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**LARGE DEVIATION LOCAL LIMIT THEOREMS
FOR RATIO STATISTICS**

by

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Abstract

LARGE DEVIATION LOCAL LIMIT THEOREMS FOR RATIO STATISTICS

Sanjeev V. Sabnis

Old Dominion University, 1987

Director: Dr. N. R. Chaganty

Let $\{T_n, n \geq 1\}$ be an arbitrary sequence of non-lattice random variables and $\{S_n, n \geq 1\}$ be another sequence of positive non-lattice random variables. Let the two sequences be independent. Let ϕ_{1n} and ϕ_{2n} be the moment generating functions of $\{T_n, n \geq 1\}$ and $\{S_n, n \geq 1\}$ respectively. Let $\{a_n\}$ be a sequence of real numbers such that $a_n \rightarrow \infty$. Let

$$\psi_{1n}(z) = \frac{1}{a_n} \log \phi_{1n}(z)$$

and

$$\psi_{2n}(z) = \frac{1}{a_n} \log \phi_{2n}(z).$$

Under some mild and easily verifiable conditions on ϕ_{1n} and ϕ_{2n} , which imply that $(T_n - E(T_n))/a_n \rightarrow 0$ in probability and $(S_n - E(S_n))/a_n \rightarrow c$ in probability for some $c > 0$, we obtain a large deviation local limit theorem for the ratio statistic $R_n = T_n/S_n$, that is, we obtain an asymptotic expression for the density, $f_n(r_n)$, of the ratio statistic $R_n = T_n/S_n$ at the point r_n where $\{r_n, n \geq 1\}$ is a bounded sequence of real numbers. This expression is given by

$$f_n(r_n) = \frac{a_n \psi'_{2n}(-r_n \tau_n)}{[2\pi a_n [\psi''_{1n}(\tau_n) + r_n^2 \psi''_{2n}(-r_n \tau_n)]]^{1/2}} \\ \times \exp(a_n [\psi_{1n}(\tau_n) + \psi_{2n}(-r_n \tau_n)]) [1 + O(1/a_n)]$$

where τ_n is such that

$$\psi'_{1n}(\tau_n) - \tau_n \psi'_{2n}(-\tau_n \tau_n) = 0.$$

When S_n is degenerate at n , our result reduces to the result of Chaganty-Sethuraman (Ann. of Probab., (1985), vol. 13, 97-113). We obtain similar asymptotic expressions for the probability density functions (p.d.f.) or probability mass functions (p.m.f.) of ratio statistics arising from different combinations of non-lattice and lattice random variables T_n and S_n .

We also present the corresponding results for the ratio of sums of i.i.d. random variables along with some interesting applications. In all of these applications, the p.d.f.'s (or p.m.f.'s) do not have closed form expressions. However, our results provide simple computable approximations involving moment generating functions for such p.d.f.'s (or p.m.f.'s). We verify our result for the F statistic, whose exact density is available. Our approximation coincides with the exact expression except for the normalizing constant. However, we show that our approximation is asymptotically equivalent to the exact density.

Let $\{T_n, n \geq 1\}$ be an arbitrary sequence of random vectors and $\{S_n > 0, n \geq 1\}$ be an arbitrary sequence of random variables. In Chapter III, we obtain an asymptotic expression for the density of the vector $R_n = T_n/S_n$. A result which obtains an asymptotic expression for the tail probability, $P(R_n > \tau_n)$, is known as a strong large deviation result. In Chapter IV, we obtain strong large deviation results for statistics which can be expressed as the ratio of two independent sequences of non-lattice as well as lattice random variables.

to
my parents

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Chapter 1

Introduction and Summary

The origin of probability theory dates back to the middle of the 17th century. Since then limit theorems have become the core of probability theory as they are quite useful in understanding the random phenomena present in all our sciences. The history of the limit theorems begins with the limit theorems of Bernoulli (1713). Then followed the limit theorems of de Moivre and Laplace. Later on, Poisson succeeded in obtaining generalizations of the theorems of Bernoulli, de Moivre and Laplace. In today's context these theorems are referred to as the laws of large numbers, one of the fundamental propositions of the theory of probability. We state these results below in Theorems 1.1 and 1.2 which are now known as Weak Law of Large Numbers (WLLN) and Strong Law of Large Numbers (SLLN) respectively. The WLLN refers to convergence in probability whereas the SLLN refers to almost sure convergence of averages of random variables.

Theorem 1.1 (WLLN). *Let $\{X_n, n \geq 1\}$ be pairwise independent and identically distributed random variables with finite mean $E(X_1)$. Then we have*

$$\frac{S_n}{n} \rightarrow E(X_1) \quad (1.1)$$

in probability.

Theorem 1.2 (SLLN). *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with finite mean $E(X_1)$. Then we have*

$$\frac{S_n}{n} \rightarrow E(X_1) \quad (1.2)$$

almost surely.

Theorems 1.1 and 1.2 are due to Khintchine and Kolmogorov respectively.

Chebyshev, a Russian Mathematician, made major contributions to the theory of probability by introducing new techniques to prove the theorems concerning sums of arbitrarily distributed independent random variables. For example, under the assumption of existence of moments of all orders (by utilizing the methods of moments), Chebyshev gave different versions of the proof of the Central Limit Theorem (CLT) (see Theorem 1.3 below for a version of the CLT) for sums of independent but arbitrarily distributed random variables. The CLT deals with convergence in distribution of normalized sums of random variables. Then around 1900, Lyapunov proved CLT with substantially weaker restrictions than those required by Chebyshev. He proved the theorem utilizing the method of characteristic func-

tions, developed by himself, rather than the method of moments used by Chebyshev. We give the simplest form of the CLT in Theorem 1.3.

Theorem 1.3 (CLT). *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $E(X_1) = \mu$ and $\text{var}(X_1) = \sigma^2$ and distribution function F . Let F_n denote the distribution function of the normalized sum, that is, the distribution function of $(S_n - n\mu)/\sqrt{n}\sigma$, where $S_n = X_1 + \cdots + X_n$. Then*

$$F_n(x) \rightarrow \Phi(x) \tag{1.3}$$

uniformly in x where $\Phi(x)$ denotes the distribution function of the standard normal random variable.

While the CLT gives the limit of a sequence of distribution functions F_n , it is quite inadequate in practice. In many applications we are interested not only in the convergence of $F_n(x)$ but also in the error of approximating $F_n(x)$ by $\Phi(x)$. This question was addressed by several authors including Cramér (1938), Berry (1941), Esseen (1943, 1945) and Chernoff (1952). Berry and Esseen established an upperbound for the error $e_n(x) = |F_n(x) - \Phi(x)|$, which is uniform in x . While the upperbound for the error $e_n(x)$ is quite sharp for moderate values of x , however, for large values of x , both $F_n(x)$ and $\Phi(x)$ are close to 1 and the upperbound overestimates the error $e_n(x)$. Chernoff (1952) initiated the study of the tail of $F_n(x)$. He showed that if X_1 has a finite moment generating function in a neighbourhood of zero, then the probability that S_n/n deviates from $E(X_1)$ by a small amount $\epsilon > 0$ tends to zero exponentially fast as $n \rightarrow \infty$. The

event $E_n = \{|S_n/n - E(X_1)| > \epsilon\}$ is known as a large deviation event since it represents the deviation of S_n/n away from the mean. The probability of occurrence of E_n is known as a large deviation probability. Therefore, Chernoff's theorem is known as a large deviation limit theorem. In recent years, limit theorems for large deviations have become a vital part of the theory of probability. The theory of large deviations forms a natural complement to the law of large numbers stated in Theorems 1.1 and 1.2.

Among the different kinds of limit theorems, the local limit theorems are also of great importance. These theorems deal with the convergence of a sequence of density functions to another density function as well as with asymptotic expansions for a sequence of density functions. One should note that the hypothesis of the CLT does not guarantee the convergence of the corresponding density functions to the standard normal density. In fact, such a convergence takes place almost everywhere only when F_n is nonsingular for some n .

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables. Let $S_n = X_1 + \cdots + X_n$ be the n^{th} partial sum. Let f_n denote the density function of $S_n/\sqrt{n}$. Richter proved an important local limit theorem for large deviations for sums of i.i.d. random variables, that is, an asymptotic expansion for $f_n(x_n)$ when x_n is allowed to increase with n . Chaganty and Sethuraman (1985) generalized Richter's theorem to an arbitrary sequence of random variables thereby increasing the applicability of Richter's theorem to new classes of statistics. We state the result of Chaganty and Sethuraman in Theorem 2.2.3 of Chapter II.

The goal of this dissertation is to generalize Theorem 2.2.3 to the ratio statistic $R_n = T_n/S_n$, where $\{S_n, n \geq 1\}$ is an arbitrary sequence of positive random variables independent of $\{T_n, n \geq 1\}$. In the special case where T_n and S_n are partial sums of independent random variables our result simplifies to the heuristic conclusion of Daniels (1954).

The problem of obtaining the limit for $\log P(T_n \geq x_n)$, where $x_n \rightarrow \infty$ at a suitable rate as $n \rightarrow \infty$, has been studied by several authors including Cramér (1938), Chernoff (1952), and Ellis (1984). Chaganty and Sethuraman chose to call these results of Cramér, Chernoff and Ellis as weak large deviation results since they give an asymptotic expansion for $\log P(T_n \geq x_n)$ rather than an approximation for $P(T_n \geq x_n)$. An approximation for $P(T_n \geq x_n)$ is known as a strong large deviation result. One of the earliest strong large deviation results when T_n is the sum of i.i.d. random variables was obtained by Bahadur and Ranga Rao (1960). Recently, Chaganty and Sethuraman (1986) extended the result of Bahadur and Ranga Rao to an arbitrary sequence of random variables $\{T_n, n \geq 1\}$. In this dissertation we present analogous results for statistics which can be expressed as the ratio of two independent sequences of non-lattice as well as the ratio of two independent sequences of lattice random variables.

This dissertation has been organised as follows:

Chapter II consists of five sections. In Section 2.1 we present the background material. In Section 2.2 we state Chaganty-Sethuraman's results for an arbitrary sequence of non-lattice random variables. We also state and outline the proof of Chaganty-Sethuraman's theorem with Condition (iii')

in Theorem 2.2.3 replaced by another simple and easily verifiable Condition (iii"). As an application we obtain a large deviation local limit theorem for the Wilcoxon signed rank statistic under the null hypothesis. Section 2.3 contains the main results of this dissertation. Let $\{T_n, n \geq 1\}$ be an arbitrary sequence of non-lattice random variables and $\{S_n, n \geq 1\}$ be another arbitrary sequence of positive non-lattice random variables. Under some mild conditions on ϕ_{1n} and ϕ_{2n} , the moment generating functions of T_n and S_n respectively, we obtain in Theorem 2.3.1 an asymptotic expression for the density of the ratio statistic $R_n = T_n/S_n$. We also obtain an asymptotic expression in Theorem 2.3.12 for the density of the ratio statistic in the case where T_n is non-lattice and S_n is lattice. In Section 2.4 we obtain similar results in Theorems 2.4.1 and 2.4.2 when T_n and S_n both are lattice random variables and when T_n is lattice and S_n is non-lattice respectively. In the case where T_n and S_n are sums of i.i.d. random variables, the conditions of the Theorems 2.3.1, 2.3.12, 2.4.1 and 2.4.2 are very much simplified. We present the corresponding results for the ratios of sums of i.i.d. random variables in Theorems 2.3.13, 2.3.14, 2.4.3 and 2.4.4 respectively. In Section 2.5 we provide three examples illustrating different occurrences of non-lattice and lattice T_n and S_n . For simplicity, in these examples we let T_n and S_n to be sums of i.i.d. random variables. We also study our approximation for the F statistic, whose exact density is available. Our approximation coincides with the exact expression except for the normalizing constant. However, our approximation is asymptotically equivalent to the exact density.

Chapter III contains large deviation local limit theorems for an arbitrary sequence of random vectors. Let $\{T_n, n \geq 1\}$ be an arbitrary sequence of random vectors and $\{S_n, n \geq 1\}$ be an arbitrary sequence of positive random variables. An asymptotic expression for the density of the ratio statistic, $R_n = T_n/S_n$ is given.

In Chapter IV, we present strong large deviation results for statistics which can be expressed as the ratio of two independent sequences of non-lattice as well as the ratio of independent sequences of lattice random variables.

Chapter 2

Large Deviation Local Limit Theorems

2.1 Introduction

The very basic probability limit theorems are the weak law of large numbers (WLLN), the strong law of large numbers (SLLN) and the Central limit theorem (CLT). The WLLN refers to convergence in probability where as the SLLN refers to almost sure convergence of the averages of random variables. The CLT deals with convergence in distribution of normalized sums of random variables. If $\{X_n, n \geq 1\}$ is a sequence of i.i.d. random variables with $E(X_1) = \mu$, $Var(X_1) = \sigma^2$ and distribution function F and further if F_n denotes the distribution function of the normalized sum $(S_n - n\mu)/\sqrt{n}\sigma$ then the CLT asserts that,

$$F_n(x) \rightarrow \Phi(x)$$

uniformly in x , where $\Phi(x)$ denotes the distribution function of the standard normal random variable.

The CLT raises a number of questions. For instance,

- 1) Do the partial sums of random variables always converge in distribution to the normal distribution ?
- 2) How much error is committed in the approximation of $F_n(x)$ by $\Phi(x)$?
- 3) Does the hypothesis of the CLT imply convergence of the corresponding density functions to the normal density ?

The answer to the first question is negative. It leads to the theory of infinite divisibility. As regards to the second question different authors have obtained asymptotic expansions, as well as bounds for the error $e_n(x) = |F_n(x) - \Phi(x)|$ for fixed x . Theorem 2.1.1 below gives an asymptotic expansion for $e_n(x)$ and Theorem 2.1.2 gives a bound on the error term $e_n(x)$. Theorems 2.1.1 and 2.1.2 are due to Esseen(1945), Berry (1941) and Esseen (1943) respectively.

Theorem 2.1.1 (Esseen). *If F is non-lattice and has a finite third moment*

$$\mu_3 = \int_{-\infty}^{\infty} x^3 dF(x),$$

then

$$e_n(x) = \frac{\mu_3(1-x^2)}{6\sigma^3\sqrt{2\pi n}} \exp(-x^2/2) + o(1/n) \quad (2.1)$$

uniformly in x .

Theorem 2.1.2 (Berry and Esseen). *Let $E(X_1) = 0$ and let $\rho = E|X_1|^3$ be finite. Then for all x and n*

$$|e_n(x)| \leq \frac{33}{4} \frac{\rho}{\sigma^3 \sqrt{n}}. \quad (2.2)$$

The constant $33/4$ was first replaced by 0.91 in Zolotarev (1967) and subsequently by 0.7975 in Van Beeck (1972). For fixed n and for large values of x , the quantities $F_n(x)$ and $\Phi(x)$ both become close to 1 and the bound given in Theorem 2.1.2 becomes impracticable. The problem of studying the error of approximation was approached differently by Cramér(1938). He obtained an estimate of the ratio of the tail probabilities,

$$\frac{[1 - F_n(x_n)]}{[1 - \Phi(x_n)]} \quad (2.3)$$

for $x_n = O(\sqrt{n})$, assuming that the moment generating function of X_1 exists. Petrov (1954) extended this result to the case of non identically distributed random variables.

A counter example given in the book of Gnedenko and Kolmogorov (1954), page 223 answers the third question in the negative. A close examination of the example reveals that the convergence fails to take place only in a neighborhood of the origin. However, a result due to Ranga Rao and Varadarajan (1960) guarantees convergence almost everywhere. We state this result in Theorem 2.1.3 below.

Theorem 2.1.3. *A necessary and sufficient condition for $f_n(x)$ to converge to $\phi(x)$ almost everywhere is that F_n is nonsingular for some n .*

Theorem 2.1.3 is known as a local limit theorem for densities.

