A Mathematical Model of the Dynamics of an Optically Pumped Four-Level Solid State Laser System

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A MATHEMATICAL MODEL OF THE DYNAMICS
OF AN OPTICALLY PUMPED FOUR-LEVEL
SOLID STATE LASER SYSTEM

by

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ABSTRACT

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This is a study of a mathematical model of the dynamics of an optically pumped four-level solid state laser system. A general mathematical model that describes the spatial and temporal evolution of the electron populations in the laser rod as well as the development of the left and right traveling photon fluxes in the cavity is developed. The model consists of a coupled set of first order semilinear partial differential equations. While the model was developed for Titanium-doped sapphire lasers, it is applicable to three and four level lasers in general.

The analysis of the model is conducted in two parts. First, by formally taking an average over the spatial variable, the system of partial differential equations is reduced to a system of ordinary differential equations describing the temporal behavior of the spatially-averaged dynamic quantities. Several qualitative properties of the solutions of this system are proved and stability of the solutions under various operating conditions is investigated. The rate equations are solved numerically and the effects on the solutions of changes in the physical parameters are discussed.

The second part of this study is concerned with the qualitative and numerical
analysis of the spatial and temporal model of a Ti: Sapphire ring laser. Several qualitative properties of the solution are established. The system of partial differential equations is solved numerically by integration along the characteristic lines using an implicit integration scheme developed for this problem. The computed solutions are compared to those obtained by using a stable finite difference approximation. The results of the comparison demonstrate that the implicit integration scheme is viable as well as efficient for numerically solving the system of partial differential equations and can be considered a useful analytical tool for studying the dynamics of this type of laser system.

All computer codes are written in FORTRAN and currently run on a DEC VAX 11/750.
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Chapter 1

Introduction

During the last several years, solid state lasers have been developed that have the potential for meeting the rigorous performance requirements for space-based remote sensing of the atmosphere. One of the materials that has shown promise in achieving the performance objectives is $\text{Ti}^{3+} : \text{Al}_2\text{O}_3$, or Titanium doped Sapphire, first demonstrated in 1982 [18].

In a solid state laser, a dopant ion substitutes for an ion in the host material. Specifically, in Ti:Sapphire, the $\text{Ti}^{3+}$ ion substitutes for the $\text{Al}^{3+}$ ion in $\text{Al}_2\text{O}_3$. Strong coupling between the optically active Titanium ion and the host lattice broaden the absorption and emission lines [5]. Narrow line width operation is achieved by using a line-narrowing technique such as injection seeding [10]. In order to design a stable and efficient laser and to understand the effect on laser output of line-narrowing techniques, it is necessary to gain an understanding of the development of the dynamical processes of the laser. This is a study

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of a mathematical model of an end-pumped Ti:Sapphire laser. The model was developed for this laser system but is sufficiently general to be applied, with minor modifications, to both three and four-level laser systems.

The model consists of a coupled set of rate equations that describe the spatial and temporal development of the dynamical variables. The model presented here allows for the introduction of an injected signal and thus the effect on laser output of injection seeding can be studied.

A general time dependent rate equation model for three and four level lasers is given in [12]. Moulton [16] has employed a rate equation approach to investigate the temporal process in a $Co: MgF_2$ laser system. The rate equations were solved numerically and experimental results were compared with the numerical solutions. Moulton also applied this approach to a study of an injection locked Ti:Sapphire ring laser [18]. Ganiel, et.al. [10], numerically solved the coupled space and time dependent rate equations for an injection locked pulsed dye laser system. The purpose of the present study is to establish general analytical properties of the solutions to both the temporal and the spatial-temporal models for a Ti:Sapphire laser. In addition, the numerical treatment of the models will be discussed in detail.

In Chapter 2, a general mathematical model that describes the spatial and temporal evolution of the electron population in the active region of the laser system as well as the development of left and right traveling photon fluxes in the laser rod and cavity is presented. The model, based on a four-level model of the
transitions between electronic levels of the Titanium ion, consists of a system of first order semilinear partial differential equations. The cavity configuration is assumed to be a simple ring with injection seeding allowed through a mirror in the cavity. Boundary conditions appropriate for this arrangement are given.

Chapter 3 deals with a simplified version of the model described in Chapter 2 obtained by taking an average over the spatial variable. By formally integrating over the spatial variable, the system of partial equations is reduced to a system of nonlinear ordinary differential equations. General qualitative results are given and a linearized analysis gives stability properties of the equilibrium solutions. The numerical treatment of the system of equations is discussed and numerical results are given for both constant and time dependent pumping rates. The effects on the computed solutions of physical parameters that cannot be directly measured are studied.

The spatial and temporal model is studied in detail in Chapter 4. Some qualitative results are established and the solutions to this model are found to exhibit some of the same qualitative properties as the corresponding elements in the spatially averaged model. The approach used to solve the system of partial differential equations numerically is motivated by reducing it to a system of ordinary differential equations along the characteristics [8]. Numerical integration in the active region is performed using a Modified Euler technique (as in [7]) and the results are compared to those obtained by using Euler's Method and a stable finite difference approximation [11].
Most of the physical parameters used can be obtained by direct measurement. Characteristics of Ti:Sapphire used in the model are summarized in Figures 2.1-2.3 ([4], [17]) which show the absorption and emission spectra and the temperature dependence of the fluorescence lifetime.

All computer codes are written in FORTRAN and currently run on a DEC VAX11/750. Two computer programs are included in the appendices. In Appendix I, the computer program is given that calculates the equilibrium solutions for the spatially averaged model with a constant pumping rate and determines the stability properties of the solution. The program also determines the nature of the linearized eigenvalues which is used to predict the occurrence of oscillations in the computed solutions of the rate equations. Appendix II contains the computer program that solves the spatial and temporal model using the Modified Euler Method to integrate along the characteristics of the system.
Chapter 2

A Description of the Model of the Laser Dynamics

2.1 The Population Rate Equation

The emission and absorption spectra of Ti:Sapphire [4], [17], superimposed on an energy level diagram for the Ti$^{3+}$ ion are shown in Figure 2.4. These spectra are the basis for an idealized four level model, Figure 2.5, of the lasing action of Titanium in Sapphire. The electron population density in the $i^{th}$ energy level at position $x$ in the crystal and time $t$ is denoted by $n_i(x,t); i = 0, 1, 2, 3$.

A narrow band pump excites the ground state electrons to a level in the absorption band which is determined by the wavelength of the pump. The electrons then undergo a rapid non-radiative transition to the upper laser level. The population density in the lasing levels, $n_2$ and $n_1$, can be changed by stimulated emission and absorption. In addition, the populations in each of the levels above the ground state can change due to non-radiative transitions to the lower levels.
Figure 2.1  Polarized fluorescence spectra and calculated gain line shape for Ti:Al$_2$O$_3$. 

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Figure 2.2 Polarized absorption cross sections for the $^2T_2 \rightarrow E$ transition in Ti:Al$_2$O$_3$.
Figure 2.3 Fluorescence lifetime versus temperature for the transition in Ti:Al$_2$O$_3$.

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Figure 2.4 Emission and absorption spectra superimposed on an energy level diagram for the $t^3$ ion.
Ground state electrons are pumped to level 3 at a rate

\[ W_{03}(x,t)n_0(x,t) \]

where \( W_{03}(x,t) \) is the pumping rate giving the number of photons per second available to pump electrons from the ground state to level 3. The pumped electrons then undergo transitions to the lower levels at a rate

\[ \frac{n_3(x,t)}{\tau_{30}} + \frac{n_3(x,t)}{\tau_{31}} + \frac{n_3(x,t)}{\tau_{32}} = \frac{n_3(x,t)}{\tau_3} \]

where \( \tau_{30}^{-1} \), \( \tau_{31}^{-1} \), \( \tau_{32}^{-1} \) represent the transition rates from level 3 to levels 0, 1, and 2, respectively. Electrons arrive at level 2 at a rate

\[ \frac{n_3(x,t)}{\tau_{32}} \]

and are transferred to the lower levels at a rate

\[ \frac{n_2(x,t)}{\tau_{31}} + \frac{n_2(x,t)}{\tau_{30}} = \frac{n_2(x,t)}{\tau_2}. \]

The \( 2 \rightarrow 1 \) transition is the lasing transition so the population at level 2 is depleted due to induced emission by the quantity

\[ \left[ n_2(x,t) - \frac{g_2}{g_1}n_1(x,t) \right] \int_0^\infty \sigma(\lambda)F(x,t;\lambda)d\lambda. \]

In this expression \( g_1 \) and \( g_2 \) are the degeneracies of electronic levels 1 and 2, and \( F(x,t;\lambda) \) is the total photon flux in both directions.
Figure 2.5 Idealized four level model for the lasing action of Titanium in Sapphire. Only those non-radiative transitions to or from the lasing levels are shown.
\[ F(x, t; \lambda) = F_+(x, t; \lambda) + F_-(x, t; \lambda). \]

The quantity

\[
\int_0^\infty \sigma(\lambda) F(x, t; \lambda) d\lambda
\]

represents the probability that stimulated emission will occur. Here, \( \sigma(\lambda) \) is the emission cross section. Electrons are transferred to level 1 at the rate

\[
\frac{n_3(x, t)}{\tau_{31}} + \frac{n_2(x, t)}{\tau_{21}} + \left[ \frac{n_2(x, t)}{g_2} - \frac{n_1(x, t)}{g_1} \right] \int_0^\infty \sigma(\lambda) F(x, t; \lambda) d\lambda
\]

by transitions from the upper levels and by stimulated emission. The electrons then undergo a decay to level 0 at a rate

\[
\frac{n_1(x, t)}{\tau_1} = \frac{n_1(x, t)}{\tau_{10}}.
\]

Finally, the ground state is populated via transitions from levels 3, 2, and 1 and electrons are lost due to pumping. Hence, the rate equations describing the rates of change of the population densities in the active medium for each of the electronic levels are given below.

\[
\begin{align*}
\frac{\partial n_3(x, t)}{\partial t} &= W_{03}(x, t)n_0(x, t) - \frac{n_3(x, t)}{\tau_3} \\
\frac{\partial n_2(x, t)}{\partial t} &= \frac{n_3(x, t)}{\tau_{32}} - \frac{n_2(x, t)}{\tau_2}
\end{align*}
\]
\[
\frac{\partial n_1(x,t)}{\partial t} = \frac{n_3(x,t)}{\tau_{31}} + \frac{n_2(x,t)}{\tau_{21}} - \frac{n_1(x,t)}{\tau_1} + [n_2(x,t) - \frac{g_2}{g_1} n_1(x,t)] \int_0^\infty \sigma(\lambda) F(x,t;\lambda) d\lambda
\]

(2.1.1)

\[
\frac{\partial n_0(x,t)}{\partial t} = \frac{n_3(x,t)}{\tau_{30}} + \frac{n_2(x,t)}{\tau_{21}} + \frac{n_1(x,t)}{\tau_1} - W_{03}(x,t)n_0(x,t)
\]

The quantities \( \tau_i \) and \( \tau_{ij}^{-1} \) represent the lifetime for the \( i^{th} \) level and the transition rate from level \( i \) to level \( j \), respectively. They are related by

\[
\tau_i^{-1} = \sum_{j=0}^{i-1} \tau_{ij}^{-1} \quad i = 1, 2, 3.
\]

The population densities are constrained by the relation

\[
n_T = n_0(x,t) + n_1(x,t) + n_2(x,t) + n_3(x,t)
\]

(2.1.2)

where \( n_T \) is the concentration of optically active ions, assumed to be homogeneous throughout the crystal and independent of time. Indeed, equations (2.1.1) give

\[
\frac{\partial n_0}{\partial t} + \frac{\partial n_1}{\partial t} + \frac{\partial n_2}{\partial t} + \frac{\partial n_3}{\partial t} = 0.
\]

The right hand sides of the equations (2.1.1) account for the changes in the population densities in each of the electronic levels due to both radiative and non-radiative transitions.
The population rate equations can be simplified by making certain assumptions that describe the physical processes in a four-level laser system. The radiationless transition from the pumped level \( n_3 \) to the upper lasing level \( n_2 \) is fast so the electrons stay at the pumped level for a very short time. Consequently the electrons do not accumulate at the pumped level. Therefore the assumptions

\[
\tau_3 \ll \tau_1, \tau_2, \quad \frac{\partial n_3}{\partial t} = 0, \quad n_3 \ll n_i \text{ for } i = 0, 1, 2, \quad \text{and} \quad n_3 \ll n_T.
\]

can be made. Under these assumptions, \( n_3(x, t) \) can be replaced by \( W_{03}(x, t)n_0(x, t)\tau_3 \). In addition, the constraint condition (2.1.2) can be approximated by

\[
n_T \approx n_0(x, t) + n_1(x, t) + n_2(x, t).
\]

It is further assumed that the dominant decay path from the pumped level is the \( 3 \rightarrow 2 \) path so that \( \tau_{32} \ll \tau_{31} \) and \( \tau_{30} \). Then

\[
\frac{1}{\tau_3} = \frac{1}{\tau_{30}} + \frac{1}{\tau_{31}} + \frac{1}{\tau_{32}} \approx \frac{1}{\tau_{32}}
\]

so that \( \tau_3 \approx \tau_{32} \).

The pumped electrons are transferred to the second level at a rate

\[
\frac{n_3(x, t)}{\tau_{32}} \approx W_{03}(x, t)n_0(x, t)\frac{\tau_3}{\tau_{32}} = W(x, t)n_0(x, t)
\]

where
The quantity \( \eta = \frac{\tau_3}{\tau_{32}} \), called the branching ratio, is expected to be close to unity.

The total decay from level 2 has both radiative and non-radiative contributions which are related by

\[
\frac{1}{\tau_2} = \frac{1}{\tau_{f_2}} + \frac{1}{\tau_{NR}}
\]

where \( \tau_{f_2} \) is the fluorescence lifetime for the material and \( \tau_{NR}^{-1} \) is the rate for the non-radiative transition from level 2. With the assumption that \( \tau_2 \approx \tau_{f_2} \) and those described above, the equations for the population densities are reduced to

\[
\frac{\partial n_2(x,t)}{\partial t} = W(x,t)n_0(x,t) - \frac{n_2(x,t)}{\tau_2} - \frac{g_2}{g_1} n_2(x,t) \int_0^\infty \sigma(\lambda) F(x,t;\lambda) d\lambda
\]

\[
\frac{\partial n_1(x,t)}{\partial t} = \frac{n_2(x,t)}{\tau_{f_2}} - \frac{n_1(x,t)}{\tau_1} + \frac{g_2}{g_1} n_1(x,t) \int_0^\infty \sigma(\lambda) F(x,t;\lambda) d\lambda
\]

(2.1.3)
subject to the constraint

\[ n_T = n_0(x,t) + n_1(x,t) + n_2(x,t). \]

Equations (2.1.3) can be rewritten in terms of the population inversion \( n(x,t) \)
which is defined by

\[ n(x,t) = n_2(x,t) - \frac{g^2}{g_1} n_1(x,t). \]

Then,

\[ \frac{\partial n_1(x,t)}{\partial t} = \frac{n(x,t)}{\tau_{\text{fe}}} + \left[ \frac{g^2}{g_1} \cdot \frac{1}{\tau_{\text{fe}}} - \frac{1}{\tau_1} \right] n_1(x,t) + n(x,t) \int_0^\infty \sigma(\lambda) F(x,t;\lambda) \, d\lambda \]

and

\[ \frac{\partial n_2(x,t)}{\partial t} = \frac{\partial n_1(x,t)}{\partial t} - \frac{g_2}{g_1} \frac{\partial n_1(x,t)}{\partial t} \]

\[ = W(x,t) n_0(x,t) - n(x,t) \int_0^\infty \sigma(\lambda) F(x,t;\lambda) \, d\lambda - \left[ n(x,t) + \frac{g_2}{g_1} n_1(x,t) \right] \frac{1}{\tau_2} \]

\[ - \frac{g^2}{g_1} n_1(x,t) + \left[ \frac{g^2}{g_1} \cdot \frac{1}{\tau_{\text{fe}}} - \frac{1}{\tau_1} \right] n_1(x,t) + n(x,t) \int_0^\infty \sigma(\lambda) F(x,t;\lambda) \, d\lambda \]

\[ = W(x,t) n_0(x,t) + \left[ - \frac{1}{\tau_2} - \frac{g^2}{g_1} \cdot \frac{1}{\tau_{\text{fe}}} \right] n(x,t) \]

\[ + \left[ - \frac{g^2}{g_1} \cdot \frac{1}{\tau_2} - \frac{g^2}{g_1} \cdot \frac{g^2}{g_1} \cdot \frac{1}{\tau_{\text{fe}}} - \frac{1}{\tau_1} \right] n_1(x,t) \]

\[ - \left[ 1 + \frac{g^2}{g_1} \right] n(x,t) \int_0^\infty \sigma(\lambda) F(x,t;\lambda) \, d\lambda. \]

Now let \( \gamma = 1 + \frac{g^2}{g_1} \) and define
Then, in terms of the population inversion, equations (2.1.3) can be written as

\[
c_{11} = \frac{1}{\tau_2} - \frac{\gamma - 1}{\tau \tau_2}; \quad c_{12} = (1 - \gamma) \left[ \frac{1}{\tau_2} + \frac{(\gamma - 1)}{\tau \tau_2} - \frac{1}{\tau_1} \right]; \\
c_{21} = \frac{1}{\tau \tau_2}; \quad c_{22} = \frac{\gamma - 1}{\tau \tau_2} - \frac{1}{\tau_1}. \quad (2.1.4)
\]

This is the form of the population rate equations that will be used in the subsequent discussion.

2.2 The Photon Equations

In this section the equations that describe the changes in the photon flux as it passes through an active medium are discussed. Consider a one-dimensional flux distribution of photons where the x-axis is taken to be defined by the propagation of the laser beam. At any position in the crystal there will be a flux

\[
\frac{\partial n(x,t)}{\partial t} = c_{11} n(x,t) + c_{12} n_1(x,t) - \gamma n(x,t) \int_0^\infty \sigma(\lambda) F(x,t;\lambda) d\lambda \\
+ W(x,t) n_0(x,t) \quad (2.1.5)
\]

\[
\frac{\partial n_1(x,t)}{\partial t} = c_{21} n(x,t) + c_{22} n_1(x,t) + n(x,t) \int_0^\infty \sigma(\lambda) F(x,t;\lambda) d\lambda
\]

where

\[
n_0(x,t) = n_T - \gamma n_1(x,t) - n(x,t).
\]

This is the form of the population rate equations that will be used in the subsequent discussion.

2.2 The Photon Equations

In this section the equations that describe the changes in the photon flux as it passes through an active medium are discussed. Consider a one-dimensional flux distribution of photons where the x-axis is taken to be defined by the propagation of the laser beam. At any position in the crystal there will be a flux
in both directions at all wavelengths denoted by \( F_+(x,t;\lambda) \) and \( F_-(x,t;\lambda) \) for right and left traveling photons, respectively. The active medium is assumed to be homogeneous with refractive index \( n \).

As photons pass through an active medium, their number is changed by interaction with the optically active ions. The equations that account for the increases in photon flux due to spontaneous and stimulated emission and decreases resulting from absorption and scattering losses are obtained as in [23] and are given below.

\[
\begin{align*}
\frac{1}{v} \frac{\partial F_+(x,t;\lambda)}{\partial t} + \frac{\partial F_+(x,t;\lambda)}{\partial x} &= [\sigma(\lambda)n(x,t) - \alpha(\lambda)]F_+(x,t;\lambda) \\
&+ S_+(x,t;\lambda) \\
\frac{1}{v} \frac{\partial F_-(x,t;\lambda)}{\partial t} - \frac{\partial F_-(x,t;\lambda)}{\partial x} &= [\sigma(\lambda)n(x,t) - \alpha(\lambda)]F_-(x,t;\lambda) \\
&+ S_-(x,t;\lambda)
\end{align*}
\]

(2.2.1)

The left hand sides of equations (2.2.1) represent the net spatial and temporal change in the corresponding photon flux as it propagates through the active medium with velocity \( v = \frac{c}{n} \), where \( c \) is the speed of light in a vacuum. On the right hand sides, the changes in the photon fluxes brought about by interaction with the material are described. These changes have three components, namely, the contribution to photon fluxes due to spontaneous emission given by \( S_\pm(x,t;\lambda) \), increases resulting from stimulated emission described by \( \sigma(\lambda)n(x,t)F_\pm(x,t;\lambda) \) and decreases to photon fluxes from absorption and scattering losses given by \( \alpha(\lambda)F_\pm(x,t;\lambda) \). \( \alpha(\lambda) \) is the absorption coefficient for the material and is a function of wavelength.
The spontaneous emission functions $S_{\pm}(x, t; \lambda)$ can be expressed as a product of three factors

$$S_{\pm}(x, t; \lambda) = \frac{n_2(x, t)}{\tau_{fl}} \cdot (8\pi^2 c \sigma(\lambda) \lambda^4 \tau_{fl}) \cdot \Omega_{\pm}(x). \quad (2.2.2)$$

The first factor represents the rate of fluorescent decay from the upper laser level and can be rewritten in terms of the population inversion

$$\frac{n_2(x, t)}{\tau_{fl}} = \frac{n(x, t) + g_2 n_1(x, t)}{g_1} \frac{1}{\tau_{fl}}.$$

The second factor in (2.2.2) gives the probability for emission at wavelength $\lambda$. The third factor is a geometric factor which gives the fraction of emitted photons which can be considered to contribute to the right or left photon flux. The geometrical term depends on the length $\ell$ of the crystal and some limiting radius $r$ which defines the field of view of the emitting ion. The latter quantity is taken to be the radius of a cylindrical crystal. The geometrical factors are given by

$$\Omega_{+}(x) = \frac{1}{2} \left[ 1 - \frac{(\ell - x)}{\sqrt{(\ell - 1)^2 + r^4}} \right]$$

and

$$\Omega_{-}(x) = \frac{1}{2} \left[ 1 - \frac{x}{\sqrt{x^2 + r^4}} \right].$$

for an Aperature Limited System and by

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\[ \Omega_+(x; \lambda) = \frac{\Delta \lambda}{4\pi} \left[ \frac{n^3}{\pi W_{\text{pump}}} \lambda^2 \right] \]

for a Diffraction Limited System [6].

In the remainder of the cavity, the propagation of the right and left traveling fluxes is described by

\[ \frac{1}{c} \frac{\partial \hat{F}_+(x, t; \lambda)}{\partial t} + \frac{\partial \hat{F}_+(x, t; \lambda)}{\partial x} = 0 \]
\[ \frac{1}{c} \frac{\partial \hat{F}_-(x, t; \lambda)}{\partial t} - \frac{\partial \hat{F}_-(x, t; \lambda)}{\partial x} = 0. \]  \hspace{1cm} (2.2.3)

### 2.3 The Pump Pulse

The laser system is assumed to be end-pumped so the pumping rate has both space and time dependence. As the pump pulse moves through the crystal, the pumping photons are absorbed by the materials. The equation that describes the propagation of the pump pulse through the material is given below.

\[ \frac{1}{v} \frac{\partial W(x, t)}{\partial t} + \frac{\partial W(x, t)}{\partial t} = -\sigma_{ab} n_e(x, t) W(x, t) \]  \hspace{1cm} (2.3.1)

Here, \( \sigma_{ab} \) is the absorption cross section at the pumping wavelength. The expression on the right hand side of (2.3.1) accounts for the absorption of the
pump pulse by the ground state electrons. In terms of the population inversion and the lower laser level population density, (2.3.1) becomes

\[
\frac{1}{v} \frac{\partial W(x,t)}{\partial t} + \frac{\partial W(x,t)}{\partial x} = -\sigma_{ab}[n_T - n(x,t) - \gamma n_1(x,t)]W(x,t) \tag{2.3.2}
\]

where \( n_T \) and \( \gamma \) are as described previously in Section 2.1

### 2.4 Normalization

The coupled system of partial differential equations defined by equations (2.1.5) and (2.2.1) constitute the spatial and temporal model of the dynamics of the active medium of the laser. Equations (2.2.3) model the behavior of right and left traveling photon fluxes in the remainder of the cavity.

The dependent variables are normalized by a quantity \( n_{\text{norm}} \). Let

\[
N(x,t) = \frac{n(x,t)}{n_{\text{norm}}}
\]

\[
N_1(x,t) = \frac{n_1(x,t)}{n_{\text{norm}}}
\]

\[
N_0(x,t) = \frac{n_0(x,t)}{n_{\text{norm}}} = \frac{n_T}{n_{\text{norm}}} - N(x,t) - \gamma N_1(x,t)
\]

be the normalized population concentrations in the crystal.

The photon fluxes are related to the photon concentrations \( \phi_{\pm}(x,t;\lambda) \), by

\[
v \phi_{\pm}(x,t;\lambda) = F_{\pm}(x,t;\lambda)
\]
in the active medium and by

\[ c\hat{\Phi}_\pm(x, t; \lambda) = \dot{\Phi}_\pm(x, t; \lambda) \]

in the remainder of the cavity. These quantities are also normalized by \( n_{\text{norm}} \), so let

\[ \hat{\Phi}_\pm(x, t; \lambda) = \frac{\phi_\pm(x, t; \lambda)}{n_{\text{norm}}} \]

be the normalized photon concentration in the active region and

\[ \hat{\Phi}_\pm(x, t; \lambda) = \frac{\phi_\pm(x, t; \lambda)}{n_{\text{norm}}} \]

be the normalized photon concentrations in the rest of the cavity. Finally, the time variable is scaled to a standard time interval, \( \tau_{\text{norm}} \).

