one compares the first (or any other) absolute moments of the sample and the population. In the special case of symmetric truncation over [-A, A], the upper limit of $E|X|$ is $A/2$ (first absolute moment of $U[-A, A]$). But $\Sigma|x_i|/n \le A$. Also for the r-th absolute moment, $r > 0$

$$E|X|^r \to A^r/(r+1) \quad \text{but} \quad (\Sigma|x_i|^r/n) \le A^r.$$

The problem with the m.l.e. of θ is not just that it doesn't exist for $\Sigma x_i^2/n > M_1$, but even when $\Sigma x_i^2/n$ is less than M_1, though sufficiently close to M_1, the m.l.e. behaves terribly. Therefore, parallel to JK's comment regarding the truncated exponential, it would seem appropriate that when θ is large, uniform

distribution will model the data better. But again, the original distribution could be genuinely normal, and with moderately large probability, we may obtain an infinite value for the estimate of θ.

If $A = -\infty$, or $B = \infty$, i.e., if we have single truncation, there is no problem of nonexistence since $g(\theta) \uparrow \infty$. Also as n increases, the probability $P[\Sigma x_i^2/n > M_1]$ would tend to be smaller and will eventually become zero for sufficiently large n. The rate at which this probability tends to zero will depend on the degree of truncation, the rate being faster for smaller degrees of truncation.

Next we consider the general case of both μ and σ being unknown.

3.2 The M.L. Estimation. Case II - Both μ and σ Unknown

Now the truncated normal density is

$$f(x;\mu,\sigma) = \frac{\exp[-(x-\mu)^2/2\sigma^2]}{\int_A^B \exp[-(t-\mu)^2/2\sigma^2]dt}, \tag{3.7}$$

$A \le x \le B$, $-\infty < \mu < \infty$, $0 < \sigma < \infty$. If $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables with density $f(x;\mu,\sigma)$, then the likelihood for the sample is

$$L(\underline{x};\mu,\sigma) = \frac{\exp\{-\Sigma(x_i-\mu)^2/2\sigma^2\}}{[\int_A^B \exp\{-(t-\mu)^2/2\sigma^2\}dt]^n}, \tag{3.8}$$

and the m.l.e.'s of μ, σ are the solutions of the equations:

$$\bar{x} = \frac{\int_A^B t \exp[-(t-\mu)^2/2\sigma^2]dt}{\int_A^B \exp[-(t-\mu)^2/2\sigma^2]dt}$$

$$= \mu_T(\mu,\sigma), \tag{3.9}$$

say, and

$$s = \frac{\int_A^B (t-\mu_T)^2 \exp[-(t-\mu)^2/2\sigma^2]dt}{\int_A^B \exp[-(t-\mu)^2/2\sigma^2]dt}$$

$$= \sigma_T^2(\mu,\sigma), \tag{3.10}$$

say, where $s^2 = \Sigma(x_i-\bar{x})^2/n$, the sample variance, and where μ_T and σ_T^2 are the mean and variance, respectively, of the random variable with density (3.7) ('T' stands for truncation). The dependence of μ_T and σ_T^2 on μ and σ is to be suppresed in the future for the sake of brevity. To examine the existence of the m.l.e.'s of μ and σ, we analyse, one at a time, the behavior of the functions involved in the equations (3.9) and (3.10). We first consider (3.9).

For fixed σ, μ_T is an increasing function of μ, since $(\partial/\partial\mu)\ \mu_T = (1/\sigma^2)\ \sigma_T^2 > 0$. Also $\lim_{\mu\to -\infty} \mu_T = A$ and $\lim_{\mu \to \infty} \mu_T = B$. Further, for fixed μ, if $\sigma^2 \downarrow 0$, the distribution degenerates at A, B or μ depending on if μ is to the left of the interval [A, B], to the right of [A, B] or within [A, B] respectively. Hence μ_T approaches A, B or μ in the above three situations. For fixed μ, if $\sigma^2 \to \infty$, $\mu_T \to (A+B)/2$.

If both μ and σ^2 vary simultaneously, then

(i) $\mu \to \infty$, $\sigma^2 \to 0$ implies $\mu_T \to B$,

(ii) $\mu \to -\infty$, $\sigma^2 \to 0$ implies $\mu_T \to A$,

(iii) $\mu \to \infty$, $\sigma^2 \to \infty$ with $(\mu/\sigma^2) \to 0$ implies $\mu_T \to (A+B)/2$,

(iv) $\mu \to \pm\infty$, $\sigma^2 \to \infty$ with $(\mu/\sigma^2) \to c$ (c being a constant, positive or negative), then the density in (3.7) approaches the truncated exponential density of the form

$$f(x) = \exp(cx)/[\int_A^B \exp(ct)dt], \qquad (3.11)$$

hence μ_T approaches the mean of the distribution given by the density (3.11).

(v) $\mu \to \infty$, $\sigma^2 \to \infty$ with $(\mu/\sigma^2) \to \infty$ implies $\mu_T \to B$,

(vi) $\mu \to -\infty$, $\sigma^2 \to \infty$ with $(\mu/\sigma^2) \to -\infty$ implies $\mu_T \to A$.

In all the cases, $A \le \mu_T \le B$, also $A \le \overline{x} \le B$, hence there is no problem with the equation (3.9).

Next, let us consider the equation (3.10). For fixed μ, σ_T^2 is an increasing function of σ. Also $\lim_{\sigma^2 \to 0} \sigma_T^2 = 0$, since the distribution given by (3.7) degenerates at A, B or μ according as μ is to the left of the interval [A, B], to the right of [A, B] or within [A, B] respectively and σ_T^2 is then the variance of a degenerate distribution. Further, $\lim_{\sigma^2 \to \infty} \sigma_T^2 = [(B-A)^2/12] = M_2$, the variance of U[A, B]. For fixed σ, μ approaching $-\infty$ or ∞, σ_T^2 approaches zero since the distribution degenerates at A or B respectively.

If both μ and σ^2 vary simultaneously, then

(i) $\mu \to \infty$, $\sigma^2 \to 0$ implies $\sigma_T^2 \to 0$,

(ii) $\mu \to -\infty$, $\sigma^2 \to 0$ implies $\sigma_T^2 \to 0$,

(iii) $\mu \to \infty$, $\sigma^2 \to \infty$ with $(\mu/\sigma^2) \to 0$ implies

$\sigma_T^2 \to (B-A)^2/12 = M_2$,

(iv) $\mu \to \infty$, $\sigma^2 \to \infty$ with $(\mu/\sigma^2) \to c$ (c being a constant, positive or negative) then the density in (3.7) approaches the truncated exponential density of the form (3.11), and σ_T^2 approaches the variance of the density in (3.11), which is less than the variance of $U[A, B]$ or M_2,

(v) $\mu \to \pm\infty$, $\sigma^2 \to \infty$ with $(\mu/\sigma^2) \to \pm\infty$ implies $\sigma_T^2 \to 0$.

In all the cases, $0 \le \sigma_T^2 \le (B-A)^2/12$, while the sample variance s^2 takes values in a wider interval, $0 \le s^2 \le (B-A)^2/4$. (The maximum value of s^2 occurs when half of the sample values are at A and rest of the half at B.) Thus, the solution of the equation (3.10) does not exist whenever s^2 exceeds $(B-A)^2/12$ or M_2. It can be seen that the likelihood $L(\underline{x};\mu,\sigma)$ in (3.8) is then maximized when

$$\hat{\sigma} = \infty, \quad \hat{\mu} = +\infty, \text{ if } \bar{x} > (A+B)/2, \text{ and}$$

$$\hat{\sigma} = \infty, \quad \hat{\mu} = -\infty, \text{ if } \bar{x} < (A+B)/2.$$

From the discussion on the behavior of μ_T in the equation (3.9), it is obvious that the case when μ is unknown and σ is known, does not have any nonexistence problems of the m.l.e., and no new results arise. Therefore, this case does not need to be considered.

3.3 The Bayes Modal Estimation. Case I - μ Known

We first consider the situation discussed in section 3.1 and the density is as given in (3.1), σ^2 still being replaced by θ. The objective is to derive estimators belonging to a new class, namely the Bayes modal estimators (as detailed in section 2.3) with one important property: they exist with probability one. Other useful properties of these estimators are examined with the help of the stochastic expansions.

The common approach for the prior density of θ is to use the chi-square distribution for $1/\theta$ which is the conjugate prior distribution for the (complete) normal with its mean known (Box and Tiao, 1973; Cox and Hinkley, 1974). It is appropriate since for a sample from the normal distribution, the distribution of nS^2/θ is chi-square. The same is not true of a sample from the truncated normal distribution. But the general form of the truncated normal likelihood being the same as of the complete normal (except for the division by a probability in the former case), we use a chi-square prior distribution for $1/\theta$. Let ν be the degrees of freedom for the chi-square. The weight function (prior density) is

$$p(\theta) = c(\nu)\, \theta^{-\frac{1}{2}(\nu-2)} \exp(-1/2\theta), \qquad \theta > 0, \tag{3.12}$$

where $c(\nu) = 2^{-\nu/2}[\Gamma(\nu/2)]^{-1}$. The modified likelihood (posterior density) is

$$L_m(\theta|\underline{x}) = \frac{c(\nu)\theta^{-\frac{1}{2}(\nu-2)}\exp[-(\Sigma x_i^2+1)/2\theta]}{[\int_A^B \exp(-t^2/2\theta)dt]^n}. \tag{3.13}$$

The mode of this modified likelihood is the solution for θ of the following equation

$$(\Sigma x_i^2+1)/n = (\theta/n)(\nu-2) + g(\theta), \tag{3.14}$$

where $g(\theta)$ is defined in (3.3). Note that the only changes in (3.14), from the m.l. equation (3.3), are the additional quantities: "$1/n$" on the left side, and "$(\theta/n)(\nu-2)$" on the right side of this equation.

Recalling $g(\theta)$ to be an increasing function of θ, it is apparent that for $\nu > 2$, the right side of the equation (3.14) is an increasing function of θ and it increases to ∞ as $\theta \to \infty$, ensuring the uniqueness of the solution of (3.14) for any given value of $\Sigma x_i^2/n$. Thus there is no more a problem of the nonexistence of the estimator for θ. We denote the Bayes modal estimator of θ by $\tilde{\theta}$.

In order to find an optimum value for the chi-square parameter ν, it needs to be pointed out that the problem of nonexistence of the m.l.e. happens more frequently when θ is large, because in this case it is more likely for $\Sigma x_i^2/n$ to exceed M_1. Therefore the modification of the m.l.e. brought out by the equation (3.14) should come into play for relatively large θ. We thus derive the value of ν which minimizes the asymptotic bias of $\tilde{\theta}$ for large values of θ. Following the treatment of section 2.3,

$$\mu_2 = (1/4\theta^4)E[X^2 - E(X^2)]^2 = (1/4\theta^4)\mathrm{Var}(X^2). \tag{3.22}$$

For μ_{11}, we need $[f''(x;\theta)/f(x;\theta)]$ which is given by

$$[f''/f] = (\partial/\partial\theta)[f'/f] + [f'/f]^2. \tag{3.23}$$

Then

$$\begin{aligned}(\partial/\partial\theta)[f'/f] &= (\partial/\partial\theta)[(1/2\theta^2)\{x^2-E(X^2)\}] \\ &= -(1/\theta^3)[x^2-E(X^2)] \\ &\quad + (1/2\theta^2)[-(\partial/\partial\theta)E(X^2)],\end{aligned} \tag{3.24}$$

where

$$\begin{aligned}(\partial/\partial\theta)E(X^2) &= (1/2\theta^2)[E(X^4)-\{E(X^2)\}^2] \\ &= (1/2\theta^2)\mathrm{Var}(X^2).\end{aligned} \tag{3.25}$$

Therefore, with (3.25) substituted into (3.24), we have

$$(\partial/\partial\theta)[f'/f] = -(1/\theta^3)[x^2-E(X^2)] - (1/4\theta^4)\mathrm{Var}(X^2). \tag{3.26}$$

Hence by (3.26), (3.23) and (3.20),

$$\begin{aligned}[f''/f] = &-(1/\theta^3)[x^2-E(X^2)] - (1/4\theta^4)\mathrm{Var}(X^2) \\ &+ (1/4\theta^4)[x^2-E(X^2)]^2,\end{aligned} \tag{3.27}$$

so that combining (3.20) and (3.27), we obtain

$$\mu_{11} = (1/8\theta^6)[-4\theta\ \mathrm{Var}(X^2) + E\{X^2-E(X^2)\}^3], \tag{3.28}$$

since $E(X^2-E(X^2)) = 0$. Substituting the values of μ_2, μ_{11} and ξ into $E(b)$ in (3.16) gives

$$E(b) = [2 - (\nu/2) + (1/2\theta) - (1/4\theta)\{E(X^2-E(X^2))^3/\text{Var}(X^2)\}]$$
$$\cdot\{4\theta^3/\text{Var}(X^2)\}. \qquad (3.29)$$

$E(b)$ is a strictly decreasing function of ν. The value of ν for which $E(b)$ is zero, say ν_0, is

$$\nu_0/2 = 2 + (1/2\theta)[1 - \{E(X^2-E(X^2))^3/2\text{Var}(X^2)\}],$$

or

$$\nu_0 = 4 + (1/\theta)[1 - \{E(X^2-E(X^2))^3/2\text{Var}(X^2)\}]. \qquad (3.30)$$

For large θ, the ratio inside the curly brackets will be the ratio of the third and second central moments of the square of a random variable following $U[A, B]$. Thus for large θ, the value of ν_0 will be close to four. This number 'four' is the value of ν which minimizes the maximum asymptotic bias of the Bayes modal estimator $\tilde{\theta}$, for large values of θ, and has been used in the simulations of sections 3.6 and 3.7.

3.4 The Bayes Modal Estimation. Case II - Both μ and σ Unknown

In this section, we derive the Bayes modal estimators when both μ and σ are unknown. The density is as given in (3.7). We will use the chi-square prior density for $1/\sigma^2$, and non-informative prior for μ (since the problem of nonexistence of the solution did not lie with the equation involving means, i.e., (3.9)).