The problem of obtaining an estimate of the ratio in (2.3) of the tail probabilities can be characterized as the problem of large deviation. If $\{T_n, n \geq 1\}$ is a sequence of random variables, an approximation to $P(T_n \geq x_n)$ where $x_n \rightarrow \infty$, is known as a Strong large deviation result whereas an approximation to $k_n(x_n)$, the probability density function of T_n at x_n , is known in the literature as the large deviation local limit theorem. The result of the latter kind was first obtained by Richter (1957) for the sums of i.i.d. random variables. This result was stated in Richter's paper in terms of Cramér series. Recently, Chaganty-Sethuraman showed that the same result can be rewritten in terms of the so called large deviation rate. Large deviation local limit theorems for sums of independent, non-identically distributed lattice and non-lattice random variables were also obtained by Moskvina (1972) and by MacDonald (1979). Chaganty-Sethuraman (1985) extended Richter's result for an arbitrary sequence of random variables, for both non-lattice and lattice cases. They have demonstrated the applicability of the large deviation local limit theorems for statistics that occur in nonparametric inference and also in establishing limit theorems for some probabilistic models occurring in statistical mechanics (see Chaganty and Sethuraman (1987)).

In this chapter we first state the local limit Theorems 2.2.1 and 2.2.3 due to Richter (1957) and Chaganty-Sethuraman (1985), respectively. As noted by Chaganty-Sethuraman (1985) Condition (iii') of Theorem 2.2.3 was difficult to verify in examples. In Theorem 2.2.8 we establish Theorem

2.2.3, replacing Condition (iii') by another easily verifiable Condition (iii''). The main goal of this chapter is to present similar large deviation local limit theorems for statistics which can be expressed as the ratio of two independent random variables. These results are obtained in Sections 2.3 and 2.4. We treat the case of the ratio of two non-lattice arbitrary random variables in Theorem 2.3.1. Theorems 2.3.12, 2.4.1 and 2.4.2 cover the cases when one of the random variables is non-lattice and the other one is lattice. The conditions of our theorems are much simplified in the case of ratios of sums of i.i.d. random variables. We state these results in Theorems 2.3.13, 2.3.14, 2.4.3 and 2.4.4. In Section 2.5 we provide a number of applications of our theorems.

2.2 Local limit theorems for arbitrary sequence of random variables

Richter (1957) proved a large deviation local limit theorem involving the Cramér series for the sequence $\{X_n, n \geq 1\}$ of i.i.d. random variables. Chaganty-Sethuraman (1985) restated Richter's result in terms of the large deviation rate and extended the result to an arbitrary sequence of random variables not necessarily sums of i.i.d. random variables. We present below statements of these theorems.

Theorem 2.2.1 (Richter). *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables having common distribution function F with $E(X_1) = 0$ and*

$Var(X_1) = \sigma^2$. Let $S_n = X_1 + X_2 + \cdots + X_n$. Let f_n be the p.d.f. of $S_n/\sqrt{n}$. Assume the following conditions:

(1) There exists a positive number A such that

$$\int_{-\infty}^{\infty} \exp(sx) dF(x) < \infty$$

for all s with $|s| < A$.

(2) There exists an n_0 such that the distribution function of $S_n = X_1 + X_2 + \cdots + X_n$ is absolutely continuous with a bounded derivative for $n \geq n_0$. Then, if $x > 1$ and $x = O(\sqrt{n})$, we have, as $n \rightarrow \infty$

$$f_n(x) = \phi(x) \exp\left(\frac{x^3}{\sqrt{n}}\right) \lambda\left(\frac{x}{\sqrt{n}}\right) \left[1 + O\left(\frac{x}{\sqrt{n}}\right)\right] \quad (2.4)$$

where $\lambda(t)$ is a power series converging for all sufficiently small values of $|t|$ and $\phi(x)$ is the density of the standard normal random variable.

The next Theorem 2.2.2 is essentially a reformulation of Theorem 2.2.1 in terms of the large deviation rate function γ defined below in Theorem 2.2.2. One should also note the conditions of Theorem 2.2.1 are weaker than those of Theorem 2.2.2. Here and throughout this dissertation $\log z$ denotes the principal logarithm of the complex number z .

Theorem 2.2.2. Let $\{X_n, n \geq 1\}$ be an i.i.d. sequence of random variables with common distribution function F and analytic characteristic function $\phi(z)$, $z \in \mathcal{C}$, where $\mathcal{C}$ is the set of all complex numbers. Let $\psi(z) = \log \phi(z)$.

Let I denote the interval $(-a, a)$, for fixed positive number a . Let m be a real number and a_1 be such that $0 < a_1 < a$. Let τ be such that $\psi'(\tau) = m$ and denote $\psi''(\tau) = \sigma^2 > 0$. Let $m_n \rightarrow m$. Assume the following conditions:

- (i) There exists $\beta > 0$ such that $|\psi(s)| < \beta$, for all $s \in I$.
- (ii) There exists $\tau_n \in (-a_1, a_1)$ such that $\psi'(\tau_n) = m_n$ for $n \geq 1$.
- (iii) There exists $A < \infty$ such that

$$\sup_{s \in I} \int_{-\infty}^{\infty} \left| \frac{\phi(s+it)}{\phi(s)} \right| dt \leq A. \quad (2.5)$$

Define $\gamma(x) = \sup_{s \in I} [xs - \psi(s)]$. Let v_n denote the probability density function of S_n/n , where $S_n = X_1 + X_2 + \cdots + X_n$. Then,

$$v_n(m_n) = \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} \exp(-n\gamma(m_n)) [1 + O(1/n)]. \quad (2.6)$$

Now, we state Chaganty-Sethuraman's theorem for an arbitrary sequence of random variables. Theorem 2.2.3 below obtains an asymptotic expansion for the density function k_n of T_n/n at the point m_n which is in the region of large deviation for T_n .

Theorem 2.2.3 (Chaganty-Sethuraman). *Let $\{T_n, n \geq 1\}$ be a sequence of non-lattice random variables. Let*

$$\phi_n(z) = E(\exp zT_n) \quad (2.7)$$

and

$$\psi_n(z) = \frac{1}{n} \log \phi_n(z) \quad (2.8)$$

for a complex number z . Denote the interval $(-a, a)$ by I and $(-a_1, a_1)$ by I_1 , where $0 < a_1 < a$. Assume that $\phi_n(z)$ is nonvanishing and analytic in $\Omega = \{z : |\operatorname{Real}(z)| < a\}$. Let $\{m_n\}$ be a sequence of real numbers. Let

$$G_{n,\tau}(t) = \psi_n(\tau) + itm_n - \psi_n(\tau + it), \quad (2.9)$$

for $\tau \in I_1$. Assume that $\{T_n, n \geq 1\}$ satisfies the following conditions:

(i') There exists $\beta > 0$ such that $|\psi_n(z)| < \beta$ for $z \in \Omega$ and $n \geq 1$.

(ii') There exist $\alpha > 0$ and $\tau_n \in I_1$ such that $\psi'_n(\tau_n) = m_n$ and $\psi''_n(\tau) \geq \alpha$, for $\tau \in I_1$ and for all $n \geq 1$.

(iii') There exists $\eta > 0$ such that for any $0 < \delta < \eta$,

$$\inf_{|t| \geq \delta} \operatorname{Real}(G_n(t)) = \min[\operatorname{Real}(G_n(\delta)), \operatorname{Real}(G_n(-\delta))] \quad (2.10)$$

for all $n \geq 1$, where $G_n(t) = G_{n,\tau_n}(t)$.

(iv') There exist $p, l > 0$ such that

$$\sup_{\tau \in I} \int_{-\infty}^{\infty} \left| \frac{\phi_n(\tau + it)}{\phi_n(\tau)} \right|^{l/n} dt = O(n^p). \quad (2.11)$$

Let $\gamma_n(u) = \sup_{s \in I} [us - \psi_n(s)]$ and k_n be the probability density function of T_n/n . Then

$$k_n(m_n) = \frac{\sqrt{n}}{\sqrt{2\pi}\sqrt{\psi_n''(\tau_n)}} \exp(-n\gamma_n(m_n))[1 + O(1/n)]. \quad (2.12)$$

The case of lattice random variables is treated in the next theorem.

Theorem 2.2.4 (Chaganty-Sethuraman). Let $\{T_n, n \geq 1\}$ be a sequence of lattice random variables taking values in the set $S = \{a_n + kh_n : k = 0, \pm 1, \pm 2, \dots\}$. Define $\phi_n(z)$, $\psi_n(z)$ as in Theorem 2.2.3. Let $\phi_n(z)$ be analytic in $\Omega = \{z : |\text{Real}(z)| < a\}$. Let $\{m_n = a_n + k_n h_n/n\}$ be a sequence of real numbers, where $\{k_n\}$ is a sequence of integers. Suppose that Conditions (i') and (ii') of Theorem 2.2.3 hold. Further, let us assume the following conditions:

(iii*) There exists $\eta > 0$ such that for any $0 < \delta < \eta$,

$$\inf_{\delta \leq |t| \leq \pi/|h_n|} \text{Real}(G_n(t)) = \min[\text{Real}(G_n(\delta)), \text{Real}(G_n(-\delta))] \quad (2.13)$$

for $n \geq 1$.

(iv*) There exist $p, l > 0$ such that

$$\sup_{\tau \in I} \int_{-\pi/h_n}^{\pi/h_n} \left| \frac{\phi_n(\tau + it)}{\phi_n(\tau)} \right|^{l/n} dt = O(n^p). \quad (2.14)$$

Let $\gamma_n(u) = \sup_{s \in I} [us - \psi_n(s)]$ and $\psi_n'(\tau_n) = m_n$. Then

$$\frac{\sqrt{n}}{|h_n|} \text{Pr}(T_n/n = m_n) = \frac{1}{\sqrt{2\pi}\sqrt{\psi_n''(\tau_n)}} \exp(-n\gamma_n(m_n))[1 + O(1/n)]. \quad (2.15)$$

Remark 2.2.5. Condition (iv') of Theorem 2.2.3 is an integrability condition which guarantees the existence of the density function k_n of T_n/n . Conditions (i') and (ii') together imply that $(T_n - E(T_n))/\sqrt{\text{var}(T_n)}$ converges in distribution to the standard normal and $(T_n - E(T_n))/n$ converges to zero in probability. Condition (iii') is satisfied for example if the characteristic functions $|\phi_n(\tau_n + it)/\phi_n(\tau_n)|$ are concave in a neighbourhood of the origin. Verification of condition (iii') in examples poses some difficulties. In Theorem 2.2.8 we show that condition (iii') can be replaced by an easily verifiable condition (iii'') and still obtain the conclusion (2.12).

Remark 2.2.6. It is easy to see that Theorem 2.2.3 simplifies to Theorem 2.2.2 when T_n is taken to be the sum of n i.i.d. random variables.

Remark 2.2.7. The proofs of Theorems 2.2.2-2.2.4 have the same pattern and they consist of three major steps. In step 1, we use the inversion formula for the characteristic functions of the conjugate distribution (see (2.26) for the definition of a conjugate distribution) to get an expression for the density functions $k_n(m_n)$ and $v_n(m_n)$. This expression is dependent on the conjugate distribution. We choose the appropriate conjugate distribution using a saddle point technique. We then split the integral I_n appearing in the inversion formula into two parts I_{n1} and I_{n2} (see (2.31) below). In step 2, we show that I_{n1} goes to zero exponentially fast and in step 3, we complete the proof by showing that $I_{n2} = 1 + O(1/n)$.

We now state and outline the proof of Chaganty-Sethuraman's theorem with Condition (iii') replaced by Condition (iii''). We continue to use the same notations introduced earlier.

Theorem 2.2.8. *Assume that $\{T_n, n \geq 1\}$ satisfies the following conditions:*

(i') *There exists $\beta > 0$ such that $|\psi_n(z)| < \beta$ for $z \in \Omega$ and $n \geq 1$.*

(ii') *There exists $\alpha > 0$ and $\tau_n \in I_1$ such that $\psi'_n(\tau_n) = m_n$ and $\psi''_n(\tau) \geq \alpha$, for $\tau \in I_1$ and for all $n \geq 1$.*

(iii'') *Given $\delta > 0$ there exists $0 < \eta < 1$ such that*

$$\limsup_n \sup_{|t| > \delta} \left| \frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)} \right|^{1/n} < \eta. \quad (2.16)$$

(iv') *There exist $p, l > 0$ such that*

$$\sup_{\tau \in I} \int_{-\infty}^{\infty} \left| \frac{\phi_n(\tau + it)}{\phi_n(\tau)} \right|^{l/n} dt = O(n^p). \quad (2.17)$$

Then,

$$k_n(m_n) = \frac{\sqrt{n}}{\sqrt{2\pi} \sqrt{\psi''_n(\tau_n)}} \exp(-n\gamma_n(m_n)) [1 + O(1/n)]. \quad (2.18)$$

We will need the following lemma in the proof of Theorem 2.2.8.

Lemma 2.2.9. *Let $\psi_n(z)$ be as defined in (2.8). Assume that the condition (i') of Theorem 2.2.8 holds. Then,*

$$|\psi_n^{(k)}(\tau)| \leq \frac{k!\beta}{(a-a_1)^k} \quad (2.19)$$

for $k \geq 1$ and for $\tau \in J_1$. Let

$$R_n(\tau + it) = \psi_n(\tau + it) - \psi_n(\tau) - it\psi_n'(\tau) - \frac{it^2}{2!}\psi_n''(\tau) - \frac{it^3}{3!}\psi_n'''(\tau). \quad (2.20)$$

Then there exists $\delta_1 > 0$ such that for $|t| < \delta_1$ and for $\tau \in J_1$,

$$|R_n(\tau + it)| \leq \frac{2\beta t^4}{(a-a_1)^4}. \quad (2.21)$$

Proof. Since ψ_n is analytic in Ω , we can apply Cauchy's theorem for derivatives to get

$$\psi_n^{(k)}(\tau) = \frac{k!}{2\pi i} \int_{C(\tau, a-a_1)} \frac{\psi_n(\omega)}{(\omega - \tau)^{k+1}} d\omega \quad (2.22)$$

for $|\tau| < a$ and for all $k \geq 1$ where $C(\tau, a-a_1) = \{\omega \in \mathbb{C} : |\tau - \omega| = (a-a_1)\}$.

We use Condition (i') to obtain

$$\begin{aligned} |\psi_n^{(k)}(\tau)| &\leq \frac{k!}{2\pi} \sup_{\omega \in C(\tau, a-a_1)} |\psi_n(\omega)| \int_{C(\tau, a-a_1)} \frac{1}{|\omega - \tau_n|^{k+1}} d\omega \\ &\leq \frac{k!\beta}{(a-a_1)^k} \end{aligned} \quad (2.23)$$

for $k \geq 1$. This proves (2.19). Next, since ψ_n is analytic in Ω and $|\tau| < a$, the following expansion is valid for all $n \geq 1$ and $|t| < \delta_1$ where $\delta_1 < (a-a_1)/2$,

$$\psi_n(\tau + it) = \psi_n(\tau) + it\psi_n'(\tau) + \frac{(it)^2}{2!}\psi_n''(\tau) + \frac{(it)^3}{3!}\psi_n'''(\tau) + R_n(\tau + it) \quad (2.24)$$

where

$$R_n(\tau + it) = \frac{(it)^4}{2\pi i} \int_{C(\tau, a-a_1)} \frac{\psi_n(\omega)}{(\omega - \tau)^4(\omega - \tau - it)} d\omega.$$

An upper bound for R_n is given by

$$\begin{aligned} |R_n(\tau + it)| &\leq \frac{t^4}{2\pi} \sup_{\omega \in C(\tau, a-a_1)} |\psi_n(\omega)| \int_{C(\tau, a-a_1)} \frac{1}{|\omega - \tau|^4 |\omega - \tau - it|} d\omega \\ &\leq \frac{2\beta t^4}{(a - a_1)^4} \end{aligned} \quad (2.25)$$

wherein we have used the relation

$|\omega - \tau - it| \geq |\omega - \tau| - |t| = (a - a_1) - |t| \geq (a - a_1)/2$. This proves the lemma.