With these changes, the equations describing the dynamics in the active region become

\[
\begin{align*}
\frac{\partial N(x, t)}{\partial t} &= c_{11}N(x, t) + c_{12}N_1(x, t) - \gamma \beta N(x, t) \int_0^\infty \sigma(\lambda)\Phi(x, t; \lambda) d\lambda \\
&\quad + W(x, t) \left[ \frac{n_T}{n_{\text{norm}}} - N(x, t) - \gamma N_1(x, t) \right] \\
\frac{\partial N_1(x, t)}{\partial t} &= c_{21}N(x, t) + c_{22}N_1(x, t) + \beta N(x, t) \int_0^\infty \sigma(\lambda)\Phi(x, t; \lambda) d\lambda \\
\frac{1}{v \cdot \tau_{\text{norm}}} \frac{\partial \Phi_\pm(x, t; \lambda)}{\partial t} \pm \frac{\partial \Phi_\pm(x, t; \lambda)}{\partial x} &= [n_{\text{norm}}\sigma(\lambda)N(x, t)] \\
&\quad - \alpha(\lambda)|\Phi_\pm(x, t; \lambda) \\
&\quad + \frac{1}{v \cdot \tau_{\text{norm}}} S_\pm(x, t; \lambda) \quad (2.4.1)
\end{align*}
\]

where
\[
\beta = n_{\text{norm}} \tau_{\text{norm}} v
\]

\[
\Phi(x, t; \lambda) = \{\Phi_+(x, t; \lambda) + \Phi_-(x, t; \lambda)\}
\]

and all other quantities are as described previously. The equation describing the propagation of the pump pulse becomes

\[
\frac{1}{v \cdot \tau_{\text{norm}}} \frac{\partial W(x, t)}{\partial t} + \frac{\partial W(x, t)}{\partial x} = -\sigma_{\text{abs}} n_{\text{norm}} \left[ \frac{n_T}{n_{\text{norm}}} - N(x, t) - \gamma N_1(x, t) \right] W(x, t)
\]

(2.4.2)

Equations (2.4.1) and (2.4.2) are subject to the constraint

\[
\frac{n_T}{n_{\text{norm}}} = N_0(x, t) + N(x, t) + \gamma N_1(x, t).
\]

In the remainder of the cavity the equations describing the propagation of the normalized photon concentrations are

\[
\frac{1}{c \cdot \tau_{\text{norm}}} \frac{\partial \Phi_\pm(x, t; \lambda)}{\partial t} \pm \frac{\partial \Phi_\pm(x, t; \lambda)}{\partial x} = 0
\]

(2.4.3)
2.5 Initial and Boundary Conditions

The initial conditions for the normalized population concentrations in the active region are given below

\[ N(x, 0) = a_1(x) \]
\[ N_1(x, 0) = a_2(x) \]

and in the entire cavity, the normalized photon concentrations have the initial profile

\[ \Phi_\pm(x, 0; \lambda) = f_\pm(x; \lambda). \]

The pump pulse is initially given by

\[ W(x, 0) = W(x). \]

If the laser starts from a quiescent state, the upper level population in the crystal and the photons in the entire cavity are negligible, so the quantities \( a_1(x), a_2(x), f_\pm(x; \lambda) \) would in that case be taken to be zero.

Since the photon concentrations and pump pulse have spatial as well as time derivatives, boundary conditions must also be specified. The boundary conditions describe the optical environment of the laser. The cavity configuration considered here is a simple ring cavity in which the only optical elements are mirrors and the crystal. A schematic diagram is presented in Figure 2.6.
The pump pulse is introduced at the position \( x = 0 \) and the positive direction is taken to be the clockwise direction in the cavity. The length of the crystal is \( \ell \) and the cavity has length \( L \). Mirrors are located at positions \( L_1, L_2, \) and \( L_3 \) and have corresponding reflection and transmission coefficients \( R_i \) and \( T_i \) for \( i = 1, 2, 3 \). The crystal has surface reflection and transmission parameters, \( T_s \) and \( R_s \), called the Fresnel coefficients for the material. \( R_s \) is given by

\[
R_s = \frac{(n - 1)^2}{(n + 1)^2}
\]

and so

\[
T_s = 1 - R_s
\]

\[
= \frac{4n}{(n + 1)^2}
\]

where \( n \) is the index of refraction.

The mirror at \( x = L_1 \) is taken to be an output coupler and at this position an injection signal, \( I_\pm(t; \lambda) \), may be introduced in the system in either direction.
Figure 2.6 A schematic diagram of a simple ring laser cavity. The laser rod has length $l$ and is end pumped (at $x=0$). Mirrors are located at $x=L_1$, $x = L - 2$, and $x = L_3$. The clockwise direction is taken to be positive.
Boundary conditions that describe the optical configuration described above are as follows

\[
\hat{\Phi}_+(0^+, t; \lambda) = T\hat{\Phi}_+(L^-, t; \lambda) + R\hat{\Phi}_-(0^+, t; \lambda)
\]

\[
\hat{\Phi}_-(L^-, t; \lambda) = T\hat{\Phi}_-(0^+, t; \lambda) + R\hat{\Phi}_+(L^-, t; \lambda)
\]

\[
\hat{\Phi}_-(\ell^-, t; \lambda) = T\hat{\Phi}_-(\ell^+, t; \lambda) + R\hat{\Phi}_+(\ell^-, t; \lambda)
\]

\[
\hat{\Phi}_+(\ell^+, t; \lambda) = T\hat{\Phi}_+(\ell^-, t; \lambda) + R\hat{\Phi}_-(\ell^+, t; \lambda)
\]

\[
\hat{\Phi}_+(L_1^+, t; \lambda) = R_1\hat{\Phi}_+ (L_1^-, t; \lambda) + I_+ (t; \lambda)
\]

\[
\hat{\Phi}_-(L_1^-, t; \lambda) = R_1\hat{\Phi}_- (L_1^+, t; \lambda) + I_-(t; \lambda)
\]

\[
\hat{\Phi}_+(L_2^+, t; \lambda) = R_2\hat{\Phi}_+ (L_2^-, t; \lambda)
\]

\[
\hat{\Phi}_-(L_2^-, t; \lambda) = R_2\hat{\Phi}_- (L_2^+, t; \lambda)
\]

\[
\hat{\Phi}_+(L_3^+, t; \lambda) = R_3\hat{\Phi}_+ (L_3^-, t; \lambda)
\]

\[
\hat{\Phi}_-(L_3^-, t; \lambda) = R_3\hat{\Phi}_- (L_3^+, t; \lambda).
\]

The initial and boundary value problem defined by the above conditions and equations (2.4.1), (2.4.2), (2.4.3) are the subject of the present study.
Chapter 3

Analysis of the Spatially Averaged Model

3.1 The Spatially Averaged Model

In this section, the rate equations (2.4.1) are simplified by a heuristic derivation of a model in which the population and photon concentrations are averaged over the spatial variable. For simplicity, only those photons at a particular wavelength, $\lambda$, will be considered. Without loss of generality, the pump pulse is assumed to have a specified temporal dependence, $W_p(t)$. It is further assumed that $n_{\text{norm}} = n_T$ and that the material is homogeneous so that emission and absorption properties of the materials do not depend on the spatial variable. As in [23] the average is taken over the length of the active medium.

Incorporating these assumptions into the population equations, they are formally integrated over the length of the active medium to obtain
\[
\frac{d}{dt} \left\{ \frac{1}{\ell} \int_0^\ell N(x,t)dx \right\} = c_{11} \left\{ \frac{1}{\ell} \int_0^\ell N(x,t)dx \right\} \\
+ c_{12} \left\{ \frac{1}{\ell} \int_0^\ell N_1(x,t)dx \right\} \\
- \gamma \beta \sigma \left\{ \frac{1}{\ell} \int_0^\ell N(x,t)\Phi(x,t)dx \right\} \\
+ W_p(t) \left\{ \frac{1}{\ell} \int_0^\ell [1 - N(x,t) - \alpha N_1(x,t)]dx \right\}
\]
\[
\frac{d}{dt} \left\{ \frac{1}{\ell} \int_0^\ell N_1(x,t)dx \right\} = c_{21} \left\{ \frac{1}{\ell} \int_0^\ell N(x,t)dx \right\} \\
+ c_{22} \left\{ \frac{1}{\ell} \int_0^\ell N_1(x,t)dx \right\} \\
+ \beta \sigma \left\{ \frac{1}{\ell} \int_0^\ell N(x,t)\Phi(x,t)dx \right\}
\] (3.1.1)

where \( \sigma \) is the emission cross section at \( \bar{\lambda} \) and all other parameters are as discussed previously.

The photon concentration equations are added, giving

\[
\frac{1}{v \cdot \tau_{\text{norm}}} \frac{\partial \Phi(x,t)}{\partial t} + \frac{\partial}{\partial x} [\Phi_+(x,t) - \Phi_-(x,t)]
\]
\[
= [n_{\text{norm}} \sigma N(x,t) - \alpha] \Phi(x,t)
\]
\[
+ \frac{1}{v \cdot \tau_{\text{norm}}} \left[ S_+(x,t) + S_-(x,t) \right].
\] (3.1.2)

Here, \( \alpha \) is the absorption coefficient at \( \bar{\lambda} \). Integrating (3.1.2) over the length of the active medium yields
The term \( [S_+(x,t) + S_-(x,t)] \) is the contribution to the photon concentration in the laser beam coming from spontaneous emission of the excited electron. The spatial average \( \frac{1}{\ell} \int_0^\ell [S_+(x,t) + S_-(x,t)] dx \) is expressed as a fraction \( S_{p0} \) of the total fluorescence decay rate, \( \tau_{ft} \), so that

\[
\frac{1}{\ell} \int_0^\ell [S_+(x,t) + S_-(x,t)] dx = \frac{S_{p0}}{\tau_{ft}} \left\{ \frac{1}{\ell} \int_0^\ell N(x,t)dx + \frac{(\gamma - 1)}{\ell} \int_0^\ell N_1(x,t)dx \right\} = c_{s1} \left\{ \frac{1}{\ell} \int_0^\ell N(x,t)dx \right\} + c_{s2} \left\{ \frac{1}{\ell} \int_0^\ell N_1(x,t)dx \right\}
\]

where \( c_{s1} = \frac{S_{p0}}{\tau_{ft}}, \quad c_{s2} = \frac{(\gamma - 1)S_{p0}}{\tau_{ft}} \), and \( S_{p0} \) depends on the geometry of the laser cavity and on the fluorescence spectrum of the lasing ion [5].

The following approximation is also made [23]
Boundary conditions that account for the presence of the mirrors in the ring laser cavity are also incorporated:

\[
\Phi_-(\ell^-, t) = R_1 R_2 R_3 T_s \Phi_-(\ell^+, t)
\]
\[
\Phi_+(0^+, t) = R_1 R_2 R_3 T_s \Phi_+(L^-, t).
\]

Equation (3.1.3) can now be written as

\[
\frac{d}{dt} \left\{ \frac{1}{\ell} \int_0^\ell N(x,t) dx \right\} = \frac{-\nu \cdot \tau_{\text{norm}}}{\ell} \left\{ \left[ \Phi_+(\ell^-, t) - R_1 R_2 R_3 T_s \Phi_-(\ell^+, t) \right] + \left[ \Phi_-(0^+, t) - R_1 R_2 R_3 T_s \Phi_+(L^-, t) \right] \right\}
\]
\[
- \alpha \nu \cdot \tau_{\text{norm}} \left\{ \frac{1}{\ell} \int_0^\ell \Phi(x,t) dx \right\} + \beta \sigma \left\{ \frac{1}{\ell} \int_0^\ell N(x,t) dx \right\} \left\{ \frac{1}{\ell} \int_0^\ell \Phi(x,t) dx \right\} + c_{31} \left\{ \frac{1}{\ell} \int_0^\ell N(x,t) dx \right\} + c_{32} \left\{ \frac{1}{\ell} \int_0^\ell N(x,t) dx \right\}.
\]

The first expression on the right hand side can be considered in the following way. In terms of a standard time scale, \( \tau_{\text{norm}} \), the photon concentration in each direction is related to the corresponding photon flux by
\[ v \cdot \tau_{\text{norm}} \Phi_{\pm}(x,t) = F_{\pm}(x,t) \]

so that

\[ v \cdot \tau_{\text{norm}} \left\{ \left[ \Phi_+ (\ell^-,t) - R_1 R_2 R_3 T_3 \Phi_- (\ell^+,t) \right] 
+ \left[ \Phi_- (0^+,t) - R_1 R_2 R_3 T_3 \Phi_+ (L^-,t) \right] \right\} = [F_+ (\ell,t) - R_1 R_2 R_3 T_3 F_- (\ell,t)] 
+ [F_- (0,t) - R_1 R_2 R_3 T_3 F_+ (0,t)] = F_{\text{out}}(t) \]

where \( F_{\text{out}}(t) \) is the total flux coming from the crystal. Now, \( F_{\text{out}}(t) \) may be written [23]

\[ F_{\text{out}}(t) = \ell \left\{ \frac{1}{\ell} \int_0^\ell F(x,t) \, dx \right\} \alpha_{\text{rad}} \]

where \( \alpha_{\text{rad}} \) is the coefficient giving the fraction of photons lost due to radiation from the crystal.

Thus equation (3.1.3) can rewritten

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\[
\frac{d}{dt} \left\{ \frac{1}{L} \int_{0}^{L} \Phi(x,t)dx \right\} = c_{31} \left\{ \frac{1}{L} \int_{0}^{L} N(x,t)dx \right\} \\
+ c_{32} \left\{ \frac{1}{L} \int_{0}^{L} N(x,t)dx \right\} \\
+ c_{33} \left\{ \frac{1}{L} \int_{0}^{L} \Phi(x,t)dx \right\} \\
+ \beta \sigma \left\{ \frac{1}{L} \int_{0}^{L} \Phi(x,t)dx \right\} \left\{ \frac{1}{L} \int_{0}^{L} N_{1}(x,t)dx \right\}
\]

where \( c_{33} = -v \cdot \tau_{\text{norm}}[\alpha_{\text{rad}} + \alpha] \) represents the fraction of photons lost by all mechanisms in the cavity including losses due to absorption, scattering, and reflection. Since a photon will have some average lifetime in the cavity before being lost by the above mechanisms, it is convenient to express the losses in terms of the cavity lifetime, \( \tau_{c} \), where

\[
c_{33} = -\frac{1}{\tau_{c}}
\]

and \( \tau_{c} = \frac{2L}{c \tau_{\text{norm}}} [2\alpha \ell + \ell_{m} - \ln(R_{1}R_{2}R_{3}T_{D})]^{-1} \) ([12], [6]) for the ring laser cavity described in Chapter 2. Here, \( \ell_{m} \) represents miscellaneous losses due to scattering and absorption by the optical elements in the cavity. All other parameters are as previously defined.

If the laser system is injection seeded, then an accounting for the photons which are introduced to the system in this way must be made. In the spatial and temporal model, this accounting was made by introducing photons at an appropriate boundary. For the spatially averaged model, the contribution from
injection seeding is assumed to have a specified temporal distribution, \( I(t) \), that gives the number of photons per time unit introduced by an injected signal.

Now let

\[
\begin{align*}
n(t) &= \frac{1}{L} \int_0^t N(x,t) \, dx \\
n_1(t) &= \frac{1}{L} \int_0^t N_1(x,t) \, dx \\
\phi(t) &= \frac{1}{L} \int_0^t \Phi(x,t) \, dx.
\end{align*}
\]

Then the spatial and temporal model given in (2.4.1) is reduced to the nonlinear system of ordinary differential equations

\[
\begin{align*}
\frac{dn(t)}{dt} &= c_{11}n(t) + c_{12}n_1(t) - \gamma \beta \sigma n(t)\phi(t) + W_p(t)[1 - n(t) - \gamma n_1(t)] \\
\frac{dn_1(t)}{dt} &= c_{21}n(t) + c_{22}n_1(t) + \beta \sigma n(t)\phi(t) \\
\frac{d\phi(t)}{dt} &= c_{31}n(t) + c_{32}n_1(t) + c_{33}\phi(t) + \beta \sigma n(t)\phi(t) + I(t)
\end{align*}
\]  

(3.1.4)

The remainder of this chapter deals with the qualitative and numerical analysis of the initial value problem defined by (3.1.4) subject to the initial conditions

\[
\begin{align*}
n(0) &= a_1 \\
n_1(0) &= a_2 \\
\phi(0) &= a_3 \quad \text{where} \quad a_i \geq 0 \quad (i = 1, 2, 3).
\end{align*}
\]  

(3.1.5)
3.2 Qualitative Properties of the Solutions

This section establishes a number of qualitative properties of the solutions to the system (3.1.5). Throughout this section the material parameters are assumed to satisfy

\[ 0 < \tau_1 << \tau_2 < \tau_{f*} \]
\[ 0 \leq S_p \]
\[ 0 < \tau_e \]
\[ 1 < \gamma. \]

**Theorem 1** Let \( W_p(t) > 0 \) and \( I(t) \geq 0 \) be continuous functions on \([0, +\infty)\).

If \( n(0) \geq 0, n_1(0) \geq 0, \) and \( \phi(0) \geq 0 \), then

(i) if \( n(0) + \gamma n_1(0) < 1 \), then \( n(t) + \gamma n_1(t) < 1 \) for all \( t \);

(ii) if \( n(0) + \gamma n_1(0) \geq 1 \), then there exists \( T \) such that \( n(t) + \gamma n_1(t) < 1 \) for all \( t > T \);

(iii) \( n(t) \geq 0, n_1(t) \geq 0, \) and \( \phi(t) \geq 0 \) for all \( t \).

**Theorem 2** Let \( W_p(t) \) and \( I(t) \) satisfy the conditions in Theorem 3.1.

(i) If \( W_p(t) \) and \( I(t) \) are integrable on \([0, +\infty)\), then \( n(t), n_1(t), \) and \( \phi(t) \) are integrable on \([0, +\infty)\).
(ii) If $W_p(t)$ and $I(t)$ are bounded on $[0, +\infty)$, then $n(t), n_1(t)$ and $\phi(t)$ are bounded on $[0, +\infty)$.

Proof of Theorem 1:
(i) Since $n(0) + \gamma n_1(0) < 1$, then by continuity, $n(t) + \gamma n_1(t) < 1$ for $t$ close to zero. Suppose there exists $T$ such that $n(T) + \gamma n_1(T) = 1$ and $n(t) + \gamma n_1(t) < 1$ for $t < T$.

Then, at $t = T$,

$$\frac{d}{dt}(n + \gamma n_1) = (c_{11} + \gamma c_{21})n + (c_{12} + \gamma c_{22})n_1 > 0. \quad (3.2.1)$$

Suppose $n(T) \geq 0$ and $n_1(T) \geq 0$. Now $n(T) + \gamma n_1(T) = 1$ so not both $n(T)$ and $n_1(T)$ are zero. Also, $c_{11} + \gamma c_{21} < 0$ and $c_{12} + \gamma c_{22} < 0$.

So, at $t = T$,

$$\frac{d}{dt}(n + \gamma n_1) < 0$$

which is a contradiction to (3.2.1).

Now suppose that $n_1(T) < 0$. Since $n_1(0) \geq 0$, there exists $T_1 < T$ such that $n_1(T_1) = 0$ and $n_1(t) \geq 0$ for $0 \leq t \leq T_1$.

Then, at $t = T_1$,

$$\frac{dn_1}{dt} = c_{21}n + \beta \sigma n\phi$$

$$= (c_{21} + \beta \sigma \phi)n$$

$$\leq 0$$
Suppose $n(T_1) < 0$ and $c_{21} + \beta \sigma \phi(T_1) \geq 0$. Then there exists $T_2 < T_1$, such that $n(T_2) = 0$ and $n(t) \geq 0$ for $0 \leq t \leq T_2$. At $t = T_2$,

$$\frac{dn}{dt} = c_{12} n_1(T_2) + W_p(T_2)(1 - \gamma n_1(T_2))$$

$$> 0$$

since $c_{12} > 0, n_1(T_2) \geq 0, W_p(T_2) > 0$, and $1 - \gamma n_1(T_2) > 0$. Hence $n$ increases to the right of $T_2$ and so $n(t)$ cannot be negative for $0 \leq t \leq T_1$. Therefore, $n(T_1) \geq 0$.

Now suppose $c_{21} + \beta \sigma \phi(T_1) \leq 0$. Since $c_{21} > 0$ and $\beta \sigma > 0$, it follows that $\phi(T_1) < 0$. Then there exists $T_3 < T_1$ such that $\phi(T_3) = 0$ and $\phi(t) \geq 0$ for $0 \leq t \leq T_3$. So, at $t = T_3$

$$\frac{d\phi}{dt} = c_{31} n + c_{32} n_1 + I(T_3)$$

$$\geq 0$$

since $I(T_3) \geq 0$ and all other quantities are nonnegative. If $I(T_3) > 0$, then $\phi(t)$ increases to the right of $T_3$ so it follows that $\phi(T_1) \geq 0$. If $I(T_3) = 0$ and at least one of $n(T_3)$ or $n_1(T_3)$ is positive, then $\phi(t)$ increases to the right of $T_3$ and so $\phi(T_1) \geq 0$. If $I(T_3) = 0$ and both $n(T_3)$ and $n_1(T_3)$ are zero, then at $t = T_3$,

$$\frac{d^2 \phi}{dt^2} = c_{31} W_p(T_3) > 0$$
since \( c_{21} > 0 \) and \( W_p(T_3) > 0 \). Therefore \( \phi \) increases to the right of \( T_3 \) and in this case it is also true that \( \phi(T_1) \geq 0 \).

Suppose \( n(T_1) = 0 \) so that at \( t = T_1 \),

\[
\frac{dn_1}{dt} = 0.
\]

Then, at \( t = T_1 \),

\[
\frac{dn}{dt} = W_p(T_1) > 0.
\]

In addition, at \( t = T_1 \),

\[
\frac{d^2n}{dt^2} = c_{21} \frac{dn}{dt} + c_{22} = \frac{dn_1}{dt} + \beta \sigma \left[ \frac{\phi}{dt} + n \frac{d\phi}{dt} \right]
\]

\[
= c_{21}W_p(T_1) + \beta \sigma \phi(T_1)W_p(T_1)
\]

\[
> 0
\]

since \( c_{21} > 0, W_p(T_1) > 0, \beta \sigma > 0 \) and \( \phi(T_1) \geq 0 \). Therefore \( n_1 \) increases to the right of \( T_1 \). It follows that \( n_1(t) \geq 0 \).

Suppose now that \( n(T) < 0 \). Choose \( T_1 < T \) such that \( n(T_1) = 0 \) and \( n(t) \geq 0 \) for \( 0 \leq t \leq T_1 \). Then at \( t = T_1 \),

\[
\frac{dn}{dt} = c_{21}n_1 + W_p(T_1)(1 - \gamma n_1)
\]

\[
> 0
\]
since $c_{21} > 0, n_1(T_1) \geq 0, W_p(T_1) > 0$ and $1 - \gamma n_1(T_1) > 0$. Hence $n$ increases to the right of $T_1$. Therefore $n(T) \geq 0$.

The above argument shows that at least one of $n(T)$ or $n_1(T)$ is strictly positive, so at $t = T$,

$$\frac{d}{dt}(n + \gamma n_1) < 0$$

which is a contradiction to (3.2.1). Thus, for all $t$, $n(t) + \gamma n_1(t) < 1$.

(ii) The proof of (i) shows that if $n(0) + \gamma n_1(0) = 1$, then $n(t) + \gamma n_1(t) < 1$ for all $t > 0$.

Suppose $n(0) + \gamma n_1(0) > 1$. Then, for $t$ close to zero,

$$\frac{d}{dt}(n + \gamma n_1) = (c_{11} + \gamma c_{21})n + (c_{12} + \gamma c_{22})n_1$$

$$< 0$$

since $c_{11} + \gamma c_{21} < 0$ and $c_{12} + \gamma c_{22} < 0$. Thus $n + \gamma n_1$ is decreasing for $t$ close to zero.

Suppose there exists $\xi \geq 1$ such that $(n + \gamma n_1) \geq \xi \geq 1$ for all $t$. Then

$$\frac{d}{dt}(n + \gamma n_1) = (c_{11} + \gamma c_{21})(n + \gamma n_1) + (c_{12} + \gamma c_{22} - \gamma(c_{11} + \gamma c_{21}))n_1$$

$$\leq (c_{11} + \gamma c_{21})\xi$$

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since $e_{12} + \gamma e_{22} - \gamma (e_{11} + \gamma e_{21}) < 0$ and $e_{11} + \gamma e_{21} < 0$. But then

$$n(t) + \gamma n_1(t) - [n(0) + \gamma n_1(0)] \leq (e_{11} + \gamma e_{21})\xi t$$

or

$$n(t) + \gamma n_1(t) \leq [n(0) + \gamma n_1(0)] + (e_{11} + \gamma e_{21})\xi t.$$

Thus, it follows that

$$\lim_{t \to +\infty} [n(t) + \gamma n_1(t)] = -\infty$$

which contradicts $n + \gamma n_1 \geq \xi \geq 1$ for all $t$.

Hence, there exists $T$ such at $n(t) + \gamma n_1(T) = 1$ and $n(t) + \gamma n_1(t) < 1$ for $t$ close to $T$. The analysis used to prove (i) now applies to show that $n(t) + \gamma n_1(t) < 1$ for all $t > T$.