The two prior densities are

$$p(\sigma^2) = c(\nu)\sigma^{-(\nu-2)}\exp(-1/2\sigma^2), \qquad \sigma > 0, \qquad (3.31)$$
$$p(\mu) \propto c_1,$$

with $c(\nu) = 2^{-\nu/2}[\Gamma(\nu/2)]^{-1}$. The posterior density is then given by

$$L(\mu,\sigma^2|\underline{x}) = \frac{c_2\sigma^{-(\nu-2)}\exp[-\{\Sigma(x_i-\mu)^2+1\}/2\sigma^2]}{[\int_A^B \exp\{-(t-\mu)^2/2\sigma^2\}dt]^n}, \qquad (3.32)$$

where c_2 involves only ν. The modes of this modified likelihood (posterior density) for μ and σ are given as the solutions of the following equations:

$$\overline{x} = \mu_T, \qquad (3.33)$$

and

$$s^2 + (1/n) = (\sigma^2/n)(\nu-2) + \sigma_T^2, \qquad (3.34)$$

where μ_T and σ_T^2 are as defined in (3.9) and (3.10).

Note that the first equation is the same as the m.l. equation (3.9) and the second equation has changed exactly as in section 3.3, i.e., on the left side there is an additional "$1/n$" while on the right the quantity "$(\sigma^2/n)(\nu-2)$" has been added. Again recalling the fact that σ_T^2 is an increasing function of σ, the right side of (3.34), with $\nu > 2$, is an increasing function of σ and increases to ∞, as σ increases to ∞. Hence no problem regarding the existence of the solution of (3.34) arises. The

solutions of the equations (3.33) and (3.34) for μ and σ are then the Bayes modal estimators, denoted by $\tilde{\mu}$ and $\tilde{\sigma}$.

Since the only modification to the m.l.equations takes place in the second equation (for variances) and also since the changes are exactly the same as in section 3.3, we retain the value of ν as four to be optimal (in the sense of minimum asymptotic bias).

3.5 The Asymptotic Variances of the M.L.E.'s

In the next section, we provide some simulation results through which we can compare the behavior of the m.l.e.'s (when they exist) and the Bayes modal estimators, principally using the simulated expected bias and mean square error of the estimators. Another criterion which can be helpful in analyzing the properties and usefulness of the Bayes modal estimators is the asymptotic variances of the m.l.e.'s.

We are already aware that the m.l.e.'s exist asymptotically with probability one, since the probability of nonexistence tends to zero as n approaches ∞. It is well known (Kendall and Stuart, 1979) that the m.l.e.'s (under certain regularity conditions, which are satisfied by the truncated normal distribution) are asymptotically efficient and have an asymptotic normal distribution. The Bayes modal estimators possess the same large sample properties as the m.l.e.'s (refer to section 2.3). At the same time, some indication of the usefulness of the Bayes modal estimators can be obtained from the comparison of the asymptotic variances

of the m.l.e.'s with the finite sample mean square error of the modal estimators.

In the present context, the property that the sufficiency and completeness are inherited by the truncated distribution from the parent (Tukey, 1949; Smith, 1957), is particularly useful. Since $\bar{x}$ and s^2 are sufficient for μ for σ^2 in the complete normal distribution, hence also in the truncated normal. We then use the equation (18.64) p. 60 of Kendall and Stuart (1979) to obtain the asymptotic variances of the m.l.e.'s for our case. Accordingly, the dispersion matrix of the m.l.e.'s of μ and σ, in large samples, is $(V^{-1})^{-1}$ where

$$(v_{rs}^{-1}) = -[\partial^2 \log L/\partial\theta_r \partial\theta_s]_{\hat{\theta} = \theta} , \qquad (3.35)$$

where θ is now a two dimensional parameter, $\hat{\theta}$ the m.l.e. of θ and L the likelihood of the sample. The results derived here are parallel to those of Harter and Moore (1966), who find asymptotic variances for doubly censored samples from the normal distribution. (Curiously, the Table 11, p. 86 in JK, taken from Harter and Moore, is misplaced because it deals with censored samples and not truncated samples, which is the topic under discussion there.)

Now the log likelihood is as follows.

$$\log L = -(1/2\sigma^2)\Sigma(x_i-\mu)^2 - n\log[\int_A^B \exp\{-(t-\mu)^2/2\sigma^2\}dt]. \quad (3.36)$$

The dispersion matrix for the m.l.e.'s of μ and σ is $(V^{-1})^{-1}$ where the elements of V^{-1} are as follows:

$$v_{11}^{-1} = -\,[\partial^2 \log L/\partial\mu^2]_{\hat{\mu} = \mu, \hat{\sigma} = \sigma'}$$

$$v_{12}^{-1} = -\,[\partial^2 \log L/\partial\sigma\partial\mu]_{\hat{\mu} = \mu, \hat{\sigma} = \sigma'} \qquad (3.37)$$

$$v_{22}^{-1} = -\,[\partial^2 \log L/\partial\sigma^2]_{\hat{\mu} = \mu, \hat{\sigma} = \sigma}.$$

We now calculate the partial derivatives of the log likelihood w.r.t. the parameters μ and σ in the following.

$$(\partial/\partial\mu)\log L = (n/\sigma^2)(\overline{x} - \mu_T); \qquad (3.38)$$

$$(\partial/\partial\sigma)\log L = (1/\sigma^3)[\Sigma(x_i-\mu)^2 - nE(X-\mu)^2]; \qquad (3.39)$$

$$\begin{aligned}(\partial^2/\partial\mu^2)\log L &= -\,(n/\sigma^4)[E\{X(X-\mu)\} - E(X)E(X-\mu)] \\ &= -\,(n/\sigma^4)[E(X^2) - \{E(X)\}^2] \\ &= -\,(n/\sigma^4)\,\mathrm{Var}(X); \qquad (3.40)\end{aligned}$$

$$\begin{aligned}(\partial^2/\partial\sigma\partial\mu)\log L &= -\,(2n/\sigma^3)\,\overline{x} + (2n/\sigma^3)\mu_T \\ &\quad -\,(n/\sigma^5)[E\{X(X-\mu)^2\} - E(X)E(X-\mu)^2] \\ &= -\,(2n/\sigma^3)(\overline{x} - \mu_T) \\ &\quad -\,(n/\sigma^5)[E(X-\mu)^3 + E(X-\mu)^2(\mu - E(X))]; \qquad (3.41)\end{aligned}$$

$$\begin{aligned}(\partial^2/\partial\sigma^2)\log L &= -\,(3/\sigma^4)\,\Sigma(x_i-\mu)^2 + (3/\sigma^4)\,nE(X-\mu)^2 \\ &\quad -\,(n/\sigma^6)[E(X-\mu)^4 - \{E(X-\mu)^2\}^2]. \qquad (3.42)\end{aligned}$$

We next need the first, second, third, and fourth moments of X about μ which are as follows.

$$E(X) - \mu = \sigma\frac{\exp(-z_1^2/2) - \exp(-z_2^2/2)}{\Phi(z_2) - \Phi(z_1)} = \sigma h_0, \qquad (3.43)$$

(from equation (79) p. 81, JK), where $z_1 = (A-\mu)/\sigma$, $z_2 = (B-\mu)/\sigma$ and

$$h_0 = \frac{\exp(-z_1^2/2) - \exp(-z_2^2/2)}{\Phi(z_2) - \Phi(z_1)},$$

as defined in section 2.1. In fact, we later need more of these h-type functions, which we define here as follows,

$$h_i = \frac{z_1^i \exp(-z_1^2/2) - z_2^i \exp(-z_2^2/2)}{\Phi(z_2) - \Phi(z_1)}, \quad i=0,1,2,3. \tag{3.44}$$

Now the other moments of X about μ follow.

$$E(X-\mu)^2 = \frac{\int_A^B (t-\mu)^2 \exp[-(t-\mu)^2/2\sigma^2]dt}{\sigma[\Phi(z_2) - \Phi(z_1)]}$$

$$= \frac{\sigma^2 \int_{z_1}^{z_2} t^2 \exp(-t^2/2)dt}{\Phi(z_2) - \Phi(z_1)}$$

$$= \frac{\sigma^2(z_1\exp(-z_1^2/2) - z_2\exp(-z_2^2/2)}{\Phi(z_2) - \Phi(z_1)} + \sigma^2$$

$$= \sigma^2(h_1 + 1), \tag{3.45}$$

$$E(X-\mu)^3 = \frac{\sigma^3 \int_{z_1}^{z_2} t^3 \exp(-t^2/2)dt}{\Phi(z_2) - \Phi(z_1)}$$

$$= \frac{\sigma^3[z_1^2e^{-z_1^2/2} - z_2^2e^{-z_2^2/2} + 2e^{-z_1^2/2} - 2e^{-z_2^2/2}]}{\Phi(z_2) - \Phi(z_1)}$$

$$= \sigma^3(h_2 + 2h_0), \tag{3.46}$$

$$E(X-\mu)^4 = \frac{\sigma^4\int_{z_1}^{z_2} t^4\exp(-t^2/2)dt}{\Phi(z_2) - \Phi(z_1)}$$

$$= \frac{\sigma^4[z_1^3e^{-z_1^2/2} - z_2^3e^{-z_2^2/2} + 3z_1e^{-z_1^2/2} - 3z_2e^{-z_2^2/2}]}{\Phi(z_2) - \Phi(z_1)} + 3\sigma^4$$

$$= \sigma^4(h_3 + 3h_1 + 3). \tag{3.47}$$

Hence

$$(\partial^2/\partial\mu^2)\log L = -(n/\sigma^4)\mathrm{Var}(X) = -(n/\sigma^2)[1+h_1-h_0^2], \tag{3.48}$$

$$(\partial^2/\partial\sigma\partial\mu)\log L = -(2n/\sigma^3)(\overline{x} - \mu_T) - (n/\sigma^2)[h_2+h_0(1-h_1)], \tag{3.49}$$

$$(\partial^2/\partial\sigma^2)\log L = -(3/\sigma^4)[\Sigma(x_i-\mu)^2-nE(X-\mu)^2] - (n/\sigma^2)[(1+h_1)(2-h_1)+h_3]. \tag{3.50}$$

[The second order partial derivatives - (3.48), (3.49), and (3.50) are derived by Cohen (1957) also in a slightly different form. There is a slight error in his expression for $(\partial^2/\partial\mu^2)\log L$. The last part of the equation (3.48) is taken from JK, p. 83.]

Finally, the elements of the dispersion matrix, given in (3.37) are found by substituting $\overline{x} - \mu_T = 0$ and $\Sigma(x_i-\mu)^2-nE(X-\mu)^2= 0$ in (3.48), (3.49) and (3.50). These elements (multiplied by σ^2/n) may be written as

$$\lim_{n \to \infty} (-\sigma^2/n)\, v_{11}^{-1} = 1 + h_1 - h_0^2, \tag{3.51}$$

$$\lim_{n \to \infty} (-\sigma^2/n)\, v_{12}^{-1} = h_2 + h_0(1-h_1), \tag{3.52}$$

$$\lim_{n \to \infty} (-\sigma^2/n)\, v_{22}^{-1} = h_3 + (1+h_1)(2-h_1). \tag{3.52}$$

A table giving the elements of the asymptotic variance-covariance matrix (multiplied by σ^2/n) of the m.l.e.'s for different degrees of truncation is given here as table 3.1. The degrees of truncation are restricted to the ones for which we have done simulations in sections 3.6 and 3.7.

The table is comparable to Table 1 of Harter and Moore (1966) for doubly censored samples. The only difference is in the interpretation of n. Our n is the number of observations actually available, while for censoring, n is the total sample size out of which the lowest r_1 and the highest r_2 sample values have been censored. The ratios r_1/n and r_2/n give the lower and upper proportions of censoring respectively. For doubly truncated samples, the degrees of truncation q_1 and q_2 are as follows:

$$q_1 = [\int_{-\infty}^{z_1} \exp(-t^2/2)dt]/\sqrt{2\pi},$$

$$q_2 = [\int_{z_2}^{\infty} \exp(-t^2/2)dt]/\sqrt{2\pi},$$

z_1 and z_2 being the standardized truncation points as defined earlier.

Table 3.1 Coefficients of σ^2/n in asymptotic variances and covariances of Maximum Likelihood Estimators of Parameters μ and σ of Normal Population from Doubly Truncated Samples of Size n

		Both parameters unknown			σ known	μ known
q_1	q_2	$n\,\mathrm{var}(\hat{\mu})/\sigma^2$	$n\,\mathrm{cov}(\hat{\mu},\hat{\sigma})/\sigma^2$	$n\,\mathrm{var}(\hat{\sigma})/\sigma^2$	$n\,\mathrm{var}(\hat{\mu})/\sigma^2$	$n\,\mathrm{var}(\hat{\sigma})/\sigma^2$
0.025	0.025	1.317798	0.000000	1.291516	1.317798	1.291516
0.050	0.050	1.605097	0.000000	2.169435	1.605097	2.169435
0.050	0.100	2.103034	0.823846	3.214491	1.891890	2.891760
0.100	0.100	2.284543	0.000000	5.047871	2.284543	5.047871
0.100	0.200	4.525141	3.760231	10.110787	3.126700	6.986168

3.6 Simulation. Case I - μ Known

In order to find the m.l.e. of θ, one has to solve the non-linear equation (3.3) for θ. Note that the equation can also be written as follows

$$\Sigma x_i^2/n = \theta[1 + \frac{A\exp(-A^2/2\theta) - B\exp(-B^2/2\theta)}{\int_A^B \exp(-t^2/2\theta)dt}]$$

$$= \theta(1+h_1), \tag{3.54}$$

where h_1 is the same as in (3.44) but with $\mu = 0$, and $z_1 = A/\sqrt{\theta}$, $z_2 = B/\sqrt{\theta}$. Further, we find the Bayes modal estimator of θ by solving (3.14), which can be alternatively written as

$$(\Sigma x_i^2+1)/n = \theta[1 + h_1 + (\nu-2)/n]. \tag{3.55}$$

In order to solve (3.54) and (3.55) iteratively, we use the Newton-Raphson method. To apply the method, we write the equations (3.54) and (3.55) as

$$F_1 = (\Sigma x_i^2/n) - \theta(1+h_1) = 0, \tag{3.56}$$

$$F_2 = [(\Sigma x_i^2+1)/n] - \theta[1+h_1+(\nu-2)/n] = 0. \tag{3.57}$$

The derivatives of F_1 and F_2 w.r.t. θ are obtained below.

