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Tze-Jang Chen
Old Dominion University

Jenn-Tsann Lin
Old Dominion University

C. H. Cooke
Old Dominion University

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The Range of the Iterated Matrix Adjoint Operator

TZE-JANG CHEN AND JENN-TSANN LIN
Department of Applied Mathematics
Feng-Chia University
Taichung, Taiwan, R.O.C.

C. H. COOKE
Department of Mathematics and Statistics
Old Dominion University
Norfolk, VA 23529, U.S.A.

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Abstract—The following inverse problem is considered: for a given \( n \times n \) real matrix \( B \), does there exist a real matrix \( A \) such that

\[
B = \underbrace{\text{adj} \ adj \ \cdots \ adj}_{m} A
\]

where the classical adjoint operation is intended? The rank of \( B \) and the number of applications of the adjoint operator determine the character of this general inverse problem for the iterated adjoint operator. Thus, for given \( B \), the question of interest is whether or not \( B \) lies in the range of the iterated matrix adjoint operator. Maple V R5 is used as an aid to obtain results indicated here. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords—Adjoint inverse problem, Matrix operator, Maple V.

1. INTRODUCTION

Wardlow [1] proposed in *Mathematics Magazine* that the matrix \( E \)

\[
E = \begin{bmatrix} 5 & 5 & 2 \\ 5 & 5 & 2 \\ 1 & 1 & 6 \end{bmatrix}
\]

is not the classical adjoint of any matrix with real entries. Cooke [2] affirmed the Wardlow conjecture by considering an inverse problem for general real matrices. In [2], conditions are found under which a real matrix \( B \) satisfies

\[
B = \text{adj} \ (A),
\]

where \( A \) and \( B \) are \( n \times n \) real matrices.
In this paper, a more general adjoint inverse problem is considered. Here,

\[ B = \underbrace{\text{adj adj} \cdots \text{adj}}_{m} A, \]  

(3)

where \( A \) and \( B \) are \( n \times n \) matrices. Conditions emerge which determine whether \( B \) is in the range of this iterated matrix adjoint operator.

2. MAIN RESULTS

According to properties of the adjoint operator, for \( A, B \) \( n \times n \) matrices, we know that if \( B \) is the classical adjoint of \( A \), then

\[ AB = BA = \alpha I, \quad \text{where} \quad \alpha = \det (A). \]  

(4)

Also,

\[ (\det A) (\det B) = (\det A)^n \]  

(5)

implies that

\[ \det B = (\det A)^{n-1} \]  

(6)

and

\[ \det A = (\det B)^{1/(n-1)}. \]  

(7)

Thus,

\[ A = \det A \cdot B^{-1} = (\det B)^{1/(n-1)} \cdot B^{-1}. \]  

(8)

Now, consider the general adjoint inverse problem (3). For the case \( m = 2k, k \in \mathbb{N} \), the following theorem is obtained.

**THEOREM 1.** Let a real matrix \( B \) have the following properties:

1. \( R_B = n \) (\( > 1 \)), where \( R_B \) is the matrix rank of \( B \),

2. \( B = \underbrace{\text{adj adj} \cdots \text{adj}}_{m} A \), and

3. \( m = 2k \), \( k \in \mathbb{N} \).

Then, nonunique solutions for \( A \) are given by

\[ A = (\det B)^{(2-n)/(n-1)^{2k}} \cdot (\det B)^{(2-n)/(n-1)^{2k-2}} \cdots (\det B)^{(2-n)/(n-1)^2} \cdot B. \]  

(9)

Thus, every \( B \) of full rank is in the range of the \( m \)-iterated adjoint operator.

**PROOF.** Observe that for \( m = 2 \) (\( k = 1 \)), \( B = \text{adj adj} \text{(A)} \), where \( C = \text{adj} \text{(A)} \). Let \( B = \text{adj} \text{(C)} \), where \( C = \text{adj} \text{(A)} \).