(iii) The arguments in (i) show that $n$ and $n_1$ are nonnegative on an interval where $n + \gamma n_1 < 1$. Hence, if $n(0) + \gamma n_1(0) \leq 1$, then $n(t) \geq 0$ and $n_1(t) \geq 0$ for all $t$.

Suppose there exists $T$ such that $n(t) \geq 0$ and $n_1(t) \geq 0$ for $0 \leq t \leq T$ and $\phi(T) < 0$. Then choose $T_1 < T$ such that $\phi(T_1) = 0$. At $t = T_1$,}

---

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\[
\frac{d\phi}{dt} = c_{31} + c_{32}n_1 + I(T_1)
\]
\[
\geq 0
\]
since \(c_{31} \geq 0, c_{32} \geq 0, n(T_1) \geq 0, n_1(T_1) \geq 0\) and \(I(T_1) \geq 0\). Hence, \(\phi\) is nondecreasing to the right of \(T_1\), which contradicts \(\phi(T) < 0\). Thus, \(\phi(t) \geq 0\) for all \(t\).

Finally, suppose \(n(0) + \gamma n_1(0) > 1\). Choose \(T\) such that \(n(T) + \gamma n_1(T) = 1\) and \(n(t) + \gamma n_1(t) < 1\) for \(0 \leq t \leq T\). Suppose \(n(T_1) < 0\) for some \(T_1 < T\). Then there exist \(T_2 < T_1\) such that \(n(T_2) = 0\) and \(n(t) \geq 0\) for \(0 \leq t \leq T_2\). Then at \(t = T_2\)

\[
\frac{dn}{dt} = c_{12}n_1 + W_p(1 - \gamma n_1)
\]
\[
> c_{12}n_1
\]
since \(W_p(T_2) > 0\) and \(1 - \gamma n_1(T_2) > 0\).

If \(n_1(T_2) < 0\) then there exist \(T_3 < T_2\) such that \(n_1(T_3) = 0\) and \(n_1(t) \geq 0\) for \(0 \leq t \leq T_3\).

Then at \(t = T_3\)

\[
\frac{dn_1}{dt} = (c_{21} + \beta \sigma \phi)n
\]
\[
\geq 0
\]
since \( c_{21} > 0, \beta \sigma > 0, \text{ and } \phi(T_3) \geq 0. \)

If \( n(T_3) = 0 \), then at \( T_3 \)

\[
\frac{d^2 n_1}{dt^2} = c_{21} W_p(T_3) + \beta \sigma \phi(T_3) > 0.
\]

Therefore \( n_1 \) increases to the right of \( T_3 \).

If \( n(T_3) > 0 \), then at \( T_3 \)

\[
\frac{dn_1}{dt} > 0.
\]

Therefore, in either case \( n_1 \) increases to the right of \( T_3 \). Hence \( n_1(t) \geq 0 \) for all \( t \).

Thus at \( t = T_2 \)

\[
\frac{dn}{dt} > c_{21} n_1 \geq 0.
\]

Therefore \( n \) is nondecreasing to the right of \( T_2 \). Hence, \( n(t) \geq 0 \) for all \( t \).

Proof of Theorem 2:

(i) Fix \( T \) so that \( n(t) + \gamma n_1(t) < 1 \) for \( t > T \). Then for \( t > T \),
\[
\frac{d}{dt}(n + \gamma n_1) = (c_{11} + \gamma c_{21})(n + \gamma n_1)
\]
\[
+ (c_{12} + \gamma c_{22} - \gamma(c_{11} + \gamma c_{21}))n_1
\]
\[
+ W_p(1 - n - \gamma n_1)
\]
\[
\leq (c_{11} + \gamma c_{21})(n + \gamma n_1)
\]
\[
+ W_p(1 - n - \gamma n_1)
\]

since \(c_{21} + \gamma c_{22} - \gamma(c_{11} - \gamma c_{21}) < 0\) and \(n_1(t) \geq 0\). Hence

\[
\frac{d}{dt} \left( e^{\int_t^T [W_p - c_{11} - \gamma c_{21}]dt'} (n + \gamma n_1) \right) \leq e^{\int_t^T [W_p - c_{11} - \gamma c_{21}]dt'} W_p.
\]

Thus,

\[
e^{\int_t^T [W_p - c_{11} - \gamma c_{21}]dt'} (n + \gamma n_1) - [n(t) - \gamma n_1(t)]
\]
\[
\leq \int_T^t e^{\int_u^T [W_p - c_{11} - \gamma c_{21}]dt} W_p(u) du.
\]

It follows that

\[
0 \leq n + \gamma n_1 \leq e^{-\int_T^t [W_p - c_{11} - \gamma c_{21}]dt'} \int_T^t e^{\int_u^T [W_p - c_{11} - \gamma c_{21}]dt'} W_p(u) du
\]
\[
+ e^{-\int_T^t [W_p - c_{11} - \gamma c_{21}]dt'} [n(T) + \gamma n_1(T)]
\]
\[
= \int_T^t e^{-\int_u^T [W_p - c_{11} - \gamma c_{21}]dt'} W_p(u) du
\]
\[
+ e^{-\int_T^t [W_p - c_{11} - \gamma c_{21}]dt'} [n(T) + \gamma n_1(T)]
\]
Hence

\[ 0 \leq \int_T^s [n(t) + \gamma n_1(t)]dt \]
\[ \leq \int_T^s \int_T^s e - \int_u^t [W_p - c_{11} - \gamma c_{21}]dt' W_p(u)dudt \]
\[ + \int_T^s e - \int_T^s [W_p - c_{11} - \gamma c_{21}]dt' [n(T) + \gamma n_1(T)]dt. \]

Note that \( e - \int_u^t W_p dt' \leq 1 \) since \( W_p(t) > 0 \) for all \( t \).

Therefore,

\[ 0 \leq \int_T^s [n(t) + \gamma n_1(t)]dt \]
\[ \leq \int_T^s W_p(u) \int_u^s e^{[c_{11} + \gamma c_{21}]}(t - u) dudt \]
\[ + \int_T^s e^{[c_{11} - \gamma c_{21}] (t - T)} [n(T) + \gamma n_1(T)]dt \]
\[ = \int_T^s W_p(u) \left[ e^{[c_{11} + \gamma c_{21}]}(s - u) - \frac{1}{c_{11} + \gamma c_{21}} \right] du \]
\[ + [n(T) + \gamma n_1(T)] \left[ e^{[c_{11} + \gamma c_{12}]}(s - T) - \frac{1}{c_{11} + \gamma c_{21}} \right]. \]

Since \( c_{11} + \gamma c_{21} < 0 \) and \( W_p(t) \) is integrable on \([0, +\infty)\) it follows that \((n + \gamma n_1)(t)\) is integrable on \([0, +\infty)\). Both \( n(t) \) and \( n_1(t) \) are nonnegative so it follows that \( n(t) \) and \( n_1(t) \) are integrable on \([0, +\infty)\).
Now, \( E(t) = \int_t^\infty [c_{33} + \beta n] dt' \).

Then

\[
\phi(t) = E(t)\phi(T) = E(t) \left[ \int_T^t e^{\int_T^u [c_{33} + \beta n] dt'} \cdot [c_{31} n + c_{32} n_1 + I(u)] du \right]
\]

so that

\[
\phi(t) = \int_T^t \int_u^t [c_{33} + \beta n] dt'[c_{31} n + c_{32} n_1 + I(u)] du + E(t)\phi(T).
\]

Hence,

\[
0 \leq \int_T^t \phi(t) dt = \int_T^t \int_T^s \int_u^t [c_{33} + \beta n] dt'[c_{31} n + c_{32} n_1 + I(u)] du dt + \int_T^t e^{\int_T^u [c_{33} + \beta n] dt'} \phi(T) dt
\]

\[
= \int_T^t [c_{31} n + c_{32} n_1 + I(u)] \int_u^t e^{c_{33}(t-u)} + \int_t^t \beta \sigma n dt' \int_u^t du
\]

\[
+ \int_T^t e^{c_{33}(t-u)} + \int_T^t \beta \sigma n dt' \phi(T) dt.
\]

Now \( n(t) \) is integrable on \([0, +\infty)\) so

\[
\beta \sigma \int_0^t n dt' \leq \beta \sigma \int_T^t n dt' \leq \beta \sigma \int_0^\infty n dt' < \infty.
\]

In addition, \( n_1(t) \) and \( I(t) \) are integrable on \([0, +\infty)\) and \( c_{33} < 0 \). Therefore \( \phi(t) \) is integrable on \([0, +\infty)\).
(ii) Previous discussion shows that \( n(t) \) and \( n_1(t) \) are bounded on \( [0, +\infty) \).

Multiplying the last equation in (3.1.5) by \( \gamma \) and adding to the first equation yields

\[
\frac{d}{dt}(n + \gamma \phi) = (c_{11} + \gamma c_{31} - W_p)n + (c_{12} + \gamma c_{32} - \gamma W_p)n_1
\]

\[+ \gamma c_{33} \phi + W_p(t) + \gamma I(t)\]

\[= (c_{11} + \gamma c_{31} - W_p)n + (c_{12} + \gamma c_{32} - \gamma W_p)n_1\]

\[+ c_{33}(n + \gamma \phi) + W_p(t) + \gamma I(t).\]

Thus

\[
\frac{d}{dt}(n + \gamma \phi) - c_{33}(n + \gamma \phi) = (c_{11} + \gamma c_{31} - W_p)n + (c_{12} + \gamma c_{32} - \gamma W_p)n_1
\]

\[+ W_p(t) + \gamma I(t).\]

and so

\[
\frac{d}{dt} (e^{-c_{33}t}(n + \gamma \phi)) = [(c_{11} + \gamma c_{31} - W_p - c_{33})n + (c_{12} + \gamma c_{32} - \gamma W_p)n_1
\]

\[+ W_p(t) + \gamma I(t)]e^{-c_{33}t}.\]

Hence,
\[ n(t) + \gamma \phi(t) - e^{c_{33} t}(n(T) + \gamma \phi(T)) \]
\[ = e^{c_{33} t} \int_t^T \{(c_{11} + \gamma c_{31} - W_p - c_{33})n + (c_{12} + \gamma c_{32} - \gamma W_p)n_1 \]
\[ + W_p(t') + \gamma I(t')\} e^{-c_{33} t'} dt' \]
\[ = \int_t^T \{(c_{11} + \gamma c_{31} - W_p - c_{33})n + (c_{12} + \gamma c_{32} - \gamma W_p)n_1 \]
\[ + W_p(t') + \gamma I(t')\} e^{-c_{33} (t - t') dt'.} \]

Therefore

\[ n(t) + \gamma \phi(t) = \int_t^T \{(c_{11} + \gamma c_{31} - W_p - c_{33})n + (c_{12} + \gamma c_{32} - \gamma W_p)n_1 \]
\[ + W_p(t') + \gamma I(t')\} e^{-c_{33} (t - t') dt'} + e^{-c_{33} t}[n(T) + \gamma \phi(T)] \quad (3.2.2) \]

Since \( c_{33} < 0, n(t), n_1(t), W_p(t) \) and \( I(t) \) are bounded for all \( t \), the right hand side of (3.2.2) is bounded on \([0, +\infty)\). Therefore \( n(t) + \gamma \phi(t) \) is bounded. The nonnegativity of \( n(t) \) and \( \phi(t) \) gives the desired result.

### 3.3 Stability Analysis

Suppose now that \( W_p(t) \) is a positive constant, \( W_p(t) = W_p \), and for simplicity, let \( S_p_0 = 0 \). In addition, consider a system in which there is no injection signal, i.e., \( I(t) = 0 \). Then (3.1.5) can be simplified to the autonomous system
\[\begin{align*}
\frac{dn}{dt} &= c_{11} n + c_{12} n_1 - \beta \sigma \gamma n \phi + W_p (1 - n - \gamma n_1) \\
\frac{dn_1}{dt} &= c_{21} n + c_{22} n_1 + \beta \sigma n \phi \\
\frac{d\phi}{dt} &= (c_{33} + \beta \sigma n) \phi
\end{align*}\]  \hfill (3.3.1)

The equilibrium points of (3.3.1) are determined by setting the derivatives equal to zero and solving the resulting algebraic system

\[\begin{align*}
0 &= (c_{11} - W_p) n + (c_{12} - \gamma W_p) n_1 - \beta \sigma \gamma n \phi + W_p \\
0 &= c_{21} n + c_{22} n_1 + \beta \sigma n \\
0 &= (c_{33} + \beta \sigma n) \phi
\end{align*}\]  \hfill (3.3.2)

The third equation in (3.3.2) implies that

\[
\phi = 0 \text{ or } (c_{33} + \beta \sigma n) = 0.
\]

If \(\phi = 0\), then

\[\begin{align*}
(c_{11} - W_p) n + (c_{12} - \gamma W_p) n_1 + W_p &= 0 \\
(c_{21} n + c_{22} n_1 &= 0)
\end{align*}\]  \hfill (3.3.3)

Then

\[n_1 = \frac{-c_{21} n}{c_{22}}.\]
Thus one of the equilibrium points of (3.3.1) is given by

\[ n^1 = \frac{-c_{22}W_p}{c_{22}(c_{11} - W_p) - c_{21}(c_{12} - \gamma W_p)} \]

\[ n_1^1 = \frac{c_{21}n^1}{c_{22}} \]

\[ \phi^1 = 0. \] (3.3.4)

Similarly, if \( n = -\frac{c_{33}}{\beta \sigma} \),

\[ (c_{11} - W_p) \left( -\frac{c_{33}}{\beta \sigma} \right) + (c_{12} - \gamma W_p)n_1 - \beta \sigma \gamma \left( -\frac{c_{33}}{\beta \sigma} \right) \phi + W_p = 0 \]

\[ c_{21} \left( -\frac{c_{33}}{\beta \sigma} \right) + c_{22}n_1 + \beta \sigma \left( -\frac{c_{33}}{\beta \sigma} \right) \phi = 0. \] (3.3.5)

Thus

\[ n_1 = \frac{\beta \sigma c_{33} \phi + c_{21}c_{33}}{\beta \sigma c_{22}} \]

and hence the second equilibrium point is

\[ n^2 = -\frac{c_{33}}{\beta \sigma} \]

\[ n_1^2 = \frac{\beta \sigma c_{33} \phi + c_{21}c_{33}}{\beta \sigma c_{22}} \]

\[ \phi^2 = \frac{\beta \sigma c_{22}W_p - c_{33}[c_{22}(c_{11} - W_p) - c_{21}(c_{12} - \gamma W_p))]}{-\beta \sigma c_{33}[(c_{12} - \gamma W_p) + \gamma c_{22}]} \] (3.3.6)

Let \( \bar{y} = (\bar{n}, \bar{n}_1, \bar{\phi}) \) be an equilibrium solution of (3.3.1).
Definition 1 The equilibrium solution $y$ is stable if for each $\epsilon > 0$ there exists $\delta > 0$ such that if $\psi(t)$ is any solution of (3.3.1) having $\| \psi(t_0) - y \| < \delta$, then the solution $\psi(t)$ exists for all $t \geq t_0$ and $\| \psi(t) - y \| < \epsilon$ for $t \geq t_0$.

Definition 2 The equilibrium solution $y$ is asymptotically stable if it is stable and if there exists a number $\delta_0 > 0$ such that if $\psi(t)$ is any solution of (3.3.1) having $\| \psi(t_0) - y \| < \delta_0$, then $\lim_{t \to \infty} \psi(t) = y$. [2]

The following theorem due to Perron and Poincaré [2] gives a useful stability result.

Theorem 3 Given the system

$$y' = Ay + f(t, y). \tag{3.3.7}$$

Suppose all eigenvalues of $A$ have negative real parts, $f(t, y)$ and

$$\left( \frac{\partial f}{\partial y_i} \right)(t, y)(i = 1, \ldots, n)$$

are continuous in $(t, y)$ for $0 \leq t < \infty$, $|y| < k$ where $k > 0$ is a constant and

$$\lim_{|y| \to 0} \frac{|f(t, y)|}{|y|} = 0$$

uniformly with respect to $t$ on $0 \leq t < \infty$. Then the solution $y \equiv 0$ of (3.3.7) is asymptotically stable.

The stability of the equilibrium points of (3.3.1) is investigated by linearizing about the equilibrium point.
Let \( n = \bar{n} + \delta n \)

\[
\begin{align*}
n_1 &= \bar{n}_1 + \delta n_1 \\
\phi &= \bar{\phi} + \delta \phi
\end{align*}
\]

where \((\bar{n}, \bar{n}_1, \bar{\phi})\) is an equilibrium point of the system (3.3.1) and \((\delta n, \delta n_1, \delta \phi)\) represents a small perturbation from the equilibrium position.

Then

\[
\begin{align*}
\frac{d\delta n}{dt} &= c_{11}\delta n + c_{12}\delta n_1 - \beta \sigma \gamma n \delta \phi - \beta \sigma \bar{n} \delta \phi - \beta \sigma \phi \delta n \\
&\quad - W_p \delta n - W_p \gamma \delta n_1 + c_{11} \bar{n} + c_{12} \bar{n}_1 - \beta \sigma \gamma \bar{n} \bar{\phi} + W_p (1 - \bar{n} - \gamma \bar{n}_1) \\
\frac{d\delta n_1}{dt} &= c_{21}\delta n + c_{22}\delta n_1 + \beta \sigma \bar{n} \delta \phi + \beta \sigma \bar{n} \delta \phi + \beta \sigma \phi \delta n \\
&\quad + c_{21} \bar{n} + c_{22} \bar{n}_1 + \beta \sigma \bar{n} \bar{\phi} \\
\frac{d\delta \phi}{dt} &= c_{31}\delta n + c_{32}\delta n_1 + c_{33}\delta \phi + \beta \sigma \bar{n} \delta \phi + \beta \sigma \bar{n} \delta \phi \\
&\quad + \beta \sigma \bar{n} \delta n + c_{33} \bar{\phi} + \beta \sigma \bar{n} \bar{\phi}
\end{align*}
\]

which reduces to

\[
\begin{align*}
\frac{d\delta n}{dt} &= (c_{11} - \beta \sigma \bar{n} - W_p) \delta n + (c_{12} - \gamma W_p) \delta n_1 - \beta \sigma \gamma \bar{n} \delta \phi - \beta \sigma \gamma n \delta \phi \\
\frac{d\delta n_1}{dt} &= (c_{21} + \beta \sigma \bar{n}) \delta n + c_{22} \delta n_1 + \beta \sigma \bar{n} \delta \phi + \beta \sigma \bar{n} \delta \phi \\
\frac{d\delta \phi}{dt} &= \beta \sigma \bar{n} \delta n + (c_{33} + \beta \sigma \bar{n}) \delta \phi + \beta \sigma \bar{n} \delta \phi
\end{align*}
\]

written in matrix form

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\[(\delta y)' = A\delta y + g(\delta y)\]

where

\[\delta y = (\delta n, \delta n_1, \delta \phi)\]

and

\[
A = \begin{pmatrix}
    c_{11} - \beta \sigma \gamma \phi - W_p & c_{12} - \gamma W_p & -\beta \sigma \gamma \bar{n} \\
    c_{21} + \beta \phi & c_{22} & \beta \sigma \bar{n} \\
    \beta \sigma \gamma \phi & 0 & c_{33} + \beta \sigma \bar{n}
\end{pmatrix}
\]

The eigenvalues of the linearized system are found by solving the characteristic equation of the matrix \(A\) which is

\[\lambda^3 + \hat{\beta} \lambda^2 + \hat{\gamma} \lambda + \hat{\sigma} = 0\]

where

\[
\hat{\beta} = -(a_{11} + a_{22} + a_{33})
\]

\[
\hat{\gamma} = -(a_{31}a_{13} - a_{35}a_{11} - a_{33}a_{22} - a_{11}a_{22} + a_{21}a_{12})
\]

\[
\hat{\sigma} = (-a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} + a_{33}a_{11}a_{22} - a_{33}a_{21}a_{12})
\]

and \(a_{ij}\) is the \(ij^{th}\) entry in the matrix \(A\).

Since the characteristic equation is cubic, the real parts of the eigenvalues will be negative if
\[ \dot{\theta} > 0, \dot{\phi} > 0, \text{ and } \dot{\theta} \dot{\phi} - \dot{\phi} > 0. \]  

(3.3.8)

If, on the other hand, \( \dot{\theta} < 0 \) then at least one eigenvalue has a positive real part which implies that the equilibrium point is unstable.

Now consider the equilibrium point \((n^1, n^1_1, \phi^1)\) given in (3.3.4). Then

\[
\dot{\theta} = -(a_{11} + a_{22} + a_{33})
\]

\[
= -(c_{11} - W_p) - c_{22} - (c_{33} + \beta \sigma n^1)
\]

\[
< 0
\]

if \( c_{33} + \beta \sigma n^1 > 0 \). There, \((n^1, n^1_1, \phi^1)\) is unstable if the condition

\[ c_{33} + \beta \sigma n^1 > 0 \]  

(3.3.9)

is satisfied.

**Theorem 4** Suppose condition (3.3.9) holds for \((n^1, n^1_1, \phi^1)\). Then the equilibrium point \((n^2, n^2_1, \phi^2)\) is asymptotically stable.

Proof: Consider the equilibrium solution \((n^2, n^2_1, \phi^2)\) given by (3.3.6).

Then

\[
\phi^2 = \frac{\beta \sigma c_{22}W_p - c_{33}[c_{22}(c_{11} - W_p) - c_{21}(c_{12} - \gamma W_p)]}{\beta \sigma c_{33}[(c_{12} - \gamma W_p) + \gamma c_{22}]}
\]

\[
= [\beta \sigma n^1 + c_{33}] \left[ \frac{n^1}{-c_{22}W_p c_{33} \beta \sigma [(c_{12} - \gamma W_p) - \gamma c_{22}]} \right]
\]
Now look at the quantity

\[ c_{12} - \gamma W_p + \gamma c_{22} = (1 - \gamma) \left( \frac{1}{\tau_2} - \frac{(\gamma - 1)}{\tau_{fe}} - \frac{1}{\tau_1} \right) - \gamma W_p - \gamma \left( \frac{\gamma - 1}{\tau_{fe}} - \frac{1}{\tau_1} \right) \]

\[ = \frac{1 - \gamma}{\tau_2} + \frac{(\gamma - 1)^2 + \gamma - 1}{\tau_{fe}} - \frac{1}{\tau_1} - \gamma W_p + \frac{\gamma(\gamma - 1)}{\tau_{fe}} - \frac{\gamma}{\tau_1} \]

\[ = \left[ -\gamma W_p - \frac{1}{\tau_1} \right] + \left[ \frac{1 - \gamma}{\tau_2} + \frac{(\gamma - 1)^2 + \gamma(\gamma - 1)}{\tau_{fe}} \right] \]

\[ < 0 \]

since \( \tau_1 \ll \tau_{fe}, \tau_2 \).

Therefore

\[ \phi^2 = [\beta \sigma n^1 - c_{33}] \left( \frac{n^1}{-c_{22} W_p c_{33} \beta \sigma [(c_{12} - \gamma W_p) + \gamma c_{22}]} \right) \]

since (3.3.9) holds, \( c_{22} < 0, c_{33} < 0 \) and \( \beta > 0, \sigma > 0, W_p > 0 \).

The coefficients \( \hat{\beta} \) and \( \delta \) of the characteristic equation are thus

\[ \hat{\beta} = -a_{11} + a_{22} + a_{33} \]

\[ = -(a_{11} + a_{22}) \]

\[ = -c_{11} + \beta \sigma \gamma \phi^2 + W_p - c_{22} \]

\[ > 0, \quad \text{and} \]

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\[ \dot{\sigma} = -[a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13}] \]

\[ = -[\beta \sigma \phi^2][\beta \sigma n^2(c_{12} - \alpha W_p \gamma) + \beta \sigma \gamma n^2 c_{22}] \]

\[ = -(\beta \sigma)^2 \phi^2[c_{12} - \gamma W_p + \gamma c_{22}] \]

\[ > 0. \]

Since condition (3.3.9) holds, then

\[ n^1 = \frac{-c_{22}W_p}{c_{22}(c_{11} - W_p) - c_{21}(c_{12} - \gamma W_p)} \]

\[ > 0. \]

This implies that \( c_{22}(c_{11} - W_p) - c_{21}(c_{12} - \gamma W_p) > 0. \)
Finally, consider the quantity

$$\beta \gamma - \sigma = (a_{11} + a_{22})(a_{31}a_{13} - a_{11}a_{22} + a_{21}a_{12}) - a_{31}a_{23}a_{13} - a_{31}a_{12}a_{23}$$

$$= a_{11}a_{31}a_{13} - a_{11}a_{12}a_{22} + a_{11}a_{21}a_{12} + a_{22}a_{31}a_{13} - a_{11}a_{22}^2 + a_{22}a_{12}a_{21}$$

$$+ a_{31}a_{23}a_{13} - a_{31}a_{12}a_{23}$$

$$= (c_{11} - \beta \sigma \gamma \phi^2 - W_p)(\beta \sigma \phi^2)(-\beta \sigma \gamma n^2) - (c_{11} - \beta \sigma \gamma \phi^2 - W_p)^2c_{22}$$

$$+ c_{22}\beta \sigma \phi^2(-\beta \sigma \gamma n^2) + (c_{11} - \beta \sigma \gamma \phi^2 - W_p)(c_{21} + \beta \sigma \phi^2)(c_{12} - \gamma W_p)$$

$$-(c_{11} - \beta \sigma \gamma \phi^2 - W_p)c_{22}^2 + c_{22}(c_{21} - \gamma W_p)$$

$$+ \beta \sigma \phi^2\beta \sigma n^2(-\beta \sigma \gamma n^2) - \beta \sigma \phi^2(c_{12} - \gamma W_p)\beta \sigma n^2$$

$$= (c_{11} - \beta \sigma \gamma \phi^2 - W_p)[c_{21}(c_{12} - \gamma W_p) - c_{22}(c_{11} - W_p)]$$

$$+ c_{22}[c_{21}(c_{12} - \gamma W_p) - c_{22}(c_{11} - W_p)]$$

$$+ (c_{11} - W_p)\beta \sigma \phi^2[\gamma c_{22} + c_{12} - \gamma W_p]$$

$$-(\beta \sigma)^2(\phi^2)^2[\gamma c_{22} + c_{12} - \gamma W_p]$$

$$-(\beta \sigma)^2\phi^2n^2[\gamma c_{22} + c_{12} - \gamma W_p]$$

$$-(c_{11} - W_p)(\beta \sigma)^2\gamma \phi^2n^2 + (\beta \sigma)^3\gamma^2(\phi^2)^2n^2$$

$$+ c_{22}\beta \sigma \gamma \phi^2 - c_{22}\beta \sigma \phi^2\gamma W_p$$

$$> 0.$$  

Since $c_{11} - \beta \sigma \gamma \phi^2 - W_p < 0, c_{21}(c_{12} - \gamma W_p) - c_{22}(c_{11} - W_p) < 0, c_{12} - \gamma W_p + \gamma c_{22} < 0.$

Thus $\hat{\beta} > 0, \sigma > 0,$ and $\hat{\beta} \gamma - \hat{\sigma} > 0$ so the linearized eigenvalues have negative real parts. Hence, the result.