$$F_1' = -(1+h_1) - \theta[(\partial/\partial\theta)h_1]$$

$$= -(1+h_1) - \theta[(1/2\theta)\{h_3-h_1(1+h_1)\}] \quad \text{(see later)}$$

$$= (1/2)[(h_1+1)(h_1-2) - h_3], \tag{3.58}$$

since

$$(\partial/\partial\theta)h_1 = (1/2\theta^2)\left[\frac{A^3e^{-A^2/2\theta} - B^3e^{-B^2/2\theta}}{\int_A^B \exp(-t^2/2\theta)dt} - h_1E(X^2)\right]$$
$$= (1/2\theta)[h_3 - h_1(1+h_1)]$$

as $E(X^2) = \theta(1+h_1)$, where h_1 and h_3 are the same as in (3.44), with $\mu = 0$. The function F_1' can be compared with the expression in (3.50), except for the first part which is zero now and for the missing '1/2' because the differentiation is now w.r.t. θ, and not σ. The function F_2' is similar and is given by

$$F_2' = (1/2)[(h_1+1)(h_1-2) - h_3] - (\nu-2)/n. \tag{3.59}$$

The algorithm for the Newton-Raphson procedure is then given by

$$\hat{\theta}_{j+1} = \hat{\theta}_j - [F_1(\hat{\theta}_j)/F_1'(\hat{\theta}_j)],$$
$$\tilde{\theta}_{j+1} = \tilde{\theta}_j - [F_2(\tilde{\theta}_j)/F_2'(\tilde{\theta}_j)].$$

Simulation results are based on 1000 samples from a given truncated normal distribution (normal samples are drawn using the IMSL routine GGNML). A few representative degrees of truncation are considered. Whenever $\Sigma x_i^2/n$ exceeds M_1, we solve only the equation (3.57) to find the Bayes modal estimator, $\tilde{\theta}$. For comparison purposes, we calculate the simulated expected bias and the simulated expected mean square error (MSE) of the m.l.e. and the Bayes modal estimator. Bias is defined as $E(\hat{\theta}-\theta)$ or $E(\tilde{\theta}-\theta)$, while the mean square error is defined to be $E(\hat{\theta}-\theta)^2$ or $E(\tilde{\theta}-\theta)^2$. The proportion of samples for which the m.l.e. does not

exist is represented by p_n. The bias and MSE for the m.l.e are based only on the samples where it exists, while for the modal estimator they are based on all of 1000 samples.

To further compare the two estimators, we find an estimate of the probability $P[|\tilde{\theta} - \theta| < |\hat{\theta} - \theta|]$, by the proportion: {number of samples with $|\tilde{\theta} - \theta| < |\hat{\theta} - \theta|$}/1000. We denote this probability by p_m. If $\hat{\theta}$ does not exist, it is assumed that $|\tilde{\theta} - \theta| < |\hat{\theta} - \theta|$. The value of four for ν is used throughout.

Another interesting estimation procedure can be devised by considering a mixture of the m.l.e. and the modal estimator, with the mixing proportion depending upon the value of $\Sigma x_i^2/n$. Since the m.l.e. behaves badly whenever $\Sigma x_i^2/n$ is slightly large, especially when it is close to M_1, we can define the mixed estimator $\hat{\theta}_m$, as

$$\hat{\theta}_m = \begin{cases} \hat{\theta} & \text{if } \Sigma x_i^2/n < \alpha M_1 \\ \tilde{\theta} & \text{if } \Sigma x_i^2/n > \alpha M_1 . \end{cases} \tag{3.60}$$

The value of α could be arbitrarily decided depending on the degree of truncation, sample size, and the simulated value of θ. Estimates of the probability similar to p_m for the modal estimator, are calculated also for the mixed estimator. This probability is defined as: $p_{mx} = P[|\hat{\theta}_m - \theta| \le |\hat{\theta} - \theta|]$. One necessary word of caution for the use and interpretation of this probability for the mixed estimator is that it should only be used in conjunction with the other criteria (like bias and MSE). This is because a mixed estimator with α of one gives p_{mx} also as one,

and would be deemed preferable to another mixed estimator with α less than one, since the value of p_{mx} will then be in general less than one. At the same time, it is obvious that the mixed estimator with α near one does not behave well because of its m.l.e. component blowing up.

Since μ is assumed known, we take symmetric truncation probabilities only (of course, nonsymmetric truncation is not much different). Also since, in most practical situations, the degree of truncation is not high, we consider only the following three cases: (a) $q_1 = 0.025$, $q_2 = 0.025$ (b) $q_1 = 0.05$, $q_2 = 0.05$, and (c) $q_1 = 0.1$, $q_2 = 0.1$. The different values of n and θ are: $n = 10, 30, 50$; $\theta = 0.25, 0.5, 1, 2, 5, 10, 20$. To compare the MSE's of the m.l.e. and the modal estimator, we find the relative efficiency, i.e. EFF = [MSE(m.l.e.)/MSE(Modal)]. Similar relative efficiency is calculated for the mixed estimator.

Tables 3.2 through 3.4 give the simulation results for the three different degrees of truncation considered here. Estimated probabilities of nonexistence of the m.l.e., p_n, are seen to increase with increasing degree of truncation. For 20% truncation ($q_1 = 0.1$, $q_2 = 0.1$), the nonexistence of the m.l.e. is seen to be in as high a proportion as 20% of the samples, for samples of size 10. The corresponding values of p_n for larger sample sizes or smaller degrees of truncation are samller but still not easily ignored.

Another dramatic indication of the problem with the m.l.e. is

Table 3.2 Simulated Expected Bias and MSE of the Maximum Likelihood, the Bayes Modal, and the Mixed Estimators of θ for the Truncated Normal Distribution (μ Known)
$q_1 = 0.025$, $q_2 = 0.025$

n	θ	Max. Lik. Estimates			Bayes Modal Estimates				Mixed Estimates (α=.7)			
		p_n	Bias	MSE	Bias	MSE	EFF	p_m	Bias	MSE	EFF	p_{mx}
10	0.25	0.040	0.048	21.755	0.131	0.048	453.2	0.61	0.066	0.049	444.0	0.91
	0.50	0.034	0.440	12.258	0.035	0.072	170.3	1.00	0.022	0.083	147.7	1.00
	1.00	0.044	1.515	544.499	-0.065	0.264	2062.5	0.59	0.002	0.269	2024.2	1.00
	2.00	0.049	1.883	111.946	-0.234	1.051	106.5	0.42	-0.017	0.956	117.1	1.00
	5.00	0.040	7.206	5629.914	-0.997	6.137	917.4	0.39	-0.360	5.127	1098.1	1.00
	10.00	0.041	12.734	7524.183	-2.036	24.092	312.3	0.38	-0.649	19.204	391.8	1.00
	20.00	0.034	19.375	30477.448	-4.418	87.549	348.1	0.37	-1.516	66.894	455.6	1.00
30	0.25	0.001	0.047	0.065	0.064	0.020	3.3	0.51	0.033	0.020	3.3	0.89
	0.50	0.004	0.194	2.481	0.055	0.063	39.4	1.00	0.053	0.066	37.6	1.00
	1.00	0.000	0.211	1.276	-0.002	0.149	8.6	0.48	0.051	0.156	8.2	1.00
	2.00	0.002	0.695	54.798	-0.031	0.660	83.0	0.44	0.119	0.668	82.0	1.00
	5.00	0.001	1.006	25.595	-0.308	3.550	7.2	0.39	0.126	3.397	7.5	1.00
	10.00	0.001	2.521	159.721	-0.477	14.906	10.7	0.43	0.476	14.414	11.1	1.00
	20.00	0.001	6.859	5213.774	-1.308	60.729	85.9	0.42	0.586	58.291	89.4	1.00
50	0.25	0.000	0.027	0.021	0.044	0.013	1.7	0.48	0.024	0.012	1.7	0.90
	0.50	0.001	0.048	0.076	0.030	0.038	2.0	1.00	0.031	0.040	1.9	1.00
	1.00	0.000	0.087	0.229	0.003	0.104	2.2	0.48	0.042	0.111	2.1	1.00
	2.00	0.000	0.240	1.198	-0.000	0.441	2.7	0.45	0.116	0.463	2.6	1.00
	5.00	0.001	0.454	6.057	-0.146	2.809	2.2	0.43	0.188	2.867	2.1	1.00
	10.00	0.001	1.145	138.496	-0.389	11.649	11.9	0.41	0.293	11.669	11.9	1.00
	20.00	0.000	1.473	105.467	-1.072	36.672	2.9	0.42	0.389	37.323	2.8	1.00

Table 3.3 Simulated Expected Bias and MSE of the Maximum Likelihood, the Bayes Modal, and the Mixed Estimators of θ for the Truncated Normal Distribution (μ Known)
$q_1 = 0.05$, $q_2 = 0.05$

n	θ	Max. Lik. Estimates			Bayes Modal Estimates				Mixed Estimates (α=.8)			
		p_n	Bias	MSE	Bias	MSE	EFF	p_m	Bias	MSE	EFF	p_{mx}
10	0.25	0.101	0.486	13.861	0.160	0.056	247.5	0.64	0.070	0.056	247.5	0.96
	0.50	0.086	1.258	313.276	0.033	0.069	4540.2	1.00	0.015	0.086	3642.7	1.00
	1.00	0.107	1.962	121.074	-0.094	0.262	462.1	0.65	-0.009	0.273	443.5	1.00
	2.00	0.104	2.732	595.950	-0.415	0.969	615.0	0.44	-0.145	0.863	690.6	1.00
	5.00	0.088	9.925	12980.133	-1.431	5.788	2242.6	0.38	-0.594	4.491	2890.3	1.00
	10.00	0.090	13.536	7505.496	-3.031	23.503	319.3	0.37	-1.217	17.624	425.9	1.00
	20.00	0.109	26.186	22636.877	-5.663	97.538	232.1	0.39	-2.061	72.161	313.7	1.00
30	0.25	0.017	0.170	1.749	0.091	0.035	50.0	0.59	0.046	0.035	50.0	0.95
	0.50	0.019	0.276	2.047	0.058	0.072	28.4	1.00	0.059	0.079	25.9	1.00
	1.00	0.016	0.488	8.004	-0.018	0.199	40.2	0.50	0.063	0.219	36.5	1.00
	2.00	0.011	2.101	298.401	-0.126	0.703	424.5	0.43	0.099	0.729	409.3	1.00
	5.00	0.019	2.893	793.901	-0.454	4.759	166.8	0.41	0.16	4.678	169.7	1.00
	10.00	0.025	4.861	1053.484	-0.911	18.826	56.0	0.42	0.465	18.979	55.5	1.00
	20.00	0.017	14.976	23124.017	-2.074	75.856	304.8	0.40	0.613	73.623	314.1	1.00
50	0.25	0.004	0.088	0.603	0.071	0.025	24.1	0.52	0.040	0.024	25.1	0.94
	0.50	0.002	0.164	2.593	0.037	0.049	52.9	1.00	0.040	0.054	48.0	1.00
	1.00	0.003	0.404	5.977	0.038	0.193	31.0	0.49	0.102	0.213	28.1	1.00
	2.00	0.001	0.481	7.377	-0.058	0.503	14.7	0.44	0.128	0.576	12.8	1.00
	5.00	0.001	0.916	41.170	-0.382	2.867	14.4	0.41	0.168	3.205	12.8	1.00
	10.00	0.011	4.364	1028.953	-0.251	22.213	46.3	0.44	0.845	23.210	44.3	1.00
	20.00	0.004	7.720	2018.100	-0.637	64.740	31.2	0.46	1.631	69.655	29.0	1.00

Table 3.4 Simulated Expected Bias and MSE of the Maximum Likelihood, the Bayes Modal, and the Mixed Estimators of θ for the Truncated Normal Distribution (μ Known)

$q_1 = 0.1$, $q_2 = 0.1$

n	θ	Max. Lik. Estimates			Bayes Modal Estimates				Mixed Estimates (α=.85)			
		p_n	Bias	MSE	Bias	MSE	EFF	p_m	Bias	MSE	EFF	p_{mx}
10	0.25	0.225	0.204	1.679	0.201	0.064	26.2	0.65	0.084	0.065	25.8	0.92
	0.50	0.203	0.635	24.083	0.031	0.059	408.2	1.00	-0.016	0.082	293.7	1.00
	1.00	0.220	1.007	108.411	-0.232	0.212	511.4	0.61	-0.181	0.211	513.8	1.00
	2.00	0.194	3.211	903.521	-0.735	1.024	882.3	0.47	-0.537	0.083	1125.2	1.00
	5.00	0.234	4.001	828.061	-2.068	7.288	113.6	0.40	-1.441	5.175	160.0	1.00
	10.00	0.206	14.546	11268.216	-4.495	31.438	358.4	0.37	-3.090	21.301	529.0	0.99
	20.00	0.199	14.358	9911.040	-9.386	125.132	79.2	0.38	-6.508	83.328	118.9	0.99
30	0.25	0.076	0.549	44.098	0.148	0.053	832.0	0.60	0.078	0.053	832.0	0.90
	0.50	0.094	1.372	236.914	0.058	0.077	3076.8	1.00	0.045	0.085	2787.2	1.00
	1.00	0.089	1.503	125.296	-0.069	0.226	554.4	0.59	-0.013	0.221	567.0	1.00
	2.00	0.101	2.050	207.289	-0.276	0.845	245.3	0.45	-0.085	0.734	282.4	1.00
	5.00	0.083	5.454	2312.176	-1.091	5.040	458.8	0.40	-0.508	3.915	590.6	1.00
	10.00	0.096	16.082	43720.897	-2.239	21.103	2071.8	0.39	-0.928	15.979	2376.1	1.00
	20.00	0.107	23.480	57441.416	-4.409	85.409	672.5	0.39	-1.883	63.264	908.0	1.00
50	0.25	0.033	0.212	3.303	0.114	0.042	78.6	0.58	0.065	0.042	78.6	0.86
	0.50	0.033	0.442	17.085	0.043	0.069	247.6	1.00	0.035	0.075	227.8	1.00
	1.00	0.059	1.809	1146.348	0.007	0.265	4325.8	0.53	0.063	0.263	4358.7	1.00
	2.00	0.048	1.593	186.351	-0.176	0.882	211.3	0.42	-0.013	0.808	193.5	1.00
	5.00	0.047	5.650	2031.339	-0.573	4.952	406.6	0.42	-0.061	4.274	471.1	1.00
	10.00	0.051	10.294	5623.626	-1.292	19.053	295.2	0.41	-0.181	15.951	352.6	1.00
	20.00	0.041	20.282	52600.997	-3.198	69.921	752.3	0.40	-0.985	56.817	925.8	1.00

the large values of MSE, a result of the blowing up of the m.l.e. near the upper boundary of its existence. In comparison, the Bayes modal estimator has very small MSE's and, therefore, high efficiencies relative to the m.l.e. While the m.l.e. tends to over-estimate θ somewhat (indicated by the positive bias), the Bayes modal estimator seems to be giving a slight under-estimate (especially for larger values of simulated θ). The modal estimator does not appear to be very impressive in light of the use of the criterion p_m, except for really small values of θ. The value of p_m stays around 40% for larger values of θ.