Next, we outline the proof of Theorem 2.2.8. The proof is given in three major steps.

Proof of Theorem 2.2.8 :

Step I. Consider the conjugate distribution H_n given by

$$dH_n(x) = \frac{\exp(\tau x)}{\phi_n(\tau)} dF_n(x) \quad (2.26)$$

for $\tau \in I$, where F_n denotes the distribution function of T_n . The characteristic function (c.f.) of the distribution function (d.f.) H_n is given by $\phi_n(\tau + it)/\phi_n(\tau)$. From (2.17) it follows that $\phi_n(\tau + it)/\phi_n(\tau)$ is absolutely integrable. Therefore the p.d.f. of H_n exists and it is given by

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\phi_n(\tau + it)}{\phi_n(\tau)} \exp(-itx) dt \quad (2.27)$$

and using (2.26) we get the p.d.f. of F_n as

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_n(\tau + it) \exp(-(\tau + it)x) dt. \quad (2.28)$$

Thus the p.d.f. of T_n/n is given by

$$\begin{aligned} k_n(x) &= \frac{n}{2\pi} \int_{-\infty}^{\infty} \phi_n(\tau + it) \exp(-n(\tau + it)x) dt \\ &= \frac{n}{2\pi} \int_{-\infty}^{\infty} \exp(n[\psi_n(\tau + it) - (\tau + it)x]) dt. \end{aligned} \quad (2.29)$$

The above equality (2.29) is valid for any $\tau \in J_1$. The appropriate choice for τ is τ_n satisfying $\psi'_n(\tau_n) = m_n$. Thus, we have

$$\begin{aligned} k_n(m_n) &= \frac{n}{2\pi} \int_{-\infty}^{\infty} \exp(n[\psi_n(\tau_n + it) - (\tau_n + it)m_n]) dt \\ &= \left[\frac{n}{2\pi\psi''_n(\tau_n)} \right]^{1/2} \exp(-n\gamma_n(m_n)) I_n \end{aligned} \quad (2.30)$$

where

$$\begin{aligned} I_n &= \left[\frac{n\psi''_n(\tau_n)}{2\pi} \right]^{1/2} \int_{-\infty}^{\infty} \exp(n[\psi_n(\tau_n + it) - \psi_n(\tau_n) - itm_n]) dt \\ &= \left[\frac{n\psi''_n(\tau_n)}{2\pi} \right]^{1/2} \left[\int_{|t| \geq \delta} \exp(n[\psi_n(\tau_n + it) - \psi_n(\tau_n) - itm_n]) dt \right. \\ &\quad \left. + \int_{|t| < \delta} \exp(n[\psi_n(\tau_n + it) - \psi_n(\tau_n) - itm_n]) dt \right], \end{aligned} \quad (2.31)$$

where δ is any positive number less than δ_1 of Lemma 2.2.9. We choose our δ small enough such that

$$M(\delta) = [\beta\delta/(a - a_1)^3 + 2\beta\delta^2/(a - a_1)^4] < \alpha/2.$$

Thus we can write

$$I_n = I_{n1} + I_{n2} \text{ (say)}. \quad (2.32)$$

We now show that I_{n1} goes to zero exponentially fast.

Step II. Consider,

$$\begin{aligned} |I_{n1}| &= \left[\frac{n\psi_n''(\tau_n)}{2\pi} \right]^{1/2} \left| \int_{|t| \geq \delta} \exp(n[\psi_n(\tau_n + it) - \psi_n(\tau_n) - itm_n]) dt \right| \\ &\leq \left[\frac{n\psi_n''(\tau_n)}{2\pi} \right]^{1/2} \int_{|t| \geq \delta} \left| \frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)} \right| dt \\ &= \left[\frac{n\psi_n''(\tau_n)}{2\pi} \right]^{1/2} \int_{|t| \geq \delta} \left| \frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)} \right|^{l/n} \left| \frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)} \right|^{(1-l/n)} dt \\ &\leq \left[\frac{n\psi_n''(\tau_n)}{2\pi} \right]^{1/2} \left[\sup_{|t| \geq \delta} \left| \frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)} \right|^{1/n} \right]^{n(1-l/n)} \int_{|t| \geq \delta} \left| \frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)} \right|^{l/n} dt \\ &\leq \left[\frac{n\psi_n''(\tau_n)}{2\pi} \right]^{1/2} \left[\sup_{|t| \geq \delta} \left| \frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)} \right|^{1/n} \right]^{n(1-l/n)} \int_{-\infty}^{\infty} \left| \frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)} \right|^{l/n} dt. \end{aligned} \quad (2.33)$$

From (2.16) and (2.17) it follows that,

$$\begin{aligned} |I_{n1}| &\leq \left[\frac{n\psi_n''(\tau_n)}{2\pi} \right]^{1/2} \eta^{n(1-l/n)} O(n^p) \\ &= O(n^{p+1/2}) \exp(-\eta_1(n-l)) \end{aligned} \quad (2.34)$$

where $\eta_1 = -\log \eta > 0$. Thus I_{n1} goes to zero exponentially fast. Next we will show that $I_{n2} = 1 + O(1/n)$ in step III below.

Step III. Let us recall that

$$I_{n2} = \left[\frac{n\psi_n''(\tau_n)}{2\pi} \right]^{1/2} \int_{|t| < \delta} \exp(n[\psi_n(\tau_n + it) - \psi_n(\tau_n) - itm_n]) dt. \quad (2.35)$$

By making a change of variable $t = s/\sqrt{n}$, we obtain

$$I_{n2} = \left[\frac{\psi_n''(\tau_n)}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{n}\delta} \exp(n[\psi_n(\tau_n + is/\sqrt{n}) - \psi_n(\tau_n) - \frac{is}{\sqrt{n}}m_n]) ds. \quad (2.36)$$

From (2.24) we get,

$$\begin{aligned} I_{n2} &= \left[\frac{\psi_n''(\tau_n)}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{n}\delta} \exp(n[\frac{-s^2}{2n}\psi_n''(\tau_n) - \frac{is^3}{6n\sqrt{n}}\psi_n'''(\tau_n) \\ &\quad + R_n(\tau_n + is/\sqrt{n})]) ds \\ &= \left[\frac{\psi_n''(\tau_n)}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{n}\delta} \exp(-\frac{s^2}{2}\psi_n''(\tau_n))[1 - \frac{is^3}{6\sqrt{n}}\psi_n'''(\tau_n) \\ &\quad + nR_n(\tau_n + is/\sqrt{n}) + L_n(s)] ds \end{aligned} \quad (2.37)$$

where

$$L_n(s) = [\exp(z_n) - 1 - z_n] \quad (2.38)$$

and

$$z_n = [-\frac{is^3}{6\sqrt{n}}\psi_n'''(\tau_n) + nR_n(\tau_n + is/\sqrt{n})]. \quad (2.39)$$

The r.h.s. of (2.37) can be written as the sum of four integrals. The first integral is equal to $1 + o(1/n)$ by Mill's ratio (see Feller (1968) page 175, (1.8)). It is clear that the second integral is zero. In order to show

that the third integral is $O(1/n)$ we will use the bound in (2.21) for R_n .

For that consider,

$$\begin{aligned}
& \left| n \left[\frac{\psi_n''(\tau_n)}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{n}\delta} \exp\left(-\frac{s^2}{2} \psi_n''(\tau_n)\right) R_n(\tau_n + is/\sqrt{n}) ds \right| \\
& \leq n \left[\frac{\psi_n''(\tau_n)}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{n}\delta} \exp\left(-\frac{s^2}{2} \psi_n''(\tau_n)\right) |R_n(\tau_n + is/\sqrt{n})| ds \\
& \leq n \left[\frac{\psi_n''(\tau_n)}{2\pi} \right]^{1/2} \frac{2\beta}{n^2 (a - a_1)^4} \int_{|s| < \sqrt{n}\delta} s^4 \exp\left(-\frac{s^2}{2} \psi_n''(\tau_n)\right) ds \\
& = O(1/n). \tag{2.40}
\end{aligned}$$

Thus

$$n \left[\frac{\psi_n''(\tau_n)}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{n}\delta} \exp\left(-\frac{s^2}{2} \psi_n''(\tau_n)\right) R_n(\tau_n + is/\sqrt{n}) ds = O(1/n). \tag{2.41}$$

Finally, to show that the fourth integral is also $O(1/n)$, we first get an upperbound for $L_n(s)$. From (2.19) and (2.21) we have for $|s| < \sqrt{n}\delta$,

$$\begin{aligned}
|z_n| &= \left| -\frac{is^3}{6\sqrt{n}} \psi_n'''(\tau_n) + n R_n(\tau_n + is/\sqrt{n}) \right| \\
&\leq \frac{|s|^3}{6\sqrt{n}} |\psi_n'''(\tau_n)| + n |R_n(\tau_n + is/\sqrt{n})| \\
&\leq \left[\frac{\beta |s|^3}{\sqrt{n} (a - a_1)^3} + \frac{2\beta s^4}{n (a - a_1)^4} \right] \\
&\leq \left[\frac{\beta \delta s^2}{(a - a_1)^3} + \frac{2\beta \delta^2 s^2}{(a - a_1)^4} \right] \\
&= M(\delta) s^2 \tag{2.42}
\end{aligned}$$

where $M(\delta) = [\beta \delta / (a - a_1)^3 + 2\beta \delta^2 / (a - a_1)^4]$. Using the inequality

$|\exp(z_n) - 1 - z_n| \leq |z_n|^2 \exp |z_n|$ and (2.42) we get

$$\begin{aligned} L_n(s) &= |\exp(z_n) - 1 - z_n| \\ &\leq \left[\frac{\beta |s|^3}{\sqrt{n} (a - a_1)^3} + \frac{2\beta s^4}{n(a - a_1)^4} \right]^2 \exp(M(\delta)s^2). \end{aligned} \quad (2.43)$$

We will use the bound in (2.43) to show that the fourth integral in the r.h.s. of (2.37) is $O(1/n)$.

Consider,

$$\begin{aligned} &\left| \left[\frac{\psi_n''(\tau_n)}{2\pi} \right] \int_{|s| < \sqrt{n}\delta} \exp\left(-\frac{s^2}{2}\psi_n''(\tau_n)\right) L_n(s) ds \right| \\ &\leq \left[\frac{\psi_n''(\tau_n)}{2\pi} \right] \int_{|s| < \sqrt{n}\delta} \exp\left(-\frac{s^2}{2}\psi_n''(\tau_n)\right) \exp(M(\delta)s^2) \\ &\quad \left[\frac{\beta |s|^3}{\sqrt{n} (a - a_1)^3} + \frac{2\beta s^4}{n(a - a_1)^4} \right]^2 ds \\ &\leq \left[\frac{\psi_n''(\tau_n)}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{n}\delta} \exp\left(-\left(\frac{\alpha}{2} - M(\delta)\right)s^2\right) \\ &\quad \left[\frac{\beta |s|^3}{\sqrt{n} (a - a_1)^3} + \frac{2\beta s^4}{n(a - a_1)^4} \right]^2 ds \\ &= O(1/n) \end{aligned} \quad (2.44)$$

since we chose δ such that $M(\delta) < \alpha/2$. Thus, from (2.37), (2.41), (2.44) it follows that $I_{n2} = 1 + O(1/n)$. The proof of Theorem 2.2.8 is now complete.

For lattice random variables T_n we have the following analogous theorem.

Theorem 2.2.10. *Let T_n take values in the set $\{a_n + kh_n : k = 0, \pm 1, \pm 2, \dots\}$. Let $\{m_n = (a_n + k_n h_n)/n\}$ be a sequence of real numbers where $\{k_n\}$ is a sequence of integers. Assume that conditions (i') and (ii') of Theorem 2.2.8 hold and replace conditions (iii'') and (iv') by the following:*

(iii**) *Given $\delta > 0$, there exists $0 < \eta < 1$ such that*

$$\limsup_n \sup_{\delta \leq |t| \leq \pi/|h_n|} \left| \frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)} \right|^{1/n} < \eta. \quad (2.45)$$

(iv**) *There exist $p, l > 0$ such that*

$$\sup_{\tau \in I} \int_{-\pi/|h_n|}^{\pi/|h_n|} \left| \frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)} \right|^{l/n} dt = O(n^p). \quad (2.46)$$

Then,

$$\frac{\sqrt{n}}{|h_n|} Pr(T_n/n = m_n) = \frac{1}{\sqrt{2\pi} \sqrt{\psi_n''(\tau_n)}} \exp(-n\gamma_n(m_n)) [1 + O(1/n)]. \quad (2.47)$$

Proof. The proof of this theorem is analogous to that of Theorem 2.2.5 except that the range of integration in (2.27) is from $-\pi/|h_n|$ to $\pi/|h_n|$ instead of the whole real line. Hence we skip the details of this proof.

As an application to large deviation local limit theorems for an arbitrary sequence of random variables Chaganty- Sethuraman (1985) have discussed

two examples in non-parametric theory. One of the examples deals with Wilcoxon signed rank statistic. We show below that the Wilcoxon signed rank statistic satisfies our new Condition (iii**).

Example 2.2.11 Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with median m . Arrange $|X_1|, |X_2|, \dots, |X_n|$ in increasing order of magnitude and assign ranks $1, 2, \dots, n$. The Wilcoxon signed-ranked statistic U_n is defined as the sum of the ranks of positive X_i 's. This statistic is used to test the null hypothesis $H_0 : m = 0$ vs $H_1 : m \neq 0$. Let $T_n = U_n/n$. The c.f. of T_n under the null hypothesis H_0 is given by

$$\phi_n(z) = \prod_{k=1}^n [(\exp(kz/n) + 1)/2], \quad z \in \mathbb{C} \quad (2.48)$$

and therefore

$$\psi_n(z) = (1/n) \sum_{k=1}^n \log[(\exp(kz/n) + 1)/2], \quad z \in \mathbb{C}. \quad (2.49)$$

In this example verification of Conditions (i'), (ii') and (iv**) of Theorem 2.2.10 does not pose any difficulties. The analysis in Chaganty and Sethuraman (1985) page 26 shows that, there exists n_0 and $\delta_1 > 0$ such that for $0 < \delta < \delta_1$,

$$\sup_{\delta \leq |t| < \pi/n} \left| \frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)} \right|^{1/n} \leq \exp(-\alpha \delta^2/4) \quad (2.50)$$

for $n \geq n_0$. This verifies Condition (iii**). Thus all the conditions of Theorem 2.2.10 are satisfied. Hence, the conclusion of Theorem 2.2.10 yields an expression for $P(T_n = nm_n)$.

2.3 Large deviation local limit theorems for ratio statistics

This section deals with the main theorems of this dissertation, namely, Theorems 2.3.1 and 2.3.12. Let $\{T_n, n \geq 1\}$ be an arbitrary sequence of random variables and $\{S_n, n \geq 1\}$ be another arbitrary sequence of positive random variables. Let the two sequences be independent. Theorem 2.3.1 obtains an asymptotic expression for the density function of $R_n = T_n/S_n$ at the point r_n in the case where both T_n and S_n are non-lattice random variables. The other Theorem 2.3.12 obtains similar result in the case where T_n is non-lattice and S_n is a lattice random variable. We treat the case where T_n and S_n both are lattice random variables later in Section 2.4.

Our Theorem 2.3.12 simplifies to the result of Chaganty- Sethuraman (see Theorem 2.2.8) in the case where T_n is taken to be the sum of n i.i.d. non-lattice random variables and S_n is taken to be degenerate at n . Thus our theorem generalizes the results obtained by Chaganty and Sethuraman.