In order for laser emission to occur
\[ \frac{d\phi}{dt} > 0 \quad (3.3.10) \]

must be satisfied. Laser threshold \[12\] is characterized by steady state population inversion and lower laser population,

\[ \frac{dn}{dt} = \frac{dn_1}{dt} = 0 \]

as well as a very small photon density. Thus (3.3.10) is equivalent to the instability condition (3.3.9) for the first equilibrium position (3.3.3). The pumping rate necessary to exceed laser threshold can be found from (3.3.9) so that

\[ \beta \sigma n^1 > -c_{33} \]

where

\[ n^1 = \frac{-c_{22}W_p}{c_{22}(c_{11} - W_p) - c_{21}(c_{12} - \gamma W_p)}. \]

This implies that

\[ W_p[c_{22}(1 - \beta \sigma \tau_e) - \gamma c_{21}] > c_{11}c_{22} - c_{21}c_{12}. \]

or in terms of the physical parameters

\[ W_p > \frac{\tau_{fe}}{\tau_1 \tau_{fe} \beta \sigma \tau_e - 1 + \tau_1 \tau_2 (1 - \gamma) \beta \sigma \tau_e}. \quad (3.3.11) \]

Now consider the system (3.1.5) in which \( W_p(t) \) and \( I(t) \) satisfy the condition of Theorem 3.2 (i). Then the following theorem from [2] can be easily applied.
Theorem 5 Consider the system

\[ y' = (A + B(t))y + f(t, y) \]

where \( A \) and \( f \) satisfy the hypothesis of Theorem 3.3. If \( B(t) \) is continuous for \( 0 \leq t < \infty \) with \( \lim_{t \to \infty} B(t) = 0 \), then the zero solution of the system is asymptotically stable.

Theorem 6 Let \( W_p(t) \) and \( I(t) \) satisfy the conditions of Theorem 3.2(i). Then (3.1.5) is asymptotically stable.

Proof: Let \((\bar{n}, \bar{n}_1, \bar{\phi})\) solve the system (3.1.5). Let

\[
egin{align*}
n &= N - \bar{n} \\
n_1 &= N_1 - \bar{n}_1 \\
\phi &= \Phi - \bar{\phi}.
\end{align*}
\]

Assuming \( \bar{n}(0) + \gamma \bar{n}_1(0) < 1 \) and \( n(0) + \gamma n_1(0) < 1 \), the following system is obtained.

\[
egin{align*}
\frac{dN}{dt} &= (c_{11} - \beta \sigma \bar{\phi} - W_p)N + (c_{12} - \gamma W_p)N_1 - \beta \sigma \gamma \bar{n} \Phi - \beta \sigma \gamma N \Phi \\
\frac{dN_1}{dt} &= (c_{21} + \beta \sigma \bar{\phi})N + c_{22}N_1 + \beta \sigma \bar{n} \Phi + \beta \sigma N \Phi \\
\frac{d\Phi}{dt} &= (c_{31} + \beta \sigma \bar{\phi})N + c_{32}N_1 + (c_{33} + \beta \sigma \bar{n}) \Phi + \beta \sigma N \Phi
\end{align*}
\]

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which has the trivial solution $N = N_1 = \Phi = 0$.

Consider the linear system

\[
\frac{dN}{dt} = c_{11}N + c_{12}N_1 \\
\frac{dN_1}{dt} = c_{21}N + c_{22}N_1 \\
\frac{d\Phi}{dt} = c_{31}N + c_{32}N_1 + c_{33}\Phi
\]

which has the coefficient matrix

\[
C = \begin{pmatrix}
c_{11} & c_{12} & 0 \\
c_{21} & c_{22} & 0 \\
c_{31} & c_{32} & c_{33}
\end{pmatrix}
\]

The characteristic equation of the matrix $C$ is

\[
(c_{33} - \lambda)(\lambda^2 - (c_{11} + c_{22})\lambda + (c_{11}c_{22} - c_{21}c_{12})) = 0.
\]

The eigenvalues of the characteristic equation will have negative real parts provided that

\[
(c_{11} + c_{22}) > 0 \quad \text{and} \quad c_{11}c_{22} - c_{21}c_{12} > 0.
\]

The first condition is satisfied so consider
Therefore the eigenvalues of \( C \) have negative real parts and so the linear system (3.3.13) is asymptotically stable. Since \( \tilde{n}, \tilde{n}_1, \tilde{\phi}, \) and \( W_p(t) \) are nonnegative and integrable, it follows from Theorem 3.5 that \( N = N_1 = \tilde{\Phi} = 0 \) is an asymptotically stable solution to the nonlinear system (3.3.12).

### 3.4 Numerical Analysis

In this section, the numerical solution to the system (3.1.5) subject to initial conditions (3.1.6) are discussed. In particular, the system is assumed initially to be quiescent, i.e. \( n(0) = n_1(0) = \phi(0) = 0. \)

The initial value problem was solved numerically on a DEC VAX 11/750 using a Runge-Kutta Fehlberg (RKF) algorithm and on a CYBER CY173 using a backward difference method [13]. Material parameters for Titanium-doped Sapphire that were used in the calculations are given in Table 3.1. Experimental parameters that are typical of a Ti:Sapphire laser system are listed in Table 3.2.
Other parameters to be considered are the cavity lifetime $\tau_c$ and the spontaneous emission parameter $S p_0$. The cavity lifetime for the ring cavity discussed in Section 2 was computed by

$$\tau_c = \frac{2L}{c \cdot \tau_{\text{norm}}} \left[ 2\alpha l - l_m - \ln[R_1 R_2 R_3 T_s^4]\right]^{-1}.$$ 

Variations in $\tau_c$ were assumed to correspond to changes in the mirror reflectivities or cavity length so the effects on the numerical solution caused by changes in $\tau_c$ were studied. The spontaneous emission parameter $S p_0$ was computed by

$$S p_0 = \left[ \left(1 + \frac{\tau}{\bar{\tau}}\right) - \sqrt{1 + \left(\frac{\tau}{\bar{\tau}}\right)^2} \right] \left[ 8\pi c \tau_{\text{norm}} n^2 \sigma \frac{\Delta \lambda}{\lambda^4} \tau_{fe} \right].$$

The model (3.1.5) allows for the introduction of an injection signal to the system. The injection signal, when present, was assumed to be a Gaussian distribution given by

$$I(t) = I \cdot f(t, t_I, \tau_I) \quad \text{(3.4.1)}$$

where
Table 3.1 Material Parameters for Titanium-Doped Sapphire

\begin{align*}
g_2 &= 2.0 \\
g_1 &= 3.0 \\
n_- &= 1.76 \\
\tau_{fr} &= 3870.0\text{ns} \\
\tau_1 &= 1.0\text{ns} \\
\tau_2 &= 3000.0\text{ns} \\
\sigma &= 3.0 \times 10^{19}\text{cm}^2 \\
\sigma_{ab} &= 6.8 \times 10^{-20}\text{cm}^2 \\
\lambda &= 780\text{nm} \\
\Delta\lambda &= 1\text{nm} \\
\alpha &= 0.02\text{cm}^{-1}
\end{align*}

Table 3.2 Experimental Parameters for Ti:Sapphire System

Crystal Parameters

\begin{align*}
\ell &= 2.5\text{cm} \\
r &= 0.05\text{cm}^{-1} \\
n_T &= 1.2 \times 10^{-19}\text{cm}^{-3}
\end{align*}

Cavity Parameters

\begin{align*}
L &= 120\text{cm} \\
\ell_m &= 0.0 \\
R_1 &= .4 \\
R_2 &= .9 \\
R_3 &= .9
\end{align*}
\[ I = \left[ \frac{E_I}{(h)_{1}} \right] \left[ \frac{1}{\tau_f} \right] \left[ \frac{1}{\pi r^2 \ell} \right] \left[ \frac{1}{n_T} \right] \]

and the function \( f \) is defined by

\[ f(t, t_0, \tau) = 2\sqrt{\frac{\ln 2}{\pi}} \exp \left[ \frac{-((\ln 2)(t - t_0)^2}{(\frac{1}{\tau})^2} \right]. \]

In the definition for \( I \),

- \( E_I = \) energy of pulse (\( \mu J \))
- \( \lambda_I = \) wavelength of the injected signal (cm)
- \( t_f = \) center of injection pulse (ns)
- \( \tau_f = \) width of injection pulse (ns).

Variations in \( I \) were assumed to correspond to variations in the energy of the injected signal. Some typical values of the injection parameters are given in Table 3.3.

Two pumping schemes were considered. In the case of a constant pump pulse,

\[ W_p(t) = W_p \]

\[ = \eta \left[ \frac{E_p}{(h)_{1}} \right] \left[ \frac{1}{\tau_p} \right] \left[ \frac{1 - e^{-2(\frac{r}{\tau_p})^2}}{\pi r^2 \ell} \right] \left[ \frac{1 - e^{-\alpha \ell}}{N_T} \right] \]

(3.4.2)
Table 3.3 Injection Pulse Parameters

\[ E_I = 10\mu J \]
\[ t_I = 60 \text{ ns} \]
\[ \tau_I = 20 \text{ ns} \]

Table 3.4 Pump Pulse Parameters

\[ E_p = 10 \text{ mJ} \]
\[ r_p = .05 \text{ cm} \]
\[ \eta = 1.0 \]
\[ \lambda_p = 532.0 \text{ nm} \]
\[ t_p = 30 \text{ ns} \]
\[ \tau_p = 10 \text{ ns} \]
where

\[ \eta = \text{branching ratio} \]

\[ E_p = \text{pumping energy (mJ)} \]

\[ \lambda_p = \text{pump wavelength (cm)} \]

\[ \alpha_p = \text{absorption coefficient (cm}^{-1}) \text{ given by } n_\tau \sigma_{ab} \]

\[ \tau_p = \text{length of pump pulse (ns)} \]

\[ r_p = \text{pump beam radius (cm)} \]

Typical values for the pump parameters are given in Table 3.4. Changes in the value of \( W_p \) were assumed to correspond to changes in the pumping energy. Effects on the numerical solution caused by changes in \( W_p \) were studied.

The second pumping scheme considered was a Gaussian distribution for the pump pulse, given by

\[ W_p(t) = W_p \cdot f(t, t_p, \tau_p) \]

where \( W_p \) is as described above and

\[ t_p = \text{center of the pump pulse (ns)} \]

All numerical computations performed on the VAX computer were in double precision. For relatively low values of \( W_p \), the RKF algorithm was used, but for large values of the parameter the backward difference method on the CYBER computer was employed.

The normalized population inversion and photon concentration are plotted in Figs. 3.1 and 3.2 as a function of time for a system in which the pumping rate
Figure 3.1 Computed curve showing the evolution of the inverted population concentration for an end-pumped laser with constant pumping rate. The population concentration is normalized to the doping concentration in the crystal.

Here, $w_p = 0.001 \text{ns}^{-1}$ and $\tau_c = 1.0 \text{ns}$.
Figure 3.2 Computed curve showing the evolution of the normalized photon concentration for the parameter values $W = 0.001 \text{ns}^{-1}$ and $\tau_c = 1.0 \text{ms}$.
is constant, \( W_p = .001 \text{ns}^{-1} \), \( I = 0 \), and \( \tau_e = 1.0 \text{ ns} \). Initially, the number of upper energy electrons is zero and the photon concentration is zero. At \( t = 0 \), the pump is turned on and the population inversion increases. The increase continues until the photons in the pump beam are absorbed. During this time the photon concentration has been steadily increasing (but not visibly on the graph). Up to this point the dynamics have been dominated by the linear part of the rate equations. As the photon concentration becomes significant, the nonlinear term \( (n \cdot \phi) \) begins to dominate the dynamics. At this point there is a rapid increase in the photon concentration along with a corresponding decrease in the population inversion. In this particular case, relaxation oscillations are present in the solution. As time passes, the oscillations die out as the system approaches its asymptotically stable equilibrium solution. In a laser system, this relaxation process is known as spiking and for some laser applications it is desirable to suppress the modulations in the output [12]. If \( \tau_e \) is changed to 5.0 ns, the numerical solutions (Figs. 3.3 and 3.4) predict that the oscillations die out more quickly.

A study of the nature of the eigenvalues of the system (3.1.5) linearized about the equilibrium solution provides a means of predicting values of the parameters that lead to relaxation oscillations.
Figure 3.3 Curve showing the effect on the normalized population inversion of holding $W_p$ constant at $0.001 \text{ns}^{-1}$ while increasing $\tau_c$ to 5.0 ns.
Figure 3.4  The normalized photon concentration for parameter values $W_p = .001 \text{ns}^{-1}$ and $\tau_c = 5.0 \text{ns}$
The characteristic equation for the coefficient matrix is cubic,

$$\lambda^3 + \hat{\beta}\lambda^2 + \gamma\lambda + \delta = 0 \quad (3.4.3)$$

where

$$\hat{\beta} = -[a_{11} + a_{22} + a_{33}]$$

$$\hat{\gamma} = -[a_{23}a_{32} + a_{21}a_{12} + a_{31}a_{13} - a_{11}a_{22} - a_{33}a_{11} - a_{23}a_{32}]$$

$$\hat{\delta} = -[a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{21}a_{12}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{33} - a_{31}a_{32}a_{11}]$$

and

$$a_{11} = c_{11} - \beta\sigma\gamma\phi_0 - W_p \quad a_{12} = c_{12} - \gamma W_p \quad a_{13} = -\beta\sigma\gamma n_0$$

$$a_{21} = c_{21} + \beta\sigma\phi_0 \quad a_{22} = c_{22} \quad a_{23} = \beta\sigma n_0$$

$$a_{31} = c_{31} + \beta\phi_0 \quad a_{32} = c_{32} \quad a_{33} = c_{33} + \beta\sigma n_0$$

where \((n_0^0, n_1^0, \phi^0)\) is an equilibrium solution.

Making the transformation

$$\lambda = \xi - \frac{\hat{\beta}}{3}$$

equation (3.4.3) becomes

$$\xi^3 + \rho\xi + q = 0$$

where
\[ P = \hat{\gamma} - \frac{1}{3}\hat{\beta}^2 \]
\[ q = \hat{\sigma} - \frac{\hat{\gamma}\hat{\beta}}{3} + \frac{2\hat{\beta}^3}{27}. \]

The cubic discriminant is defined

\[ D = -4p^3 - 27q^3. \]

If \( D < 0 \), then there is one real and two complex roots to equation (3.4). If \( D \geq 0 \), then all roots of (3.4.3) are real and if \( D = 0 \), then there is multiplicity of the roots.

The computer code given in Appendix I determines the nature of the eigenvalues of the linearized system using the cubic discriminant as well as the stability properties by using the criteria (3.3.8). Figure 3.5 is a plot of the cubic discriminant as a function of \( \tau_e \) holding \( W_p \) constant, \( W_p = .001\text{ns}^{-1} \). The plot predicts oscillations in the solutions will occur for an approximate range \(.5 \leq \tau_e \leq 28.6 \). After \( \tau_e \) exceeds 28.6ns the oscillations should not be present in the solutions. Figs. 3.6 and 3.7 give the numerical solution for the population inversion and photon concentration, respectively, for \( \tau_e = 30\text{ns} \). Figs. 3.8, 3.9 Figs. 3.8, 3.9 and 3.10 give the phase portraits for \( \tau_e = 1.0\text{ns}, \tau_e = 5.0\text{ns} \), and \( \tau_e = 30\text{ns} \) which display the asymptotic behavior of the system.
Figure 3.5 The discriminant of the cubic characteristic equation for the linearized eigenvalues computed for $W = 0.001 \text{ns}^{-1}$ as a function of $\tau_c$. 

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Figure 3.6 Curve demonstrating the absence of relaxation oscillations in the normalized population inversion for the parameter values $W_p = .001 \text{ns}^{-1}$ and $\tau_c = 30.0 \text{ons}$
Figure 3.7 The normalized photon concentration computed for the value $W_p = 0.001\text{ns}^{-1}$ and $\tau_c = 30.0\text{ns}$.
Figure 3.8 Phase plane portrait showing the asymptotic behavior of the system for the parameter values $W_p = 0.001 \text{ns}^{-1}$ and $\tau_c = 1.0 \text{ns}$.
Figure 3.9 Phase plane portrait emphasizing the reduction in oscillatory behavior for the system in which $W_p = 0.001 \text{ns}^{-1}$ and $\tau_c = 5.0 \text{ns}$
Figure 3.10 Phase plane portrait demonstrating the absence of oscillations for the system in which $W_\text{p}$ to .01ns$^{-1}$ and $\tau_c$=30.0ns.
A similar study was conducted with \( \tau_e \) held constant and varying \( W_p \). Figure 3.11 displays the cubic discriminant for \( \tau_e = 5.0\text{ns} \) and \(.005\text{ns}^{-1} \leq W_p \leq 0.1\text{ns}^{-1} \). Oscillations are predicted for an approximate range \(.005^{-1} \leq W_p \leq .04\text{ns}^{-1} \). Figs. 3.12-3.15 give the computer results for \( \tau_e = 5.0\text{ns} \) and \( W_p = .01\text{ns}^{-1} \) and \( W_p = .1\text{ns}^{-1} \), respectively. Phase portraits are given in Figs. 3.16 and 3.17.

Figure 3.18 shows a typical computation for a system in which the pump pulse has a Gaussian time dependence and no injection signal is present. In this case \( W_p = .01\text{ns}^{-1} \). Initially, the system is in a quiescent state and at \( t = 0 \), the pump is turned on. The population inversion follows the leading edge of the pump pulse and increases until the active ions have absorbed the photons in the pump beam. When the photon concentration in the active medium is sufficiently large, there is a rapid decrease in the population inversion and stimulated emission occurs. In this case the pump is negligible after the laser pulse so there is no further growth of excited state ions and the system approaches the quiescent state once again. This type of dynamic behavior of the solutions is predicted by Theorem 3.5 for the case of an integrable time dependence of the pump pulse.

The value of \( W_p \) was varied from \(.05\text{ns}^{-1} \) to \(1.0\text{ns}^{-1} \) by factors of two. It is evident from Figure 3.19 that the larger the value of \( W_p \), the earlier stimulated
Figure 3.11 The cubic discriminant of a system in which $\tau_c$ is fixed at 5.0ns as a function of $W_p$. 

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Figure 3.12 The reduction in oscillatory behavior in the normalized population inversion by increasing $W_p$ to $0.1\text{ns}^{-1}$. $\tau_c$ is fixed at $5.0\text{ns}$.
Figure 3.13 The evolution of the normalized photon concentration for the system in which $\tau_c = 5.0\text{ns}$ and $W = 0.01\text{ns}^{-1}$
Figure 3.14 Computed curve showing the absence of oscillations in the normalized population inversion for the parameter values \( \tau_c = 5.0 \text{ns} \) and \( W_p = 1 \text{ns}^{-1} \)

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Figure 3.15 The normalized photon concentration for system in which $\tau_c=5.0\text{ns}$ and $W_p=1\text{ns}^{-1}$.
Figure 3.16 Phase plane portrait for type system in which $\tau_c = 5.0 \text{ns}$ and $W_p = 0.01 \text{ns}^{-1}$.
Figure 3.17 Phase plane portrait for the system in which $\tau_c = 5.0 \text{ns}$ and $W_p = 0.1 \text{ns}^{-1}$
Figure 3.18 Typical computation for Gaussian pump pulse. Here, $W_p = 0.1\text{ns}^{-1}$.
Figure 3.19 Curves showing the effect of increasing the intensity of the pump beam. The strength of the output pulse increases with pump energy.
emission will occur and the stronger it will be. The efficiency of the RKF algorithm was studied for the different pumping rates. The algorithm required 822 evaluations of the derivatives for the case when $W_p = .01ns^{-1}$ but as $W_p$ was increased, the number of derivative evaluations increased significantly to 1944 for $W_p = .1ns^{-1}$ and 6612 evaluations for $W_p = 1.0ns^{-1}$. The growth in the number of derivative evaluations as a function of pumping rate is shown in Figure. 3.20.

For larger values of $W_p$, the RKF algorithm required an unreasonable number of iterations. In excess of 67000 derivative evaluations for the case of $W_p = 500ns^{-1}$ were required to compute solutions from $t = 0$ to $t = 44ns$. A backward difference method significantly improved computing effort, requiring 17000 derivative evaluations and 3700 Jacobean evaluations to compute solutions over the interval $0 \leq t \leq 200ns$ (Figure 3.21). For high values of $W_p$, the system of differential equations becomes stiff, resulting in considerable effort to compute solutions.

The effect of introduction of an injected signal into the system was studied by varying the value of $I$. Figure 3.22 shows the results of varying $I$ from $1.0ns^{-1}$ to $1.0 \times 10^{-4}ns^{-1}$ to $1.0 \times 10^{-6}ns^{-1}$. It is clear from the plot that the presence of an injected beam speeds up the onset of the stimulated emission. The time at which the peak of the photon concentration occured as a function of increasing injection strength is displayed in Fig. 3.23 The efficiency of the RKF algorithm did not appear to be adversely affected by increasing the value of $I$. The solutions
required no more than 846 derivative evaluations for these cases.
Figure 3.20 Stiffening of the equations with increasing pump energy displayed in terms of the number of derivative evaluations required in the Runge-Kutta-Fehlberg algorithm
Figure 3.21 Extreme values of pump energy computed using a backward difference algorithm on the CYBER CY173
Figure 3.22 Curves showing the variations in the population inversion and photon concentration resulting from varying the injection energy. Increasing the strength of the injection pulse speeds up the onset of stimulated emission.
Figure 3.23 Data showing the decrease in time required for the photon pulse to reach its peak as a function of increased injection strength.
Chapter 4

Analysis of the Spatial and Temporal Model

4.1 The Spatial and Temporal Model

In this chapter the model of the spatial and temporal dynamics in a ring laser cavity is considered. The model consists of the system of partial differential equations (2.4.1) and (2.4.3) together with the initial conditions (2.5.1), (2.5.2) and boundary conditions (2.5.4). Qualitative properties of the solutions are analyzed and the numerical procedure for generating approximate solutions is developed. The well-posedness of the problem is not addressed.

It is assumed that the emission cross section, $\sigma(\lambda)$, tends to zero sufficiently fast so that the integral

$$\int_{0}^{\infty} \sigma(\lambda) \Phi(x; t; \lambda) d\lambda$$

(4.1.1)

exists for $0 \leq x \leq \ell$ and all $t$. In order to study the system of equations, the significant part of the emission spectrum of Ti:Sapphire is divided into $m$ intervals,
each having length $\Delta \lambda$. Then the integral in (4.1.1) is approximated by

$$
\int_0^\infty \sigma(\lambda) \Phi(x, t; \lambda) d\lambda \sim \int_{\lambda_0}^{\lambda_m} \sigma(\lambda) \Phi(x, t; \lambda) d\lambda
$$

$$
\sim \sum_{j=1}^m \frac{1}{2} [\sigma(\lambda_{j-1}) + \sigma(\lambda_j)] \Phi(x, t; \bar{\lambda}_j) \Delta \lambda
$$

where $\bar{\lambda}_i = \frac{1}{2} [\lambda_{i-1} + \lambda_i]$ for $i = 1, \ldots, m$. Then the model is modified to

$$
\frac{\partial n(x, t)}{\partial t} = c_{11} N(x, t) + c_{12} N_1(x, t) - \gamma \beta N(x, t) \sum_{i=1}^m \sigma_i \Phi(x, t; \bar{\lambda}_i) \Delta \lambda
$$

$$
+ W(x, t) \left[ \frac{n_T}{n_{\text{norm}}} - \gamma N_1(x, t) + N(x, t) \right]
$$

$$
\frac{\partial n(x, t)}{\partial t} = c_{21} N(x, t) + c_{22} N_1(x, t) - \gamma \beta N(x, t) \sum_{i=1}^m \sigma_i \Phi(x, t; \bar{\lambda}_i) \Delta \lambda
$$

$$
\frac{1}{v \cdot r_{\text{norm}}} \frac{\partial \phi_{\pm}(x, t; \bar{\lambda}_i)}{\partial t} \pm \frac{\partial \phi_{\pm}(x, t; \bar{\lambda}_i)}{\partial x} = \frac{1}{v \cdot r_{\text{norm}}} \left[ f_\pm(x, t; \bar{\lambda}_i)[N(x, t) + (\gamma - 1)N_1(x, t)] \right]
$$

$$
+ [n_{\text{norm}} \sigma_i N(x, t) - a_i] \phi_{\pm}(x, t; \bar{\lambda}_i)
$$

for $0 \leq x \leq \ell$ and $i = 1, \ldots, m$.