The mixed estimator does not appear to be performing strikingly better than the modal estimator, as evidenced by the comparable values of the two efficiencies. Theoretically, one can always find a best value of α for the mixed estimator in a given case, but in the interest of finding a generally acceptable value of α, we have considered three mixed estimators with α of 0.7 , 0.8 and 0.85, and the best among them is reported. The mixed estimator still behaves remarkably well in the light of two criteria :

(i) Bias - It is consistently smaller than that of both the m.l.e. and the modal estimator, in absolute value.

(ii) p_{mx} - This estimate of the probability is always one except for very small values of θ. Even for small θ, this probability is between 85% and 90%. Given the earlier caution against the use of p_{mx} for α near one, these values of p_{mx} are still very good because they are generated by values of α from 0.7 to 0.85, not so close to one.

3.7 Simulation. Case II - Both μ and σ Unknown

The m.l.e.'s of μ and σ are given by the solution of the equations (3.9) and (3.10), while the Bayes modal estimators are the solution of the equations (3.33) and (3.34). We attempted to use the two-dimensional modification of the Newton-Raphson method in order to solve the two pairs of equations, but since the Newton-Raphson procedure (in two dimensions) is only locally convergent, the iterations failed to converge for a large number of samples. (The failure of convergence of the Newton-Raphson algorithm, in a very small number of cases, could also be due to theoretical reasons. Since our nonexistence conditions are only sufficient, for some rare samples, the solution may not actually be possible.) We therefore, use the IMSL routine ZSPOW to solve the pairs of equations iteratively. This routine is based on a variation of the Newton-Raphson method and takes precautions to avoid large step sizes or increased residuals.

The pairs of equations, given in section 3.2 are:

i) For m.l. estimation

$$F(1,1) = \overline{x} - \mu_T = \overline{x} - (\mu+\sigma h_0) = 0, \tag{3.61}$$

$$F(1,2) = s^2 - \sigma_T^2 = s^2 - \sigma^2(1+h_1-h_0^2) = 0, \tag{3.62}$$

where h_i's $(i=0,1,2,3)$ are defined in (3.44);

ii) For Bayes modal estimation

$$F(2,1) = \bar{x} - \mu_T = -(\mu+\sigma h_0) + \bar{x} = 0, \qquad (3.63)$$

$$F(2,2) = s^2 + (1/n) - (1/n)(\nu-2)\sigma^2 - \sigma_T^2$$

$$= s^2 + (1/n) - \sigma^2[1+h_1-h_0^2+(\nu-2)/n] = 0. \qquad (3.64)$$

The two pairs of equations have to be solved for μ and σ. For the m.l.e., the equations can be solved only if $s^2 < M_2 = (B-A)^2/12$, but due to the explosiveness of the solutions near the boundary M_2 of s^2, the algorithm often has difficulty giving convergence. Therefore, we find the m.l.e.'s only for samples with $s^2 < (.95)M_2$. Of course, modal estimators are calculated for all 500 samples. Similar to the μ known case, we find the mixed estimators for both μ and σ also.

Simulated expected bias and MSE of the m.l.e.'s, the modal estimators, and the mixed estimators can be used for comparing the three classes of estimators. Bias is defined as $E(\hat{\theta}_e-\theta)$ and the MSE as $E(\hat{\theta}_e-\theta)^2$, where θ is used as a general representation for either μ or σ, and $\hat{\theta}_e$ stands for any one of these estimators. The proportion of samples for which s^2 exceeds $(.95)M_2$ is denoted by p_n.

As in the μ known case, we also compare the three estimators by computing the estimates of the probabilities $P[|\tilde{\theta} - \theta| < |\hat{\theta} - \theta|]$ and $P[|\hat{\theta}_m - \theta| \le |\hat{\theta} - \theta|]$, where $\tilde{\theta}$ stands for the Bayes modal estimator, $\hat{\theta}_m$ is the mixed estimator and $\hat{\theta}$ is the m.l.e.

Since μ is the location parameter and the equations involve only $(\bar{x} - \mu)$, the simulation does not differ for different values of μ, and only one value of μ needs to be considered. The range of values of σ and n are as follows: $n = 10, 30, 50$; $\sigma = 0.5, 1, 3, 5$. We now take nonsymmetric truncation into account also, since μ is unknown. The degrees of truncation considered are, again based on practical applications, taken to be low. They are (a) $q_1 = 0.05$, $q_2 = 0.05$ (b) $q_1 = 0.05$, $q_2 = 0.1$ (c) $q_1 = 0.1$, $q_2 = 0.1$, and (d) $q_1 = 0.1$, $q_2 = 0.2$. For comparing the MSE's of the m.l.e. and the Bayes modal estimator, we find the relative efficiency: EFF = [MSE(m.l.e.)/MSE(modal)]. Similarly for the mixed estimator, the relative efficiencies are calculated. Also to assess the efficiency of the modal or the mixed estimators, their MSE's can be compared with the asymptotic variances of the m.l.e.'s calculated in section 3.5, table 3.1.

Tables 3.5 through 3.8 contain the simulation results for the four different degrees of truncation considered in this section. Similar to the simulations in section 3.5, the probabilities of nonexistence of the m.l.e.'s are again moderately large. The bias and MSE of σ are now comparable to the bias and MSE of θ in section 3.5, except for the square-root transformation. This step down from θ to σ seems to be one reason for relatively less dramatic values for the MSE of the m.l.e. At the same time, the MSE for the m.l.e. of μ is quite large, especially for smaller samples, giving large relative efficiencies for the modal and the mixed estimator.

Table 3.5 Simulated Expected Bias and MSE of the Maximum Likelihood, the Bayes Modal, and the Mixed Estimators of μ and σ for the Truncated Normal Distribution $q_1 = 0.05$, $q_2 = 0.05$

n	σ	Max. Lik. Estimates p_n	Bias	MSE	Bayes Modal Estimates Bias	MSE	EFF	p_m	Mixed Estimates (α=.85) Bias	MSE	EFF	p_{mx}
ESTIMATION OF μ												
10	0.5	0.12	0.149	3.534	0.006	0.100	35.3	0.25	0.070	1.423	2.5	1.00
	1.0	0.08	0.368	25.052	0.020	0.159	157.6	0.74	0.186	7.867	3.2	1.00
	3.0	0.10	1.159	147.347	0.119	0.991	148.7	1.00	0.160	2.416	61.0	1.00
	5.0	0.10	-0.393	70.207	0.025	2.848	24.7	1.00	-0.398	44.164	1.6	1.00
30	0.5	0.02	0.006	0.018	0.010	0.020	0.9	0.10	0.009	0.018	1.0	1.00
	1.0	0.03	-0.001	0.130	0.008	0.061	2.1	0.94	0.007	0.076	1.7	1.00
	3.0	0.02	-0.027	1.441	-0.048	0.410	3.5	1.00	-0.064	0.607	2.4	1.00
	5.0	0.02	-0.019	2.690	-0.022	1.286	2.1	1.00	-0.021	1.854	1.5	1.00
50	0.5	0.01	0.001	0.016	-0.001	0.012	1.3	0.07	-0.000	0.010	1.5	1.00
	1.0	0.01	-0.001	0.047	-0.002	0.033	1.4	0.97	-0.001	0.040	1.2	1.00
	3.0	0.00	0.030	0.412	0.025	0.284	1.5	1.00	0.028	0.364	1.1	1.00
	5.0	0.00	0.062	1.408	0.030	0.687	2.0	1.00	0.033	0.859	1.6	1.00
ESTIMATION OF σ												
10	0.5	0.12	0.036	0.204	0.116	0.032	6.4	0.70	0.029	0.091	2.2	1.00
	1.0	0.08	0.098	1.073	-0.141	0.068	15.8	0.77	-0.042	0.262	4.1	1.00
	3.0	0.10	0.229	9.034	-0.658	0.877	10.3	1.00	-0.177	0.957	9.4	1.00
	5.0	0.10	0.224	10.727	-1.123	2.505	4.3	1.00	-0.261	4.363	2.5	1.00
30	0.5	0.02	0.009	0.021	0.059	0.017	1.2	0.56	0.010	0.019	1.1	1.00
	1.0	0.03	0.080	0.172	-0.026	0.050	3.4	0.94	0.028	0.064	2.7	1.00
	3.0	0.02	0.098	1.142	-0.315	0.404	2.8	1.00	-0.036	0.475	2.4	1.00
	5.0	0.02	0.259	2.996	-0.469	1.135	2.6	1.00	0.023	1.349	2.2	1.00
50	0.5	0.01	0.013	0.017	0.045	0.013	1.3	0.53	0.011	0.013	1.3	1.00
	1.0	0.01	0.046	0.082	-0.019	0.035	2.3	0.97	0.025	0.047	1.7	1.00
	3.0	0.00	0.005	0.475	-0.244	0.250	1.9	1.00	-0.037	0.300	1.6	1.00
	5.0	0.00	0.086	1.672	-0.369	0.755	2.2	1.00	-0.016	0.893	1.9	1.00

Table 3.6 Simulated Expected Bias and MSE of the Maximum Likelihood, the Bayes Modal, and the Mixed Estimators of μ and σ for the Truncated Normal Distribution
$q_1 = 0.05$, $q_2 = 0.10$

n	σ	Max. Lik. Estimates p_n	Bias	MSE	Bayes Modal Estimates Bias	MSE	EFF	p_m	Mixed Estimates (α=.85) Bias	MSE	EFF	p_{mx}
ESTIMATION OF μ												
10	0.5	0.14	0.382	8.743	0.051	0.135	64.8	0.36	0.021	0.108	81.0	0.98
	1.0	0.10	0.720	30.044	-0.015	0.157	191.4	0.72	0.162	6.265	4.8	0.99
	3.0	0.15	2.966	292.071	-0.057	0.988	295.6	0.82	0.782	60.244	4.8	0.99
	5.0	0.14	1.312	420.826	-0.239	3.008	139.9	0.85	-0.167	43.701	9.6	0.99
30	0.5	0.04	0.098	0.808	0.047	0.032	25.3	0.36	0.031	0.028	28.9	0.99
	1.0	0.07	0.123	0.629	0.005	0.071	8.8	0.72	0.033	0.133	4.7	0.97
	3.0	0.04	0.049	2.359	-0.122	0.432	5.5	0.69	-0.051	0.594	4.0	0.99
	5.0	0.06	0.707	68.803	-0.067	1.573	43.7	0.73	0.100	2.610	26.4	0.98
50	0.5	0.02	0.018	0.025	0.022	0.017	1.5	0.41	0.011	0.015	1.7	0.98
	1.0	0.02	0.056	0.621	-0.002	0.042	14.8	0.69	0.011	0.051	12.2	0.98
	3.0	0.02	0.180	1.157	0.018	0.348	3.2	0.65	0.103	0.533	2.2	0.98
	5.0	0.02	0.273	2.904	-0.013	0.870	3.3	0.65	0.120	1.235	2.4	0.99
ESTIMATION OF σ												
10	0.5	0.14	0.051	0.336	0.131	0.033	10.2	0.71	0.005	0.049	6.9	0.99
	1.0	0.10	0.121	1.315	-0.160	0.072	18.3	0.74	-0.060	0.278	4.7	1.00
	3.0	0.15	0.263	11.840	-0.784	1.050	11.3	1.00	-0.276	2.655	4.5	1.00
	5.0	0.14	0.247	24.493	-1.371	3.006	8.1	1.00	-0.495	4.337	5.6	1.00
30	0.5	0.04	0.037	0.090	0.084	0.024	3.8	0.59	0.022	0.027	3.3	0.98
	1.0	0.07	0.088	0.236	-0.038	0.049	4.8	0.93	0.026	0.067	3.5	1.00
	3.0	0.04	0.071	1.368	-0.401	0.484	2.8	1.00	-0.115	0.415	3.3	1.00
	5.0	0.06	0.357	7.052	-0.556	1.456	4.8	1.00	-0.021	1.498	4.7	1.00
50	0.5	0.02	0.025	0.025	0.064	0.018	1.4	0.56	0.021	0.018	1.4	0.99
	1.0	0.02	0.052	0.152	-0.035	0.042	3.6	0.95	0.010	0.049	3.1	1.00
	3.0	0.02	0.116	0.931	-0.255	0.327	2.8	1.00	-0.009	0.360	2.6	1.00
	5.0	0.02	0.209	2.743	-0.441	0.967	2.8	1.00	-0.018	1.038	2.6	1.00