The conditions of the Theorems 2.3.1, 2.3.12 are very much simplified when T_n and S_n are taken to be the sums of i.i.d. random variables. These results are stated in Theorems 2.3.13 and 2.3.14. The conclusion of Theorem 2.3.13 agrees with the heuristic result of Daniels (1954).

Like the theorems in the previous section, the theorems in this section are also obtained under some mild and easily verifiable conditions on the moment generating functions ϕ_{1n} and ϕ_{2n} of T_n and S_n respectively. These conditions are such that they give bounds on the means of T_n/a_n

and S_n/a_n and variances of $T_n/\sqrt{a_n}$ and $S_n/\sqrt{a_n}$ which in turn imply that T_n/a_n converges to zero in probability and S_n/a_n , in view of (2.54), converges in probability to some positive constant. Thus by Slutsky's theorem $R_n = T_n/S_n$ converges to zero in probability and therefore for any $r_n > 0$, the asymptotic expressions for the density function of R_n at r_n obtained in this section become a large deviation local limit theorem. We proceed with some notations.

Let $\{T_n, n \geq 1\}$ be an arbitrary sequence of non-lattice random variables and let $\{S_n, n \geq 1\}$ be a sequence of positive non-lattice valued random variables. Let the two sequences be independent. Let ϕ_{1n} and ϕ_{2n} be the moment generating functions of $\{T_n, n \geq 1\}$ and $\{S_n, n \geq 1\}$ respectively. Let us further assume that $\phi_{1n}(z)$ is analytic in $\Omega_1 = \{z \in \mathbb{C} : |z| < c_1\}$ and $\phi_{2n}(z)$ is analytic in $\Omega_2 = \{z \in \mathbb{C} : |z| < c_2\}$ where $\mathbb{C}$ denotes the set of all complex numbers and c_1, c_2 are some positive constants. Let $\{a_n\}$ be a sequence of real numbers such that $a_n \rightarrow \infty$.

Let

$$\psi_{1n}(z) = \frac{1}{a_n} \log \phi_{1n}(z), z \in \Omega_1. \quad (2.51)$$

$$\psi_{2n}(z) = \frac{1}{a_n} \log \phi_{2n}(z), z \in \Omega_2. \quad (2.52)$$

Let $J_1 = (-b_1, b_1)$, $0 < b_1 < c_1$ and $J_2 = (-b_2, b_2)$, $0 < b_2 < c_2$. Let $\{r_n\}$ be a bounded sequence of real numbers such that there exists a sequence $\{\tau_n\}$ contained in J_1 satisfying

$$\psi'_{1n}(\tau_n) - r_n \psi'_{2n}(-r_n \tau_n) = 0 \quad (2.53)$$

for $r_n \tau_n \in J_2$ for all $n \geq 1$. Let there exist positive constants α_1 and α_2

such that

$$\psi''_{1n}(\tau) > \alpha_1 \text{ and } \psi'_{2n}(\tau') > \alpha_2 \quad (2.54)$$

and $\tau' \in J_2$.

We now state the main theorem of this section. Theorem 2.3.1 below obtains a large deviation local limit theorem for the ratio statistic $R_n = T_n/S_n$ at the point r_n .

Theorem 2.3.1. *Assume that the two sequences $\{T_n, n \geq 1\}$ and $\{S_n, n \geq 1\}$ satisfy the following conditions:*

(A1) *There exist $\beta_1 < \infty$ and $\beta_2 < \infty$ such that*

$$|\psi_{1n}(z)| < \beta_1 \text{ for } z \in \Omega_1 \quad (2.55)$$

and

$$|\psi_{2n}(z)| < \beta_2 \text{ for } z \in \Omega_2, \text{ for all } n \geq 1. \quad (2.56)$$

(A2) *For any given $\delta > 0$, there exist $0 < \eta < 1$ and $q > 0$ such that*

$$\limsup_n \sup_{|t| > \delta} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \right|^{1/a_n} < \eta \quad (2.57)$$

and

$$\sup_{|t| > \delta} |\psi'_{2n}(-r_n(\tau_n + it))| = O(a_n^q). \quad (2.58)$$

(A3) *There exist $p, l > 0$ such that*

$$\sup_{\tau \in J_1} \int_{-\infty}^{\infty} \left| \frac{\phi_{1n}(\tau + it)}{\phi_{1n}(\tau)} \right|^{l/a_n} dt = O(a_n^p). \quad (2.59)$$

Then an asymptotic expansion for the density function g_n of T_n/S_n at the point r_n is given by

$$g_n(r_n) = \frac{\sqrt{a_n} \psi'_{2n}(-r_n \tau_n)}{[2\pi[\psi''_{1n}(\tau_n) + r_n^2 \psi''_{2n}(-r_n \tau_n)]]^{1/2}} \times \exp(a_n[\psi_{1n}(\tau_n) + \psi_{2n}(-r_n \tau_n)])[1 + O(1/a_n)]. \quad (2.60)$$

The proof of this theorem is deferred until the end of Lemma 2.3.11. We now make some remarks about the Conditions (A1), (A2) and (A3) of the above theorem.

Remark 2.3.2. Condition (A1) of Theorem 2.3.1 requires that ψ_{1n} and ψ_{2n} be bounded uniformly in n in a circle around the origin in the complex plane. In Lemma 2.3.7 we show that Condition(A1) implies that the means of T_n/a_n and S_n/a_n are bounded and the variances of $T_n/\sqrt{a_n}$ and $S_n/\sqrt{a_n}$ are also bounded. Applying Chebyshev's inequality we can verify that $(T_n - E(T_n))/a_n \rightarrow 0$ in probability and $(S_n - E(S_n))/a_n \rightarrow c$ in probability for some c positive. The application of Slutsky's theorem shows that $R_n = T_n/S_n$ goes to zero in probability.

Remark 2.3.3. Since $|\phi_{1n}(\tau_n + it)/\phi_{1n}(\tau_n)|$ is the characteristic function of a non-lattice random variable, for each $\delta > 0$ we can always find $0 < \eta_n < 1$ such that

$$\sup_{|t| > \delta} |\phi_{1n}(\tau_n + it)/\phi_{1n}(\tau_n)|^{1/a_n} < \eta_n.$$

Condition (A2) requires that the $\limsup \eta_n = \eta < 1$. We use Condition (A2) mainly in Lemma 2.3.9 to show that the term I_{n1} defined in (2.68) goes exponentially fast to zero.

Remark 2.3.4. Condition (A3) guarantees the existence of the density function of T_n and permits the use of inversion formula to get an expression for the p.d.f. of T_n . This condition is also used to show that the term I_{n1} in (2.68) goes exponentially fast to zero.

Remark 2.3.5. It is interesting to note that the conclusion of the Theorem 2.3.1 remains the same if ϕ_{1n} is replaced by ϕ_{2n} in (2.57) and (2.59) of Conditions (A2) and (A3).

Remark 2.3.6. Selection of an appropriate sequence $\{r_n\}$ of real numbers guarantees the existence of a saddle point $(\tau_n, 0)$ for the function $Real(\psi_{1n}(\tau + it) + \psi_{2n}(-r_n(\tau + it)))$ for each n . We use this saddle point to get an asymptotic expression for the integral in (2.107).

We will need the following Lemmas 2.3.7 thru 2.3.10 in the proof of Theorem 2.3.1. Lemma 2.3.7 obtains bounds which are independent of n for $\psi_{1n}^{(k)}$ and $\psi_{2n}^{(k)}$ for $k \geq 1$ and also for the remainder terms $R_{1n}(\tau + it)$ and $R_{2n}(\tau + it)$ defined below.

Lemma 2.3.7. *Let ψ_{1n} and ψ_{2n} be as defined in (2.51) and (2.52). Assume that Condition (A1) of Theorem 2.3.1 holds. Then the following holds:*

$$\sup_{\tau \in J_j} |\psi_{jn}^{(k)}(\tau)| \leq \frac{k! \beta_j}{(c_j - b_j)^k} \quad (2.61)$$

for $j = 1, 2$ and for $k \geq 1$. Let

$$R_{jn}(\tau + it) = \psi_{jn}(\tau + it) - \psi_{jn}(\tau) - (it)\psi'_{jn}(\tau) - \frac{(it)^2}{2!}\psi''_{jn}(\tau) - \frac{(it)^3}{3!}\psi'''_{jn}(\tau) \quad (2.62)$$

for $\tau \in J_j$, $j = 1, 2$. Then there exists $\delta_j < (c_j - b_j)/2$ such that

$$\sup_{\tau \in J_j} |R_{jn}(\tau + it)| \leq \frac{2\beta_j t^4}{(c_j - b_j)^4} \quad (2.63)$$

for $|t| < \delta_j$ and for $j = 1, 2$.

Proof. The proof of this lemma is similar to the proof of Lemma 2.2.9 and hence it is omitted.

Remark 2.3.8. Let $\sup |r_n| = r$. Let $\delta_3 = \delta_2/r$. Since $\psi'_{2n}(z)$ is analytic in Ω_2 the following expansion is valid for all $n \geq 1$ and $|t| < \delta_3$:

$$\begin{aligned} \psi'_{2n}(-r_n(\tau_n + it)) &= \psi'_{2n}(-r_n\tau_n) + (-ir_nt)\psi''_{2n}(-r_n\tau_n) + \frac{(-ir_nt)^2}{2!}\psi'''_{2n}(-r_n\tau_n) \\ &\quad + R(-r_n(\tau_n + it)) \end{aligned} \quad (2.64)$$

where

$$R(-r_n(\tau_n + it)) = \frac{(-ir_nt)^3}{2\pi i} \int_{C_2} \frac{\psi'_{2n}(\omega)}{(\omega - (-r_n\tau_n))^3 (\omega - (-r_n\tau_n) + ir_nt)} d\omega \quad (2.65)$$

and $C_2 = C(-r_n\tau_n, c_2 - b_2)$. Therefore, for $|t| < \delta_3$ we have,

$$|R(-r_n(\tau_n + it))| \leq \frac{2\beta_2 r^3 t^3}{(c_2 - b_2)^3}. \quad (2.66)$$

Lemma 2.3.9 shows that the term I_{n1} (see (2.113)) appearing in the proof of Theorem 2.3.1 goes exponentially fast to zero.

Lemma 2.3.9. *Assume that the Conditions (A2) and (A3) of Theorem 2.3.1 are satisfied. Let*

$$f_n(z) = \psi_{1n}(z) + \psi_{2n}(-r_n z). \quad (2.67)$$

Then

$$\begin{aligned} I_{n1} &= \left[\frac{a_n f_n''(\tau_n)}{2\pi} \right]^{1/2} \int_{|t| \geq \delta} \exp([a_n(f_n(\tau_n + it) - f_n(\tau_n))]) \\ &\quad \times \frac{\psi'_{2n}(-r_n(\tau_n + it))}{\psi'_{2n}(-r_n \tau_n)} dt \end{aligned} \quad (2.68)$$

goes exponentially fast to zero for all $\delta > 0$ such that $\delta < \min(\delta_1, \delta_2, \delta_3)$ where δ_1 and δ_2 are as in Lemma 2.3.7 and δ_3 is as in Remark 2.3.8.

Proof: Consider

$$\begin{aligned} |I_{n1}| &= \left| \left[\frac{a_n f_n''(\tau_n)}{2\pi} \right]^{1/2} \int_{|t| \geq \delta} \exp([a_n(f_n(\tau_n + it) - f_n(\tau_n))]) \frac{\psi'_{2n}(-r_n(\tau_n + it))}{\psi'_{2n}(-r_n \tau_n)} dt \right| \\ &\leq \left[\frac{a_n f_n''(\tau_n)}{2\pi} \right]^{1/2} \int_{|t| \geq \delta} |\exp([a_n(f_n(\tau_n + it) - f_n(\tau_n))])| \left| \frac{\psi'_{2n}(-r_n(\tau_n + it))}{\psi'_{2n}(-r_n \tau_n)} \right| dt. \end{aligned} \quad (2.69)$$

Replacing $f_n(\tau_n)$ by $\psi_{1n}(\tau_n) + \psi_{2n}(-r_n \tau_n)$ inside the integral, we get

$$|I_{n1}| \leq \left[\frac{a_n f_n''(\tau_n)}{2\pi} \right]^{1/2} \int_{|t| \geq \delta} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \frac{\phi_{2n}(-r_n(\tau_n + it))}{\phi_{2n}(-r_n \tau_n)} \right| \left| \frac{\psi'_{2n}(-r_n(\tau_n + it))}{\psi'_{2n}(-r_n \tau_n)} \right| dt$$

$$\begin{aligned}
&\leq \left[\frac{a_n f_n''(\tau_n)}{2\pi} \right]^{1/2} \int_{|t| \geq \delta} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \right| \left| \frac{\psi'_{2n}(-r_n(\tau_n + it))}{\psi'_{2n}(-r_n \tau_n)} \right| dt \\
&\leq \left[\frac{a_n f_n''(\tau_n)}{2\pi} \right]^{1/2} \sup_{|t| \geq \delta} \left| \frac{\psi'_{2n}(-r_n(\tau_n + it))}{\psi'_{2n}(-r_n \tau_n)} \right| \\
&\quad \int_{|t| \geq \delta} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \right|^{l/a_n} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \right|^{1-l/a_n} dt. \tag{2.70}
\end{aligned}$$

Using the fact that $\psi'_{2n}(\tau) > \alpha_2$, for $\tau \in J_2$ and Condition (A2) we get,

$$|I_{n1}| \leq \left[\frac{a_n f_n''(\tau_n)}{2\pi} \right]^{1/2} \frac{1}{\alpha_2} O(a_n^q) \eta^{a_n(1-l/a_n)} \int_{-\infty}^{\infty} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \right|^{l/a_n} dt.$$

Finally, Condition (A3) implies that,

$$|I_{n1}| \leq O(a_n^{p+q+1/2}) \exp(-\eta_1(a_n - l)), \tag{2.71}$$

where $\eta_1 = -\log \eta > 0$. Hence I_{n1} goes exponentially fast to zero.

We need the following lemma to prove Lemma 2.3.11.