Here

$$
\Phi(x, t; \bar{\lambda}_j) = \phi_+(x, \bar{\lambda}_j) + \phi_-(x, \bar{\lambda}_j) \text{ for each } j
$$

$$
\sigma_i = \frac{1}{2} [\sigma(\lambda_{i-1}) + \sigma(\lambda_i)] \quad i = 1, \ldots, m
$$

$$
\alpha_i = \frac{1}{2} [\alpha(\lambda_{i-1}) + \alpha(\lambda_i)] \quad j = 1, \ldots, m, \text{ and}
$$

$$
f_\pm(x, \bar{\lambda}_i)[N(x, t) + (\gamma - 1)N_1(x, t)] = S_{\pm}(x, t; \bar{\lambda}_i) \text{ for each } i.
$$

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In the remainder of the cavity,

\[ \frac{1}{c \cdot \tau_{\text{norm}}} \frac{\partial \Phi_\pm(x, t; \lambda_i)}{\partial t} \pm \frac{\partial \Phi_\pm(x, t; \lambda_i)}{\partial x} = 0 \quad \text{for} \quad i = 1, ..., m \] (4.1.2)

describe the development of the photon density.

Let \( u = (N, N_1, W, \phi_{+1}, \phi_{+2}, ..., \phi_{+m}, \phi_{-1}, \phi_{-2}, ..., \phi_m)^T \) be the 2m+3 dimensional column vector that contains the dependent variables in the active medium,

\[ 0 \leq x \leq \ell. \]

Here \( \phi_{\pm i}(x, t; \lambda_i) \) for \( i = 1, ..., m \) and \( 0 \leq x \leq \ell \). Then the system (4.1.1) along with (2.4.2) can be written in matrix form

\[ u_t + Au_x - B = 0 \] (4.1.3)

where \( A \) is the diagonal matrix whose elements are \( 0, 0, v \cot \tau_{\text{norm}}, ..., v \cdot \tau_{\text{norm}}, -v \cdot \tau_{\text{norm}}, ..., -v \cdot \tau_{\text{norm}} \) and \( B \) is the column vector whose elements are given by the right hand sides of the equations in (4.1.1) and (2.4.2). The matrix \( A \) does not depend on \( u \), so the system of partial differential equations is semilinear.

Since \( A \) is diagonal and hence symmetric, the system is symmetric hyperbolic and \( A \) has a full set of \( 2m + 3 \) characteristic values, \( \xi_j, \quad j = 1, 2, ..., 2m + 3 \) [8]. Each equation is of the form

\[ \frac{\partial u_j}{\partial t} + \xi_j \frac{\partial u_j}{\partial x} = b_j, \quad j = 1, 2, ..., 2m + 3. \]

The \( 2m + 3 \) solutions of

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\[ \det(A - \xi I) = 0 \]

where \( \xi = \frac{dx}{dt} \) defines the characteristic directions of (4.1.3) at the point \((x, y)\)[11]. Thus if \( s \) is a parameter along a characteristic curve, then \( u_j \) must satisfy the ordinary differential equation

\[ \frac{du_j}{ds} = b_j \]

for each \( j \) along the characteristic.

A number of qualitative properties for the solutions of (4.1.3) can be determined by investigating the solutions along their respective characteristics. In addition, integrating along the characteristics provides a natural approach to solving the initial and boundary value problem numerically. The qualitative results and numerical analysis of the system are discussed in sections 4.2 and 4.3, respectively.

### 4.2 Some Qualitative Results

In the active region, \( 0 \leq x \leq \ell \), consider equations (4.1.1) for the laser dynamics and (2.4.2) for the pump pulse. Without loss of generality, assume that

\[ n_{\text{norm}} = n_T \]

and recall that

\[ 
c_{11} = - \left( \frac{1}{\tau_2} + \frac{\gamma - 1}{\tau_{fl}} \right), \quad c_{12} = (1 - \gamma) \left[ \frac{1}{\tau_2} + \frac{\gamma - 1}{\tau_{fl}} - \frac{1}{\tau_1} \right], \\
c_{21} = \frac{1}{\tau_{fl}}, \quad c_{22} = \frac{\gamma - 1}{\tau_{fl}} - \frac{1}{\tau_1}. 
\]
Assume that the physical parameters satisfy

\[ 0 < \sigma_j, \alpha_j < \infty \quad j = 1, \ldots, m \]

\[ f_{\pm}(x, \lambda_j) > 0 \quad 0 \leq x \leq \ell \quad j = 1, \ldots, m \]

\[ \gamma > 1 \quad (4.2.1) \]

\[ 0 < r_1 << r_2 < r_{fr} \]

\[ \sigma_{ab}, \beta > 0. \]

The initial conditions are assumed to satisfy

\[ N(x, 0) \geq 0 \]

\[ N_1(x, 0) \geq 0 \]

\[ \phi_{\pm}(x, 0; \lambda_j) \geq 0 \quad j = 1, \ldots, m \quad (4.2.2) \]

\[ W(0, x) > 0 \]

for \(0 \leq x \leq \ell\) and

\[ \Phi_{\pm}(x, 0; \lambda_j) \geq 0 \quad j = 1, \ldots, m \]

for \(\ell \leq x \leq L\). The boundary conditions obey
\[
\begin{align*}
\phi_+(0^+, t; \lambda_j) &= A_1 \phi_+(L^-, t; \lambda_j) \quad 0 < A_1 \leq 1 \\
\phi_-(L^-, t; \lambda_j) &= A_2 \phi_-(0^+, t; \lambda_j) \quad 0 < A_2 \leq 1 \\
\phi_+(\ell^+, t; \lambda_j) &= A_3 \phi_+(\ell^-, t; \lambda_j) \quad 0 < A_3 \leq 1 \\
\phi_-(\ell^-, t; \lambda_j) &= A_4 \phi_-(\ell^+, t; \lambda_j) \quad 0 < A_4 \leq 1
\end{align*}
\]

for \( t \geq 0 \) and \( j = 1, \ldots, m \).

Lemma 1  Let \( W(0, t) > 0 \) for \( 0 \leq t < \infty \) and

\[ W(x, 0) > 0 \text{ for } 0 \leq x \leq \ell. \text{ Then for any } x, \ 0 \leq x \leq \ell, \]

\[ W(x, t) > 0, \quad 0 \leq t < \infty. \]

Proof: \( W(x, t) \) satisfies the partial differential equation

\[
\frac{\partial W}{\partial t} + v \cdot \tau_{\text{norm}} \frac{\partial W}{\partial x} = -c[1 - \gamma N_1 - N]W, \quad \text{where } c > 0. \quad (4.2.4)
\]

If \( s \) is a parameter along a characteristic, i.e. \( x = x(s) \) and \( t = t(s) \) where

\[
\frac{dt}{ds} = 1 \quad \text{and} \quad \frac{dx}{ds} = v \cdot \tau_{\text{norm}},
\]

then \( W \) satisfies the ordinary differential equation

\[
\frac{dW}{ds} = -c[1 - \gamma N_1 - N]W
\]

along the characteristics. Then

\[
W(s) = W(s_0) \exp\left\{-c \int_{s_0}^s \left[1 - \gamma N_1(x(r), t(r)) - N(x(r), t(r))\right] dr\right\}
\]

where
\[ W(s_0) = W(x(s_0), t(s_0)) \]

and \((x(s_0), t(s_0))\) is a point on the initial line or on the boundary.

Therefore, since \(W(s_0) > 0\) it follows that \(W(s) > 0\) for all \(s\). Then \(W(x, t) > 0\) for all \(t \geq 0\) and \(0 \leq x \leq \ell\), which was to be proved.

The following results can now be shown.

Theorem 7 \textit{Given the system of partial differential equations defined by (4.1.1) and (4.2.5) along with initial and boundary conditions given in (4.2.8), (4.2.9) and (4.2.4). If the physical parameters also satisfy (4.2.1), then}

(i) if \(N(x, 0) + \gamma N_1(x, 0) < 1\) for \(0 < x < \ell\), then \(N(x, t) + \gamma N_1(x, t) < 1\) for all \(t \geq 0\) and \(0 \leq x \leq \ell\).

(ii) if \(N(x, 0) + \gamma N_1(x, 0) \geq 1\) for \(0 < x < \ell\), then there exists \(T\) such that \(N(x, t) + \gamma N_1(x, t) < 1\) for all \(t > T\) and \(0 \leq x \leq \ell\).

(iii) \(N(x, t) \geq 0, N_1(x, t) \geq 0\) for all \(t \geq 0\) and \(0 \leq x \leq \ell\) and

\[ \phi_{\pm}(x, t; \lambda_j) \geq 0 \text{ for } j = 1, 2, \ldots, m \text{ and } t \geq 0, \ 0 \leq x \leq \ell. \]

Proof: (i) Suppose \(N(x, 0) + \gamma N_1(x, 0) < 1\), \(0 \leq x \leq \ell\). Then for each \(x\), \(N(x, t) + \gamma N_1(x, t) < 1\) for \(t\) close to zero.

Suppose there exists \(X \leq \ell\) and \(T\) such that

\[ N(X, T) + \gamma N_1(X, T) = 1 \]

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and \( N(x,t) + \gamma N_1(x,t) < 1 \) for \( 0 \leq t < T \) and \( 0 \leq x \leq \ell \).

Then, along the characteristic \( x = X \), \( \frac{\partial (N + \gamma N_1)}{\partial t} = \frac{d(N + \gamma N_1)}{dt} \) and so, at \((T,X)\),

\[
\frac{d}{dt}(N + \gamma N_1) = (c_1 + \gamma c_{21})N + (c_{12} + \gamma c_{22})N_1 \geq 0.
\]

Suppose \( N(X,T) \geq 0 \) and \( N_1(X,T) \geq 0 \). Since \( N(X,T) + \gamma N_1(X,T) = 1 \), \( N(X,T) \) and \( N_1(X,T) \) are not both zero. Now, \( c_{11} + \gamma c_{21} < 0 \) and \( c_{12} + \gamma c_{22} < 0 \), so it follows that at \((T,X)\),

\[
\frac{d}{dt}(N + \gamma N_1) < 0,
\]

which contradicts

\[
N(X,t) + \gamma N_1(X,t) < 1 \text{ for } 0 \leq t < T.
\]

Now suppose that \( N_1(X,t) < 0 \). From the initial condition, \( N_1(x,0) \geq 0 \) for \( 0 \leq x \leq \ell \), so consider the two cases

a) \( N_1(x,0) > 0 \) for \( 0 \leq x \leq \ell \) and

b) \( N_1(x,0) = 0 \) for some \( x, \ 0 \leq x \leq \ell \).

a) If \( N_1(x,0) > 0 \), then for each \( x \)
\(N_1(x,t) > 0\) for \(t\) sufficiently close to zero. Then there exists a point \((X_1, T_1),\ T_1 < T,\ X_1 < X\) such that \(N_1(X_1, T_1) = 0\) and \(N_1(x,t) > 0\) for \(0 < x \leq l\) and \(0 < t < T_1\).

Then, at \((X_1, T_1),\)

\[
\frac{dN_1}{dt} = c_{21} + \beta \sum_{j=1}^{m} \sigma(\lambda_j)[\phi_{+j} + \phi_{-j}] \Delta \lambda \leq 0.
\]

Suppose that \(N(X_1, T_1) < 0\) and

\[
\left[ c_{21} + \beta \sum_{j=1}^{m} \sigma(\lambda_j)[\phi_{+j} + \phi_{-j}] \Delta \lambda \right]_{(x_1, x_3)} \geq 0.
\]

Then, since \(N(X_1, 0) \geq 0\), there exists \(T_2 < T_1\) such that \(N(X_1, T_2) = 0\) and \(N(X_1,t) \geq 0\) for \(0 < t < T_2\).

So, at \((T_2, X_1)\)

\[
\frac{dN}{dt} = c_{12}N_1 + W[1 - \gamma N_1] > 0
\]

since \(c_{12} > 0,\ N_1(X_1, T_2) > 0,\ W(X_1, T_2 > 0\) and \(1 - \gamma N_1(X_1, T_2) > 0\) which contradicts \(N(X_1,t) \geq 0\) for \(0 < t < T_2\). So \(N(X_1, T_1) > 0\).

Suppose there exists a point \((\hat{X}, \hat{T})\) with \(\hat{T} < T_1\) and \(\hat{X} \leq X_1\) such that \(N(\hat{X}, \hat{T}) < 0\). Then since \(N(\hat{X}, 0) \geq 0\) there exists \(\tilde{T} < \hat{T}\) such that \(N(\hat{X}, \tilde{T}) = 0\) for and \(N(\hat{X}, t) \geq 0\) for \(0 \leq t < \tilde{T}\).

Then, at \((\hat{X}, \tilde{T}),\frac{dN}{dt} > 0\). So by similar argument, \(N(\hat{X}, \tilde{T}) \geq 0\). It follows that \(N(x, t) \geq 0\) on the rectangle \([0, l] \times [0, T_1]\).
On the rectangle \([0, \xi] \times (0, T), N_1 > 0, N \geq 0\) and \(N + \gamma N_1 < 1\), so now consider the case where

\[
c_{21} + \beta \sum_{j=1}^{m} \sigma(\lambda_j) [\phi_+(X_1, T_1; \lambda_j) + \phi_-(X_1, T_1; \lambda_j)] \Delta \lambda \leq 0. \tag{4.2.5}
\]

Since \(c_{21} > 0, \ \beta > 0, \ \sigma(\lambda_j) > 0\) and \(\Delta \lambda > 0\) for \(j = 1, \ldots, m\), (4.2.5) can be true only if for some \(j\),

\[
\phi_+(X_1, T_1; \lambda_j) < 0 \quad \text{or} \quad \phi_-(X_1, T_1; \lambda_j) < 0.
\]

Suppose \(\phi_+(X_1, T_1; \lambda_j) < 0\). Then there exists \(\hat{T} < T_1\) and \(\hat{X} < X_1\) such that \(\phi_+(\hat{X}, \hat{T}, \lambda_j) = 0\) and \(\phi_+(x, t; \lambda_j) < 0\) for \(\hat{T} < t < T_1\) and \(\hat{X} < X < X_1\). Then if \(s\) is a parameter along the characteristic through \((\hat{X}, \hat{T})\), at \((\hat{X}, \hat{T})\)

\[
\frac{d\phi_+}{ds} = f(\hat{X}; \lambda_j)[N + (\gamma - 1)N_1] > 0
\]

since \(f(\hat{X}; \lambda_j) > 0, N(\hat{X}, \hat{T}) \geq 0, (\gamma - 1)N_1(\hat{X}, \hat{T}) > 0\) which contradicts \(\phi_+(x, t; \lambda_j) < 0\) for \(\hat{T} < t < T_1\) and \(\hat{X} < X < x_1\). Therefore, \(\phi_+(X_1, T_1; \lambda_j) \geq 0\).

Now suppose \(\phi_-(X_1, T_1, \lambda_j) < 0\). Then there exists \((\hat{X}, \hat{T})\), with \(\hat{T} < T_1\) and \(\hat{X} > X_1\) such that

\[
\phi_-(\hat{X}, \hat{T}; \lambda_j) = 0 \quad \text{and} \quad \phi_-(x, t; \lambda_j) < 0 \quad \text{for} \quad \hat{T} < t \leq T_1 \quad \text{and} \quad X_1 \leq x < \hat{X}. \quad \text{Then at} \ (\hat{X}, \hat{T}),
\]

\[
\frac{d\phi_-}{ds} = f_-(\hat{X}, \lambda_j)[N + (\gamma - 1)N_1] > 0,
\]

where \(s\) is a parameter along the characteristic through \((\hat{X}, \hat{T})\). This contradicts \(\phi_-(x, t; \lambda_j) < 0\) for \(\hat{T} < t \leq t_1\) and \(\hat{X} < t \leq X_1\).
Therefore, \( \phi_-(X_1, T; \lambda_j) \geq 0 \).

Suppose now that \( N(X_1, T_1) = 0 \). Then, at \( (X_1, T_1) \),

\[
\begin{align*}
\frac{dN}{dt} &= W > 0 \\
\frac{dN_1}{dt} &= 0.
\end{align*}
\]

Then look at

\[
\frac{d^2 N_1}{dt^2}(x, t) = c_{21} \frac{dN}{dt}(x, t) + c_{22} \frac{dN_1}{dt}(x, t) \\
+ \beta \left[ \frac{dN}{dt}(x, t) \sum_{j=1}^{m} \sigma(\lambda_j) [\phi_{+j} + \phi_{-j}] \Delta \lambda \\
+ N(x, t) \sum_{j=1}^{m} \frac{d}{dt}[\phi_{+j} + \phi_{-j}] \Delta \lambda \right]
\]

At \( (X_1, T_1) \)

\[
\frac{d^2 N_1}{dt^2} = c_{21} W + \beta W \sum_{j=1}^{m} \sigma(\lambda_j) [\phi_{+j} + \phi_{-j}] \Delta \lambda > 0
\]

since

\[
c_{21} > 0, \ W(X_1, T_1) > 0, \beta > 0, \ \sum_{j=1}^{m} \sigma(\lambda_j) [\phi_{+j}(X_1, T_1, \lambda_j) + \phi_{-j}(X_1, T_1, \lambda_j)] \Delta \lambda \geq 0.
\]

This contradicts \( N_1(X_1, t) \geq 0 \) for \( 0 \leq t \leq T_1 \). Therefore, \( N_1(X, T) \geq 0 \).

Suppose that \( N(X, T) < 0 \). Then there exists \( \hat{T} < T \) such that \( N(X, \hat{T}) = 0 \) and \( N(X, t) \geq 0 \) for \( 0 \leq t < \hat{T} \). At \( (X, \hat{T}) \),

\[
\frac{dN}{dt} - c_{21} N_1 + W[1 - \gamma N_1] > 0
\]

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which contradicts the fact that $N(X, t) \geq 0$ for $0 \leq t \leq \hat{T}$.

Therefore $N(X, T) \geq 0$.

At least one of $N_1(X, T)$ and $N(X, T)$ is strictly positive, so suppose $N_1(X, T) > 0$ and $N(X, T) = 0$. Then, at $(X, T)$,

$$\frac{d(N + \gamma N_1)}{dt} = (c_{12} + \gamma c_{22})N_1 < 0$$

since $c_{12} + \gamma c_{22} < 0$.

Likewise, if $N(X, T) > 0$ and $N_1(X, T) = 0$

$$\frac{d(N + \gamma N_1)}{dt} = (c_{11} + \gamma c_{21})N < 0$$

since $c_{11} + \gamma c_{21} < 0$.

In either case,

$$\frac{d(N + \gamma N_1)}{dt} < 0$$

which is a contradiction to $N(X, t) + \gamma N_1(X, t) < 1$ for $0 \leq t < T$.

Therefore,

$$N(x, t) + \gamma N_1(x, t) < 1 \text{ for } 0 \leq t < \infty \text{ and } 0 \leq x \leq \ell$$

when $N(x, 0) + \gamma N_1(x, 0) < 1$ and $N_1(x, 0) > 0$ for $0 \leq x \leq \ell$.

(b) Now suppose $N_1(x, 0) = 0$ for some $x$, $0 \leq x \leq \ell$.

Then

$$\frac{dN_1(x, 0)}{dt} = N(x, 0)[c_{21} + \beta \sum_{j=1}^{m} \sigma(\lambda_j)[\phi_+(x, 0; \lambda_j) + \phi_-(x, 0; \lambda_j)]\Delta \lambda$$

$$\geq 0$$

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since $c_{21} > 0$, $\beta > 0$, $\sigma(\lambda_j) > 0$ for $j = 1, \ldots, m$ and $\Delta \lambda > 0$.

Since $c_{21} + \beta \sum_{j=1}^{m} \sigma(\lambda_j)[\phi_+(x,0;\lambda_j) + \phi_-(x,0;\lambda_j)] \Delta \lambda > 0$, it follows that

$$\frac{dN_1(x,0)}{dt} = 0 \quad \text{if and only if} \quad N(x,0) = 0.$$ 

Now, if $N(x,0) = 0$, then

$$\frac{dN_1(x,0)}{dt} = 0 \quad \text{and}$$

$$\frac{d^2N_1}{dt^2} = c_{21}W + \beta W \sum_{j=1}^{m} \sigma(\lambda_j)[\phi_+(x,0;\lambda_j) + \phi_-(x,0;\lambda_j)] \Delta \lambda > 0$$

In addition, it is clear that $\frac{dN_1}{dt} > 0$ at $(x,0)$ if $N(x,0) > 0$.

Therefore, if $N_1(x,0) = 0$, then $N_1(x,t) > 0$ for $t$ close to zero. Then the arguments in the previous case can be used to obtain the desired result.

(ii) The proof of (i) shows that if $N(x,0) + \gamma N_1(x,0) = 1$ for $0 \leq x \leq \ell$, then $N(x,t) + \gamma N_1(x,t) < 1$ for all $t > 0$ and $0 \leq x \leq \ell$.

Suppose for some $x, N(x,0) + \gamma N_1(x,0) > 1$. Then for some $t$ close to zero

$$\frac{d}{dt}[N + \gamma N_1] = (c_{11} + \gamma c_{21})N + (c_{12} + \gamma c_{22})N_1 < 0$$

since $c_{11} + \gamma c_{21} < 0$, $c_{12} + \gamma c_{22} < 0$, and $N(x,0) + \gamma N_1(x,0) > 1$ implies that at least one of $N(x,0) + \gamma N_1(x,0)$ is strictly positive.
Now suppose there exists \( \xi \geq 1 \) since that \( N(x,t) + \gamma N_1(x,t) \geq \xi \geq 1 \) for all \( t \). Then

\[
\frac{d(N + \gamma N_1)}{dt} = (c_{11} + \gamma c_{21})(N + \gamma N_1) + (c_{12} + \gamma c_{22} - \gamma(c_{11} + \gamma c_{21}))N_1 \\
\leq (c_{11} + \gamma c_{21})\xi
\]

since \( c_{12} + \gamma c_{22} - \gamma(c_{11} + \gamma c_{21}) < 0 \) and \( c_{11} + \gamma c_{21} < 0 \).

Integrating along the characteristic,

\[
N(x,t) + \gamma N_1(x,t) - [N(x,0) + \gamma N_1(x,0)] \leq (c_{11} + \gamma c_{21})\xi t
\]

so

\[
N(x,t) + \gamma N_1(x,t) \leq (c_{11} + \gamma c_{21})\xi t + [N(x,0) + \gamma N_1(x,0)].
\]

Thus

\[
\lim_{t \to \infty} [N(x,t) + \gamma N_1(x,t)] = -\infty
\]

which contradicts \( N(x,t) + \gamma N_1(x,t) \geq \xi \geq 1 \) for all \( t \).

Hence, there exists \( T \) such that \( N(x,T) + \gamma N_1(x,T) = 1 \) and \( N(x,t) + \gamma N_1(x,t) < 1 \) for \( t \) close \( T \). The proof of (i) can now be used to establish the desired result.

(iii) The arguments in (i) show that \( N \) and \( N_1 \) are nonnegative whenever \( N + \gamma N_1 < 1 \). Therefore if \( N(x,t) + \gamma N_1(x,t) < 1 \) for \( 0 \leq x \leq \ell \), it follows that \( N(x,t) \geq 0 \) and \( N_1(x,t) \geq 0 \) for all \( t \) and \( 0 \leq x \leq \ell \).
Suppose there exists \((X,T)\) such that \(N(x,t) \geq 0\) and \(N_1(x,t) \geq 0\) for \(0 \leq x \leq X\) and \(0 \leq t \leq T\) and also \(\phi_+(X,T;\lambda_j) < 0\) for some \(j\). Then there exists \((X_1,T)\) such that

\[
\phi_+(X_1,T;\lambda_j) = 0 \quad \text{and} \quad \phi_+(x,t;\lambda_j) < 0 \quad \text{for} \quad T_1 < t < T \quad \text{and} \quad x_1 < x < X.
\]

If \(s\) is a parameter along the characteristic through \((X_1,T_1)\), at \((X_1,T_1)\),

\[
\frac{d\phi_+}{ds} = f_+(X_1;\lambda_j)[N + (\gamma - 1)N_1] \geq 0
\]

which is a contradiction. Therefore

\[
\phi_+(x,t;\lambda_j) \geq 0 \quad \text{for} \quad j = 1, \ldots, 0 \leq x \leq \ell, \quad \text{and} \quad 0 \leq t < \infty.
\]

Similarly, if there exist \((X,T)\) such that \(\phi_-(X,T;\lambda_j) < 0\) for some \(j\), then there exists \((X_1,T_1)\) such that

\[
\phi_-(X_1,T_1;\lambda_j) = 0 \quad \text{and} \quad \phi_-(x,t;\lambda_j) < 0 \quad \text{for} \quad T_1 < t < T \quad \text{and} \quad X < x < X_1.
\]

Then if \(s\) is a parameter along the characteristic through \((X_1,T_1)\), at \((X_1,T_1)\),

\[
\frac{d\phi_-}{ds} = f_-(X_1;\lambda_j)[N + (\gamma - 1)N_1] \geq 0
\]

which is a contradiction. Therefore \(\phi_-(x,t;\lambda) \geq 0\) for \(0 \leq x \leq \ell\) and all \(t\).
Finally, suppose that

\[ N(x,0) + \gamma N_1(x,0) > 1 \] for some \( x \). Then choose \( T \) so that

\[ N(x,T) + \gamma N_1(x,T) = 1 \] and \( N(x,t) + \gamma N_1(x,t) > 1 \) for \( 0 < t < T \). Suppose there exists \( T_1 \) such that \( N(x,T_1) < 0 \). Then choose \( T_2 < T_1 < T \) such that

\[ N(x,T_2) = 0 \] and \( N(x,t) \geq 0 \) for \( 0 < t \leq T_2 \). Then, at \( (x,T_2) \),

\[ \frac{dN}{dt} = c_{12}N > 0 \]

since \( c_{12} > 0 \) and \( N_1(x,T_2) > 0 \), which is a contradiction. Therefore \( N(x,t) \geq 0 \) for all \( t \).