Table 3.7 Simulated Expected Bias and MSE of the Maximum Likelihood, the Bayes Modal, and the Mixed Estimators of μ and σ for the Truncated Normal Distribution $q_1 = 0.10$, $q_2 = 0.10$

n	σ	Max. Lik. Estimates p_n	Bias	MSE	Bayes Modal Estimates Bias	MSE	EFF	p_m	Mixed Estimates (α=.85) Bias	MSE	EFF	p_{mx}
ESTIMATION OF μ												
10	0.5	0.18	0.133	2.998	-0.002	0.144	20.8	0.27	-0.014	0.181	16.6	0.96
	1.0	0.20	0.093	17.669	-0.025	0.161	109.7	0.64	-0.063	0.638	27.7	1.00
	3.0	0.17	0.496	136.514	0.002	1.045	130.6	1.00	-0.016	3.566	38.3	1.00
	5.0	0.20	0.312	447.442	0.081	2.911	153.7	1.00	0.959	220.855	2.0	1.00
30	0.5	0.13	0.025	0.715	-0.012	0.049	14.6	0.25	-0.015	0.055	13.0	0.95
	1.0	0.14	-0.035	0.718	-0.011	0.072	10.0	0.86	-0.008	0.089	8.1	1.00
	3.0	0.10	0.803	67.635	0.069	0.541	125.0	1.00	0.075	0.725	93.3	1.00
	5.0	0.12	0.155	17.556	0.027	1.366	12.9	1.00	0.008	1.941	9.0	1.00
50	0.5	0.06	-0.015	0.023	-0.014	0.025	0.9	0.18	-0.012	0.020	1.2	0.91
	1.0	0.07	0.025	0.064	0.011	0.036	1.8	0.91	0.013	0.042	1.5	1.00
	3.0	0.08	0.005	0.857	-0.008	0.336	2.6	1.00	-0.011	0.421	2.0	1.00
	5.0	0.10	0.281	44.501	0.036	0.931	47.8	1.00	0.045	1.151	38.7	1.00
ESTIMATION OF σ												
10	0.5	0.18	-0.045	0.211	0.141	0.032	6.6	0.68	-0.031	0.074	2.9	0.96
	1.0	0.20	-0.032	1.078	-0.180	0.082	13.1	0.72	-0.169	0.210	5.1	1.00
	3.0	0.17	-0.084	8.648	-0.953	1.223	7.1	1.00	-0.651	1.659	5.2	1.00
	5.0	0.20	-0.202	26.533	-1.618	3.643	7.3	1.00	-0.951	10.744	2.5	1.00
30	0.5	0.13	0.014	0.092	0.106	0.031	3.0	0.68	0.029	0.037	2.5	0.95
	1.0	0.14	0.016	0.191	-0.076	0.053	3.6	0.86	-0.041	0.056	3.4	1.00
	3.0	0.10	0.117	4.372	-0.525	0.618	7.1	1.00	-0.297	0.506	8.6	1.00
	5.0	0.12	0.010	4.865	-0.860	1.738	2.8	1.00	-0.441	1.221	4.0	1.00
50	0.5	0.06	0.016	0.027	0.083	0.023	1.2	0.62	0.027	0.025	1.1	0.91
	1.0	0.07	0.001	0.083	-0.059	0.049	1.7	0.91	-0.024	0.049	1.7	1.00
	3.0	0.08	0.050	0.961	-0.326	0.458	2.1	1.00	-0.127	0.365	2.6	1.00
	5.0	0.10	0.317	5.553	-0.464	1.187	4.7	1.00	-0.128	0.905	6.1	1.00

Table 3.8 Simulated Expected Bias and MSE of the Maximum Likelihood, the Bayes Modal, and the Mixed Estimators of μ and σ for the Truncated Normal Distribution $q_1 = 0.10$, $q_2 = 0.20$

n	σ	Max. Lik. Estimates p_n	Bias	MSE	Bayes Modal Estimates Bias	MSE	EFF	p_m	Mixed Estimates (α=.85) Bias	MSE	EFF	p_{mx}
ESTIMATION OF μ												
10	0.5	0.23	0.207	2.864	0.068	0.273	10.5	0.40	0.034	0.458	6.3	0.96
	1.0	0.25	0.629	20.511	-0.064	0.205	100.0	0.69	0.126	4.468	4.6	0.98
	3.0	0.22	2.173	198.428	-0.253	1.027	193.2	0.75	0.716	65.132	3.0	0.98
	5.0	0.23	2.965	583.432	-0.520	2.623	222.4	0.77	0.586	118.012	4.9	0.98
30	0.5	0.22	0.218	2.056	0.090	0.091	22.6	0.50	0.088	0.469	4.4	0.88
	1.0	0.22	0.348	7.987	-0.044	0.089	89.7	0.72	-0.031	0.106	75.3	0.94
	3.0	0.20	0.993	57.491	-0.194	0.575	100.0	0.71	-0.094	1.038	55.4	0.95
	5.0	0.21	2.138	220.529	-0.354	1.585	139.1	0.72	-0.144	3.224	68.4	0.95
50	0.5	0.18	0.049	0.422	0.053	0.053	7.9	0.50	0.026	0.045	9.4	0.87
	1.0	0.19	0.330	6.259	-0.017	0.063	99.3	0.71	0.006	0.159	39.4	0.93
	3.0	0.16	0.715	35.333	-0.097	0.506	69.8	0.64	-0.034	0.611	57.8	0.92
	5.0	0.15	0.804	81.109	-0.232	1.167	69.5	0.62	-0.107	1.582	51.3	0.91
ESTIMATION OF σ												
10	0.5	0.23	-0.107	0.170	0.167	0.034	5.0	0.70	-0.049	0.084	2.0	0.95
	1.0	0.25	-0.122	0.922	-0.232	0.087	10.6	0.73	-0.240	0.311	3.0	1.00
	3.0	0.22	-0.475	8.546	-1.188	1.673	5.1	1.00	-0.887	3.993	2.1	1.00
	5.0	0.23	-0.624	33.836	-2.061	4.947	6.8	1.00	-1.836	10.649	3.2	1.00
30	0.5	0.22	0.006	0.118	0.146	0.036	3.3	0.65	0.053	0.054	2.2	0.87
	1.0	0.22	-0.001	0.434	-0.122	0.060	7.2	0.78	-0.108	0.064	6.8	1.00
	3.0	0.20	-0.093	4.116	-0.713	0.850	4.8	1.00	-0.533	0.792	5.2	1.00
	5.0	0.21	0.164	13.488	-1.135	2.155	6.3	1.00	-0.792	1.786	7.6	1.00
50	0.5	0.18	-0.001	0.055	0.126	0.034	1.6	0.65	0.054	0.040	1.4	0.83
	1.0	0.19	0.050	0.438	-0.072	0.053	8.3	0.84	-0.050	0.059	7.4	1.00
	3.0	0.16	0.124	3.048	-0.478	0.605	5.0	1.00	-0.323	0.433	7.0	1.00
	5.0	0.15	-0.055	6.232	-0.916	1.826	3.4	1.00	-0.635	1.296	4.8	1.00

The over-estimation by the m.l.e. and under-estimation by the Bayes modal estimator for σ is still very much evident, though the same is not true for the estimation of μ. The Bayes modal estimators for μ and σ are significantly better in the light of the criterion p_m now, compared to the ones for θ in section 3.6. The only values of p_m which are very small, are for the estimation of μ with simulated value of σ of 0.5 (for all n's). In almost all other cases, p_m is at least 50%, indicating the modal estimation to be reasonably good even with the use of p_m criterion.

The mixed estimators follow roughly the same pattern as in the μ known case, giving p_{mx} almost always one (even with α of 0.85 which is not very close to one). Against the usual expectation of a significant improvement over the Bayes modal estimator in terms of the bias and MSE, the mixed estimators are not much better (in some cases even worse). We considered α values of 0.85 and 0.90, and the better of the two (always 0.85) has been reported. One can always find an α which will be better in a given case, but for a generally acceptable value of α, 0.85 seems to be reasonable (especially with the p_{mx} values being almost always one).

Further, the mixed estimators do not exhibit a consistent trend of under- or over-estimating the parameters, as the m.l.e.'s and the modal estimators do. Actually, the bias of the mixed estimator for estimating σ is smaller, in absolute value, than

that of both the m.l.e. and the modal estimator. The same is generally true for the bias of the mixed estimator of μ when compared with the m.l.e.

4. THE TRUNCATED GAMMA AND WEIBULL DISTRIBUTIONS

This chapter deals with the truncated gamma and Weibull distributions, for which we discuss the question of existence of the m.l.e.'s, when both the parameters (shape and scale) are unknown (sections 4.1 and 4.4). Since the proof of the nonexistence of the m.l.e.'s for the truncated gamma, when both the parameters are unknown, is incomplete, the Bayes modal estimators are derived only for the shape parameter known case (section 4.2). Note that this includes the truncated exponential as a special case. This also completes the truncated Weibull case for the shape parameter known, because of its direct relationship with the truncated exponential. Bayes modal estimation for the truncated Weibull, when both the parameters are unknown, is discussed in section 4.5. Simulation results comparing the m.l.e.'s, the modal estimators, and the mixed estimators for the two distributions are to be found in sections 4.3 and 4.6.

4.1 The Gamma Case. M.L. Estimation - Both α and θ Unknown

The truncated exponential case has been dealt with in detail by Blumenthal and Marcus (1975), including the Bayes modal estimation, using a conjugate prior density for the parameter θ (or $1/\theta$). For the truncated gamma with known shape parameter, it is already

known that the m.l.e. of θ does not exist whenever $(\bar{x}/T)$ exceeds $\alpha/(\alpha+1)$, with reference to the density in (2.8) (cf. Broeder, 1955). The Bayes modal estimation, for this case, is considered in the next section.

We first consider the m.l.e.'s for the scale and shape parameters of the gamma distribution from truncated samples of size n with density (2.8). There is a slight simplification, if we employ the approach of taking the interval, within which the sample observations are available, to be [0, 1]. As is pointed out in section 2.2, this can be done without loss of generality, through a simple transformation, $Y = X/T$. Therefore, in the future analysis, we assume $T = 1$.

Given a sample of size n from the density (2.8), the likelihood and the log likelihood are given as follows.

$$L(\underline{x};\alpha,\theta) = \frac{\{\Pi\, x_i^{\alpha-1}\}\exp(-\Sigma x_i/\theta)}{[\int_0^1 t^{\alpha-1}\exp(-t/\theta)dt]^n}, \qquad 0 \le x_i \le 1, \qquad (4.1)$$

$$\log L = -n(\bar{x}/\theta) + (\alpha-1)\Sigma \log x_i - n\log[\int_0^1 t^{\alpha-1}\exp(-t/\theta)dt], \quad (4.2)$$

where the product and the summations range over $1 \le i \le n$. The m.l.e.'s of α and θ are the solutions of the equations:

$$\frac{\Sigma \log x_i}{n} = \frac{\int_0^1 t^{\alpha-1}\log t\, \exp(-t/\theta)dt}{\int_0^1 t^{\alpha-1}\exp(-t/\theta)dt} = E[\log X], \qquad (4.3)$$

$$\bar{x} = \frac{\int_0^1 t^{\alpha}\exp(-t/\theta)dt}{\int_0^1 t^{\alpha-1}\exp(-t/\theta)dt} = E(X). \qquad (4.4)$$

$$(\partial/\partial\theta)E[\log X] = \frac{\int_0^1 (t^{\alpha}/\theta^2)\log t\ \exp(-t/\theta)dt}{\int_0^1 t^{\alpha-1}\exp(-t/\theta)dt}$$

$$- \frac{\int_0^1 t^{\alpha-1}\log t\ \exp(-t/\theta)dt . \int_0^1 (t^{\alpha}/\theta^2)\exp(-t/\theta)dt}{[\int_0^1 t^{\alpha-1}\exp(-t/\theta)dt]^2}$$

$$= (1/\theta^2)[E(X\log X)-E(X)E(\log X)]. \qquad (4.6)$$

It is easily seen that $x\log x$ is a convex function of x, implying through the use of Jensen's inequality, that

$$E(X\log X) \geq E(X)\log E(X) . \qquad (4.7)$$

Also $\log x$ is a concave function of x hence (again by Jensen's inequality) we have

$$E[\log X] \leq \log E(X) . \qquad (4.8)$$

Combining (4.7) and (4.8), we obtain

$$E(X\log X) \geq E(X)E[\log X], \qquad (4.9)$$

and therefore using (4.6), we get

$$(\partial/\partial\theta)E[\log X] \geq 0, \qquad (4.10)$$

indicating that for fixed α, $E[\log X]$ is an increasing function of θ. The limiting values of $E[\log X]$ on both the ends of the range of values of θ are as follows:

(a) $\lim_{\theta \downarrow 0} E[\log X] = -\infty$, since the distribution represented by

the density in (2.8) becomes degenerate at 0 as $\theta \downarrow 0$;

(b) $\lim_{\theta \downarrow 0} E[\log X] = -1/\alpha$, since the limiting distribution is now given by the density in (2.12), for which $E[\log X] = -1/\alpha$.

The fact that $-\infty \le E[\log X] \le -1/\alpha$, again suggests that the equation (4.3) may not have any solution for certain given values of α, but for both α and θ unknown, it does not exclude the possibility of a value for the pair (α,θ) satisfying (4.3).

Further, for fixed θ, $E[\log X]$ is an increasing function of α, since

$$\begin{aligned}(\partial/\partial\alpha)E[\log X] &= \frac{\int_0^1 t^{\alpha-1}(\log t)^2 \exp(-t/\theta)dt}{\int_0^1 t^{\alpha-1}\exp(-t/\theta)dt} \\ &\quad - \left\{\frac{\int_0^1 t^{\alpha-1}\log t\, \exp(-t/\theta)dt}{\int_0^1 t^{\alpha-1}\exp(-t/\theta)dt}\right\}^2 \\ &= [E(\log X)^2 - \{E(\log X)\}^2] \\ &= \mathrm{Var}[\log X] \ge 0. \qquad (4.11)\end{aligned}$$

Also in this case, $-\infty \le E[\log X] \le 0$, which is a result of the following arguments:

(a) $\lim_{\alpha \downarrow 0} E[\log X] = -\infty$, since the distribution represented by the density in (2.8) becomes degenerate at 0 as $\alpha \downarrow 0$;

(b) $\lim_{\alpha \uparrow \infty} E[\log X] = 0$, since the distribution represented by the density in (2.8) becomes degenerate at one as $\alpha \uparrow \infty$.

Collecting all the facts together, we need to solve the pair

of non-linear equations,

$$\overline{x} - E(X) = 0,$$

$$\overline{\log x} - E[\log X] = 0,$$

where $\overline{\log x} = \Sigma \log x_i/n$, for α and θ, when every partial derivative of $E(X)$ and $E[\log X]$ is nonnegative. For a fixed α,

$$0 \le E(X) \le \alpha/(\alpha + 1),$$

$$-\infty \le E[\log X] \le -1/\alpha.$$

Thus for fixed α, there may be some α values for which each of the above equations may fail to provide a solution. For a fixed θ,

$$0 \le E(X) \le 1,$$

$$-\infty \le E[\log X] \le 0.$$

The above analysis, however, is not sufficient to prove if the solution to equations (4.3) and (4.4) always exists or not. Further work is required to investigate this point.