Lemma 2.3.10. *Let ψ_{1n} and ψ_{2n} be as defined in (2.51) and (2.52). Assume that Condition (A1) of Theorem 2.3.1 holds.*

Let

$$d_n = \psi''_{1n}(\tau_n) + r_n^2 \psi''_{2n}(-r_n \tau_n) \tag{2.72}$$

and

$$L_n(s) = \left[\exp(z_n) \frac{\psi'_{2n}(-r_n(\tau_n + i \frac{s}{\sqrt{a_n}}))}{\psi'_{2n}(-r_n \tau_n)} - 1 - z_n \right] \tag{2.73}$$

where

$$z_n = \left[\frac{-is^3}{6\sqrt{a_n}} \psi_{1n}'''(\tau_n) + \frac{ir_n^3 s^3}{6\sqrt{a_n}} \psi_{2n}'''(-r_n \tau_n) \right. \\ \left. + a_n R_{1n}(\tau_n + i \frac{s}{\sqrt{a_n}}) + a_n R_{2n}(-r_n(\tau_n + i \frac{s}{\sqrt{a_n}})) \right]. \quad (2.74)$$

Then there exists $\delta > 0$ such that

$$\left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-s^2}{2} d_n\right) L_n(s) ds = O(1/a_n). \quad (2.75)$$

Proof. Let δ be as in Lemma 2.3.9. Then

$$Q_n = \left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-s^2}{2} d_n\right) \left[\exp(z_n) \frac{\psi_{2n}'(-r_n(\tau_n + i \frac{s}{\sqrt{a_n}}))}{\psi_{2n}'(-r_n \tau_n)} - 1 - z_n \right] ds. \quad (2.76)$$

We can write Q_n as the sum of four integrals as follows:

$$Q_n = \left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-s^2}{2} d_n\right) \left[\exp((z_n) - 1 - z_n) \frac{\psi_{2n}'(-r_n(\tau_n + i \frac{s}{\sqrt{a_n}}))}{\psi_{2n}'(-r_n \tau_n)} \right. \\ \left. + \left(\frac{\psi_{2n}'(-r_n(\tau_n + i \frac{s}{\sqrt{a_n}}))}{\psi_{2n}'(-r_n \tau_n)} - 1 \right) - (z_n) + \left(z_n \frac{\psi_{2n}'(-r_n(\tau_n + i \frac{s}{\sqrt{a_n}}))}{\psi_{2n}'(-r_n \tau_n)} \right) \right] ds \\ = I_{n1}' + I_{n2}' + I_{n3}' + I_{n4}' \text{ (say)} \quad (2.77)$$

where

$$I_{n1}' = \left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-s^2}{2} d_n\right) \exp((z_n) - 1 - z_n) \frac{\psi_{2n}'(-r_n(\tau_n + i \frac{s}{\sqrt{a_n}}))}{\psi_{2n}'(-r_n \tau_n)} ds \quad (2.78)$$

$$I'_{n2} = \left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-s^2}{2} d_n\right) \left(\frac{\psi'_{2n}(-r_n(\tau_n + i \frac{s}{\sqrt{a_n}}))}{\psi'_{2n}(-r_n \tau_n)} - 1 \right) ds \quad (2.79)$$

$$I'_{n3} = \left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-s^2}{2} d_n\right) (-z_n) ds \quad (2.80)$$

$$I'_{n4} = \left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-s^2}{2} d_n\right) z_n \frac{\psi'_{2n}(-r_n(\tau_n + i \frac{s}{\sqrt{a_n}}))}{\psi'_{2n}(-r_n \tau_n)} ds. \quad (2.81)$$

The proof is completed by showing that $I'_{ni} = O(1/a_n)$ for $i = 1, 2, 3, 4$. In order to show that $I'_{n1} = O(1/a_n)$, we get an upper bound for $(\exp(z_n) - 1 - z_n)$, by first obtaining an upperbound for z_n . For $|s| < \sqrt{a_n} \delta$, using Condition (A1) and (2.61) we get that

$$\begin{aligned} |z_n| &\leq \frac{|s|^3 \beta_1}{\sqrt{a_n}(c_1 - b_1)^3} + \frac{|r_n|^3 |s|^3 \beta_2}{\sqrt{a_n}(c_2 - b_2)^3} + \frac{2\beta_1 s^4}{a_n(c_1 - b_1)^4} + \frac{2\beta_2 r_n^4 s^4}{a_n(c_2 - b_2)^4} \\ &\leq \frac{\beta_1 \delta |s|^2}{(c_1 - b_1)^3} + \frac{\beta_2 |r_n|^3 \delta |s|^2}{(c_2 - b_2)^3} + \frac{2\beta_1 \delta^2 s^2}{(c_1 - b_1)^4} + \frac{2\beta_2 r_n^4 \delta^2 s^2}{(c_2 - b_2)^4} \\ &= \left[\frac{\beta_1 \delta}{(c_1 - b_1)^3} + \frac{\beta_2 |r_n|^3 \delta}{(c_2 - b_2)^3} + \frac{2\beta_1 \delta^2}{(c_1 - b_1)^4} + \frac{2\beta_2 r_n^4 \delta^2}{(c_2 - b_2)^4} \right] s^2. \end{aligned} \quad (2.82)$$

Using the boundedness of r_n in (2.82), we get

$$|z_n| \leq M(\delta) s^2 \quad (2.83)$$

where

$$\begin{aligned} M(\delta) &= \left[\beta_1 \delta / (c_1 - b_1)^3 + \beta_2 r^3 \delta / (c_2 - b_2)^3 \right. \\ &\quad \left. + 2\beta_1 \delta^2 / (c_1 - b_1)^4 + 2\beta_2 r^4 \delta^2 / (c_2 - b_2)^4 \right]. \end{aligned} \quad (2.84)$$

We choose our δ such that $M(\delta) < \alpha_1/2$. Using the simple inequality

$|\exp(z_n) - 1 - z_n| \leq |z_n|^2 \exp(|z_n|)$ we obtain,

$$\begin{aligned} |\exp(z_n) - 1 - z_n| &\leq \left[\frac{|s|^3 \beta_1}{\sqrt{a_n}(c_1 - b_1)^3} + \frac{r^3 |s|^3 \beta_2}{\sqrt{a_n}(c_2 - b_2)^3} \right. \\ &\quad \left. + \frac{2\beta_1 s^4}{a_n(c_1 - b_1)^4} + \frac{2\beta_2 r^4 s^4}{a_n(c_2 - b_2)^4} \right]^2 \exp(M(\delta)s^2). \end{aligned} \quad (2.85)$$

We are now in a position to show that $I'_{n1} = O(1/a_n)$. Consider

$$\begin{aligned} |I'_{n1}| &= \left| \left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n}\delta} \exp\left(\frac{-s^2}{2} d_n\right) (\exp(z_n) - 1 - z_n) \frac{\psi'_{2n}(-r_n(\tau_n + i\frac{s}{\sqrt{a_n}}))}{\psi'_{2n}(-r_n\tau_n)} ds \right| \\ &\leq \left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n}\delta} \exp\left(\frac{-s^2}{2} d_n\right) |(\exp(z_n) - 1 - z_n)| \left| \frac{\psi'_{2n}(-r_n(\tau_n + i\frac{s}{\sqrt{a_n}}))}{\psi'_{2n}(-r_n\tau_n)} \right| ds \\ &\leq \left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n}\delta} \exp\left(\frac{-s^2}{2} d_n\right) \exp(M(\delta)s^2) \left[\frac{|s|^3 \beta_1}{\sqrt{a_n}(c_1 - b_1)^3} + \frac{r^3 |s|^3 \beta_2}{\sqrt{a_n}(c_2 - b_2)^3} \right. \\ &\quad \left. + \frac{2\beta_1 s^4}{a_n(c_1 - b_1)^4} + \frac{2\beta_2 r^4 s^4}{a_n(c_2 - b_2)^4} \right]^2 \left| \frac{\psi'_{2n}(-r_n(\tau_n + i\frac{s}{\sqrt{a_n}}))}{\psi'_{2n}(-r_n\tau_n)} \right| ds \\ &\leq \frac{1}{a_n} \left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n}\delta} \exp\left(\frac{-s^2}{2}(\alpha_1 - 2M(\delta))\right) \left[\frac{\beta_1 |s|^3}{(c_1 - b_1)^3} + \frac{r^3 |s|^3 \beta_2}{(c_2 - b_2)^3} \right. \\ &\quad \left. + \frac{2\beta_1 s^4}{\sqrt{a_n}(c_1 - b_1)^4} + \frac{2\beta_2 r^4 s^4}{\sqrt{a_n}(c_2 - b_2)^4} \right]^2 \left| \frac{\psi'_{2n}(-r_n(\tau_n + i\frac{s}{\sqrt{a_n}}))}{\psi'_{2n}(-r_n\tau_n)} \right| ds. \end{aligned} \quad (2.86)$$

Using Taylor series expansion for ψ'_{2n} and the bounds (2.61) and (2.63) obtained in Lemma 2.3.7 we can show that $|\psi'_{2n}(-r_n(\tau_n + i\frac{s}{\sqrt{a_n}}))/\psi'_{2n}(-r_n\tau_n)|$ is bounded in n . Since $M(\delta) < \alpha_1/2$ we get that $I'_{n1} = O(1/a_n)$. Thus

$$I'_{n1} = O(1/a_n). \quad (2.87)$$

Next we show that $I'_{n2} = O(1/a_n)$. Consider

$$\begin{aligned}
I'_{n2} &= \left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-s^2}{2} d_n\right) \left(\frac{\psi'_{2n}(-r_n(\tau_n + i \frac{s}{\sqrt{a_n}}))}{\psi'_{2n}(-r_n \tau_n)} - 1 \right) ds \\
&= \left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-s^2}{2} d_n\right) \left[-i r_n \frac{s}{\sqrt{a_n}} \frac{\psi''_{2n}(-r_n \tau_n)}{\psi'_{2n}(-r_n \tau_n)} \right. \\
&\quad \left. - r_n^2 \frac{s^2}{a_n} \frac{\psi'''_{2n}(-r_n \tau_n)}{\psi'_{2n}(-r_n \tau_n)} + \frac{R(-r_n \tau_n - i r_n \frac{s}{\sqrt{a_n}})}{\psi'_{2n}(-r_n \tau_n)} \right] ds \tag{2.88}
\end{aligned}$$

where $R(-r_n(\tau_n + it))$ is as given in (2.65). It is easy to see that the first integral on the right hand side of I'_{n2} is zero. The use of bound on $\psi'''_{2n}(-r_n \tau_n)$ (see (2.61)) and the fact that $\psi'_{2n}(-r_n \tau_n) > \alpha_2$, it follows that the second integral is $O(1/a_n)$. Similarly (2.66) and the fact that $\psi'_{2n}(-r_n \tau_n) > \alpha_2$ imply that the third integral is $O(1/a_n^2)$. Therefore,

$$I'_{n2} = O(1/a_n). \tag{2.89}$$

Next, consider

$$\begin{aligned}
I'_{n3} &= \left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-s^2}{2} d_n\right) (-z_n) ds \\
&= (-1) \left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-s^2}{2} d_n\right) \left[\frac{-is^3}{6\sqrt{a_n}} \psi'''_{1n}(\tau_n) + \frac{ir_n^3 s^3}{6\sqrt{a_n}} \psi'''_{2n}(-r_n \tau_n) \right. \\
&\quad \left. + a_n R_{1n}(\tau_n + i \frac{s}{\sqrt{a_n}}) + a_n R_{2n}(-r_n(\tau_n + i \frac{s}{\sqrt{a_n}})) \right] ds. \tag{2.90}
\end{aligned}$$

The first two integrals on the right hand side of I'_{n3} are zero and the third and fourth integrals using (2.63) can be seen to be equal to $O(1/a_n)$.

Thus

$$I'_{n3} = O(1/a_n). \quad (2.91)$$

Lastly, consider

$$\begin{aligned} I'_{n4} &= \left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-s^2}{2} d_n\right) z_n \frac{\psi'_{2n}(-r_n(\tau_n + i\frac{s}{\sqrt{a_n}}))}{\psi'_{2n}(-r_n\tau_n)} ds \\ &= \left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-s^2}{2} d_n\right) \left[\frac{-is^3}{6\sqrt{a_n}} \psi'''_{1n}(\tau_n) + \frac{i\tau_n^3 s^3}{6\sqrt{a_n}} \psi'''_{2n}(-r_n\tau_n) \right. \\ &\quad \left. + a_n R_{1n}(\tau_n + i\frac{s}{\sqrt{a_n}}) + a_n R_{2n}(-r_n(\tau_n + i\frac{s}{\sqrt{a_n}})) \right] \left[1 - i\tau_n \frac{s}{\sqrt{a_n}} \frac{\psi''_{2n}(-r_n\tau_n)}{\psi'_{2n}(-r_n\tau_n)} \right. \\ &\quad \left. - r_n^2 \frac{s^2}{a_n} \frac{\psi'''_{2n}(-r_n\tau_n)}{\psi'_{2n}(-r_n\tau_n)} + \frac{R(-r_n\tau_n - i\tau_n \frac{s}{\sqrt{a_n}})}{\psi'_{2n}(-r_n\tau_n)} \right] ds. \end{aligned} \quad (2.92)$$

I'_{n4} can be easily shown to be $O(1/a_n)$. Therefore, it follows from (2.86), (2.88), (2.90) and (2.91) that $Q_n = O(1/a_n)$. This completes the proof of the lemma.

The next lemma shows that the term I_{n2} appearing in the proof of the main Theorem 2.3.1, (see (2.111)) is $1 + O(1/a_n)$.

Lemma 2.3.11. *Let $f_n(t)$ be as defined in (2.67). Assume that Conditions (A1) and (A2) of Theorem 2.3.1 hold. Let δ be as in Lemma 2.3.10. Then*

$$\begin{aligned} I_{n2} &= \left[\frac{a_n d_n}{2\pi} \right]^{1/2} \int_{|t| < \delta} \exp(a_n[f_n(\tau_n + it) - f_n(\tau_n)]) \frac{\psi'_{2n}(-r_n(\tau_n + it))}{\psi'_{2n}(-r_n\tau_n)} dt \\ &= 1 + O(1/a_n). \end{aligned} \quad (2.93)$$

Proof. Consider

$$I_{n2} = \left[\frac{a_n d_n}{2\pi} \right]^{1/2} \int_{|t| < \delta} \exp(a_n [\psi_{1n}(\tau_n + it) - \psi_{1n}(\tau_n) + \psi_{2n}(-r_n(\tau_n + it)) - \psi_{2n}(-r_n\tau_n)]) \frac{\psi'_{2n}(-r_n(\tau_n + it))}{\psi'_{2n}(-r_n\tau_n)} dt. \quad (2.94)$$

Making a change of variable $t = s/\sqrt{a_n}$, we get,

$$I_{n2} = \left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n}\delta} \exp \left(a_n \left[\psi_{1n}(\tau_n + i\frac{s}{\sqrt{a_n}}) - \psi_{1n}(\tau_n) + \psi_{2n}(-r_n(\tau_n + i\frac{s}{\sqrt{a_n}})) - \psi_{2n}(-r_n\tau_n) \right] \right) \frac{\psi'_{2n}(-r_n(\tau_n + i\frac{s}{\sqrt{a_n}}))}{\psi'_{2n}(-r_n\tau_n)} ds. \quad (2.95)$$

Using the Taylor's series expansions for ψ_{1n} and ψ_{2n} (see (2.62)), we get

$$\begin{aligned} I_{n2} &= \left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n}\delta} \exp \left(-a_n \left[\frac{s^2}{2a_n} \psi''_{1n}(\tau_n) + \frac{r_n^2 s^2}{2a_n} \psi''_{2n}(-r_n\tau_n) \right. \right. \\ &\quad \left. \left. + \frac{is^3}{6a_n\sqrt{a_n}} \psi'''_{1n}(\tau_n) - \frac{ir_n^3 s^3}{6a_n\sqrt{a_n}} \psi'''_{2n}(-r_n\tau_n) - R_{1n}(\tau_n + i\frac{s}{\sqrt{a_n}}) \right. \right. \\ &\quad \left. \left. - R_{2n}(-r_n(\tau_n + i\frac{s}{\sqrt{a_n}})) \right] \right) \frac{\psi'_{2n}(-r_n(\tau_n + i\frac{s}{\sqrt{a_n}}))}{\psi'_{2n}(-r_n\tau_n)} ds \\ &= \left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n}\delta} \exp \left(\frac{-s^2}{2} [\psi''_{1n}(\tau_n) + r_n^2 \psi''_{2n}(-r_n\tau_n)] \right) \\ &\quad \times \exp \left(\frac{-is^3}{6\sqrt{a_n}} \psi'''_{1n}(\tau_n) + \frac{ir_n^3 s^3}{6\sqrt{a_n}} \psi'''_{2n}(-r_n\tau_n) + a_n R_{1n}(\tau_n + i\frac{s}{\sqrt{a_n}}) \right. \\ &\quad \left. + a_n R_{2n}(-r_n(\tau_n + i\frac{s}{\sqrt{a_n}})) \right) \frac{\psi'_{2n}(-r_n(\tau_n + i\frac{s}{\sqrt{a_n}}))}{\psi'_{2n}(-r_n\tau_n)} ds. \quad (2.96) \end{aligned}$$

Noting that $\psi''_{1n}(\tau_n) + r_n^2 \psi''_{2n}(-r_n \tau_n)$ is d_n defined in (2.72), we get that

$$I_{n2} = \left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp \left(\frac{-d_n s^2}{2} \right) \left[1 - \frac{-is^3}{6\sqrt{a_n}} \psi'''_{1n}(\tau_n) + \frac{ir_n^3 s^3}{6\sqrt{a_n}} \psi'''_{2n}(-r_n \tau_n) \right. \\ \left. + a_n R_{1n}(\tau_n + i \frac{s}{\sqrt{a_n}}) + a_n R_{2n}(-r_n(\tau_n + i \frac{s}{\sqrt{a_n}})) + L_n(s) \right] ds \quad (2.97)$$

where d_n , z_n and $L_n(s)$ are as defined in (2.72), (2.74) and (2.73) respectively. The right hand side of (2.96) involves six integrals. Let us analyze these integrals.