Since \( A \text{ adj} \text{(A)} = \text{adj} \text{(A)} A = (\det A) I \) and \( AC = CA = \det (A) I \), then \( A = (\det A) C^{-1} \), similarly \( C = (\det C) B^{-1} \).

Hence, \( A = (\det A) (\det C)^{-1} B \).

If \( R_B = n \), then \( \det A \neq 0 \) and \( \det C \neq 0 \)

\[ \det C \det A = (\det A)^n, \]

\[ \det C \det B = (\det C)^n. \]

Hence,

\[ \det A = (\det C)^{1/(n-1)}, \]

\[ \det C = (\det B)^{1/(n-1)}. \]
Thus, \( \det A = (\det B)^{(1/(n-1))^2} \), then

\[
A = (\det A) (\det C)^{-1} B \\
= (\det C)^{1/(n-1)} (\det C)^{-1} B \\
= (\det C)^{(1-n+1)/(n-1)} B \\
= (\det B)^{(2-n)/(n-1)^2} B.
\]

For the induction hypothesis, assume that for \( m = 2k, k \in N \), that \( B \) is in the range of the iterated adjoint operator, and we have

\[
A = (\det B)^{(2-n)/(n-1)^2k} \cdot (\det B)^{(2-n)/(n-1)^{2k-2}} \cdots (\det B)^{(2-n)/(n-1)^2} B.
\]

Now to check that the hypothesis holds when \( k \) is replaced by \( k + 1 \), since \( m = 2k + 2, k \in N \), we have

\[
B = \operatorname{adj} \operatorname{adj} \cdots \operatorname{adj} A.
\]

Let \( C = \operatorname{adj} \operatorname{adj} \cdots \operatorname{adj} A \) and \( B = \operatorname{adj} \operatorname{adj} (C) \), then

\[
A = (\det C)^{(2-n)/(n-1)^{2k}} \cdot (\det C)^{(2-n)/(n-1)^{2k-2}} \cdots (\det C)^{(2-n)/(n-1)^2} C,
\]

so that

\[
A = \left\{ \det \left[ (\det B)^{(2-n)/(n-1)^2} \cdot B \right] \right\}^{(2-n)/(n-1)^{2k}} \cdot \left\{ \det \left[ (\det B)^{(2-n)/(n-1)^2} \cdot B \right] \right\}^{(2-n)/(n-1)^{2k-2}} \cdots (\det B)^{(2-n)/(n-1)^2} B
\]

since

\[
\det \left[ (\det B)^{(2-n)/(n-1)^2} \cdot B \right] = (\det B)^{(2n-n^2)/(n-1)^2} \cdot \det B
\]

\[
= (\det B)^{1/(n-1)^2}.
\]

Thus, the hypothesis holds when \( m = 2k + 2, k \in N \), and the theorem is established by mathematical induction.

As was shown in [2], multiple matrices \( A \) in the operator domain may map to a single \( B \) satisfying the conditions indicated. For small \( n \) one might possibly predict this number; but for large \( n \) the combinatorial explosion of possibilities makes this impractical. The number of such matrices \( A \) is not considered as relevant as the question of whether any such \( A \) exist, which has been answered here. This same observation holds true for Theorem 2 below.

**Example 1.** Consider the nonsingular matrix \( B = \operatorname{adj} \operatorname{adj} (A) \) with

\[
B = \begin{bmatrix}
1296 & 3888 & 2592 & 5184 \\
6480 & 3888 & 2592 & 5184 \\
1296 & 3888 & 6480 & 9072 \\
2592 & 3888 & 5184 & 6480
\end{bmatrix}
\]
and $\det(B) = 10155995668416 \neq 0$. So, by Theorem 1, we have ($n = 4$ and $k = 1$)
\[
A = (\det B)^{-2/3} \cdot B = (10155995668416)^{-2/9}.
\]
\[
\begin{bmatrix}
1296 & 3888 & 2592 & 5184 \\
6480 & 3888 & 2592 & 5184 \\
1296 & 3888 & 6480 & 9072 \\
2592 & 3888 & 5184 & 6480
\end{bmatrix}
\]
Thus,
\[
A = \begin{bmatrix}
1 & 3 & 2 & 4 \\
5 & 3 & 2 & 4 \\
1 & 3 & 5 & 7 \\
2 & 3 & 4 & 5
\end{bmatrix}
\]
Theorem 1 can also be extended to the case where $m = 2k + 1$, $k \in N$, is an odd integer. Clearly,

$\det A = (\det B)^{2-n}/(n-1) \cdot B^{-1}$.