4.2 The Gamma Case. Bayes Modal Estimation - α Known

Here we deal with the Bayes modal estimation for the single parameter case when α is assumed known. The m.l. estimation and the Bayes modal estimation for the two parameter case (when both α and θ are unknown) are rather complicated to deal with numerically, because of the instability of the integral $\int_0^1 t^{\alpha-1}\log t \exp(-t/\theta)dt$ near the limit zero, and there is no close-form solution possible for the integral. For this reason, and also

because of a lack of clear knowledge regarding the existence or nonexistence of the m.l.e.'s, we do not deal with the gamma case when both the parameters α and θ are unknown.

For the α known case, the m.l.e. of θ is the solution of the equation (4.4) for $\bar{x} < \alpha/(\alpha + 1)$. When $\bar{x} > \alpha/(\alpha + 1)$, $\hat{\theta}$ is ∞.

In order to carry out the Bayes modal estimation, the prior density for $1/\theta$ is taken to be a gamma density as follows

$$p(\theta) = a^{b+1}\theta^{-b}\exp(-a/\theta)/\{\Gamma(b+1)\}, \qquad \theta > 0, \tag{4.12}$$

$a > 0$, $b > -1$. This is the conjugate prior density for $1/\theta$ and was also used by Blumenthal and Marcus (1975) when dealing with the Bayes modal estimation in the truncated exponential case.
The modified likelihood and its log are given by

$$L_m = \frac{a^{b+1}\,\theta^{-b}\,\Pi x_i^{\alpha-1}\exp[-(\Sigma x_i+a)/\theta]}{\Gamma(b+1)[\int_0^1 t^{\alpha-1}\exp(-t/\theta)dt]^n}, \tag{4.13}$$

$$\log L_m = c(a,b,x,\alpha) - b\log\theta - [(\Sigma x_i+a)/\theta] - n\log[\int_0^1 t^{\alpha-1}\exp(-t/\theta)dt]. \tag{4.14}$$

The mode of the modified likelihood L_m, in (4.13), is the solution for θ of the equation $(\partial/\partial\theta)\log L_m = 0$, which gives the Bayes modal estimator $\tilde{\theta}$ of θ to be the solution of the equation

$$\bar{x} + (a/n) = E(X) + (b/n)\theta . \tag{4.15}$$

For $b > 0$, the right hand side of (4.15) is an increasing function of θ, increasing to ∞ as θ approaches ∞. Therefore,

the solution of the equation (4.15) always exists. Note that the modifications in (4.15) from the m.l. equation are the addition of terms of order (1/n) on both sides of the equation.

We next focus our attention on finding optimum values of the prior density parameters a and b, which is done as in section 3.3 (for the normal case) namely, by minimizing the maximum asymptotic bias of $\tilde{\theta}$. The asymptotic bias of $\tilde{\theta}$, as derived in (2.23), is

$$E(\text{bias}) = \mu_2^{-2}\ [(-\mu_{11}/2) + \mu_2\xi],$$

where μ_2, μ_{11} and ξ are defined in (3.17), (3.18) and (3.19) respectively.

Now

$$[f'/f] = (1/\theta^2)[x - \frac{\int_0^1 t^{\alpha}\exp(-t/\theta)dt}{\int_0^1 t^{\alpha-1}\exp(-t/\theta)dt}]$$

$$= (1/\theta^2)[x - E(X)]; \qquad (4.16)$$

$$\xi = -(b/\theta) + (a/\theta^2); \qquad (4.17)$$

$$\mu_2 = E[f'/f]^2 = E[\{x-E(X)\}/\theta^2]^2 = \text{Var}(X)/\theta^4. \qquad (4.18)$$

For μ_{11}, we first need $[f''/f]$, which is given by (3.23). In order to determine $[f''/f]$ using (3.23), we find the partial derivative w.r.t. θ of $[f'/f]$ which is given as follows.

$$(\partial/\partial\theta)[f'/f] = -(2/\theta^3)[x-E(X)] + (1/\theta^2)[-(\partial/\partial\theta)E(X)]$$
$$= -(2/\theta^3)[x-E(X)] - (1/\theta^4)\text{Var}(X), \qquad (4.19)$$

since

$$(\partial/\partial\theta)E(X) = (1/\theta^2)\mathrm{Var}(X).$$

Hence, using (4.19) and (4.16), we get

$$[f''/f] = -(2/\theta^3)[x-E(X)] - (1/\theta^4)\mathrm{Var}(X) + (1/\theta^4)[x-E(X)]^2. \tag{4.20}$$

Substituting from (4.20) and (4.16) into the definition of μ_{11}, we obtain

$$\mu_{11} = (1/\theta^6)[-2\theta\mathrm{Var}(X) + E\{X-E(X)\}^3]. \tag{4.21}$$

Finally, substituting μ_{11}, μ_2 and ξ from (4.21), (4.18) and (4.17) into the expression for asymptotic bias, we get

$$E(\text{bias}) = [\theta\mathrm{Var}(X)(1-b) + a\,\mathrm{Var}(X) - (1/2)E\{X-E(X)\}^3] \times[\theta^2/\{\mathrm{Var}(X)\}^3]. \tag{4.22}$$

Comparison to zero of the coefficient of θ and the constant term inside the square brackets in (4.22) yields

$$b = 1, \qquad a = [E\{X-E(X)\}^3]/[2\mathrm{Var}(X)], \tag{4.23}$$

as the optimum values for the prior density parameters a and b. The value of a is one-half of the ratio of the third and second central moments of the truncated gamma distribution, and depends on both α and θ (in turn, on the truncation probability).

The recommended values for a and b for the exponential case by Blumenthal and Marcus (1975) are $a = 1/2$ and $b = 1$. We

compare two different values of a, namely, 1/4 and 1/2, and keep the value of b as One.

4.3 The Gamma Case. Simulation - α Known

In order to find the m.l.e. of θ, one has to solve the non-linear equation (4.4) for $\overline{x} < \alpha/(\alpha+1)$. The equation can also be written as

$$\overline{x} = \alpha\theta - \theta\exp(1/\theta)/[\int_0^1 t^{\alpha-1}\exp(-t/\theta)dt]. \tag{4.24}$$

Further, we find the Bayes modal estimator of θ by solving (4.15), which can be alternatively written as follows

$$\begin{aligned}\overline{x} + (a/n) &= (\alpha + b/n)\theta \\ &\quad - \theta\exp(-1/\theta)/[\int_0^1 t^{\alpha-1}\exp(-t/\theta)dt].\end{aligned} \tag{4.25}$$

We use the Newton-Raphson algorithm to solve each of the two equations (4.24) and (4.25) iteratively. To apply the method, we write the equations (4.24) and (4.25) as follows.

$$F_1 = \overline{x} - E(X) = 0, \tag{4.26}$$

$$F_2 = \overline{x} + \{(a-b\theta)/n\} - E(X) = 0. \tag{4.27}$$

The derivatives of F_1 and F_2 w.r.t. θ are derived next.

$$\begin{aligned}F_1' &= -(1/\theta^2)\mathrm{Var}(X) \\ &= -(1/\theta^2)[E(X)\{1+(\alpha+1)\theta-E(X)\} - \alpha\theta],\end{aligned} \tag{4.28}$$

since

$$E(X^2) = \frac{\int_0^1 t^{\alpha+1}\exp(-t/\theta)dt}{\int_0^1 t^{\alpha-1}\exp(-t/\theta)dt}$$

$$= (\alpha+1)\theta E(X) - \theta\exp(-1/\theta)/[\int_0^1 t^{\alpha-1}\exp(-t/\theta)dt]$$

$$= (\alpha+1)\theta E(X) + E(X) - \alpha\theta \qquad \text{(using (4.5))}$$

$$= E(X)[1 + (\alpha+1)\theta] - \alpha\theta, \qquad (4.29)$$

and therefore

$$\text{Var}(X) = E(X)[1+(\alpha+1)\theta-E(X)] - \alpha\theta. \qquad (4.30)$$

Similarly

$$F_2' = -(1/\theta^2)\text{Var}(X) - (b/n)$$
$$= -(1/\theta^2)[E(X)\{1+(\alpha+1)\theta-E(X)\} - \alpha\theta] - (b/n). \qquad (4.31)$$

The forms for F_1' and F_2' given in (4.28) and (4.31) are best suited for numerical calculations, since $E(X)$ can be first calculated using the incomplete gamma integral (with IMSL routine MDGAM) in the expression (4.5). Next the same value of $E(X)$ can be used to obtain F_1' and F_2'.

Simulations are carried out by taking 500 samples from a truncated gamma distribution using the IMSL routine GGAMR. The gamma case is different from the normal one in the sense that the interval of truncation here has been fixed in advance to be [0, 1]. In the normal case, the degree of truncation is first decided and the intervals of truncation are obtained subsequently for various values of the parameter. The probability of truncation, i.e., the probability

of the complete gamma distribution outside the interval [0, 1] is denoted by q.

Several values of α and θ to simulate gamma samples are used with the restriction that the gamma probability within the interval [0,1] should not be less than 0.8. Thus, as before, we do not deal with high probabilities of truncation. Simulated expected bias and MSE of the m.l.e. (when it exists), the modal estimator and the mixed estimator (as defined in section 3.6) have been found and reported in the accompanying tables. Results on the estimates of the probability of closeness of the modal and the mixed estimators to the actual simulated value of θ as compared to that of the m.l.e. (as in sections 3.6 and 3.7) are also reported.

Tables 4.1 through 4.4 provide the results of simulation for different sample sizes and the prior density parameter values considered here. The comparison for the two values of a are carried out in tables 4.1 and 4.2 for samples of size 10. The Bayes modal estimators with a of $\frac{1}{4}$ do not behave very well, which is clear from many of the efficiencies being less than one. This is not the case with a of $\frac{1}{2}$ where none of the efficiencies is less than one. In other comparisons, the efficiencies of the modal and the mixed estimators with $a = \frac{1}{4}$ are only rarely better than the corresponding efficiencies with $a = \frac{1}{2}$. Considering these facts, the value of $\frac{1}{2}$ for a has been used exclusively in later simulations for the samples of sizes 30 and 50.

Table 4.1 Simulated Expected Bias and MSE of the Maximum Likelihood, the Bayes Modal, and the Mixed Estimators of θ for the Truncated Gamma Distribution (α Known) $n = 10,\ a = \frac{1}{2}$

α	θ	q	Max. Lik. Estimates p_n	Bias	MSE	Bayes Modal Estimates Bias	MSE	EFF	p_m	Mixed Estimates (α=.85) Bias	MSE	EFF	p_{mx}
0.20	0.25	0.001	0.00	0.044	0.108	0.103	0.030	3.6	0.58	0.040	0.080	1.4	1.00
	0.50	0.013	0.07	0.714	16.772	0.044	0.076	220.7	1.00	0.085	0.198	84.7	1.00
	0.75	0.032	0.14	1.608	179.854	-0.111	0.138	1303.3	0.82	-0.403	0.247	728.2	1.00
	1.00	0.052	0.19	0.270	15.240	-0.291	0.251	60.7	0.75	-0.173	0.334	45.6	1.00
	2.00	0.121	0.34	-0.190	16.849	-1.106	1.452	11.6	0.62	-0.958	1.239	13.6	0.98
	3.00	0.170	0.36	0.823	250.117	-2.055	4.524	55.3	0.65	-1.940	4.135	60.5	0.95
0.40	0.25	0.003	0.00	0.003	0.015	0.055	0.012	1.2	0.51	0.003	0.015	1.0	1.00
	0.50	0.033	0.03	-0.078	0.042	-0.049	0.029	1.4	1.00	-0.071	0.041	1.0	1.00
	0.75	0.076	0.10	-0.262	0.105	-0.220	0.085	1.2	0.66	-0.229	0.098	1.1	0.95
	1.00	0.119	0.18	-0.457	0.246	-0.401	0.205	1.2	0.51	-0.404	0.214	1.1	0.84
0.60	0.25	0.007	0.00	-0.003	0.008	0.035	0.007	1.1	0.46	-0.003	0.008	1.0	1.00
	0.50	0.060	0.04	-0.115	0.030	-0.084	0.024	1.3	1.00	-0.103	0.031	1.0	1.00
	0.75	0.131	0.11	-0.297	0.105	-0.258	0.086	1.2	0.67	-0.268	0.095	1.1	0.86
	1.00	0.197	0.16	-0.513	0.281	-0.463	0.239	1.2	0.59	-0.470	0.248	1.1	0.79
0.80	0.25	0.011	0.00	-0.010	0.007	0.021	0.006	1.2	0.50	-0.009	0.007	1.0	1.00
	0.50	0.094	0.04	-0.133	0.028	-0.106	0.022	1.3	1.00	-0.122	0.028	1.0	1.00
	0.75	0.194	0.10	-0.327	0.117	-0.295	0.099	1.2	0.76	-0.305	0.107	1.1	0.79
1.00	0.25	0.180	0.00	-0.014	0.006	0.011	0.005	1.2	0.53	-0.014	0.006	1.0	0.99
	0.50	0.135	0.03	-0.145	0.029	-0.125	0.023	1.3	1.00	-0.139	0.028	1.0	1.00
1.50	0.25	0.046	0.00	-0.026	0.003	-0.008	0.002	1.4	0.66	-0.026	0.003	1.0	0.99
2.00	0.25	0.092	0.00	-0.034	0.003	-0.019	0.002	1.4	0.73	-0.033	0.003	1.0	0.97
2.50	0.25	0.156	0.00	-0.043	0.003	-0.030	0.002	1.5	0.87	-0.042	0.003	1.0	0.94

Table 4.2 Simulated Expected Bias and MSE of the Maximum Likelihood, the Bayes Modal, and the Mixed Estimators of θ for the Truncated Gamma Distribution (α Known)
$n = 10$, $a = \frac{1}{4}$