The 1st integral

$$\left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp \left(\frac{-s^2}{2} d_n \right) ds = 1 - 2\Phi(-\sqrt{a_n} \delta \sqrt{d_n}) \\ = 1 + o(1/a_n) \quad (2.98)$$

follows from Mill's Ratio (see Feller (1968) page 175,(1.8)).

The 2nd integral

$$\left[\frac{d_n}{2\pi} \right]^{1/2} \frac{i\psi''_{1n}(\tau_n)}{6\sqrt{a_n}} \int_{|s| < \sqrt{a_n} \delta} s^3 \exp \left(\frac{-s^2}{2} d_n \right) ds = 0. \quad (2.99)$$

The 3rd integral

$$\left[\frac{d_n}{2\pi} \right]^{1/2} \frac{ir_n^3 \psi''_{2n}(-r_n \tau_n)}{6\sqrt{a_n}} \int_{|s| < \sqrt{a_n} \delta} s^3 \exp \left(\frac{-s^2}{2} d_n \right) ds = 0. \quad (2.100)$$

For the 4th integral consider

$$\left| \left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} a_n R_{1n}(\tau_n + i \frac{s}{\sqrt{a_n}}) \exp \left(\frac{-s^2}{2} d_n \right) ds \right|$$

$$\begin{aligned}
&\leq a_n \left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp \left(\frac{-s^2}{2} d_n \right) \left| R_{1n} \left(\tau_n + i \frac{s}{\sqrt{a_n}} \right) \right| ds \\
&\leq \frac{1}{a_n} \left[\frac{d_n}{2\pi} \right]^{1/2} \frac{2\beta_1}{(c_1 - b_1)^4} \int_{|s| < \sqrt{a_n} \delta} s^4 \exp \left(\frac{-s^2}{2} d_n \right) ds \\
&= O(1/a_n). \tag{2.101}
\end{aligned}$$

Thus

$$\left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} a_n R_{1n} \left(\tau_n + i \frac{s}{\sqrt{a_n}} \right) \exp \left(\frac{-s^2}{2} d_n \right) ds = O(1/a_n). \tag{2.102}$$

Similarly, the 5th integral can be shown to be equal to $O(1/a_n)$, that is,

$$\begin{aligned}
&\left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} a_n R_{2n} \left(-\tau_n + i \frac{s}{\sqrt{a_n}} \right) \exp \left(\frac{-s^2}{2} d_n \right) ds \\
&= O(1/a_n). \tag{2.103}
\end{aligned}$$

The 6th integral

$$\left[\frac{d_n}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp \left(\frac{-s^2}{2} d_n \right) L_n(s) ds$$

is nothing but Q_n defined by (2.76) and therefore, by Lemma 2.3.10 it is equal to $O(1/a_n)$. Thus

$$I_{n2} = 1 + O(1/a_n).$$

This completes the proof of Lemma 2.3.11.

We now proceed with the proof of the main Theorem 2.3.1.

Proof. Let G_n be the distribution function of $R_n = T_n/S_n$. Then

$$G_n(r) = P \left[\frac{T_n}{S_n} \leq r \right] = \int_0^\infty F_{1n}(ry) dF_{2n}(y). \quad (2.104)$$

The density function, g_n , of G_n is given by

$$g_n(r) = \int_0^\infty y f_{1n}(ry) dF_{2n}(y) \quad (2.105)$$

where f_{1n} is the p.d.f. of T_n .

Using the conjugate distribution technique (see (2.26), (2.27) and (2.28)), we get that

$$f_{1n}(ry) = \frac{1}{2\pi} \int_{-\infty}^\infty \phi_{1n}(\tau + it) \exp(-(\tau + it)ry) dt \quad (2.106)$$

which is true for all $\tau \in J_1$. Later we will choose τ appropriately. Substituting (2.105) in (2.104) we obtain

$$\begin{aligned} g_n(r) &= \int_0^\infty y \left[\frac{1}{2\pi} \int_{-\infty}^\infty \phi_{1n}(\tau + it) \exp(-(\tau + it)ry) dt \right] dF_{2n}(y) \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty \phi_{1n}(\tau + it) \left[\int_0^\infty y \exp(-(\tau + it)ry) dF_{2n}(y) \right] dt \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty \phi_{1n}(\tau + it) \phi'_{2n}(-r(\tau + it)) dt \\ &= \frac{a_n}{2\pi} \int_{-\infty}^\infty \exp(a_n[\psi_{1n}(\tau + it) + \psi_{2n}(-r(\tau + it))]) \psi'_{2n}(-r(\tau + it)) dt. \end{aligned}$$

Saddle point technique suggests that when r is replaced by r_n an appropriate choice of τ is τ_n where τ_n is such that

$$\psi'_{1n}(\tau_n) - r_n \psi'_{2n}(-r_n \tau_n) = 0 \quad (2.107)$$

for $\tau_n \in J_1$ and $r_n \tau_n \in J_2$. Therefore, we have,

$$\begin{aligned} g_n(r_n) &= \frac{a_n}{2\pi} \int_{-\infty}^{\infty} \exp(a_n[\psi_{1n}(\tau_n + it) + \psi_{2n}(-r_n(\tau_n + it))]) \psi'_{2n}(-r_n(\tau_n + it)) dt \\ &= \frac{\sqrt{a_n} \psi'_{2n}(-r_n \tau_n) \exp(a_n[\psi_{1n}(\tau_n) + \psi_{2n}(-r_n \tau_n)])}{[2\pi [\psi''_{1n}(\tau_n) + r_n^2 \psi''_{2n}(-r_n \tau_n)]]^{1/2}} I_n \end{aligned} \quad (2.108)$$

where

$$\begin{aligned} I_n &= \left[\frac{a_n [\psi''_{1n}(\tau_n) + r_n^2 \psi''_{2n}(-r_n \tau_n)]}{2\pi} \right]^{1/2} \int_{-\infty}^{\infty} \exp(a_n[\psi_{1n}(\tau_n + it) - \psi_{1n}(\tau_n) \\ &\quad + \psi_{2n}(-r_n(\tau_n + it)) - \psi_{2n}(-r_n \tau_n)]) \frac{\psi'_{2n}(-r_n(\tau_n + it))}{\psi'_{2n}(-r_n \tau_n)} dt. \end{aligned} \quad (2.109)$$

Let $f_n(z)$ and d_n be as defined in (2.67) and (2.72) respectively. We have,

$$\begin{aligned} I_n &= \left[\frac{a_n d_n}{2\pi} \right]^{1/2} \int_{-\infty}^{\infty} \exp(a_n[f_n(\tau_n + it) - f_n(\tau_n)]) \frac{\psi'_{2n}(-r_n(\tau_n + it))}{\psi'_{2n}(-r_n \tau_n)} dt \\ &= \left[\frac{a_n d_n}{2\pi} \right]^{1/2} \left[\int_{|t| \geq \delta} \exp(a_n[f_n(\tau_n + it) - f_n(\tau_n)]) \frac{\psi'_{2n}(-r_n(\tau_n + it))}{\psi'_{2n}(-r_n \tau_n)} dt \right. \\ &\quad \left. + \int_{|t| < \delta} \exp(a_n[f_n(\tau_n + it) - f_n(\tau_n)]) \frac{\psi'_{2n}(-r_n(\tau_n + it))}{\psi'_{2n}(-r_n \tau_n)} dt \right] \end{aligned} \quad (2.110)$$

where δ is chosen such that $\delta < \min(\delta_1, \delta_2, \delta_3)$ and $M(\delta) < \alpha_1/2$ where δ_1, δ_2 are as in Lemma 2.3.7 and δ_3 is as in Remark 2.3.8 and

$$\begin{aligned} M(\delta) &= [\beta_1 \delta / (c_1 - b_1)^3 + \beta_2 r^3 \delta / (c_2 - b_2)^3 \\ &\quad + 2\beta_1 \delta^2 / (c_1 - b_1)^4 + 2\beta_2 r^2 \delta^2 / (c_2 - b_2)^4]. \end{aligned} \quad (2.111)$$

Thus

$$I_n = I_{n1} + I_{n2} \quad (2.112)$$

where

$$I_{n1} = \left[\frac{a_n d_n}{2\pi} \right]^{1/2} \int_{|t| \geq \delta} \exp(a_n [f_n(\tau_n + it) - f_n(\tau_n)]) \frac{\psi'_{2n}(-r_n(\tau_n + it))}{\psi'_{2n}(-r_n \tau_n)} dt \quad (2.113)$$

and

$$I_{n2} = \left[\frac{a_n d_n}{2\pi} \right]^{1/2} \int_{|t| < \delta} \exp(a_n [f_n(\tau_n + it) - f_n(\tau_n)]) \frac{\psi'_{2n}(-r_n(\tau_n + it))}{\psi'_{2n}(-r_n \tau_n)} dt. \quad (2.114)$$

Lemma 2.3.9 shows that the term I_{n1} goes exponentially fast to zero and

Lemma 2.3.11 shows that $I_{n2} = 1 + O(1/a_n)$.

Thus

$$I_n = 1 + O(1/a_n). \quad (2.115)$$

Hence the proof of the theorem is complete.

Next, we obtain similar asymptotic expression for the density function of ratio statistic $R_n = T_n/S_n$ for the case where $\{T_n, n \geq 1\}$ is a sequence of non-lattice random variables and $\{S_n, n \geq 1\}$ is a sequence of positive lattice random variables. This result is stated below in Theorem 2.3.12.

Theorem 2.3.12. *Let $\{T_n, n \geq 1\}$ be a sequence of non-lattice random variables with distribution functions F_{1n} and let $\{S_n, n \geq 1\}$ be a sequence of positive lattice random variables with distribution functions F_{2n} . Let the two sequences $\{T_n, n \geq 1\}$ and $\{S_n, n \geq 1\}$ be independent. Let S_n take values in the set $S = \{a_n + kh_n : a_n \text{ and } h_n \text{ are such that } a_n + kh_n > 0\}$. Let ψ_{1n} and ψ_{2n} be as defined in (2.51) and (2.52) respectively. Assume that (2.53) is satisfied for an appropriate sequence r_n . Let $\{T_n\}$ and $\{S_n\}$ further*

satisfy the Conditions (A1), (A2) and (A3) of Theorem 2.3.1. Then, an asymptotic expression for the p.d.f., g_{2n} , of T_n/S_n at the point r_n is given by

$$g_{2n}(r_n) = \frac{a_n \psi'_{2n}(-r_n \tau_n)}{[2\pi a_n [\psi''_{1n}(\tau_n) + r_n^2 \psi''_{2n}(-r_n \tau_n)]]^{1/2}} \exp(a_n [\psi_{1n}(\tau_n) + \psi_{2n}(-r_n \tau_n)]) [1 + O(1/a_n)]. \quad (2.116)$$

The proof of this theorem runs parallel to that of earlier Theorems 2.3.1. Hence, we will give first few steps of the proof of this theorem. Proof. Let G_{2n} be the distribution function of T_n/S_n . Then,

$$\begin{aligned} G_{2n}(r) &= P\left(\frac{T_n}{S_n} \leq r\right) \\ &= \sum_{y \in S} P(T_n \leq ry) P(S_n = y) \\ &= \sum_{y \in S} F_{1n}(ry) P(S_n = y). \end{aligned} \quad (2.117)$$

Therefore the p.d.f. of T_n/S_n is given by

$$\begin{aligned} g_{2n}(r) &= \sum_{y \in S} y f_{1n}(ry) P(S_n = y) \\ &= \sum_{y \in S} y P(S_n = y) f_{1n}(ry). \end{aligned} \quad (2.118)$$

Using the inversion formula for the conjugate distribution as in (2.109) we can get an expression for $f_{1n}(ry)$. Thus we have

$$g_{2n}(r) = \sum_{y \in S} y P(S_n = y) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{1n}(\tau + it) \exp(-(\tau + it)ry) dt \right]$$

$$\begin{aligned}
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{1n}(\tau + it) \left[\sum_{y \in S} y P(S_n = y) \exp(-(\tau + it)ry) \right] dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{1n}(\tau + it) \phi'_{2n}(-r(\tau + it)) dt \\
&= \frac{a_n}{2\pi} \int_{-\infty}^{\infty} \exp(a_n[\psi_{1n}(\tau + it) + \psi_{2n}(-r(\tau + it))]) \psi'_{2n}(-r(\tau + it)) dt.
\end{aligned} \tag{2.119}$$

Replacing r by r_n and τ by τ_n we get

$$g_{2n}(r_n) = \frac{a_n}{2\pi} \int_{-\infty}^{\infty} \exp(a_n[\psi_{1n}(\tau_n + it) + \psi_{2n}(-r_n(\tau_n + it))]) \psi'_{2n}(-r_n(\tau_n + it)) dt. \tag{2.120}$$

The rest of the proof is similar to that of Theorem 2.3.1. Hence

$$\begin{aligned}
g_{2n}(r_n) &= \frac{a_n \psi'_{2n}(-r_n \tau_n)}{[2\pi a_n [\psi''_{1n}(\tau_n) + r_n^2 \psi''_{2n}(-r_n \tau_n)]]^{1/2}} \\
&\quad \times \exp(a_n[\psi_{1n}(\tau_n) + \psi_{2n}(-r_n \tau_n)]) [1 + O(1/a_n)]. \tag{2.121}
\end{aligned}$$

Theorems 2.3.13 and 2.3.14 below obtain asymptotic expressions for the density of the ratio of sums of i.i.d. random variables. Theorem 2.3.13 deals with the occurrence of non-lattice T_n and non-lattice S_n where as Theorem 2.3.14 deals with the occurrence of non-lattice T_n and lattice S_n . When T_n and S_n are sums of n i.i.d. random variables not only that the conditions of Theorems 2.3.13 and 2.3.14 become simpler, but, part 1 of condition (A2) ((2.57)) is automatically satisfied. We skip the proofs of these theorems as they can easily be deduced from Theorems 2.3.1 and 2.3.12.

Theorem 2.3.13. Let $\{X_n, n \geq 1\}$ and $\{Y_n > 0, n \geq 1\}$ be two sequences of non-lattice random variables with distribution functions F_1 and F_2 respectively. Let the two sequences be independent. Let ϕ_1 and ϕ_2 denote the characteristic functions of $\{X_n, n \geq 1\}$ and $\{Y_n > 0, n \geq 1\}$ respectively. Let $\psi_i(z) = \log \phi_i(z)$ for $i = 1, 2$. Let $\phi_i(z)$ be non-vanishing and analytic in $\Omega_i = \{z \in \mathbb{C} : |z| < c_i\}$ for $i = 1, 2$. Let J_i denote the interval $(-b_i, b_i)$ where $0 < b_i < c_i$ for $i = 1, 2$. Further, let $\{\tau_n\}$ be a sequence of real numbers such that there exists $\{\tau_n\} \in J_1$ and

$$\psi_1'(\tau_n) - r_n \psi_2'(-r_n \tau_n) = 0 \quad (2.122)$$

for $r_n \tau_n \in J_2$. Let there exist $\alpha_1 > 0$ and $\alpha_2 > 0$ such that $\psi_1''(\tau) > \alpha_1$ for $\tau \in J_1$ and $\psi_2'(\tau) > \alpha_2$ for $\tau \in J_2$.