Now, if there exists \( T_1 \) such that \( N_1(x,T_1) < 0 \), then choose \( T_2 < T_1 < T \) such that \( N_1(x,T_2) = 0 \) and \( N_1(x,t) \geq 0 \) for \( 0 < t \leq T_2 \). Then, at \( (x,T_2) \),

\[ \frac{dN_1}{dt} = N[c_{21} + \beta \sum_{j=1}^{m} \sigma(\lambda_j)[\phi_{+j} + \phi_{-j}]]\Delta \lambda > 0 \]

since \( N(x,T_2) > 0 \) and \( c_{21} + \beta \sum_{j=1}^{m} \sigma(\lambda_j)[\phi_{+j} + \phi_{-j}]\Delta \lambda > 0 \). Therefore, by similar reasoning, \( N_1(x,t) \geq 0 \) for all \( t \).

The following corollary can now be proved.

**Corollary 1** Let \( W \) be as in Lemma 4.1. If \( W(0,t) \) is an integrable function of \( t \), then for fixed \( x, \ 0 \leq x \leq \ell \), \( W(x,t) \) is an integrable function of \( t \).

Proof: Fix \( x = \hat{x}, 0 \leq \hat{x} \leq \ell \). Then \( W(\hat{x},t) \) is a continuous function of \( t \).

Let \( T \) be such that \( N(\hat{x},T) + \gamma N_1(\hat{x},T) < 1 \) for \( t > T \). If \( T > \frac{1}{v \cdot \tau_{\text{norm}}} \hat{x} \), then consider the integral
\[
\int_0^t W(\hat{x}, r) dr = \int_0^t W(\hat{x}, r) dr + \int_0^T W(\hat{x}, r) dr + \int_T^t W(\hat{x}, r) dr.
\]

Otherwise consider
\[
\int_0^t W(\hat{x}, r) dr = \int_0^t W(\hat{x}, r) dr + \int_0^T W(\hat{x}, r) dr.
\]

By continuity \( \int_0^t W(\hat{x}, r) dr \) and \( \int_0^T W(\hat{x}, r) dr \) exist.

So look at \( \int_0^t W(\hat{x}, r) dr \). By the lemma, \( W(x) = W(s_0) \exp \left[ -c \int_{s_0}^t [1 - \gamma N_1 - N] dr \right] \)

where \( s \) is a parameter along the characteristic through \((\hat{x}, t)\) and \( s_0 \) is a parameter along the boundary. Since every characteristic to the right of the line \( x = (v \cdot \text{norm}) t \) intersects the \( t \) axis, choose \( s_0 = t_0 \). Then \( \frac{dt}{ds} = 1 \) implies \( t - t_0 = s - s_0 \) so that \( t = s \) and \( \frac{dx}{ds} = v \cdot \text{norm} \) implies \( x - x_0 = v \cdot \text{norm}(s - s_0) \)

so that \( x = v \cdot \text{norm}(t - t_0) \) since \( x_0 = 0 \). Therefore \( t_0 = t - \frac{1}{v \cdot \text{norm}} \hat{x} \)

and so it follows that

\[
W(t, \hat{x}) = W(0, t_0) \exp \left[ -c \int_{t_0}^t [1 - \gamma N_1 - N] dr \right]
\]

Since \( 0 < 1 - \gamma N_1 - N \leq 1 \) for \( t \geq T \) and \( 0 \leq x \leq \ell \), it follows that for \( t \geq T \)

\[
\exp \left[ -c \int_{t - \frac{1}{v \cdot \text{norm}} \hat{x}}^t [1 - \gamma N_1 - N] dr \right] < 1.
\]
Therefore, if $T < \frac{1}{\nu \text{norm}}$

$$\int_{0}^{t} W(\hat{x}, r) dr = \int_{0}^{1} \left( W(\hat{x}, r) dr + \int_{0}^{T} W(\hat{x}, r) dr \right) + \int_{T}^{t} W(\hat{x}, r) dr$$

and so $W(\hat{x}, t)$ is an integrable function of $t$.

On the other hand, if $T > \frac{1}{\nu \text{norm}}$

$$\int_{0}^{t} W(\hat{x}, r) dr = \int_{0}^{1} \left( W(\hat{x}, r) dr + \int_{0}^{T} W(\hat{x}, r) dr \right) + \int_{T}^{t} W(\hat{x}, r) dr$$

which again implies that $W(\hat{x}, t)$ is an integrable function of $t$.

Theorem (4.2) gives a general qualitative result for the populations $N$ and $N_{1}$ provided that $W(0, t)$ is an integrable function of $t$.

Theorem 8 Let the conditions in (4.4.2) hold and let $N(x, 0) + \gamma N_{1}(x, 0) < 1$ for $0 \leq x \leq \ell$. In addition, let $W(0, t)$ be an integrable function of $t$. Then for fixed $x$, $0 \leq x \leq \ell$, $N(x, t)$ and $N_{1}(x, t)$ are integrable functions of $t$.

Proof: Since $N(x, 0) + \gamma N_{1}(x, 0) < 1$ for $0 \leq x \leq \ell$, then for fixed $x = \hat{x}$,
\[
\frac{d(N + \gamma N_1)}{dt} = (c_{11} + \gamma c_{21})N + (c_{12} + \gamma c_{22})N_1 + W[1 - \gamma N_1 - N] \\
= (c_{11} + \gamma c_{21})(N + \gamma N_1) - \gamma(c_{11}\gamma c_{21})N_1 + (c_{12} + \gamma c_{22})N_1 \\
+ W[1 - \gamma N_1 - N] \\
\leq (c_{11} + \gamma c_{21})(N + \gamma N_1) + W[1 - \gamma N_1 - N]
\]

since \(c_{12} - \gamma c_{22} - \gamma(c_{22} + \gamma c_{21}) < 0\) and \(N_1(\hat{x}, t) \geq 0\).

Therefore,

\[
\frac{d(N + \gamma N_1)}{dt} \leq (c_{11} + \gamma c_{21} - W)(N + \gamma N_1) + W.
\]

Then

\[
e^\int_0^t [W - \epsilon_{11} - \gamma c_{21}]dr\frac{d(N + \gamma N_1)}{dt} \leq e^\int_0^t [W - \epsilon_{11} - \gamma c_{21}]dr[c_{11} + \gamma c_{21} - W](N + \gamma N_1) \\
\leq e^\int_0^t [W - \epsilon_{11} - \gamma c_{21}]drW(\hat{x}, t),
\]

or

\[
\frac{d}{dt} \left[e^\int_0^t [W - \epsilon_{11} - \gamma c_{21}]dr(N + \gamma N_1)\right] \leq e^\int_0^t [W - \epsilon_{11} - \gamma c_{21}]drW(\hat{x}, t).
\]

So

\[
e^\int_0^t [W - \epsilon_{11} - \gamma c_{21}]dr[N + \gamma N_1] - [N(\hat{x}, 0) + \gamma N_1(\hat{x}, 0)] \leq \int_0^t e^\int_0^r [W - \epsilon_{11} - \gamma c_{21}]dr'W(\hat{x}, \tau)dr,
\]

and thus
\[
N + \gamma N_1 \leq e^{\int_0^t [c_{11} + \gamma c_{21} - W]dr} \int_0^t e^{\int_r^t [c_{11} + \gamma c_{21}]dr'} W(\hat{z}, r) dr \\
+ e^{\int_0^t [c_{11} + \gamma c_{21} - W]dr} \int_0^t e^{\int_r^t [c_{11} + \gamma c_{21}]dr'} W(\hat{z}, r) dr \\
= \int_0^t e^{\int_0^t [c_{11} + \gamma c_{21}]dr'} W(\hat{z}, r) dr \\
+ e^{\int_0^t [c_{11} + \gamma c_{21} - W]dr} \int_0^t e^{\int_r^t [c_{11} + \gamma c_{21}]dr'} W(\hat{z}, r) dr \\
+ e^{\int_0^t [c_{11} + \gamma c_{21} - W]dr} [N(\hat{z}, 0) + \gamma N_1(\hat{z}, 0)]
\]

Now, \(W(\hat{z}, t) \geq 0\) so \(e^{-\int_0^t W dr} \leq 1\) so

\[
N + \gamma N_1 \leq \int_0^t e^{\int_0^t [c_{11} + \gamma c_{21}]dr'} W(\hat{z}, r) dr + e^{\int_0^t [c_{11} + \gamma c_{21}]dr} [N(\hat{z}, 0) + \gamma N_1(\hat{z}, 0)]
\]

So now,

\[
\int_0^t [N(\hat{z}, t) + \gamma N_1(\hat{z}, t)] dt \leq \int_0^t \int_0^t e^{(c_{11} + \gamma c_{21})(t-r)} W(\hat{z}, r) dr dt \\
+ \int_0^t e^{(c_{11} + \gamma c_{21})(t-r)} [N(\hat{z}, 0) + \gamma N_1(\hat{z}, 0)] dt \\
= \int_0^t W(\hat{z}, r) \int_r^t e^{(c_{11} + \gamma c_{21})(t-r)} dt dr \\
+ [N(\hat{z}, 0) + \gamma N_1(\hat{z}, 0)] \left[ \frac{e^{(c_{11} + \gamma c_{21})t'} - 1}{c_{11} + \gamma c_{21}} \right] \\
= \int_0^t W(\hat{z}, r) \left[ \frac{e^{(c_{11} + \gamma c_{21})(t'-1)} - 1}{c_{11} + \gamma c_{21}} \right] dr \\
+ [N(\hat{z}, 0) + \gamma N_1(\hat{z}, 0)] \left[ \frac{e^{(c_{11} + \gamma c_{21})t'} - 1}{c_{11} + \gamma c_{21}} \right]
\]

Since \(c_{11} + \gamma c_{21} < 0\) and \(W(\hat{z}, t)\) is an integrable function of \(t\), it follows that

\([N + \gamma N_1](\hat{z}, t)\) is an integrable function of \(t\). Both \(N\) and \(N_1\) are nonnegative.
so \( N(\hat{z}, t) \) and \( N_1(\hat{z}, t) \) are integrable function of \( t \). Therefore, for fixed \( x \),

\[ 0 \leq x \leq \ell, \quad N(x, t) \text{ and } N_1(x, t) \text{ are integrable functions of } t. \]

### 4.3 Numerical Analysis

The spatial and temporal model presents a number of problems which must be considered in choosing a numerical scheme for solving the system. The laser dynamics in the active region, taken to be \( 0 \leq x \leq \ell \), and the rest of the cavity, \( \ell \leq x \leq L \), are represented by two different sets of partial differential equations which are coupled via boundary conditions at \( x = 0 \) (which corresponds to \( x = L \)) and at \( x = \ell \). In addition, the solutions must satisfy other conditions that describe the optical environment of the active medium. For example, in the simple ring cavity, the effects of the mirrors must be taken into account. It is expected that the dynamic variables will evolve in a similar way, qualitatively, as those in the spatially averaged version, so there may be regions where the time derivatives are large and in order to capture the correct behavior of the solutions, small time steps and large amounts of computer time may be needed. In this section the discretization of the problem as well as an effective algorithm for the numerical solution of the initial and boundary value problem are given. Results obtained from this algorithm are compared to those obtained from a stable finite difference algorithm. Effects on the solutions of changes in the pump pulse and injection seeding are presented.

The numerical computations were done in double precision on a DEC VAX
The emission spectrum of Ti:Sapphire from 650 to 950 nm was divided into 24 intervals of length $\Delta \lambda = 12.5$ nm. The physical parameters used were the same as for the spatially averaged version of the model and so typical values are given in Tables 3.1 and 3.2.

The pump pulse at the boundary $x = 0$ is assumed to obey a specified temporal dependence and as in Chapter 3, the cases of a Gaussian pump profile as well as a constant pump pulse are considered. For the Gaussian pump pulse,

$$W(0, t) = W_p \cdot f(t, t_p, \tau_p)$$

where

$$W_p = \eta \left[ \left( \frac{E_p}{h c} \right) \right] \left[ \frac{1}{\tau_p} \right] \left[ \frac{\sigma_{ab}}{\pi \tau_p^2} \right]$$

and

$$f(t, t_p, \tau_p) = \sqrt{\frac{\ln 2}{\pi}} \exp \left( -\frac{\ln 2}{\pi} \left[ \frac{(t - t_0)^2}{\frac{1}{2} \tau_p^2} \right] \right).$$

For the constant pump case

$$W(0, t) = W_p.$$

In each of the above expressions, the parameters are as defined previously.

The model also allows for the introduction of an injected signal. Injection seeding is accomplished by a reverse wave suppressor mirror external to the ring cavity [3,6] (Fig. 4.1). The contribution to the photon concentration by injection seeding is incorporated into the boundary condition at the output coupler, $x = L_1$ and given by
Figure 4.1 Schematic of a ring laser cavity which allows for injection seeding. A fraction of the laser output is reintroduced into the system through the output mirror.
\[ \Phi_+ (L_1, t; \lambda_j) = \Phi_+ (L_1, t; \lambda_j) + R_I \Phi_- \text{out}(t; \lambda_j) \]

where \( R_I \) is the reflectively of the external mirror and \( \Phi_- \text{out}(t; \lambda_j) \) is the fraction of the reverse wave which is transmitted through the output mirror.

The numerical procedure used to solve the initial and boundary value problem is motivated by the "method of characteristics" [8,11] and exploits the particularly simple form that the characteristics of the system enjoy. Since the characteristics do not depend on the solution and in fact are known in advance to be three families of lines in the \( x - t \) plane, the system of partial differential equations can be reduced to a system of ordinary differential equations along the relevant characteristics in a suitable small region. For example, assume that initial conditions (Fig. 4.2) are specified along a segment of the \( x \)-axis and that the point \( P(\xi, \eta) \) is a point in a small vicinity of the \( x \)-axis. The solutions at \( P(\xi, \eta) \) are found by integrating each ordinary differential equation along the appropriate characteristic and applying the initial conditions.

By introducing a lattice of mesh points that coincide with points of intersection of the characteristic lines, at each lattice point the system of partial differential equations together with the initial and boundary conditions is reduced to an initial value problem with the corresponding system of ordinary differential equations along the characteristics.

Consider the ring cavity configuration in Fig. 4.1 For simplicity, the effect of the second and third mirrors are incorporated into the boundary conditions.
Figure 4.2 Determination of solutions at a point \( P(\xi, \eta) \) of the partial differential equations by integration along the relevant characteristics.
of $\phi_+$ at $x = 0$ and for $\phi_-$ at the location of the first mirror, $x = L_1$. As in the figure, it is assumed that the cavity is equilateral, however, the assumption is without loss of generality.

Let $M$ be the number of spatial increments in the crystal. Then the length of each increment is given by $\Delta x_{cr} = \frac{x}{M}$. The time increment will be the length of time necessary for a photon to travel a distance of $\Delta x_{cr}$. Thus

$$\Delta t = \frac{1}{v} \Delta x_{cr}$$

which implies that

$$\Delta x_{cav} = c \Delta t.$$

It is assumed for convenience that the locations of the first mirror and edge of the crystal at $x = L$ coincide with spatial grid points. To approximate this situation, let $M_1$ be the number of spatial increments between the edge of the crystal $x = \ell$ and the mirror at $x = L_1$, defined by

$$M_1 = \text{nearest integer } \left\lfloor \frac{L_1 - \ell}{\Delta x_{cav}} \right\rfloor.$$

Also, let $M_2$ be the number of spatial increments between the first mirror at $x = L_1$ and the edge of the crystal at $x = L$, defined by

$$M_2 = \text{nearest integer } \left\lfloor \frac{L - L_1}{\Delta x_{cav}} \right\rfloor.$$

Then $M_T = M + M_1 + M_2$ defines the number of spatial increments in the cavity.

Now consider the discrete functions $N^r_s, N^{r*}_{1s}, W^r_s$, and $\phi^r_{xjs}$ defined by
\[
N'_r = N(x_r, t_s) \\
N'_{1s} = N_1(x_r, t_s) \\
W'_r = W(x_r, t_s) \\
\phi_{\pm j_0}^r = \phi_{\pm}(x_r, t_s; \lambda_j) \quad j = 1, ..., m
\]

for \( r = 0, 1, ..., M \) and \( s = 0, 1, ... \), which correspond to the solution in the active region at the mesh points. Corresponding to the solutions at the mesh point in the rest of the cavity, the discrete function \( \Phi_{\pm j_0}^r \), \( j = 1, ..., m \) are defined similarly for \( r = M, M + 1, ..., M_T \) and \( s = 0, 1, ... \).

In terms of the discrete functions, the initial conditions are

\[
N'_0 = N'_{10} = 0 \quad r = 0, 1, ..., M \\
\Phi_{\pm j_0}^r = 0 \quad r = 0, 1, ..., M \\
\Phi_{\pm j_0}^r = 0 \quad r = M + 1, ..., M_T \quad j = 1, ..., m. \quad (4.3.1)
\]

The boundary conditions at \( x = 0 \) and \( x = L \) become

\[
\phi_{+j_0}^0 = T_2 R_2 R_9 \Phi_{+j_0}^{M_T} \\
\phi_{-j_0}^0 = T_3 \phi_{-j_0}^0, \quad (4.3.2)
\]

at the edge of the crystal \( x = \ell \),
\[ \Phi_{+j_s}^M = T_s \Phi_{+j_s}^M \]
\[ \phi_{-j_s}^M = T_s \Phi_{-j_s}^M , \quad (4.3.3) \]

and at the location of the first mirror \( x = L_1 \)

\[ \phi_{+j_s}^{M_1+1} = R_1 \Phi_{+j_s}^{M_1} + R_{j_1} T_1 \Phi_{-j_s}^{M_1+1} \]
\[ \Phi_{-j_s}^{M_1-1} = R_1 R_2 R_3 \Phi_{-j_s}^{M_1} , \quad j = 1, \ldots, M \quad (4.3.4) \]

where \( R_1, R_2, R_3, R_{j_1}, \) and \( T_1 \) are as previously defined and \( s = 1, 2, \ldots, \).

The discrete pump function has initial and boundary conditions given by

\[ W'^0_s = W_p \cdot f(t_s, t_p, \tau_p) \text{ for } s = 0, 1, \ldots, \text{ where } W_p \text{ and } f(t, t_p, \tau_p) \text{ are as specified before} \]
or

\[ W'^0_s = W_p. \]

The homogeneous equations in the region \( x = \ell \) to \( x = L \) are particularly easy to integrate, for if \( t \) is the independent variable along a characteristic,

\[ \frac{d \Phi_{\pm}(x(t), t; \lambda_j)}{dt} = 0 \]

which implies that

\[ \Phi_{+j_s+1}^{r+1} = \Phi_{+j_s}^r \quad (4.3.5) \]
for } r = M, \ldots, M_1 - 1 \text{ and } r = M_1 + 1, \ldots, M_T \\
and \\

\Phi_r^{-1} = \Phi_{r,s} \\
(4.3.6) \quad \\

\text{for } r = M_T, M_T - 1, \ldots, M_1 + 1 \text{ and } r = M_1 - 1, M_1 - 2, \ldots, M. \\

Note that \\
\Phi_{r,s}^{M_1+1} \text{ and } \Phi_{r,s}^{M_1-1} \text{ are specified by (4.3.4).} \\

The nonlinear ordinary differential equations along the characteristics in the active region must be iterated by an approximate method. Euler's Method is a first order accurate [as in 7] procedure that can be implemented in the following way. For the population equations, integration is along the vertical lines } x = x_r, \\
so that \\

N_{s+1}^r = N_s^r + \Delta t \left[ \frac{dN(x_r,t_s)}{dt} \right] \\
N_{1s+1}^r = N_{1s}^r + \Delta t \left[ \frac{dN_1(x_r,t_s)}{dt} \right], \quad r = 0, \ldots, M \\
and s = 0, 1, \ldots, \\
(4.3.7) \\

The functions } W \text{ and } \phi_+ \text{ are integrated along the lines parallel to } t = \frac{1}{\epsilon} x, \text{ so} \\

W_{s+1}^r = W_s^r + \Delta t \left[ \frac{dW(x_r,t_s)}{dt} \right] \\
\phi_{s+1}^{r+1} = \phi_s^r + \Delta t \left[ \frac{d\phi_+(x_r,t_s; \lambda_s)}{dt} \right], \quad r = 0, \ldots, M \\
and s = 0, 1, \ldots, \\
(4.3.8)
Similarly, $\phi_-$ is integrated along the lines parallel to $t = \frac{1}{c} x$, so

$$
\phi_{r-1,j+1}^{s+1} = \phi_{r,j}^{s+1} + \Delta t \left[ \frac{d\phi_{-}(x_r,t_s;\lambda_j)}{dt} \right] \quad r = M, \ldots, 1 \quad \text{and} \quad s = 0, 1, \ldots ,
$$

For illustrative purposes, the pump energy was taken to be large, $E_p = 200\text{mJ}$ and no injection present in the system.

Figures 4.3-4.6 display the time evolution of the numerical solution at the first interior grid point in the crystal using $M=5$. If $M$ is increased by a factor of two, the qualitative behavior of the solutions is basically unchanged as shown by Figs 4.7-4.10, but quantitatively the solutions are slightly different. While increased accuracy is expected from a smaller step size, it is at the expense of tremendously increased computational effort necessary to calculate the solutions.

The accuracy of Euler's method can be improved to second order by implementing the Modified Euler Method [as in 7] in the following way. The values of the discrete functions are predicted at the new grid points by the Euler relations (4.3.7) - (4.3.8). Then an improved estimate is obtained by iteration using the relations

\[
\begin{align*}
\mathcal{N}_{r+1}^s &= N_r^s + \Delta t \left[ \frac{1}{2} \left( \frac{dN(x_r,t_{s+1})}{dt} + \frac{dN(x_r,t_s)}{dt} \right) \right] \\
\mathcal{N}_r^{s+1} &= N_r^s + \Delta t \left[ \frac{1}{2} \left( \frac{dN(x_r,t_{s+1})}{dt} + \frac{dN(x_r,t_s)}{dt} \right) \right] \\
W_{r+1}^s &= W_r^s + \Delta t \left[ \frac{1}{2} \left( \frac{dW(x_{r+1},t_{s+1})}{dt} + \frac{dW(x_r,t_s)}{dt} \right) \right] \\
\phi_{r+1,j}^s &= \phi_r^s + \Delta t \left[ \frac{1}{2} \left( \frac{d\phi_+(x_{r+1},t_{s+1};\lambda_j)}{dt} + \frac{d\phi_-(x_r,t_s;\lambda_j)}{dt} \right) \right] \\
\phi_{r-1,j}^s &= \phi_r^s + \Delta t \left[ \frac{1}{2} \left( \frac{d\phi_+(x_{r-1},t_{s+1};\lambda_j)}{dt} + \frac{d\phi_-(x_r,t_s;\lambda_j)}{dt} \right) \right]
\end{align*}
\]
Figure 4.3 The computed pump pulse ($E_p=200\text{mJ}$) using Euler's Method for integration with $M=5$. The plot displays the results at the first interior grid point.
Figure 4.4 The computed population inversion at the first interior grid point. $E_p = 200 \text{ mJ}$.
Figure 4.5 The computed lower laser level population at the first interior grid point. $E_p = 200$ mJ.
Figure 4.6 The computed total photon concentration at the first grid point. $E_p=200$ mJ.
Figure 4.7 The computed pump pulse \((E_p = 200 \text{ MJ})\) using Euler's Method for integration with \(M=10\). The plot displays the results at the second interior grid point.

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Figure 4.8 The computed population inversion at the second inter grid point. $E_p = 200 \text{ mJ}$.
Figure 4.9 The computed lower laser level concentration at the second interior grid point. $E_p = 200$ mJ.
Figure 4.10 The computed total photon concentration at the second interior grid point. $E_p = 200 \text{ mJ}$. 

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Figure 4.11 is a plot of the computed solution for the population inversion at the first interior grid point using one iteration of the Modified Euler procedure. By using two iterations, a significant improvement over Euler's method with the same number of grid points in the crystal is shown by comparing Fig. 4.11 and Fig. 4.12. In fact, doubling the number of grid points in the Modified Euler procedure with two iterations does not significantly change the solution quantitatively as is evident from Fig. 4.13. While the calculated solutions are shown for the population inversion, the same comparison applies to the other dynamic variables.