α	θ	q	Max. Lik. Estimates p_n	Max. Lik. Estimates Bias	Max. Lik. Estimates MSE	Bayes Modal Estimates Bias	Bayes Modal Estimates MSE	Bayes Modal Estimates EFF	Bayes Modal Estimates p_m	Mixed Estimates (0.85) Bias	Mixed Estimates (0.85) MSE	Mixed Estimates (0.85) EFF	Mixed Estimates (0.85) p_{mx}
0.20	0.25	0.001				0.044	0.015	7.2	1.00	0.039	0.079	1.3	1.00
	0.50	0.013				-0.076	0.065	258.0	0.65	0.059	0.172	97.5	1.00
	0.75	0.032				-0.240	0.161	1117.1	0.57	-0.084	0.218	825.0	1.00
	1.00	0.052				-0.426	0.321	47.5	0.54	-0.224	0.317	48.1	1.00
	2.00	0.121				-1.257	1.776	9.5	0.50	-1.040	1.357	12.4	0.98
	3.00	0.170		SAME		-2.209	5.141	48.7	0.52	-2.033	4.444	56.3	0.95
0.40	0.25	0.003				0.001	0.009	1.7	1.00	0.003	0.014	1.0	1.00
	0.50	0.033		AS		-0.110	0.037	1.1	0.47	-0.079	0.038	1.1	1.00
	0.75	0.076				-0.283	0.114	0.92	0.25	-0.245	0.097	1.1	0.92
	1.00	0.119				-0.466	0.259	0.95	0.23	-0.428	0.224	1.1	0.84
0.60	0.25	0.007		IN		-0.003	0.006	1.4	1.00	-0.003	0.008	1.0	1.00
	0.50	0.060				-0.125	0.032	0.94	0.34	-0.107	0.030	1.0	1.00
	0.75	0.131		TABLE		-0.301	0.109	0.96	0.17	-0.279	0.097	1.1	0.86
	1.00	0.197				-0.506	0.279	1.0	0.19	-0.487	0.259	1.1	0.79
0.80	0.25	0.011		4.1		-0.008	0.005	1.2	1.00	-0.009	0.007	0.98	1.00
	0.50	0.094				-0.137	0.029	0.95	0.22	-0.126	0.027	1.0	0.99
	0.75	0.194				-0.327	0.119	0.98	0.14	-0.315	0.111	1.1	0.79
1.00	0.25	0.180				-0.013	0.005	1.2	1.00	-0.014	0.005	1.0	1.00
	0.50	0.135				-0.150	0.030	0.97	0.18	-0.143	0.028	1.0	0.88
1.50	0.25	0.046				-0.024	0.003	1.1	1.00	-0.026	0.003	1.0	1.00
2.00	0.25	0.092				-0.032	0.003	1.1	1.00	-0.034	0.003	1.0	1.00

Table 4.3 Simulated Expected Bias and MSE of the Maximum Likelihood, the Bayes Modal, and the Mixed Estimators of θ for the Truncated Gamma Distribution (α Known)
$n = 30$, $a = \frac{1}{2}$

α	θ	q	Max. Lik. Estimates p_n	Bias	MSE	Bayes Modal Estimates Bias	MSE	EFF	p_m	Mixed Estimates (0.85) Bias	MSE	EFF	p_{mx}
0.20	0.25	0.001	0.00	0.020	0.019	0.055	0.016	1.2	0.51	0.020	0.019	1.0	1.00
	0.50	0.013	0.01	0.193	1.206	0.042	0.087	13.9	1.00	0.098	0.191	6.3	1.00
	0.75	0.032	0.03	0.785	15.056	-0.015	0.130	115.8	0.76	0.105	0.242	62.2	1.00
	1.00	0.052	0.09	1.024	46.009	-0.119	0.269	171.0	0.61	0.016	0.332	138.6	1.00
	2.00	0.121	0.22	4.748	5717.991	-0.841	1.109	5156.0	0.47	-0.688	0.875	6534.8	0.97
	3.00	0.170	0.31	3.210	699.607	-1.649	3.265	214.3	0.49	-1.529	2.856	245.0	0.88
0.40	0.25	0.003	0	-0.002	0.005	0.019	0.005	1.1	0.48	-0.002	0.005	1.0	1.00
	0.50	0.033	0.00	-0.068	0.020	-0.061	0.017	1.2	1.00	-0.068	0.020	1.0	1.00
	0.75	0.076	0.02	-0.211	0.065	-0.210	0.063	1.0	0.51	-0.209	0.064	1.0	0.96
	1.00	0.119	0.07	-0.399	0.179	-0.385	0.172	1.0	0.29	-0.381	0.170	1.1	0.82
0.60	0.25	0.007	0.00	-0.005	0.003	0.009	0.003	1.1	0.51	-0.005	0.003	1.0	1.00
	0.50	0.060	0.00	-0.095	0.017	-0.090	0.015	1.1	1.00	-0.096	0.017	1.0	1.00
	0.75	0.131	0.02	-0.264	0.078	-0.258	0.075	1.0	0.60	-0.260	0.077	1.0	0.86
	1.00	0.197	0.07	-0.473	0.232	-0.460	0.222	1.0	0.41	-0.461	0.222	1.0	0.76
0.80	0.25	0.011	0	-0.012	0.002	-0.001	0.002	1.2	0.56	-0.012	0.002	1.0	1.00
	0.50	0.094	0	-0.126	0.020	-0.121	0.018	1.1	1.00	-0.126	0.020	1.0	1.00
	0.75	0.194	0.03	-0.309	0.101	-0.302	0.096	1.1	0.78	-0.304	0.099	1.0	0.83
1.00	0.25	0.018	0	-0.016	0.0018	-0.007	0.0016	1.2	0.63	-0.016	0.0018	1.0	1.00
	0.50	0.135	0	-0.142	0.023	-0.137	0.021	1.1	1.00	-0.142	0.023	1.0	1.00
1.50	0.25	0.046	0	-0.025	0.0015	-0.018	0.0012	1.3	0.77	-0.025	0.0015	1.0	1.00
2.00	0.25	0.092	0	-0.035	0.0018	-0.030	0.0014	1.2	0.92	-0.035	0.0018	1.0	1.00
2.50	0.25	0.156	0	-0.045	0.0024	-0.040	0.0020	1.2	0.98	-0.045	0.0024	1.0	1.00

Table 4.4 Simulated Expected Bias and MSE of the Maximum Likelihood, the Bayes Modal, and the Mixed Estimators of θ for the Truncated Gamma Distribution (α Known)
$n = 50$, $a = \frac{1}{2}$

			Max. Lik. Estimates			Bayes Modal Estimates				Mixed Estimates (0.85)			
α	θ	q	p_n	Bias	MSE	Bias	MSE	EFF	p_m	Bias	MSE	EFF	p_{mx}
0.20	0.25	0.001	0	0.008	0.010	0.032	0.009	1.1	0.51	0.008	0.010	1.0	1.00
	0.50	0.013	0	0.072	0.157	0.026	0.047	3.4	1.00	0.056	0.100	1.6	1.00
	0.75	0.032	0.01	0.416	3.462	0.027	0.134	25.8	0.69	0.142	0.250	13.8	1.00
	1.00	0.052	0.05	1.360	282.873	-0.050	0.242	1168.9	0.57	0.107	0.342	827.1	1.00
	2.00	0.121	0.18	6.221	3496.821	-0.588	0.929	3764.4	0.47	-0.432	0.737	4744.7	0.99
	3.00	0.170	0.25	2.464	788.799	-1.440	2.835	278.2	0.39	-1.297	2.372	332.5	0.86
0.40	0.25	0.003	0	-0.008	0.0028	0.005	0.0026	1.1	0.54	-0.008	0.0030	1.0	1.00
	0.50	0.033	0	-0.068	0.014	-0.064	0.012	1.1	1.00	-0.068	0.014	1.0	1.00
	0.75	0.076	0	-0.208	0.056	-0.211	0.056	1.0	0.43	-0.209	0.056	1.0	0.98
	1.00	0.119	0.02	-0.391	0.168	-0.393	0.169	1.0	0.21	-0.389	0.167	1.0	0.83
0.60	0.25	0.007	0	-0.009	0.002	-0.001	0.002	1.1	0.56	-0.009	0.002	1.0	1.00
	0.50	0.006	0	-0.097	0.014	-0.094	0.013	1.1	1.00	-0.097	0.014	1.0	1.00
	0.75	0.131	0.00	-0.260	0.073	-0.258	0.072	1.0	0.54	-0.259	0.073	1.0	0.90
	1.00	0.197	0.02	-0.468	0.225	-0.465	0.223	1.0	0.34	-0.465	0.223	1.0	0.77
0.80	0.25	0.011	0	-0.010	0.0015	-0.003	0.0013	1.1	0.57	-0.010	0.0015	1.0	1.00
	0.50	0.094	0	-0.121	0.017	-0.118	0.016	1.1	1.00	-0.121	0.017	1.0	1.00
	0.75	0.194	0.00	-0.305	0.097	-0.304	0.095	1.0	0.81	-0.305	0.096	1.0	0.85
1.00	0.25	0.018	0	-0.016	0.0013	-0.010	0.0011	1.2	0.67	-0.016	0.0013	1.0	1.00
	0.50	0.135	0	-0.142	0.022	-0.139	0.021	1.0	1.00	-0.142	0.002	1.0	1.00
1.50	0.25	0.046	0	-0.024	0.0012	-0.020	0.0010	1.2	0.82	-0.024	0.0012	1.0	1.00
2.00	0.25	0.092	0	-0.037	0.0017	-0.034	0.0015	1.2	0.98	-0.037	0.0017	1.0	1.00
2.50	0.25	0.156	0	-0.044	0.0022	-0.042	0.0020	1.1	1.00	-0.044	0.0022	1.0	1.00

Estimates of the probability of nonexistence of the m.l.e. of θ are found to be as high as 0.36 for samples of size 10, and the truncation probability of only 0.170. Even for a sample of size 50, the corresponding estimate is 0.25.

The comparison of the MSE's of the m.l.e. with that of the modal estimator yields relative efficiencies that are quite large, especially for smaller values of α. The comparison of the MSE's is not so dramatic when higher values of α are considered. The same observation can be made for comparing the MSE's of the m.l.e. and the mixed estimator.

The Bayes modal estimator consistently under-estimates θ, except for smaller values of simulated θ, a trend which was observed for the truncated normal distribution also. The m.l.e. overestimates θ only when α is 0.20, otherwise generally underestimating, though only slightly.

The modal estimators are reasonably good, in light of the p_m criterion, giving estimates of p_m between o.34 and 1.00, the values only rarely going below 50%, and staying in the vicinity of 60% or higher very often. The mixed estimators prove even better with the similar p_{mx} criterion, giving the estimates of probability almost always one, and never below 77%. The mixed estimator, very much like the other two estimators mostly underestimates θ, except when α is 0.20. For α of 0.20, the bias is negative for larger values of θ and positive otherwise. Actually the m.l.e. and the mixed estimator are virtually the same for higher α's and higher sample sizes (30 or 50) since the probability

of nonexistence of the m.l.e. is then very close to zero. This is further evident from the relative efficiency of the mixed estimator, which is almost uniformly one for values of α other than 0.20.

4.4 The Weibull Case. M.L. Estimation - Both α and θ Unknown

We have already discussed, in section 2.2, the nonexistence of the m.l.e. of θ for the truncated Weibull distribution, when α is assumed known. Here we consider in detail the problem of finding the m.l.e.'s when both α and θ are unknown.

Similar to the truncated gamma distribution, considerable simplification of the analysis can be achieved, if we standardize the problem by considering the truncation interval to be [0, 1] in place of [0, T]. Therefore, in the future analysis, we assume T to be one.

Given a sample of size n from the density (2.9), the likelihood and the log likelihood are given as follows.

$$L(x;\alpha,\theta) = \frac{\alpha^n \,\Pi\, x_i^{\alpha-1}\exp(-\Sigma x_i^\alpha/\theta)}{\theta^n[1-\exp(-1/\theta)]^n}, \qquad 0 \le x_i \le 1, \tag{4.32}$$

$$\log L = n\log\alpha + (\alpha-1)\Sigma\log x_i - (\Sigma x_i^\alpha/\theta) - n\log\theta - n\log[1-\exp(-1/\theta)]. \tag{4.33}$$

The product and the summations extend over the range $1 \le i \le n$. The m.l.e.'s of α and θ are then the solutions of the pair of equations,

$$[\Sigma \log x_i/n] = [\Sigma x_i^{\alpha} \log x_i/n\theta] - (1/\alpha), \tag{4.34}$$

$$[\Sigma x_i^{\alpha}/n] = \theta - [\exp(1/\theta)-1]^{-1}. \tag{4.35}$$

Note that the standardization of the truncation interval to [0, 1] leaves no α in the last term of (4.33), hence the right side of the equation (4.35) does not also involve any α. At the same time, equation (4.34) simplifies due to the dropping out of a term which involves both α and θ. One advantage of this is that now θ can be written, from (4.34), as an explicit function of just the x's, α and n as follows

$$\theta = \frac{\Sigma x_i^{\alpha} \log x_i}{(n/\alpha) + \Sigma \log x_i}, \tag{4.36}$$

which gives the m.l.e. of θ, say $\hat{\theta}$, as a function of the m.l.e. of α. This m.l.e. of θ is well defined only when $\hat{\theta} > 0$, or when

$$(n/\alpha) + \Sigma \log x_i < 0, \qquad (\text{since } \Sigma x_i^{\alpha} \log x_i < 0)$$

or when

$$\alpha > -[n/(\Sigma \log x_i)] = \alpha_0, \tag{4.37}$$

say. The quantity α_0 depends only on the sample and is positive. Under the condition $\alpha > \alpha_0$, $\hat{\theta}$ is a decreasing function of α, since

$$(\partial\theta/\partial\alpha) = \frac{[(n/\alpha)+ \log x_i]\{\Sigma x_i^{\alpha}(\log x_i)^2\} + \{\Sigma x_i^{\alpha}\log x_i\}(n/\alpha^2)}{[(n/\alpha) + \Sigma \log x_i]^2} \qquad (4.38)$$

$$< 0,$$

due to both of the terms in the numerator being negative. The limiting values of $\hat{\theta}$ for various values of α are given by

$$\lim_{\alpha \downarrow \alpha_0} \hat{\theta} = \infty, \qquad \lim_{\alpha \uparrow \infty} \hat{\theta} = 0.$$

Next, the function of θ on the right side of the equation (4.35) (this function is the same as in the equation (2.10) for the truncated exponential, with $T = 1$) is an increasing function of θ and increases to 1/2 as θ approaches ∞. Substituting θ from (4.36) into (4.35), we obtain one equation, to be solved for α, given by

$$[\Sigma x_i^{\alpha}/n] = \frac{\Sigma x_i^{\alpha} \log x_i}{(n/\alpha) + \Sigma \log x_i} - \left[\exp\left\{ \frac{(n/\alpha) + \Sigma \log x_i}{\Sigma x_i^{\alpha} \log x_i} \right\} - 1 \right]^{-1} . \qquad (4.39)$$

The solution of this equation gives the m.l.e. of α, say $\hat{\alpha}$.