Assume the following conditions:

(B1) There exist $\beta_i < \infty$ such that

$$|\psi_i(z)| < \beta_i \text{ for all } z \in J_i \text{ for } i = 1, 2. \quad (2.123)$$

(B2) Given $\delta > 0$, there exists $q > 0$, such that

$$\sup_{|t| > \delta} |\psi_2'(-r_n(\tau_n + it))| = O(n^q). \quad (2.124)$$

(B3) There exists $B < \infty$ such that

$$\sup_{\tau \in J_1} \int_{-\infty}^{\infty} \left| \frac{\phi_1(\tau + it)}{\phi_1(\tau)} \right| < B. \quad (2.125)$$

Let q_{1n} denote the probability density function of T_n/S_n where $T_n = X_1 + \dots + X_n$ and $S_n = Y_1 + \dots + Y_n$. Then

$$q_{1n}(\tau_n) = \frac{\sqrt{n} \psi_2'(-r_n \tau_n)}{[2\pi [\psi_1''(\tau_n) + r_n^2 \psi_2''(-r_n \tau_n)]]^{1/2}}$$

$$\times \exp (n[\psi_1(\tau_n) + \psi_2(-r_n\tau_n)]) [1 + O(1/n)]. \quad (2.126)$$

Theorem 2.3.14. *Let $\{X_n, n \geq 1\}$ be a sequence of non-lattice random variables with common distribution function F_1 and let $\{Y_n > 0, n \geq 1\}$ be a sequence of lattice valued random variables with common distribution function F_2 . Let the two sequences be independent. Let ϕ_1 and ϕ_2 denote the characteristic functions of $\{X_n, n \geq 1\}$ and $\{Y_n > 0, n \geq 1\}$ respectively. Let $\phi_i(z)$ be non-vanishing and analytic in $\Omega_i = \{z \in \mathbb{C} : |z| < c_i\}$ for $i = 1, 2$. Let J_i denote the interval $(-b_i, b_i)$ where $0 < b_i < c_i$ for $i = 1, 2$. Further, let $\{\tau_n\}$ be a sequence of real numbers such that there exists $\{\tau_n\} \in J_1, r_n\tau_n \in J_2$ and*

$$\psi_1'(\tau_n) - r_n \psi_2'(-r_n\tau_n) = 0. \quad (2.127)$$

Also, let there exist $\alpha_1 > 0$ such that $\psi_1''(\tau) > \alpha_1$.

Assume Conditions (B1), (B2) and (B3) of Theorem 2.3.13. Let q_{2n} denote the probability density function of T_n/S_n where $T_n = X_1 + \dots + X_n$ and $S_n = Y_1 + \dots + Y_n$. Then,

$$q_{2n}(r_n) = \frac{\sqrt{n} \psi_2'(-r_n\tau_n)}{[2\pi[\psi_1''(\tau_n) + r_n^2 \psi_2''(-r_n\tau_n)]]^{1/2}} \\ \times \exp (n[\psi_1(\tau_n) + \psi_2(-r_n\tau_n)]) [1 + O(1/n)]. \quad (2.128)$$

2.4 Large deviation local limit theorems for ratio statistics of lattice random variables

In this section we obtain similar local limit theorems for ratio statistic $R_n = T_n/S_n$ when

- (i) T_n and S_n both are lattice random variables and
- (ii) when T_n is lattice and S_n is non-lattice.

First, we consider case (i) where both T_n and S_n are lattice random variables. We continue to use the same notations introduced in Section 2.3.

Theorem 2.4.1 *Let $\{T_n, n \geq 1\}$ be a sequence of lattice random variables with distribution functions F_{1n} . Let T_n take values in the set $S_1 = \{a_n + kh_n : a_n \text{ and } h_n \text{ are real numbers and } k \text{ is an integer}\}$. We assume that the two sequences are independent. Let $\{S_n > 0, n \geq 1\}$ be a sequence of positive lattice random variables taking values in the set $S_2 = \{a'_n + kh'_n : a'_n \text{ and } h'_n \text{ are such that } a'_n + kh'_n > 0\}$. Let r_n be an appropriate sequence of real numbers such that there exist $\tau_n \in J_1$ such that*

$$\psi'_{1n}(\tau_n) - r_n \psi'_{2n}(-r_n \tau_n) = 0 \quad (2.129)$$

for $r_n \tau_n \in J_2, n \geq 1$. Let α_1, α_2 be positive real numbers such that

$$\psi''_{1n}(\tau) > \alpha_1 \text{ and } \psi'_{2n}(\tau') > \alpha_2 \quad (2.130)$$

for $\tau \in J_1$ and $\tau' \in J_2$. Assume that these two sequences satisfy the following conditions: (D1) There exist $\beta_1 < \infty$ and $\beta_2 < \infty$ such that

$$|\psi_{jn}(z)| < \beta_j \text{ for } z \text{ with } |z| < A_j \text{ for } j = 1, 2 \text{ and } n \geq 1. \quad (2.131)$$

(D2) Given $\delta > 0$, there exists $0 < \eta < 1$ such that

$$\limsup_n \sup_{\delta < |t| \leq \pi/|h_n|} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \right|^{1/a_n} < \eta. \quad (2.132)$$

(D3) There exist $p, l > 0$ such that

$$\sup_{\tau \in I} \int_{-\pi/|h_n|}^{\pi/|h_n|} \left| \frac{\phi_{1n}(\tau + it)}{\phi_{1n}(\tau)} \right|^{l/a_n} dt = O(a_n^p). \quad (2.133)$$

Let $P_n(r) = P(T_n = rS_n)$. Then

$$\begin{aligned} \frac{\sqrt{a_n}}{|h_n|} P_n(r_n) &= \left[\frac{1}{2\pi \{ \psi_{1n}''(\tau_n) + r_n^2 \psi_{2n}''(-r_n \tau_n) \}} \right]^{1/2} \\ &\times \exp(a_n \{ \psi_{1n}(\tau_n) + \psi_{2n}(-r_n \tau_n) \}) [1 + O(1/a_n)]. \end{aligned} \quad (2.134)$$

Proof. Consider

$$\begin{aligned} P_n(r) &= P(T_n = rS_n) \\ &= \sum_{y \in S_2} P(T_n = ry) P(S_n = y). \end{aligned} \quad (2.135)$$

Using the conjugate density technique and condition (D4), we get that

$$P(T_n = ry) = \frac{|h_n|}{2\pi} \int_{-\pi/|h_n|}^{\pi/|h_n|} \phi_{1n}(\tau + it) \exp(-(\tau + it)ry) dt. \quad (2.136)$$

Substituting this in (2.134), we have,

$$\begin{aligned}
P_n(r) &= \sum_{y \in S_2} P(S_n = y) \left[\frac{|h_n|}{2\pi} \int_{-\pi/|h_n|}^{\pi/|h_n|} \phi_{1n}(\tau + it) \exp(-(\tau + it)ry) dt \right] \\
&= \frac{|h_n|}{2\pi} \int_{-\pi/|h_n|}^{\pi/|h_n|} \phi_{1n}(\tau + it) \sum_{y \in S_2} P(S_n = y) \exp(-(\tau + it)ry) dt \\
&= \frac{|h_n|}{2\pi} \int_{-\pi/|h_n|}^{\pi/|h_n|} \phi_{1n}(\tau + it) \phi_{2n}(-r(\tau + it)) dt \\
&= \frac{|h_n|}{2\pi} \int_{-\pi/|h_n|}^{\pi/|h_n|} \exp(a_n[\psi_{1n}(\tau + it) + \psi_{2n}(-r(\tau + it))]) dt. \quad (2.137)
\end{aligned}$$

Replacing r by r_n and τ by τ_n we can rewrite $P_n(r_n)$ as follows:

$$\begin{aligned}
\frac{\sqrt{a_n}}{|h_n|} P_n(r_n) &= \frac{\sqrt{a_n}}{2\pi} \int_{-\pi/|h_n|}^{\pi/|h_n|} \exp(a_n[\psi_{1n}(\tau_n + it) + \psi_{2n}(-r_n(\tau_n + it))]) dt \\
&= \left[\frac{1}{2\pi[\psi_{1n}''(\tau_n) + r_n^2 \psi_{2n}''(-r_n \tau_n)]} \right]^{1/2} \exp(a_n[\psi_{1n}(\tau_n) + \psi_{2n}(-r_n \tau_n)]) J_n \quad (2.138)
\end{aligned}$$

where

$$\begin{aligned}
J_n &= \left[\frac{a_n[\psi_{1n}''(\tau_n) + r_n^2 \psi_{2n}''(-r_n \tau_n)]}{2\pi} \right]^{1/2} \int_{-\pi/|h_n|}^{\pi/|h_n|} \exp(a_n[\psi_{1n}(\tau_n + it) \\
&\quad - \psi_{1n}(\tau_n) + \psi_{2n}(-r_n(\tau_n + it)) - \psi_{2n}(-r_n \tau_n)]) dt. \quad (2.139)
\end{aligned}$$

We can easily verify that J_n is equal to $1 + O(1/a_n)$.

Thus,

$$\begin{aligned}
\frac{\sqrt{a_n}}{|h_n|} P_n(r_n) &= \left[\frac{1}{2\pi[\psi_{1n}''(\tau_n) + r_n^2 \psi_{2n}''(-r_n \tau_n)]} \right]^{1/2} \\
&\quad \times \exp(a_n[\psi_{1n}(\tau_n) + \psi_{2n}(-r_n \tau_n)]) [1 + O(1/a_n)]. \quad (2.140)
\end{aligned}$$

For case (ii) we have the following.

Theorem 2.4.2. *Let $\{T_n, n \geq 1\}$ be a sequence of lattice random variables with distribution functions F_{1n} and let $\{S_n > 0, n \geq 1\}$ be a sequence of non-lattice random variables with distribution functions F_{2n} . Let T_n take values in the set $S = \{a_n + kh_n : a_n \text{ and } h_n \text{ are such that } a_n + kh_n > 0\}$. Let r_n be a bounded sequence of real numbers satisfying (2.138). Suppose that Conditions (D1), (D2) and (D3) hold. If g_{2n} is the p.d.f. of T_n/S_n , then*

$$\begin{aligned} \frac{\sqrt{a_n}}{|h_n|} g_{2n}(r_n) &= \left[\frac{1}{2\pi[\psi''_{1n}(\tau_n) + r_n^2 \psi''_{2n}(-r_n \tau_n)]} \right]^{1/2} \\ &\times \exp(a_n[\psi_{1n}(\tau_n) + \psi_{2n}(-r_n \tau_n)]) [1 + O(1/a_n)]. \end{aligned} \quad (2.141)$$

Proof of this theorem runs parallel to the proof of the preceding theorems and hence is omitted.

Theorems 2.4.3 and 2.4.4 stated below consider cases (i) and (ii) for i.i.d. random variables.

Theorem 2.4.3. *Let $\{X_n, n \geq 1\}$ be a sequence of independent lattice valued random variables with common distribution function F_1 and $\{Y_n > 0, n \geq 1\}$ be another sequence of independent and identically distributed lattice valued random variables with distribution F_2 . Let the two sequences $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ be independent of each other. Let ϕ_1 and ϕ_2 denote the characteristic functions of X_1 and Y_1 respectively. Let $J_i =$*

$(-b_i, b_i)$, where $0 < b_i < c_i$ for some c_i , $i = 1, 2$. Let $\{\tau_n\}$ be a proper sequence of real numbers such that there exist $\tau_n \in J_1$, $\alpha_1 > 0$ and

$$\psi_1'(\tau_n) - r_n \psi_2'(-r_n \tau_n) = 0 \quad (2.142)$$

and

$$\psi_1''(\tau) > \alpha_1. \quad (2.143)$$

Assume that $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ satisfy the following conditions:

(E1). There exist $\beta_i < \infty$ such that

$$|\psi_i(z)| < \beta_i \text{ for all } z \in J_i \text{ for } i = 1, 2. \quad (2.144)$$

(E2). There exists $B < \infty$ such that

$$\sup_{\tau \in J_1} \int_{-\pi/|h|}^{\pi/|h|} \left| \frac{\phi_1(\tau + it)}{\phi_1(\tau)} \right| dt < B. \quad (2.145)$$

Let $T_n = X_1 + \cdots + X_n$ and $S_n = Y_1 + \cdots + Y_n$. Then

$$\frac{\sqrt{n}}{|h|} P(T_n = r_n S_n) = \left[\frac{1}{2\pi \{\psi_1''(\tau_n) + r_n^2 \psi_2''(-r_n \tau_n)\}} \right]^{1/2} \quad (2.146)$$

$$\exp(n[\psi_1(\tau_n) + \psi_2(-r_n \tau_n)])[1 + O(1/n)].$$

Theorem 2.4.4 Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed lattice valued random variables with distribution function F_1 . Let $\{Y_n > 0, n \geq 1\}$ be another sequence of non-lattice and independent random variables with common distribution function F_2 . Assume that

the two sequences satisfy the conditions of Theorem 2.4.3. The probability density function g_{2n} of T_n/S_n is given by

$$\begin{aligned} \frac{\sqrt{n}}{|h|} g_{2n}(r_n) &= \left[\frac{1}{2\pi\{\psi_1''(\tau_n) + r_n^2\psi_2''(-r_n\tau_n)\}} \right]^{1/2} \\ &\times \exp(n[\psi_1(\tau_n) + \psi_2(-r_n\tau_n)]) [1 + O(1/n)]. \quad (2.147) \end{aligned}$$

2.5 Applications

In this section we present a number of examples to illustrate large deviation local limit theorems for the ratio statistic $R_n = T_n/S_n$ obtained in Sections 2.3 and 2.4. Our examples cover all the four combinations of non- lattice and lattice cases for T_n and S_n . One should note that in all these examples the exact density does not have a closed form, however our theorems provide a simple asymptotic expressions. These examples clearly demonstrate the wide applicability of our theorems. To simplify matters in all the examples we choose T_n and S_n to be the sum of n i.i.d. random variables. The first Example 2.5.1 considers the case of non-lattice for both T_n and S_n . In this example we choose T_n to be the sum of n i.i.d. $N(0,1)$ random variables and S_n to be the sum of n i.i.d. χ^2 random variables with one degree of freedom. We verify easily all the conditions of Theorem 2.3.13. Thus the conclusion (2.125) of Theorem 2.3.13 yields a simple expression for the density of the random variable which is the ratio of $N(o,n)$ and χ^2 with n degrees of freedom. The second Example 2.5.2 considers the case of non-lattice for T_n and lattice for S_n . Here we choose T_n to be the same as in Example 2.5.1 but S_n is chosen to be the sum of n i.i.d. Poisson random variables with mean equal to 1. Proceeding as in Example 2.5.1 we can easily verify that all the conditions (B1), (B2) and (B3) of Theorem 2.3.14 are satisfied. Thus we obtain in (2.153) an asymptotic expression for the density of the ratio statistic $N(0,n)$ over Poisson (n) at a suitable point r_n . In the third Example 2.5.3 we consider the case of lattice random variables T_n and S_n . We chose both T_n and S_n to be the sum of n independent Poisson

random variables each with means λ_1 and λ_2 respectively. As mentioned earlier the probability mass function of T_n/S_n can only be written as an infinite series and does not have a closed form. However, we easily verify all the conditions of Theorem 2.4.3 and thus we obtain a simple closed form asymptotic expression for the probability mass function of T_n/S_n .

Finally we obtain an approximation for the density function of consider the F statistic with n degrees of freedom for both the numerator and denominator. We assume that n is large. In this case we show that our approximation for the density agrees with the exact expression except for the normalizing constant. However, we show that the ratio of two constants converges to one as $n \rightarrow \infty$, directly instead of appealing to Theorem 2.3.13.