While the Modified Euler Method with two iterations requires more computational effort than Euler's Method alone, it was necessary to increase the number of spatial increments in the crystal to $M=20$ in order to achieve comparable accuracy with $M=5$ in the Modified Euler scheme. Therefore, using the Modified Euler Method with a smaller number of mesh points is more efficient than using Euler's Method with four times the number of mesh points.

To verify the results obtained by using the method of characteristics, a stable first order finite difference approximation [11] was employed. Figs. 4.14-4.15 display the results for the population inversion for $M = 10$ and $M = 50$ at corresponding grid points in the crystal. The qualitative behavior for the latter case ($M = 50$) compares favorably to the results obtained by the method of characteristics but the computer time required to compute the first order solutions was several times greater than with the Modified Euler Method. The Modified
Figure 4.11 Curve demonstrating the improvement in accuracy in the computed population inversion using one iteration of the Modified Euler Method for integration. Results are shown for $M=5$ at the first interior grid point with $E_p = 2300 \text{ mJ}$. 

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Figure 4.12 Further improvement in accuracy in the computed population inversion using two iterations of the Modified Euler scheme. Results are shown for $M=5$ at the first interior grid point with $E_p=200$ mJ.
Figure 4.13 Computed population inversion using two iterations of the Modified Euler Method with \( M = 10 \). Results are shown at the second interior grid point with \( E = 200 \text{ mJ} \). No significant improvement over the case with \( M = 5 \) is demonstrated.
Figure 4.14 Computed population inversion using a finite difference algorithm. Here, $M=10$ and $E_p=200$ mJ. Results are shown at the second interior grid point.
Figure 4.15 Computed population inversion using a finite difference algorithm. In this case, $M=50$ and $E_p=200$ mJ. Results are shown at the tenth interior grid point.
Euler Method with $M = 5$ required one-fourth the CPU time necessary to run the first order finite difference method with $M = 10$. The Modified Euler procedure, with $M = 5$ was almost forty times faster in terms of CPU time than the finite difference approximation for $M = 50$ and only then are the solutions comparable. It is evident that integration along the characteristics using a Modified Euler Algorithm with a spatial and temporal discretization described here represents a significant improvement in the accuracy of the computed solutions while providing a significant savings in computational effort over that required for comparable solutions using the finite difference algorithm.

One observation that can be made from the plots of the population inversion is that this quantity appears to decay very slowly, unlike the corresponding situation in the spatially averaged version of the model. It was proved in 4.2 that the population inversion is integrable as a function of $t$ for fixed position $\bar{x}$. To see that this result is not violated by the numerical approximation of the solution, note that it is evident that for some time $T$, the quantities $N_1(\bar{x}, t), \phi_{\pm}(\bar{x}, t), \text{and } W(\bar{x}, t)$ are negligible for $t \geq T$ so that the differential equation for $N$ is approximated by

$$\frac{dN}{dt} \approx c_{11}N$$

and so

$$N(\bar{x}, t) \sim N(\bar{x}, T)e^{c_{11}(t-T)} \quad \text{for } t > T$$

where $c_{11}$ is small in magnitude and negative.
To study the effects on the solutions of changes in the pump pulse, the Modified Euler Method was used to compute the numerical solutions. Figs. 4.16 and 4.17 show the temporal development of the population inversion and photon concentrations at each interior grid points for a system in which no injection is present and the pump pulse is constant with \( E_p = 10mJ \). The laser crystal is assumed to be end pumped so the pump pulse is absorbed as it moves through the crystal. Consequently, it is expected that at a position in the crystal closer to the pump, the inverted population density will be greater than at a position further from the pump source. It is also expected that the photon density will increase due to stimulated emission as photons propagate through the crystal and that density of the clockwise traveling photons will dominate the counterclockwise traveling photons so that the total photon concentration will increase as one moves through the crystal. These qualitative properties are observed in the numerical results as is evident from Fig. 4.16 and 4.17. The oscillatory behavior is consistent with the results from the spatially averaged version of the model assuming similar conditions. If the pump energy is increased to \( E_p = 20mJ \), (Fig. 4.18 and 4.19) the oscillations appear to be damped out more quickly, which is also consistent with the results of spatially averaged version.

Figs. 4.20 and 4.21 display the effect of introducing an injected signal of the type described earlier. In these examples, the pump pulse at the boundary of the crystal is a Gaussian pulse with \( E_p = 10mJ \) and \( E_p = 20mJ \). Again, the
results are consistent with the dynamics of the spatially averaged model.
Figure 4.16 Population inversion computed at each of the interior grid points in a system with constant pump, $E_p = 10 \text{ mJ}$. 

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Figure 4.17  Total photon concentration at each interior grid point in a system with constant pump, $E_p = 10$ mJ.
Figure 4.18 Population inversion at each interior grid point in a system with constant pump, $E_p = 20$ mJ.
Figure 4.19 Total photon concentration at each interior grid point in a system with constant pump, $E_p = 20$ mJ.
Figure 4.20 Curves showing the effect on the population inversion of the presence of an injection signal. Results are shown for varying amounts of output pulse reintroduced into the system. Here, $E = 20\text{mJ}$ and Gaussian boundary distribution of the pump pulse.
Figure 4.21 Curves showing the effect on the total photon concentration of the presence of an injection signal. Results are shown for varying amounts of output pulse reintroduced into the system. Here, $E_p=20$ mJ and Gaussian boundary distribution of the pump pulse.
Chapter 5

Concluding Remarks

The analytical study of both the spatial and temporal model and the spatially averaged version of the model verify that the solutions satisfy physically realistic qualitative properties under realistic assumptions about the laser material and the operating environment of the laser.

While the spatially averaged version of the model does not offer specific information on spatial variations in the laser dynamics, it does provide insight into the effects on the solutions of changes in the physical parameters, and pump pulse. It allows the introduction of an injection signal so that injection seeding as a line narrowing technique may be studied. The stability analysis provides estimates on laser threshold conditions that must be exceeded in order for stimulated emission to occur.

The spatial and temporal model is complete in the sense that it provides the capability of analyzing the dynamics at any point in the laser cavity. It also
includes the dependence of the photons on discrete wavelengths in the emission spectrum. While the study was based on a simple ring cavity configuration, the numerical procedure developed here gives approximate solutions which agree favorably with the results from a stable finite difference algorithm that required many more grid points and hence much more computer effort to obtain comparable accuracy. The solutions generated by the Modified Euler Method in conjunction with the method of characteristics also agreed favorably with anticipated qualitative properties.

The spatial and temporal model, together with the qualitative and numerical analysis here, provide a powerful analytical tool for the study of an optically pumped four-level solid state laser system.
References


13. Langley Research Center Implementation of LSODE, and ODE software package based on the GEAR and GEARB packages (A.C. Hindmarsh, Lawrence Livermore Lab.)


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Appendix I

Computer Program: Equilibrium
THIS PROGRAM CALCULATES THE EQUILIBRIUM POINTS
OF THE FOUR-LEVEL THREE EQUATION MODEL. IT
DETERMINES THE STABILITY PROPERTIES OF THE
EQUILIBRIUM POINTS. IN ADDITION, IT ALSO FINDS
THE EIGENVALUES OF THE LINEARIZED SYSTEM BY
NEWTON'S METHOD TO FIND THE FIRST EIGENVALUE
THEN FACTORS THE REMAINING
QUADRATIC TO DETERMINE
THE OTHER TWO EIGENVALUES.

PROGRAM EQUILIBRIUM
IMPLICIT DOUBLE PRECISION(A-H,O-Z)

OPEN DATA FILES:
EQUILPT4.DAT EQUILIBRIUM POINTS
DISPT1.DAT STABILITY PROPERTIES AND
NATURE OF THE LINEARIZED
EIGENVALUES
DISPT2.DAT STABILITY PROPERTIES AND
EIGENVALUES
OPEN(UNIT=1,FILE='EQUILPT4.DAT',STATUS='NEW')
OPEN(UNIT=2,FILE='DISPT2.DAT',STATUS='NEW')
OPEN(UNIT=3,FILE='DISPT1.DAT',STATUS='NEW')

C*****TI:SAPPHIRE PHYSICAL PARAMETERS*****
G21=2.0D0/3.0D0
SIGMA=3.0D-19
C=3.0D10
DEX=1.76D0
V=C/DEX
TOTAL=1.0D18
TAUFL=3800.0D0
TAU1=1.0D0
TAUC=16.5D0
TAUN=1.0D-09
TAU2=3000.0D0

C*****PUMP PARAMETER**********
WP=.001

C*****SPONTANEOUS EMISSION PARAMETER*************
SPO=1.0D-04

C*****DIFFERENTIAL EQUATION COEFFICIENTS*******
ALPHA=SIGMA*V*TOTAL*TAUN
C11=-(1.0D0/TAU2+G21/TAUFL)
C12=-G21*(-C11-1.0D0/TAU1)
C21=1.0D0/TAUFL
C22=G21/TAUFL-1.0D0/TAU1
C31=SPO/TAUFL
C32=G21*C31
C33=-1.0D0/TAUC

C*****CALCULATION OF EQUILIBRIUM POINTS******
D21=C31-C21

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D22 = C32 - C22
B11 = C11 - WP
B12 = C12 - GAMMA * WP
D11 = B11 + GAMMA * C21
D12 = B12 + GAMMA * C22
DET = D11 * D22 - D12 * D21
A = ALPHA * D12 * C33
B = C33 * (C21 * B12 - C22 * B11) - ALPHA * WP * D22
C = WP * (C22 * C31 - C21 * C32)
DISCR = B * B - 4.0D0 * A * C

IF (DISCR.LT.0.0D0) THEN
  TYPE 201
  GO TO 30
ELSE
  D = 2.0D0 * A

CCCCCCBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBB
ITFLAG=0
IROOT=0
IEIGEN=0
MFLAG=0
ISTAB=0

C CALCULATION OF CUBIC DISCRIMINANT FOR FIRST EQUIL. PT

A111=C11-ALPHA*GAMMA*Z1-WP
A121=C12-WP*GAMMA
A131=-ALPHA*GAMMA*X1
A211=C21+ALPHA*Z1
A221=C22
A231=ALPHA*X1
A311=C31+ALPHA*Z1
A321=C32
A331=C33+ALPHA*X1

BETA1=-(A111+A221+A331)
GAMMA1=-(A231*A321+A211*A121+
1 A311*A131-A111*A221-A331*A111-
2 A331*A221)

SIGMA1=-(A111*A231*A321-A211*A121*A331
1 +A211*A321*A131+A311*
2 A121*A231-A311*A131*A221)
P=-(1.0D0/3.0D0)*BETA1*BETA1+GAMMA1
Q=-GAMMA1*BETA1/3.0D0+SIGMA1+2.0D0*BETA1

DIS=-27.0D0*Q*Q-4.0D0*P*P*P

TEST=BETA1*GAMMA1-SIGMA1

C STABILITY CHECK FOR FIRST EQUILIBRIUM POINT

IF(BETA1.GT.0.0D0. AND.SIGMA1.GT.0.0D0
1 .AND.TEST.GT.0.0D0)THEN
ISTAB=1
WRITE(1,101)ISTAB
WRITE(2,200)WP,DIS,ISTAB
ELSE
ISTAB=0
WRITE(1,101)ISTAB
WRITE(2,200)WP,DIS,ISTAB
END IF

C CHECK FOR NATURE OF EIGENVALUES

IF(DIS.LT.0.0D0) THEN
IEIGEN=1
WRITE(1,102)IEIGEN
ELSE
IF(DIS.GT.0.0D0) THEN
IEIGEN=0
WRITE(1,102)IEIGEN
ELSE
IEIGEN=0
MFLAG=1

155
WRITE(1,103)IEIGEN,MFLAG
END IF
END IF

CALCULATION OF EIGENVALUES—FIRST DETERMINED BY
NEWTON’S METHOD. REMAINING TWO DETERMINED FROM
REDUCED QUADRATIC.

CALL ANEWTON(BETA1,GAMMA1,SIGMA1,E1,
1
   ITFLAG)
   IF(ITFLAG.EQ.1)GO TO 10
   CALL ROOTS(E1,BETA1,GAMMA1,SIGMA1
1
   ,E2R,E2I,E3R,E3I,IROOT)
   IF(IROOT.EQ.1)GO TO 10
   WRITE(1,105)E1,E2R,E2I,E3R,E3I
10 CONTINUE
   WRITE(1,104)ITFLAG,IROOT
   WRITE(1,150)WP,X2,Y2,Z2

CHECK FOR STABILITY OF SECOND EQUILIBRIUM POINT
CHECK FOR NATURE OF EIGENVALUES

ITFLAG=0
IROOT=0
IEIGEN=0
MFLAG=0
ISTAB=0
A112=C11-ALPHA*GAMMA*Z2-WP
A122=C12-WP*GAMMA
A132=-ALPHA*GAMMA*X2
A212=C21+ALPHA*Z2
A222=C22
A232=ALPHA*X2
A312=C31+ALPHA*Z2
A322=C32
A332=C33+ALPHA*X2
BETA2=-(A112+A222+A332)
GAMMA2=-(A112*A222*A332-
1
   A112*A232*A322-A212*A122*A332
   +A212*A322*A132+
3
   A312*A122*A232-A312*A132*A222)
P=-(1.0D0/3.0D0)*BETA2*BETA2+GAMMA2
Q=-GAMMA2*BETA2/3.0D0+SIGMA2+
1
   2.0D0*BETA2*BETA2/27.0D0
DIS=-27.0D0*Q-Q-4.0D0*P*P*P
TEST=BETA2*GAMMA2-SIGMA2
IF(BETA2.GT.0.0D0.AND.SIGMA2.GT.0.0D0.
1
   AND.TEST.GT.0.0D0)THEN
   ISTAB=1
   WRITE(1,101)ISTAB
   WRITE(3,200)WP,DIS,ISTAB
ELSE
   ISTAB=0
   WRITE(1,104)ITFLAG,IROOT
   WRITE(1,150)WP,X2,Y2,Z2

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WRITE(1,101)ISTAB
WRITE(3,200)WP,DIS,ISTAB
END IF
IF(DIS.LT.0.0D0) THEN
IEIGEN=1
WRITE(1,102)IEIGEN
ELSE
IF(DIS.GT.0.0D0) THEN
IEIGEN=0
WRITE(1,102)IEIGEN
ELSE
IEIGEN=0
MFLAG=1
WRITE(1,103)IEIGEN,MFLAG
END IF
END IF
CALL ANEWTON(BETA2,GAMMA2,SIGMA2,
E1,ITFLAG)
IF(ITFLAG.EQ.1)GO TO 20
CALL ROOTS(E1,BETA2,GAMMA2,SIGMA2,
E2R,E2I,E3R,E3I,IROOT)
IF(IROOT.EQ.1)GO TO 20
WRITE(1,105)E1, E2R,E2I,E3R,E3I
END IF
20 CONTINUE
WRITE(1,104)ITFLAG,IROOT
30 CONTINUE
45 CONTINUE
100 FORMAT(' WP=',G22.9,/, ' 2ND EQUILIBRIUM POINT',3G15.8)
101 FORMAT(' STABILITY FLAG',I10)
102 FORMAT(' COMPLEX EIGENVALUE FLAG',I10)
103 FORMAT(' COMPLEX EIGENVALUE FLAG',I10,/,
1 ' MULTIPLICITY FLAG',I10)
104 FORMAT(' ITERATION FLAG',I10,' ROOT CHECK FLAG',I10)
105 FORMAT(' EIGENVALUES',/,G22.8,/,2(2G22.8,/)1
150 FORMAT(' WP=',G22.8,/, ' 1ST EQUILIBRIUM POINT',3G15.8),
200 FORMAT(2G22.8,I10)
STOP
END

ccccccccccccccccccccccccccccccccccccccccccccccccccc
SUBROUTINE ANEWTON(B,C,D,X,ITFLAG)
INPUT:
C
B,C,D=COEFFICIENTS OF CUBIC
C
OUTPUT:
C
X=COMPUTED ROOT
C
ITFLAG=ITERATION FLAG
IMPLICIT DOUBLE PRECISION(A-H,O-Z)
ITFLAG=0
TOI=1.0D-12
MAXIT=200
X1=B*.5D0
DO 10 I=1,MAXIT
   DENOM=DERIV(B,C,X1)
   DNUMER=FUNCT(B,C,D,X1)
   DNUX=X1-DNUMER/DENOM
   TEST=FUNCT(B,C,D,DNUX)
   TE=DABS(DNUX-X1)
   IF(DABS(TE).LT.TOL.AND.TEST.LT.TOL)GO TO 20
   X1=DNUX
10  CONTINUE
   ITFLAG=1
   GO TO 30
20  CONTINUE
   X=DNUX
30  CONTINUE
RETURN

FUNCTION DERIV(B,C,X)
C FUNCTION CALLED BY NEWTON'S METHOD TO COMPUTE
C DERIVATIVE
IMPLICIT DOUBLE PRECISION(A-H,O-Z)
BETAM=3.0D0*X+2.0D0*B
DERIV=X*BETAM+C
RETURN

FUNCTION FUNCT(B,C,D,X)
C FUNCTION CALLED BY NEWTON TO EVALUATE CUBIC
IMPLICIT DOUBLE PRECISION(A-H,O-Z)
BETAF=X+B
GAMMAF=X*BETAF+C
FUNCT=X*GAMMAF+D
RETURN

FUNCTION ROOTS( X ,B,C,D, R1R, R1I,R2R, R2I,IROOT)
C FUNCTION COMPUTES ROOTS OF REDUCED QUADRATIC
IMPLICIT DOUBLE PRECISION(A-H,O-Z)
IROOT=0
BETAR=X+B
GAMMAR=X*BETAR+C
TEST=X*GAMMAR+D
IF(TEST.LT.1.0D-12) THEN
   DISC=BETAR*BETAR-4.0D0*GAMMAR
   IF(DISC.LT.0.0D0)THEN
      R1R=-BETAR/2.0D0
      R2R=R1R
      R1I=DSQRT(-DISC)/2.0D0
      R2I=-R1I
   ELSE
      R1I=0.0D0
      R2I=R1I
END IF
R1R = -BETAR/2.0D0 + DSQRT(DISC)/2.0D0
R2R = -BETAR/2.0D0 - DSQRT(DISC)/2.0D0
END IF
ELSE
  IROOT = 1
END IF
RETURN
END
Appendix II

Computer Program: Ring
PROGRAM RING
C RING COMPUTES APPROXIMATE SOLUTIONS TO THE PDE MODEL
C OF A RING LASER USING THE MODIFIED EULER METHOD
C THE RING LASER CAVITY IS ASSUMED TO BE EQUILATERAL
C IF THIS IS NOT THE CASE SOME MODIFICATION TO THE
C GRID WILL BE NECESSARY.
C
C SUBROUTINES:
C SWITCH
C PHICAV
C DQ1
C DQ2
C PUMP
C
C DISCRETIZATION PARAMETERS
C
C NXCR=#SPATIAL INCREMENTS IN CRYSTAL
C NXTOTAL=MAXIMUM NUMBER OF SPATIAL INCREMENTS IN
C CAVITY
C NL=NUMBER OF WAVELENGTHS IN EMISSION SPECTRUM OF
C CONCERN
C
C DEPENDENT VARIABLES: (CURRENT VALUES)
C X— ARRAY OF LENGTH NXCR+1. POPULATION INVERSION
C X1-ARRAY OF LENGTH NXCR+1. LOWER LEVEL POPULATION
C W— ARRAY OF LENGTH NXCR+1. PUMP
C PHI-ARRAY OF DIMENSIONS 2xNXTOTL+1xNL. RIGHT AND
C LEFT TRAVELING PHOTON CONCENTRATIONS IN THE
C ACTIVE REGION AND IN THE CAVITY
C TEMPORARY VALUES OF THE DEPENDENT VARIABLES ARE
C STORED IN THE ARRAYS
C XTMP,X1TMP,WTMP,PHITMP.
C PREVIOUS VALUES OF THE DEPENDENT VARIABLES ARE
C STORED IN THE ARRAYS
C XOLD,X1OLD,WOLD,PHIOLD.
C NEWLY COMPUTED VALUES OF THE DEPENDENT VARIABLES
C ARE STORED IN THE ARRAYS
C XNEW,X1NEW,WNEW,PHINEW.
C VALUES OF THE DIRECTIONAL DERIVATIVES

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PARAMETER(NXCR=5,NXTOTL=1000,NL=24)

C tägliches DONE IN REAL DOUBLE PRECISION
IMPLICIT DOUBLE PRECISION(A-H,L,O-Z)

DIMENSION X (0:NXCR), XI(0:NXCR), XTMP(0:NXCR),
1 X1TMP(0:NXCR), XNEW(0:NXCR), X1NEW(0:NXCR),
2 W (0:NXCR), WNEW(0:NXCR), WTMP(0:NXCR),
3 PHI(2,0:NXTOTL,NL), PHINEW(2,0:NXTOTL,NL),
4 DWOLD(0:NXCR), DXOLD(2,0:NXCR), DW(0:NXCR),
5 DX(2,0:NXCR), LAM(NL), OUTPUT(2,NL),
6 DPHI(2,0:NXCR,NL), PHITMP(2,0:NXTOTL,NL),
7 XONEW(0:NXCR), DPHOLD(2,0:NXCR, NL),
8 DWTMP(0:NXCR), DXTMP(2,0:NXCR),
9 DPTMP(2,0:NXCR,NL), SIG(NL),
A WOLD(0:NXCR), PHIOLD(2,0:NXTOTL,NL),
B XOLD(0:NXCR), XIOLD(0:NXCR)

C WAVELENGTHS IN MICRONS
DATA LAMBDA/.650D0,.675D0,.6875D0,.700D0,.712D0,
1 .725D0,.7375D0,.750D0,.7625D0,.775D0,
2 .7875D0,.800D0,.812D0,
3 .825D0,.8375D0,.850D0,
4 .8625D0,.875D0,.8875D0,.900D0,.9125D0,
5 .925D0,.9375D0,.950D0/

C CROSS SECTIONS IN CM^2
DATA SIGMA/3.48D-20,8.0D-20,1.2D-19,1.6D-19,2.0D-19,
1 2.4D-19,2.8D-19,3.22D-19,
2 3.52D-19,3.79D-19,3.88D-19,
3 3.94D-19,3.88D-19,3.8D-19,3.48D-19,
4 3.272D-19,3.12D-19,2.9D-19,2.62D-19,
5 2.432D-19,2.28D-19,2.0D-19,1.86D-19,
6 1.6D-19/

C ABSORPTION COEFFICIENTS IN 1/CM
DATA ALPHA/.07D0,.07D0,.07D0,.07D0,.07D0,.07D0,
1 .07D0,.07D0,.07D0,.07D0,.07D0,.07D0,.
2 .07D0,.07D0,.07D0,.07D0,.07D0,.07D0,.
3 .07D0,.07D0,.07D0,.07D0,.07D0,.07D0,.