The right side of (4.39) is a decreasing function of α. The maximum value of this function is 1/2 when $\alpha = \alpha_0$. The left side in (4.39) is also decreasing in α but can be such that it is greater than 1/2 at α_0, i.e. $[\Sigma x_i^{\alpha_0}/n] > 1/2$. Whenever $[\Sigma x_i^{\alpha_0}/n]$ exceeds $\frac{1}{2}$, we need to show that (4.39) has no solution for α, hence a solution for θ does not exist either.

Thus the existence of solution of the pair of equations (4.34) and (4.35) for (α,θ) is as follows:

(a) If $[\Sigma x_i^{\alpha_0}/n] < 1/2$, $\hat{\alpha}$ is found as a solution of the equation (4.39) and then $\hat{\theta}$ is obtained from (4.36).

(b) If $[\Sigma x_i^{\alpha_0}/n] > 1/2$, no $\hat{\alpha}$ greater than α_0, satisfying (4.39) can be found. Then the likelihood in (4.32) takes its maximum for $\hat{\alpha} = \alpha_0$, $\hat{\theta} = \infty$.

Now, the proof of the fact that whenever $[\Sigma x_i^{\alpha_0}/n]$ exceeds 1/2, no solution for α (greater than α_0) of the equation (4.39) exists, still needs to be worked out. A large number of values of α (from very small to moderately large) to generate Weibull samples and the above result is found to be numerically true in every case.

4.5 The Weibull Case. Bayes Modal Estimation - Both α and θ Unknown

To deal with the Bayes modal estimation in the two parameter case, when both α and θ are unknown, we take the gamma prior for $1/\theta$ as in section 4.2 (for the truncated gamma distribution), for which the density is as given in (4.12). The prior for α is assumed to be noninformative. The modified likelihood and the log of the modified likelihood are then given by

$$L_m = \frac{c_1 a^{b+1}\theta^{-b}\alpha^n \,\Pi\, x_i^{\alpha-1}\exp[-(\Sigma x_i^{\alpha}+a)/\theta]}{[\Gamma(b+1)]\theta^n[1-\exp(-1/\theta)]^n}, \qquad (4.40)$$

$$\log L_m = c(a,b,x) + n\log\alpha + (\alpha-1)\Sigma\log x_i - (b+n)\log\theta$$
$$- [(\Sigma x_i^{\alpha}+a)/\theta] - n\log[1-\exp(-1/\theta)]. \qquad (4.41)$$

The modes of the likelihood in (4.40) are given as the solutions for α and θ of the following equations

$$[\Sigma\log x_i/n] = [\Sigma x_i^{\alpha} \log x_i/n\theta] - (1/\alpha), \qquad (4.42)$$

and

$$[(\Sigma x_i^{\alpha}+a)/n] = [(b+n)\theta/n] - [\exp(1/\theta)-1]^{-1}. \qquad (4.43)$$

Note that the first equation has no changes from the corresponding m.l. equation (4.34), while the second has additional terms of order $(1/n)$ on both sides of the equation, as compared to (4.35). Further, the right side of (4.43) is an increasing function of θ for $b > 0$, and has no finite upper bound for $\theta > 0$ (unlike that in the equation (4.35), where the function on the right increased to a finite limit of $1/2$ as θ increased to ∞).

The problem can again be reduced to the consideration of one parameter at a time. First, the Bayes modal estimator of α, say $\tilde{\alpha}$ is found as a solution of (4.43) with θ from (4.36) substituted into it. Then the modal estimator of θ is found by substituting $\tilde{\alpha}$ in (4.36).

Without a formal proof of the statement regarding the nonexistence of a solution of (4.35) under the condition $[\Sigma x_i^{\alpha_0}/n] > 1/2$, one can visualize that the problem with the equation (4.35) stems from the finite upper bound of $1/2$ for the function on the right

side of the equation. Avoiding such a pitfall in equation (4.43) allows for a solution for (α, θ) which should exist with probability one. This is again without proof, but a fairly large number of values of α was tried and the result was found to be numerically true in every case.

At this stage, it is imperative to note that the whole analysis for the optimum value(s) of the prior density parameter(s), by minimizing the maximum asymptotic bias of the Bayes modal estimator, (first reproduced in section 2.3 and subsequently repeated in sections 3.3 and 4.2) assumes a one-dimensional parameter, which is not the case here. Extension of this analysis to a two-dimensional parameter is beyond the scope of this study. We, therefore, retain the "optimum" values of a and b derived for the truncated gamma distribution in section 4.2. Thus we use $b = 1$ and $a = \frac{1}{4}, \frac{1}{2}$ for the present case also.

4.6 The Weibull Case. Simulation - Both α and θ Unknown

The truncated Weibull is the only two-parameter case under study in this dissertation, where we could find the m.l.e.'s, as well as the Bayes modal estimators of the parameters by solving one equation at a time. The m.l.e.'s of α and θ are found by first finding $\hat{\alpha}$ (greater than α_0) by solving the equation (4.39) for α (except when $[\Sigma x_i^{\alpha_0}/n] > 1/2$), and then calculating $\hat{\theta}$ by substituting $\hat{\alpha}$ into (4.36). Similarly, the Bayes modal estimators are found by first calculating $\tilde{\alpha}$ (greater than α_0) by solving

the following equation for α,

$$[(\Sigma x_i^{\alpha}+a)/n] = (1+b/n)\, \frac{\Sigma x_i^{\alpha} \log x_i}{(n/\alpha) + \Sigma \log x_i}$$

$$- \left[\exp \left\{ \frac{(n/\alpha) + \Sigma \log x_i}{\Sigma x_i^{\alpha} \log x_i} \right\} - 1 \right]^{-1}, \qquad (4.44)$$

and then evaluating $\tilde{\theta}$ by substituting $\tilde{\alpha}$ into (4.36).

Since the only nonlinear equations to be solved are (4.39) and (4.44), it would have been ideal to solve each of these equations by using the Newton-Raphson algorithm. But the form of the functions being fairly complicated in α, the derivatives are quite messy. Also, the numerical calculations based on these derivatives are bound to have large approximation errors because of these derivatives involving a number of terms. Therefore, we have used the IMSL routine ZSPOW to solve each of the two equations (4.39) and (4.44).

Simulation results are based on 500 samples from a given Weibull distribution. The connection between the Weibull and the exponential distribution has been utilized to draw Weibull samples. We first draw sample values, Y_i, from an exponential distribution with the scale parameter θ (using the IMSL routine GGEXN) and then transform $Y^{1/\alpha} = X$ to obtain the Weibull sample observations from the distribution with shape parameter θ and scale parameter α. Different values of α and θ are used with the restriction that the probability within the interval [0, 1] for

the Weibull density is not less than 0.8.

We calculate the simulated expected bias and MSE of the m.l.e.'s, the Bayes modal estimators and the mixed estimators of α and θ. The bias and MSE for the m.l.e.'s are based on the samples for which $[\Sigma x_i^{\alpha_0}/n]$ is less than 1/2, while for the modal and the mixed estimators, they are based on all 500 samples. The proportion of samples for which the m.l.e.'s do not exist, is represented by p_n. Estimates of the probability of closeness of the modal and the mixed estimators to the actual simulated values of θ compared to that of the m.l.e.'s (p_m and p_{mx}) are also calculated as before.

One numerical problem, which is discovered in the simulations for the truncated Weibull distribution, is due to the fact that our condition for the nonexistence of the m.l.e., namely $[\Sigma x_i^{\alpha_0}/n]$ greater than 1/2, is only a sufficient condition. Thus even when $[\Sigma x_i^{\alpha_0}/n]$ is less than 1/2, (presumably when it is much smaller than 1/2), the m.l.e.'s of α and θ may not exist. Whenever a solution to equation (4.39) for α does not exist, the iterations either converge to a value less than α_0, or tend towards ∞ ($\alpha = \infty$ makes both sides of (4.39) zero). Any sample with $[\Sigma x_i^{\alpha_0}/n]$ less than 1/2 and exhibiting such behavior for iterations of equation (4.39) as described above, is excluded from consideration.

For this reason, the simulations have to take considerably more samples to obtain the 500 samples that give proper convergence. These extra samples needed increase the computer time manifold, more so when the sample size is 50. Therefore, even one such case takes disproportionately large computer time. For this reason, we

have carried out the simulations for sample sizes 10 and 30 only, assuming that the patterns from these sample sizes will indicate the behavior for the samples of size 50 also. Also instead of considering both $\frac{1}{4}$ and $\frac{1}{2}$ for the values of a, we use only $\frac{1}{2}$ throughout. Since the simulated relative efficiencies for the Bayes modal and the mixed estimators always come out to be one or greater, this choice of a seems reasonable.

The simulation results are available in Tables 4.5 and 4.6 for the two sample sizes considered. The relative efficiencies for the modal and the mixed estimators of θ are quite large even though the probability of nonexistence of the m.l.e. is not very big. The relative efficiencies for estimating α do not follow the pattern of the relative efficiencies of θ, since the m.l.e. of α remains finite and does not blow up near the upper boundary of its existence. As noted in section 4.4, the m.l.e. of α is close to α_0, a finite quantity, when the m.l.e. of θ tends to increase indefinitely.

There is no definite trend of over- or under-estimation discernible in the bias, when estimating α, but all three estimators over-estimate θ. The bias in estimating θ, is still much smaller for the modal and the mixed estimators than that for the m.l.e. This is also generally true when estimating α.

The Bayes modal and the mixed estimators both give moderately large estimates of the probabilities p_m and p_{mx}. Thus considering the bias, the MSE (or the relative efficiency), as well as the

Table 4.5 Simulated Expected Bias and MSE of the Maximum Likelihood, the Bayes Modal, and the Mixed Estimators of α and θ for the Truncated Weibull Distribution
$n = 10$, $a = \frac{1}{2}$

θ	q	α	Max. Lik. Estimates			Bayes Modal Estimates				Mixed Estimates (0.9)			
			p_n	Bias	MSE	Bias	MSE	EFF	p_m	Bias	MSE	EFF	p_{mx}
ESTIMATION OF α													
0.3	0.036	0.5	0.01	0.018	0.018	-0.019	0.011	1.4	0.62	0.021	0.015	1.0	0.91
		1.5	0.05	-0.172	0.137	-0.185	0.129	1.1	0.64	-0.159	0.126	1.1	0.64
0.4	0.082	0.5	0.06	0.038	0.023	0.013	0.017	1.4	0.69	0.040	0.022	1.0	0.86
		1.5	0.10	-0.058	0.149	-0.050	0.128	1.2	0.56	-0.030	0.133	1.1	0.57
0.5	0.135	0.5	0.11	0.045	0.027	0.031	0.020	1.4	0.72	0.049	0.025	1.1	0.82
		1.5	0.13	0.023	0.188	0.050	0.174	1.1	0.51	0.066	0.182	1.0	0.51
ESTIMATION OF θ													
0.3	0.036	0.5	0.01	0.202	2.760	0.086	0.032	86.3	0.66	0.039	0.035	78.9	0.94
		1.5	0.05	0.416	3.163	0.218	0.073	43.3	0.42	0.198	0.070	45.2	0.83
0.4	0.082	0.5	0.06	0.464	10.450	0.075	0.051	204.9	0.84	0.041	0.059	177.1	0.91
		1.5	0.10	0.492	2.792	0.176	0.072	38.8	0.75	0.163	0.073	38.2	0.88
0.5	0.135	0.5	0.11	0.769	32.328	0.050	0.059	547.4	1.00	0.027	0.068	475.4	1.00
		1.5	0.13	0.677	9.159	0.112	0.059	155.2	1.00	0.101	0.062	147.7	1.00

Table 4.6 Simulated Expected Bias and MSE of the Maximum Likelihood, the Bayes Modal, and the Mixed Estimators of α and θ for the Truncated Weibull Distribution
$n = 30$, $a = \frac{1}{2}$

θ	q	α	Max. Lik. Estimates p_n	Bias	MSE	Bayes Modal Estimates Bias	MSE	EFF	p_m	Mixed Estimates (0.9) Bias	MSE	EFF	p_{mx}
ESTIMATION OF α													
0.3	0.036	0.5	0	0.016	0.007	-0.000	0.006	1.2	0.60	0.016	0.007	1.0	0.90
		1.5	0	-0.139	0.061	-0.154	0.060	1.0	0.79	-0.138	0.058	1.1	0.79
0.4	0.082	0.5	0.00	0.018	0.010	0.012	0.008	1.3	0.69	0.021	0.009	1.1	0.83
		1.5	0.00	-0.072	0.060	-0.069	0.050	1.2	0.57	-0.059	0.051	1.2	0.57
0.5	0.135	0.5	0.02	0.031	0.012	0.028	0.010	1.2	0.75	0.033	0.012	1.0	0.81
		1.5	0.03	-0.007	0.077	0.006	0.064	1.2	0.44	0.013	0.066	1.2	0.44
ESTIMATION OF θ													
0.3	0.036	0.5	0	0.010	0.020	0.026	0.012	1.7	0.60	0.007	0.014	1.4	0.87
		1.5	0	0.147	0.076	0.138	0.031	2.5	0.21	0.126	0.029	2.6	0.74
0.4	0.082	0.5	0.00	0.131	0.814	0.041	0.032	25.4	0.77	0.029	0.034	23.9	0.82
		1.5	0.00	0.266	1.078	0.118	0.043	25.1	0.72	0.112	0.043	25.1	0.77
0.5	0.135	0.5	0.02	0.416	33.677	0.018	0.059	570.8	1.00	0.018	0.067	502.6	1.00
		1.5	0.03	0.426	8.494	0.093	0.061	139.2	1.00	0.089	0.062	137.0	1.00

probability criterion, the methods of modal and the mixed estimation work very well.

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