**Theorem 2.** Given a real matrix $B$ with the following properties.

1. $R_B = n (> 1)$.
2. $B = \text{adj} \text{adj} \cdots \text{adj} A$.
3. $m = 2k + 1$, $k \in N$.

Then, the nonunique solutions for $A$ are given by

$A = (\det B)^{(2-n)/(n-1)^{2k+1}} \cdot (\det B)^{(2-n)/(n-1)^{2k-1}} \cdots (\det B)^{(2-n)/(n-1)^3} \cdot (\det B)^{1/(n-1)} \cdot B^{-1}$.

**Proof.** For $m = 3$ ($k = 1$), consider the relation

$B = \text{adj} \text{adj} \text{adj} (A)$.

Let $C = \text{adj} \text{adj} (A)$ and $B = \text{adj} (C)$, and then

$A = (\det C)^{(2-n)/(n-1)^2} \cdot C$,

$C = (\det B)^{1/(n-1)} \cdot B^{-1}$

and

$A = \left\{ \det \left[ (\det B)^{1/(n-1)} \cdot B^{-1} \right] \right\}^{(2-n)/(n-1)^2} \cdot (\det B)^{1/(n-1)} \cdot B^{-1}$

$= \left[ (\det B)^{n/(n-1)} \cdot (\det B)^{-1} \right]^{(2-n)/(n-1)^2} \cdot (\det B)^{1/(n-1)} \cdot B^{-1}$

$= (\det B)^{(2-n)/(n-1)^3} \cdot (\det B)^{1/(n-1)} \cdot B^{-1}$.

Thus, for $m = 3$ ($k = 1$) it is true that $B$ is in the range of the iterated adjoint operator. For induction hypothesis, assume that for $m = 2k - 1$, $k \in N$, $B$ is in the range of the iterated adjoint operator, that is, there exists $A$ such that

$B = \text{adj} \text{adj} \cdots \text{adj} A$. 

Then
\[ A = \left( \text{det} B \right)^{(2-n)/(n-1)^{2k-1}} \cdot \left( \text{det} B \right)^{(2-n)/(n-1)^{2k-3}} \cdots \left( \text{det} B \right)^{(2-n)/(n-1)^3} \cdot \left( \text{det} B \right)^{1/(n-1)} \cdot B^{-1}. \]

Now to show the hypothesis holds for \( m = 2k + 1, k \in \mathbb{N} \), we have
\[ B = \text{adj} \text{adj} \cdots \text{adj} A. \]