C COMMON BLOCKS
COMMON/COEFF/C11,C12,C21,C22,GAMMA,V,TOTALN,
1 XNORM, TNORM, BETA, BETAG, PFACR, XNORML,
COMMON/SP/L,R,DEX,DELTAX,DLAMDBA(NL),
COMMON/REFL/TS,R1,R2,R3,T1,FRACTN,AREA,
COMMON/DISCRT/N1,N2,MIDX
G2G1=RATIO OF DEGENERACIES
DEX=INDEX OF REFRACTION
TAUFL=FLUORESCENCE LIFETIME (NS)
TAU2
TAU1
TS=FRESNEL TRANSMISSION COEFFICIENT
G2G1=2.0D0/3.0D0
DEX=1.76D0
TAUFL=3870.0D0
TAU2=3150.0D0
TAU1=1.0D0
TS=4.0D0*DEX/((DEX+1.0D0)*(DEX+1.0D0))
TOTALN=CONCENTRATION OF DOPANT IONS (/CM^3)
L=LENGTH (CM)
R=RADIUS (CM)
TOTALN=1.26D19
L=2.5D0
R=.05D0
ETOTL=ENERGY OF PUMP (mJ)
TAU, T0=GAUSSIAN BEAM PARAMETERS (NS)
BWAIST=BEAM WAIST (CM)
ETAP=BRANCHING RATIO
PLMBDA=PUMP WAVELENGTH (MICRON)
SIGMAA=ABSORPTION CROSS SECTION (CM^2)
ETOTL=200.0D0
TAU=10.0D0
T0=30.0D0
BWAIST=.05D0
ETAP=1.0D0
PLMBDA=.532D0
PLMBDA=PLMBDA*1.0D-04
SIGMAA=5.12D-20
LP=LENGTH OF CAVITY
R1=REFLECTION OF 1ST MIRROR
T1=TRANSMISSION OF 1ST MIRROR
R2=REFLECTION OF 2ND MIRROR
R3=REFLECTION OF 3RD MIRROR
LP=120.0D0
R1=.4D0
T1=.5D0
R2=0.9D0
R3=0.9D0
BEGIN-END TIMES (NS)

TBEGIN=0.0D0
TEND=130.0D0

REFLECTIVITY OF REVERSE WAVE SUPRESSOR MIRROR

FRACTN=0.0D0

OPEN DATA OUTPUT FILES (USER DEPENDENT)

FOR EXAMPLE: THE FOLLOWING MAY BE OPENED TO PRINT THE VALUES OF THE POPULATION INVERSION, PHOTON CONCENTRATION AND PUMP AT A PARTICULAR GRID POINT:

OPEN (4, FILE = 'N.DAT', STATUS = 'NEW')
OPEN (5, FILE = 'PHI.DAT', STATUS = 'NEW')
OPEN (6, FILE = 'PUMP.DAT', STATUS = 'NEW')

NUMBER OF WAVELENGTHS IN THIS EXAMPLE

NLMBDA MUST BE LESS THAN OR EQUAL TO NL-1

NLMBDA=NL-1

COMPUTE DLAMBDAS AND XSECTION AND WAVELENGTH AVERAGES

DO 5 I = 1, NLMBDA
    DLAMBA(I) = LAMBDA(I+1) - LAMBDA(I)
    SIG(I) = 0.5D0*(SIGMA(I+1) + SIGMA(I))
    LAM(I) = 0.5D0*(LAMBDA(I+1) + LAMBDA(I))
5 CONTINUE

RENAMED SIG AND LAM—CONVERT LAMBDAS AND DLAMBDAS TO CM

DO 8 I = 1, NLMBDA
    SIGMA(I) = SIG(I)
    LAMBDA(I) = LAM(I) * 1.0D-04
    DLAMBA(I) = DLAMBA(I) * 1.0D-04
8 CONTINUE

NORMALIZING CONSTANTS

XNORM=TOTALN
TNORM=1.0D-09

OUTPUT PARAMETER—WILL PRINT EVERY 10th VALUE TO DATA FILE

IO=1

PHYSICAL CONSTANTS

C=3.0D10
H=6.6262D-31

COMPUTED CONSTANTS

PI=4.0D0*DATAN(1.0D0)
AREA=PI*BWAIST*BWAIST

COEFFICIENTS IN RATE EQUATIONS
**C**

\[
\begin{align*}
\text{GAMMA} &= 1.0D0 + G2G1 \\
V &= C/DEX \\
C11 &= 1.0D0/TAU2 - G2G1/TAUFL \\
C12 &= -G2G1*(-C11-1.0D0/TAU1) \\
C21 &= 1.0D0/TAUFL \\
C22 &= G2G1/TAUFL - 1.0D0/TAU1 \\
PFACTR &= -SIGMAA*XNORM \\
BETA &= V*XNORM*TNORM \\
BETAG &= BETA*GAMMA \\
XNORML &= TOTALN/XNORM \\
\end{align*}
\]

**CC**

**T** = TBEGIN

**TMAX** = TEND

**CC**

GRID MESH SELECTION

MUST BE MODIFIED IF CAVITY IS NOT EQUILATERAL

**C**

NXCR = # SPATIAL INCREMENTS IN CRYSTAL

N1 = # SPATIAL INCREMENTS FROM EDGE OF CRYSTAL TO 1ST MIRROR

N2 = # SPATIAL INCREMENTS FROM 1ST MIRROR TO EDGE OF CRYSTAL

CAV = LP/3.0D0

L1 = .5D0*(CAV+L)

L2 = 2.0D0*CAV+L1

DENOM = DBLE(NXCR)

DELTAT = L/(V*DENOM*TNORM)

DELTAX = L/DENOM

DELTAXBY2 = DELTAX/2.0D0

DENOM1 = C*DELTAT*TNORM

N1 = IDNINT((L1-L)/DENOM1)

N2 = IDNINT((L2-L)/DENOM1)

**CC**

INITIAL CONDITIONS

**C**

DO 20 I=0,NXCR
   X0(I) = XNORML
   X(I) = 0.0D0
   XI(I) = 0.0D0

20 CONTINUE

**C**

PUMP

W(0) = PUMP(T)

DO 25 I=1,NXCR
   W(I) = 0.0D0

25 CONTINUE

**C**

PHOTONS + AND -

DO 30 I=0,NXCR+N1+N2
   DO 29 J=1,NLMBDA
      PHI(1, I, J) = 0.0D0
      PHI(2, I, J) = 0.0D0

29 CONTINUE

30 CONTINUE
C CALCULATE DERIVATIVES AT THE INITIAL POINTS
CALL DQ1(T,X,X1,W,PHI,DW,DX)
CALL DQ2(T,X,X1,PHI,DPHI)
C FIRST VALUES COMPUTED BY EULER’S METHOD AND BOUNDARY
C CONDITIONS
M=1
IOUT=MOD(M,IO)
T=T+DELTAT
DO 40 I=0,NXCR
  IF(I.EQ.0)THEN
    WNEW(0)=PUMP(T)
  ELSE
    WNEW(I)=W(I-1)+DELTAX*DW(I-1)
  END IF
  XNEW(I)=X(I)+DELTAT*DX(1,1)
  X1NEW(I)=X1(I)+DELTAT*DX(2,1)
  DO 50 J=1,NLMBDA
    IF(I.EQ.0)THEN
      PHINEW(1,1,J)=TS*R2*R3*
        PHI(1,NXCR+N1+N2-1,J)
    ELSE
      PHINEW(1,I,J)=PHI(1,I-1,J)+
        DELTAX*DPHI(1,1,J)
    END IF
  50 CONTINUE
40 CONTINUE
CALL PHICAV(PHI,PHINEW,OUTPUT)
C PHIMINUS IN CRYSTAL
DO 55 I=NXCR,1,-1
  DO 60 J=1,NLMBDA
    PHINEW(2,I,J)=PHI(2,I,J)+
      DELTAX*DPHI(2,I,J)
  60 CONTINUE
55 CONTINUE
C CALCULATES PHIMINUS GOING OUT OF CRYSTAL
DO 70 J=1,NLMBDA
  PHINEW(2,NXCR+N1+N2,J)=TS*PHINEW(2,0,J)
70 CONTINUE
C SWITCH VALUES
CALL SWITCH(W,WNEW,X,XNEW,X1,X1NEW,PHI,
  PHI, DW,DX,DPHI,WOLD,XOLD,X1OLD,
  PHIOLD,DWOLD,DXOLD,DPHOLD,T)
IF(IOUT.EQ.0)GO TO 400
150 CONTINUE
M=M+1
IOUT=MOD(M,IO)
C CALCULATE DERIVATIVES AT CURRENT TIME
CALL DQ1(T,X,X1,W,PHI,DW,DX)
CALL DQ2(T,X,X1,PHI,DPHI)
C EULER’S METHOD IS USED FOR THE FIRST GUESS AT NEW TIME
T=T+DELTAT
IF (T.GT.TMAX) GO TO 500  
C LOOP 80 COMPUTES FIRST GUESS FOR THE PUMP AND  
C POPULATIONS  
C INTEGRATION FOR PUMP ALONG CHARACTERISTIC PARALLEL TO  
C X=VT  
C INTEGRATION FOR POPULATIONS ALONG CHARACTERISTICS  
C X=CONSTANT  
DO 80 I=0,NXCR  
C BOUNDARY CONDITIONS FOR THE PUMP  
IF (I.EQ.0) THEN  
   WTMP(0) = PUMP(T)  
ELSE  
   WTMP(I) = W(I-1) + DELTAX*DW(I-1)  
END IF  
XTMP(I) = X(I) + DELTAT*DX(1,I-1)  
X1TMP(I) = X1(I) + DELTAT*DX(2,I-1)  
80 CONTINUE  
C BOUNDARY CONDITIONS FOR PHOTONS CLOCKWISE AND  
C CCLOCKWISE  
DO 85 J=1,NLMBDA  
   PHITMP(1,0,J) = TS*R2*R3*PHI(1,NXCR+N1+N2-1,J)  
   PHITMP(2,NXCR,J) = TS*PHI(2,NXCR+1,J)  
C LOOP 90 COMPUTES FIRST GUESS FOR THE CLOCKWISE  
C PHOTONS  
C INTEGRATION ALONG CHARACTERISTIC PARALLEL TO X=VT  
DO 90 I=1,NXCR  
   PHITMP(I,I,J) = PHI(1,I-1,J) + DELTAX*(DPHI(1,I-1,J))  
90 CONTINUE  
C LOOP 95 COMPUTES FIRST GUESS FOR CCLOCKWISE PHOTONS  
C INTEGRATION ALONG CHARACTERISTIC PARALLEL TO X=-VT  
DO 95 I=NXCR-1,0,-1  
   PHITMP(2,I,J) = PHI(2,I+1,J) + DELTAX*(DPHI(2,I+1,J))  
95 CONTINUE  
85 CONTINUE  
C CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC  
C IMPROVEMENTS ON FIRST GUESS  
C BY MODIFIED EULER'S METHOD  
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC  
C EVALUATE DERIVATIVES AT THE PREDICTED VALUES  
CALL DQ1(T,XTMP,X1TMP,WTMP,PHITMP,DWTMP,DXTMP)  
CALL DQ2(T,XTMP,X1TMP,PHITMP,DPTMP)  
C MAKE IMPROVEMENT  
C LOOP 310 CORRECTS PUMP AND POPULATIONS  
DO 310 I=0,NXCR  
IF (I.EQ.0) THEN  
   WNEW(I) = WTMP(I)  
ELSE  
   WNEW(I) = W(I-1) + DXBY2*(DW(I-1))  
END IF  
310 CONTINUE  
K = K+1  
C CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC  
167  
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XNEW(I) = X(I) + DTBY2*(DXTMP(1, I) + DX(1, I))
X1NEW(I) = X1(I) + DTBY2*(DXTMP(2, I) + DX(2, I))

310 CONTINUE
DO 320 J=1, NLMBDA

C LOOP 325 CORRECTS CLOCKWISE PHOTONS
DO 325 I=0, NXCR
IF (I.EQ.0) THEN
  PHINEW(1, I, J) = PHITMP(1, I, J)
ELSE
  PHINEW(1, I, J) = PHI(1, I-1, J) + DXBY2*(DPTMP(1, I, J) + DPHI(1, I-1, J))
END IF

325 CONTINUE

C LOOP 330 CORRECTS CCLOCKWISE PHOTONS
DO 330 I=NXCR, 1, -1
IF (I.EQ.NXCR) THEN
  PHINEW(2, I, J) = PHITMP(2, I, J)
ELSE
  PHINEW(2, I-1, J) = PHI(2, I, J) + DXBY2*(DPTMP(2, I-1, J) + DPHI(2, I, J))
END IF

330 CONTINUE

320 CONTINUE

C SECOND ITERATION IN MODIFIED EULER'S METHOD
C REPLACE TEMPORARY VALUES WITH NEW VALUES CALCULATED
C ABOVE
C GO THROUGH IMPROVEMENTS AGAIN.
IF (K.EQ.1) THEN
  XTMP(I) = XNEW(I)
  X1TMP(I) = X1NEW(I)
  WTMP(I) = WNEW(I)
  DO 345 J=1, NLMBDA
    PHITMP(1, I, J) = PHINEW(1, I, J)
    PHITMP(2, NXCR-I, J) = PHINEW(2, NXCR-I, J)
  345 CONTINUE
  340 CONTINUE
  GO TO 300
END IF

420 CONTINUE

C CALCULATE PHOTONS IN THE REST OF THE CAVITY
CALL PHICAV(PHI, PHINEW, OUTPUT)

C SWITCH VALUES
CALL SWITCH(W, WNEW, X, XNEW, X1, X1NEW, PHI, PHINEW, DW, DX, DPHI, WOLD, XOLD, X1OLD, PHIOLD, DWOLD, DXOLD, DPHOLD, T)

C IF M IS A MULTIPLE OF IO THEN WRITE TO DATA FILES
IF (IOUT.EQ.0) THEN
  WRITE(4, 700) T, X(1)
  WRITE(6, 700) T, W(1)
  SUM1 = 0.0D0
  DO 410 J=1, NLMBDA
  410 CONTINUE
`SUM1=SUM1+PHI(1,1,J)+PHI(2,1,J)

410 CONTINUE
   WRITE(5,700)T,SUM1
   GO TO 150
500 STOP
700 FORMAT(2E18.6)
END`

```
C SUBROUTINE TO SWITCH VALUES
C
C ARGUMENTS: W=PUMP (ON INPUT CURRENT VALUES FOR PUMP. ON OUTPUT NEW VALUES FOR PUMP)
C WNEW=NEWLY CALCULATED VALUES FOR PUMP
C WOLD=ON INPUT OLD VALUES FOR PUMP. ON OUTPUT CURRENT VALUES FOR PUMP
C X=POPULATION INVERSION (ON INPUT CURRENT VALUES FOR POP. INVERSION. ON OUTPUT NEW VALUES FOR PUMP)
C XNEW=NEWLY CALCULATED VALUES FOR POP. INV.
C XOLD=ON INPUT OLD VALUES FOR POP. INV. ON OUTPUT CURRENT VALUES FOR POP. INV.
C X1=LOWER LASER LEVEL POPULATION. ON INPUT CURRENT VALUES FOR L.L.POP. ON OUTPUT NEW VALUES FOR L.L.POP.
C X1NEW=NEWLY CALCULATED VALUES FOR L.L.POP
C X10LD=ON INPUT OLD VALUES FOR L.L.POP.ON OUTPUT CURRENT VALUES FOR L.L.POP.
C XI=LOWER LASER LEVEL POPULATION. ON INPUT CURRENT VALUES FOR L.L.POP. ON OUTPUT NEW VALUES FOR L.L.POP.
C PHI=PHOTON CONCENTRATIONS. ON INPUT CURRENT VALUES FOR PHOTON CONC. ON OUTPUT NEW VALUES FOR PHOTON CONC.
C PHINEW=NEWLY CALCULATED VALUES FOR PHOTON CONC.
C PHIOLD=ON INPUT OLD VALUES FOR PHOTON CONC. ON OUTPUT CURRENT VALUES FOR PHOTON CONC.
C DX=CURRENT DERIVATIVE VALUES OF POPULATIONS
C DXOLD=ON INPUT OLD DERIVATIVE VALUES. ON OUTPUT CURRENT DERIVATIVES OF POP.
C DW=CURRENT DERIVATIVES OF PUMP
C DWOLD=ON INPUT OLD DERIVATIVES OF PUMP. ON OUTPUT CURRENT DERIV. OF PUMP
C DPHI=CURRENT DERIVATIVES OF PHOTON.
C DPHIO=ON INPUT OLD DERIV. OF PHOTON. ON OUTPUT CURR. DERIV. OF PHOTON.

SUBROUTINE SWITCH(W,WNEW,X,XNEW,X1,X1NEW,PHI,PHINEW,
1 DW,DX,DPHI,WOLD,XOLD,X1OLD,PHIOLD,DWOLD,
2 DPHIO,T)
PARAMETER(NXCR=5,NL=24,NXTOTL=1000)
IMPLICIT DOUBLE PRECISION(A-H,L,O-Z)
DIMENSION W(0:NXCR),WNEW(0:NXCR),X(0:NXCR),
1 XNEW(0:NXCR),X1NEW(0:NXCR),PHI(2,0:NXTOTL,NL),
2 X1(0:NXCR),PHINEW(2,0:NXTOTL,NL),
```

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DO 15 I = 0, NXCR
  XOLD(I) = X(I)
  X10LD(I) = X(I)
  WOLD(I) = W(I)
  X(I) = XNEW(I)
  W(I) = WNEW(I)
  DXOLD(1,1) = DX(1,1)
  DXOLD(2,I) = DX(2,I)
  DWOLD(I) = DW(I)
15 CONTINUE
DO 30 K = 1, 2
  DO 20 J = 1, NLMBDA
    DO 25 I = 0, NXCR+N1+N2
      PHIOLD(K,I,J) = PHI(K,I,J)
      PHI(K,I,J) = PHINEW(K,I,J)
25 CONTINUE
20 CONTINUE
30 CONTINUE
RETURN
END

C SUBROUTINE TO SOLVE HOMOGENEOUS PDES IN CAVITY

SUBROUTINE PHICAV(PHI,PHINEW,OUTPUT)
PARAMETER(NXCR=5, NXTOTL=1000, NL=24)
IMPLICIT DOUBLE PRECISION (A-H,L,O-Z)
DIMENSION PHI(2,0:NXTOTL,NL), PHINEW(2,0:NXTOTL,NL),
  OUTPUT(  2, NL) ,  Z (2, NL)
COMMON/REFL/TS, R1, R2, R3, T1,FRACTN, AREA
COMMON/DISCRT/N1, N2, MIDX
COMMON/WAVL/SIGMA(NL), ALPHA(NL), NLMBDA, LAMBDA(NL)
C CALCULATES PHIPLUS AT THE BOUNDARY X=L
DO 60 J = 1, NLMBDA
  PHINEW(1,NXCR,J) = TS*PHINEW(1,NXCR,J)
C CALCULATES PHIPLUS BETWEEN CRYSTAL AND OUTPUT COUPLER
DO 65 I = NXCR,NXCR+N1-1
  PHINEW(1,I+1,J) = PHI(1,I,J)
65 CONTINUE
C CALCULATES PHIMINUS GOING OUT OF CRYSTAL
  PHINEW(2,NXCR+N1+N2,J) = TS*PHINEW(2,0,J)
C CALCULATES PHIMINUS FROM EDGE OF CRYSTAL TO OUTPUT
C COUPLER
DO 75 I=NXCR+N1+N2, NXCR+N1+1, -1
   PHINEW(2,I-1,J)=PHI(2,I,J)
75 CONTINUE
C CALCULATES OUTPUT (BOTH PHI+ AND PHI-)
DO 80 I=1,2
   OUTPUT(I,J)=T1*PHINEW(I,NXCR+N1,J)*AREA
80 CONTINUE
C CALCULATES INJECTION TERM
   Z(1,J)=T1*FRACTN*OUTPUT(1,J)/AREA
   Z(2,J)=0.0D0
C CALCULATES PHIPLUS AND PHIMINUS AT FIRST MIRROR
   PHINEW(1,NXCR+N1,J)=R1*PHINEW(1,NXCR+N1,J)+Z(1,J)
   PHINEW(2,NXCR+N1,J)=R1*R2*R3*
1   PHINEW(2,NXCR+N1,J)+Z(2,J)
C CALCULATES PHIPLUS FROM OUTPUT COUPLER TO EDGE OF CRYSTAL
DO 85 I=NXCR+N1, NXCR+N1+N2-1
   PHINEW(1,I+1,J)=PHI(1,1,J)
85 CONTINUE
C CALCULATES PHIPLUS GOING INTO THE CRYSTAL
   PHINEW(1,0,J)=TS*R2*R3*PHINEW(1,NXCR+N1+N2,J)
C CALCULATES PHIMINUS FROM OUTPUT COUPLER TO EDGE OF CRYSTAL
DO 90 I=NXCR+N1,NXCR+1,-1
   PHINEW(2,1-1,J)=PHI(2,I,J)
90 CONTINUE
C CALCULATES PHIMINUS AT EDGE OF CRYSTAL
   PHINEW(2,NXCR,J)=TS*PHINEW(2,NXCR,J)
60 CONTINUE
RETURN
END

SUBROUTINES TO COMPUTE DERIVATIVES

SUBROUTINE DQ1(T,X,Y,W,PH,DW,DX)
PARAMETER( NXCR=5, NL=24, NXTOTL=1000)
IMPLICIT DOUBLE PRECISION(A-H,L,0-Z)
DIMENSION W(0:NXCR),X(0:NXCR),PH(2,0:NXTOTL,NL),
1 Y(0:NXCR),PH(2,0:NXTOTL,NL),
2 DW(0:NXCR),DX(2,0:NXCR)
COMMON/COEFF/C11,C12,C21,C22,GAMMA,V,TOTALN,XNORM,
1 TNORM,BETA,BETAG,PFACTR,XNORML

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COMMON/WAVL/SIGMA(NL),ALPHA(NL),NLMBDA,LAMBDA(NL)
DO 10 I=0,NXCR
    A=XNORML-X(I)-GAMMA*Y(I)
    IF(A.GT.0.0D0)THEN
        X0=A
    ELSE
        X0=0.0D0
    END IF
    WP=X0*W(I)
    DW(I)=PFACTR*WP
    SUM=0.0D0
10 CONTINUE
    J=1,NLMBDA
    SUM=SUM+(PH(1,I,J)+PH(2,I,J))*SIGMA(J)
    CONTINUE
    DX(1,I)=C11*X(I)+C12*Y(I)-BETAG*X(I)*SUM+WP
    DX(2,I)=C21*X(I)+C22*Y(I)+BETA*X(I)*SUM
15 CONTINUE
 RETURN
END

PHOTON DERIVATIVES IN CRYSTAL
X=(INPUT) POPULATION INVERSION
Y=(INPUT) LOWER LASER LEVEL POPULATION
PH=(INPUT) PHOTON CONCENTRATIONS
DP=(OUTPUT) DERIVATIVES OF PHOTON CONCENTRATIONS

SUBROUTINE DQ2(T, X, Y, PH, DP)
PARAMETER(NXCR=5, NL24, NXTOTL=1000)
IMPLICIT DOUBLE PRECISION(A-H,L,O-Z)
DIMENSION X(0:NXCR), Y(0:NXCR),
1 PH(2,0:NXTOTL,NL),DP(2,0:NXCR,NL),
2 S(2,0:NXCR,NL)
COMMON/COEFF/C11,C12,C21,C22,GAMMA,
1 V,TOTALN,XNORM,TNORM,
2 BETA,BETAG,PFACTR,XNORML
COMMON/WAVL/SIGMA(NL),ALPHA(NL),NLMBDA,LAMBDA(NL)
COMMON/SP/L,R,DEX,DELTAX,DLAMBDA(NL)
COMMON/SPUMP/SPMAA,BWAIST,ETAP,PLMBDA,ETOTL,TAUP,T0
COMMON/SPUMP/SPMAA,BWAIST,ETAP,PLMBDA,ETOTL,TAUP,T0

CALCULATION OF SPONTANEOUS EMISSION CONTRIBUTION
PI=4.0D0*DATAN(1.0D0)
T4=2.0D0*DEX*DEX*DEX/(PI*BWAIST*BWAIST)
DO 20 J=1,NLMBDA
    LMDBA2=LAMBDA(J)*LAMBDA(J)
    T2=T4*SIGMA(J)*DLAMBDA(J)/LMDBA2
    DO 10 I=0,NXCR
        T1=X(I)+(GAMMA-1.0D0)*Y(I)
        T3=T1*T2
        S(1,I,J)=T3
        S(2,I,J)=T3
10 CONTINUE
20 CONTINUE

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C       CALCULATION OF PHOTON DERIVATIVES
DO 60 J=1,NLMBDA
   SIG=SIGMA(J)*XNORM
DO 50 I=0,NXCR
   DP(1,I,J)=(SIG*X(I)-ALPHA(J))*PH(1,I,J)+S(1,I,J)
1   CONTINUE
DO 55 I=NXCR,0,-1
   DP(2,I,J)=(SIG*X(I)-ALPHA(J))*PH(2,I,J)+S(2,I,J)
1   CONTINUE
60 CONTINUE
RETURN
END

cccccccccccccccccccccccccccccccccccccccccccccccccccccccc
C PUMP FUNCTION
cccccccccccccccccccccccccccccccccccccccccccccccccccccccc
FUNCTION PUMP(T)
IMPLICIT DOUBLE PRECISION(A-H,L,O-Z)
COMMON/WPUMP/SIGMAA,BWAIST,ETAP,PLMBDA,ETOTL,TAUP,TO
cccccccccccccccccccccccccccccccccccccccccccccccccccccccc
C C IS IN CM/SEC
C H IS IN mJ-SEC
C BWAIST IS IN CM
C ETOTL IS IN MILLIJOULES
C PLMBDA IS IN CM
C SIGMAA IS IN CM^2
C TAUP,TO ARE IN NS
cccccccccccccccccccccccccccccccccccccccccccccccccccccccc
C=3.0D10
PI=4.0D0*DATAN(1.0D0)
FACTOR=SIGMAA/(PI*BWAIST*BWAIST)
H=6.6262E-31
PUMP=ETAP*(ETOTL/TAUP)*(PLMBDA/(H*C))*FACTOR
BDLOG=DLOG(2.0D0)
PUMP=PUMP*2.0D0*DSQRT(BDLOG/PI)*
1   EXP(-BDLOG*((T-TO)/(.5D0*TAUP))**2)
RETURN
END
Biographical Statement

Lila Freeman Roberts was born in Asheville, North Carolina on December 21, 1954. As an undergraduate student, she attended Western Carolina University in Cullowhee, N.C. in 1973 and North Carolina State University in Raleigh, N.C. from 1974 to 1977.

After receiving a Bachelor of Science Degree in Mathematics Education from North Carolina State University in May, 1977, Ms. Roberts taught secondary school Mathematics with the Virginia Beach City Public Schools in Virginia Beach Virginia from 1977 to 1981 and with the Norfolk City Public Schools in Norfolk, Virginia in 1982.

Ms Roberts attended Old Dominion University in Norfolk, Virginia from 1983 to 1988 and received her Master of Science Degree in Computational and Applied Mathematics in August, 1986. She was a teaching assistant from 1983 to 1986. In 1986, she was elected to the honor society of Phi Kappa Phi. During the years 1985 to 1988, she conducted research at NASA Langley Research Center in Hampton, Virginia under a NASA Graduate Student Fellowship.
Currently, Ms Roberts is Assistant Professor of Mathematics at Randolph-Macon Woman's College in Lynchburg, Virginia. She resides in Madison Heights, Virginia with her husband. She will receive her Doctor of Philosophy Degree in Computational and Applied Mathematics from Old Dominion University in December 1988.