Let \( C = \text{adj} \text{adj} \cdots \text{adj} A \) and \( B = \text{adj} \text{adj} (C) \), then
\[ A = \left( \text{det} C \right)^{(2-n)/(n-1)^{2k-1}} \cdot \left( \text{det} C \right)^{(2-n)/(n-1)^{2k-3}} \cdots \left( \text{det} C \right)^{(2-n)/(n-1)^3} \cdot \left( \text{det} C \right)^{1/(n-1)} \cdot C^{-1} \]
\[ C = \left( \text{det} B \right)^{(2-n)/(n-1)^2} \cdot B, \]
so that
\[ A = \left\{ \text{det} \left[ \left( \text{det} B \right)^{(2-n)/(n-1)^2} \cdot B \right] \right\}^{(2-n)/(n-1)^{2k-1}} \]
\[ \cdot \left\{ \text{det} \left[ \left( \text{det} B \right)^{(2-n)/(n-1)^2} \cdot B \right] \right\}^{(2-n)/(n-1)^{2k-3}} \]
\[ \cdots \left\{ \text{det} \left[ \left( \text{det} B \right)^{(2-n)/(n-1)^2} \cdot B \right] \right\}^{(2-n)/(n-1)^3} \cdot \left\{ \text{det} \left[ \left( \text{det} B \right)^{(2-n)/(n-1)^2} \cdot B \right] \right\}^{1/(n-1)} \]
\[ \left[ \left( \text{det} B \right)^{(2-n)/(n-1)^2} \right]^{-1} \]
\[ = \left( \text{det} B \right)^{(2-n)/(n-1)^{2k+1}} \cdot \left( \text{det} B \right)^{(2-n)/(n-1)^{2k-1}} \cdots \left( \text{det} B \right)^{(2-n)/(n-1)^3} \cdot \left( \text{det} B \right)^{1/(n-1)} \cdot B^{-1}, \]
and thus,
\[ A = \left( \text{det} B \right)^{(2-n)/(n-1)^{2k+1}} \cdot \left( \text{det} B \right)^{(2-n)/(n-1)^{2k-1}} \cdots \left( \text{det} B \right)^{(2-n)/(n-1)^3} \cdot \left( \text{det} B \right)^{1/(n-1)} \cdot B^{-1}, \]
since
\[ \frac{1}{(n-1)^3} + \frac{n-2}{(n-1)^2} = \frac{2-n}{(n-1)^3} + \frac{1}{n-1}. \]
Thus, the hypothesis holds when \( m = 2k + 1, k \in \mathbb{N} \) is true, and Theorem 2 is established by mathematical induction.

**Example 2.** Consider the nonsingular matrix \( C = \text{adj} \text{adj} \text{adj} (A) \) with
\[
C = \begin{bmatrix}
-19591041024 & 19591041024 & 0 & 0 \\
45712429056 & -19591041024 & -52242776064 & 52242776064 \\
-32651735040 & -19591041024 & -26121388032 & 78364164096 \\
6530347008 & 19591041024 & 52242776064 & -78364164096 
\end{bmatrix}
\]
and \( \text{det} (C) = 1047532535594334222593508922191671036215296 \neq 0 \). By Theorem 2, we have for \( n = 4 \) and \( k = 1 \)
\[ A = \left( \text{det} C \right)^{-2/3} \cdot \left( \text{det} C \right)^{1/3} \cdot C^{-1} \]
\[ = \left( \text{det} C \right)^{-2/27} \cdot \left( \text{det} C \right)^{1/3} \cdot \begin{bmatrix}
1 & 1 & 1 & 1 \\
78364164096 & 26121388032 & 39182082048 & 19591041024 \\
5 & 1 & 1 & 1 \\
78364164096 & 26121388032 & 78364164096 & 19591041024 \\
1 & 1 & 1 & 1 \\
39182082048 & 26121388032 & 19591041024 & 78364164096 
\end{bmatrix}. \]
Thus, we have

\[
A = \begin{bmatrix}
1 & 3 & 2 & 4 \\
5 & 3 & 2 & 4 \\
1 & 3 & 5 & 7 \\
2 & 3 & 4 & 5 \\
\end{bmatrix}.
\]

However, for rank \(B\) ranging between 1 and \(n\), a contrary result is obtained.

**Theorem 3.** If the incompatibility condition

\[
1 < R_B < n
\]  

holds, then

\[
B = \overbrace{\text{adj adj} \cdots \text{adj} (A)}^{m}
\]

has no solution for \(A\).

**Theorem 4.** If \(R_B \leq 1\), equation (3) has infinitely many solutions, except when \(B = 0\) and \(n < 3\). In this case \(A = 0\) is the only solution.

The proofs of Theorems 3 and 4 are virtually word-for-word the same as those of corresponding theorems in [2]; consequently, they will be omitted.

### 3. CONCLUSIONS

An inverse problem for the iterated matrix adjoint operator has been investigated in this research, and several interesting theorems have been obtained. Clarifying examples have been given. The present research extends results of Cooke [2].

### REFERENCES