Example 2.5.1. Let $\{X_n, n \geq 1\}$ be a sequence of independent normal random variables with mean 0 and variance 1. Let $\{Y_n, n \geq 1\}$ be a sequence of independent random variables with common distribution function χ^2 with one degree of freedom. The c.f. of X_1 is given by $\phi_1(z) = e^{z^2/2}$ and that of Y_1 is given by $\phi_2(z) = (1 - 2z)^{-1/2}$ for $|z| < 1/2$. Let r_n be a bounded sequence of real numbers such that $0 < \underline{r} < r_n < \bar{r}$. Let

$$\tau_n = \frac{-1 + \sqrt{1 + 8r_n^2}}{4r_n}. \quad (2.148)$$

Let $b_1 = (-1 + \sqrt{1 + 8\bar{r}^2})/2\underline{r}$ and $c_1 = 2b_1$. Further, let $(-1 + \sqrt{1 + 8\bar{r}^2})/4 < b_2 < \min\{(-1 + \sqrt{1 + 8\bar{r}^2})/2, 1/2\}$ and $c_2 = 1/2$. We now verify all the conditions of Theorem 2.3.13.

Condition (B1). Let $\beta_1 = c_1^2$ and $\beta_2 = \sqrt{[\log 4]^2 + 4\pi^2}$. Then

$$|\psi_1(z)| = \frac{|z|^2}{2} < \beta_1, \text{ for } |z| < c_1 \quad (2.149)$$

and

$$|\psi_2(z)| = |\log(1 - 2z)| < \beta_2, \text{ for } |z| < 1/2. \quad (2.150)$$

Condition (B2). Let $\delta > 0$. Then

$$\begin{aligned} \sup_{|t|>\delta} |\psi'_2(-r_n(\tau_n + it))| &= \sup_{|t|>\delta} \left| \frac{1}{[1 - 2(-r_n(\tau_n + it))]} \right| \\ &= \sup_{|t|>\delta} \frac{1}{\sqrt{(1 + 2r_n\tau_n)^2 + 4r_n^2 t^2}} \\ &< q \\ &< \infty \end{aligned} \quad (2.151)$$

where $q = \sup_n q_n$ and $q_n = 1/\sqrt{(1 + 2r_n\tau_n)^2 + 4r_n^2 \delta^2}$.

Condition (B3). This condition is trivially satisfied.

Thus we have verified all the conditions of Theorem 2.3.13. An asymptotic expression for the density function of $R_n = T_n/S_n$ where $T_n = X_1 + \dots + X_n$ and $S_n = Y_1 + \dots + Y_n$ is given by

$$q_{1n}(r_n\tau_n) = \sqrt{\frac{n}{2\pi}} \frac{(1 + 2r_n\tau_n)^{-n/2}}{[2 + (1 + 2r_n\tau_n)^2]^{1/2}} \exp\left(\frac{n\tau_n^2}{2}\right) [1 + O(1/n)]. \quad (2.152)$$

Example 2.5.2. Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with common distribution function $N(0, 1)$. Let $\{Y_n, n \geq 1\}$ be a sequence of Poisson random variables with parameter $\lambda = 1$. Let $\phi_1(z) = e^{z^2/2}$ and $\phi_2(z) = e^{(e^z-1)}$. We choose $c_i > 0$ such that ϕ_i is non-vanishing and analytic in $\Omega_i = \{z \in \mathbb{C} : |z| < c_i\}$, for $i = 1, 2$. Let $J_i = (-b_i, b_i)$,

$0 < b_i < c_i$ for $i = 1, 2$. We choose b'_i 's in such a fashion so that for a bounded sequence r_n we can always find $\tau_n \in J_1$ satisfying

$$\tau_n e^{r_n \tau_n} = r_n \quad (2.153)$$

and $r_n \tau_n \in J_2$. In this example we can easily verify Conditions (B1), (B2) and (B3) of Theorem 2.3.14. Thus the density function q_{2n} of T_n/S_n at the point r_n is given by

$$q_{2n}(r_n) = \frac{\sqrt{n} e^{-r_n \tau_n}}{[2\pi[1 + r_n^2 e^{-r_n \tau_n}]]^{1/2}} \exp\left(n\left[\frac{2e^{\tau_n} + \tau_n^2 - 2}{2}\right]\right)[1 + O(1/n)] \quad (2.154)$$

Example 2.5.3. Let $\{X_n, n \geq 1\}$ be a sequence of Poisson random variables with parameter λ_1 . Let $\{Y_n, n \geq 1\}$ be another sequence of Poisson random variables with parameter λ_2 . Let $T_n = X_1 + \cdots + X_n$ and $S_n = Y_1 + \cdots + Y_n$. Then $\phi_i(z) = e^{\lambda_i(e^z - 1)}$ and $\psi_i(z) = \lambda_i(e^z - 1)$ for $i = 1, 2$. Note that $\phi_1(z)$ and $\phi_2(z)$ are analytic in all the complex plane $\mathbb{C}$. Let $J_i = (-b_i, b_i)$ for $0 < b_i < c_i$ for $i = 1, 2$. Let us choose a bounded sequence of real numbers $\{r_n\}$ such that $0 < \underline{r} < r_n < \bar{r}$ and there exist τ'_n 's satisfying

$$\tau_n = \frac{\log r_n + \log \lambda_2 - \log \lambda_1}{(r_n + 1)}. \quad (2.155)$$

Let $b_1 = 2\log \bar{r} + \log \lambda_2 - \log \lambda_1/(\underline{r} + 1)$ and $c_1 = 2b_1$. Let $b_2 = \log \bar{r} + \log \lambda_2 - \log \lambda_1$ and $c_2 = 2b_2$. With $\alpha_1 = \lambda_1^{(\bar{r}+1)/2}/2|\underline{r}|\lambda_2^{(\underline{r}+1)}$ we have $\psi_1''(\tau) = \lambda_1 e^\tau > \alpha_1$. We now proceed to verify all the conditions of Theorem 2.4.3.

Condition (E1). Let $\beta_i = 2\lambda_i(e^c + 1)^2$ for $i = 1, 2$. For this β_i it is easy to check that $|z| < c_i$ implies $|\psi_i(z)| < \beta_i$ for $i = 1, 2$.

Condition (E2) is trivially satisfied.

Thus from Theorem 2.4.3 we can write an asymptotic expression for $q_{3n}(r_n) = P(T_n = rS_n)$ which is given by

$$q_{3n}(r_n) = \frac{1}{[2\pi n[\lambda_1 e^{\tau_n} + \lambda_2 r_n^2 e^{-r_n \tau_n}]]^{1/2}} \times \exp n[\lambda_1(e^{\tau_n} - 1) + \lambda_2(e^{-r_n \tau_n} - 1)][1 + O(1/n)]. \quad (2.156)$$

Let $F_{n,n}$ denote the F statistic with n, n degrees of freedom. In the next example we compare, for large values of n , exact density of $F_{n,n}$ statistic with the asymptotic expression of the density given by (2.60).

Example 2.5.4. Let T_n and S_n be the sums of n i.i.d. χ^2 random variables with one degree of freedom. The c.f. of T_n is given by $\phi_1(z) = (1 - 2z)^{-n/2}$ and that of S_n is given by $\phi_2(z) = (1 - 2z)^{-n/2}$. Both $\phi_1(z)$ and $\phi_2(z)$ are analytic and non-vanishing in $\Omega = \{z \in \mathbb{C} : |z| < 1/2\}$. Let $J_i = (-b_i, b_i)$ for $0 < b_i < 1/2$. We now briefly verify conditions of Theorem 2.3.13.

Condition (A1). Let $\beta_i = \sqrt{[\log 4]^2 + 4\pi^2}$ for $i = 1, 2$. Then for $|z| < 1/2$ it follows that, $|\psi_i(z)| < \beta_i$ for $i = 1, 2$.

Condition (A2). This is verified as in Example 2.5.1.

Condition (A3). Consider

$$\begin{aligned} \left| \frac{\phi_1(\tau + it)}{\phi_1(\tau)} \right| &= \left| \frac{1 - 2\tau}{(1 - 2(\tau + it))} \right|^{1/2} \\ &= \frac{(1 - 2\tau)^{1/2}}{[(1 - 2\tau)^2 + 4t^2]^{1/4}}. \end{aligned} \quad (2.157)$$

Thus

$$\begin{aligned}
\sup_{\tau \in J_1} \int_{-\infty}^{\infty} \left| \frac{\phi_1(\tau + it)}{\phi_1(\tau)} \right| dt &= \sup_{\tau \in J_1} \int_{-\infty}^{\infty} \frac{(1-2\tau)^{1/2}}{[(1-2\tau)^2 + 4t^2]^{1/4}} dt \\
&= \sup_{\tau \in J_1} (1-2\tau)^{1/2} \int_{-\infty}^{\infty} \frac{1}{[(1-2\tau)^2 + 4t^2]^{1/4}} dt \\
&\leq \sqrt{2} \pi.
\end{aligned} \tag{2.158}$$

Thus we have verified all the conditions of Theorem 2.3.13 and we now apply the same theorem (see (2.60)). On simplification of the numerator and denominator in the expression (2.60) we get

$$q_{1n}(r_n) = \frac{n^{1/2} 2^{n-1/2}}{\sqrt{2\pi}} \frac{r_n^{n/2-1}}{(r_n + 1)^n} \tag{2.159}$$

where $\{r_n\}$ is an appropriate sequence of real numbers. The exact density of the ratio T_n/S_n at point r is given by

$$f(r_n|n, n) = \frac{(n-1)!}{(\frac{n}{2}-1)! (\frac{n}{2}-1)!} \frac{r_n^{n/2-1}}{(r_n + 1)^n}. \tag{2.160}$$

On comparing (2.159) and (2.160) we see that both the expressions agree except for a constant. We can use Stirling's formula and show that the ratio of the two constants occurring in (2.159) and (2.160) tends to 1 as $n \rightarrow \infty$.

Chapter 3

Large deviation local limit theorems for random vectors

In this chapter we shall extend the local limit theorems for ratio statistics of random variables in Chapter 2 to random vectors. We begin with a few definitions and notations.

3.1 Definitions and Preliminaries.

We denote the k -dimensional complex plane by $\mathcal{C}^k$. The points in $\mathcal{C}^k$ are denoted by $z = (z_1, z_2, \dots, z_k)$ where $z_i \in \mathcal{C}$, $i = 1, 2, \dots, k$. If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ is a k -dimensional vector with nonnegative integer components, we shall use the following abbreviations:

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_k^{\alpha_k} \quad (3.1)$$

$$dz = dz_1 dz_2 \cdots dz_k \quad (3.2)$$

$$\langle z, \xi \rangle = z_1 \xi_1 + z_2 \xi_2 + \cdots + z_k \xi_k \quad (3.3)$$

for all $z, \xi \in \mathbb{C}^k$.

$$|\alpha| = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_k|. \quad (3.4)$$

When f is a complex valued function defined on $\mathbb{C}^k$ and i, j, l are positive integers we write

$$D_{ij}f(z) = \frac{D^2 f(z)}{dz_i dz_j},$$

$$D_{ijl}f(z) = \frac{D^3 f(z)}{dz_i dz_j dz_l},$$

and

$$D^\alpha f(z) = \frac{D^{|\alpha|} f(z)}{dz_1^{\alpha_1} dz_2^{\alpha_2} \cdots dz_k^{\alpha_k}}.$$

Definition 3.1.1. A polydisc $s(z, r)$ of radius $r = (r_1, r_2, \dots, r_k)$ around a point z is defined as

$$s(z, r) = s(z_1, r_1) \times s(z_2, r_2) \times \cdots \times s(z_k, r_k) \quad (3.5)$$

where $s(z_i, r_i) = \{z'_i \in \mathbb{C} : |z'_i - z_i| < r_i\}$ for $i = 1, 2, \dots, k$.

Definition 3.1.2. A closed polydisc $\bar{s}(z, r)$ of radius $r = (r_1, r_2, \dots, r_k)$ around a point z is defined as

$$\bar{s}(z, r) = \bar{s}(z_1, r_1) \times \bar{s}(z_2, r_2) \times \cdots \times \bar{s}(z_k, r_k) \quad (3.6)$$

where $\bar{s}(z_i, r_i) = \{z'_i \in \mathbb{C} : |z'_i - z_i| \leq r_i\}$ for $i = 1, 2, \dots, k$.

Definition 3.1.3. A complex valued function f is said to be holomorphic at a point $z_0 \in \mathbb{C}^k$ if in some neighbourhood of z_0 , it is the sum of an absolutely convergent power series

$$f(z) = \sum_{|\alpha| \geq 0} a_\alpha (z - z_0)^\alpha. \quad (3.7)$$

Let $(z + re^{i\theta}) = (z_1 + r_1 e^{i\theta_1}, \dots, z_k + r_k e^{i\theta_k})$ for $r = (r_1, r_2, \dots, r_k)$ and $\theta = (\theta_1, \theta_2, \dots, \theta_k)$. The following theorem can be found in Vladimirov (1966) pp. 30-31.

Theorem 3.1.4 (Cauchy). Suppose that a function $f(z)$ is holomorphic and that it is bounded in the closed polydisc $\bar{s}(z_0, r)$ then the coefficient a_α in the expansion of f is given by

$$a_\alpha = \frac{1}{(2\pi)^k} \int_0^{2\pi} \dots \int_0^{2\pi} \frac{f(z_0 + r e^{i\theta})}{r^\alpha} \exp(-i \langle \theta, \alpha \rangle) d\theta. \quad (3.8)$$

Consequently,

$$|a_\alpha| \leq \frac{1}{r^\alpha} \max_{z \in \bar{s}(z_0, r)} |f(z)|. \quad (3.9)$$

3.2 Local limit theorems for random vectors

Let $\{T_n, n \geq 1\}$ be a sequence of non-lattice random vectors in $\mathbb{R}^k$ and let $\{S_n > 0, n \geq 1\}$ be a sequence of non-lattice random variables. Let us assume that S_n is independent of T_n for $n \geq 1$. Let the moment generating function $\phi_{1n}(z) = E(\exp \langle z, T_n \rangle)$ of T_n be holomorphic in Ω_1^k , where $\Omega_1 = \{x + iy : x \in I = (-c_1, c_1) \text{ and } y \in \mathbb{R}\}$ for some $c_1 > 0$. Let $\phi_{2n}(z) = E(\exp(zS_n))$ be analytic in $\Omega_2 = \{z \in \mathbb{C} : |z| < c_2\}$ for some

$c_2 > 0$. Let $\mathcal{C}$ denote the set of all complex numbers. Let $J_1 = (-b_1, b_1)$, $0 < b_1 < c_1$ and $J_2 = (-b_2, b_2)$, $0 < b_2 < c_2$. Let $\{a_n\}$ be a sequence of real numbers such that $a_n \rightarrow \infty$.

Let

$$\psi_{1n}(z) = \frac{1}{a_n} \log \phi_{1n}(z), \text{ for } z \in \Omega_1^k. \quad (3.10)$$

$$\psi_{2n}(z) = \frac{1}{a_n} \log \phi_{2n}(z), \text{ for } z \in \Omega_2. \quad (3.11)$$

Let $\nabla \psi_{1n}(z) = (D_1 \psi_{1n}(z), D_2 \psi_{1n}(z), \dots, D_k \psi_{1n}(z))$ be the vector of the first order partial derivatives and $\nabla^2 \psi_{1n}(z)$ denote the matrix of second order partial derivatives, that is,

$$\nabla^2 \psi_{1n}(z) = (D_{ij} \psi_{1n}(z)). \quad (3.12)$$

The determinant of the matrix $\nabla^2 \psi_{1n}(z)$ is denoted by $|\nabla^2 \psi_{1n}(z)|$.

Let $d_n = \nabla^2 (\psi_{1n}(\tau) + \psi_{2n}(-\langle r, \tau \rangle))$ be positive definite. Further, let the eigen values of d_n be bounded below by $\alpha_1 > 0$, for all $\tau \in J_1^k$ and $n \geq 1$. Let $\{r_n\}$ be a sequence of vectors in R^k such that $|r_{jn}| < \overline{r_j} < \infty$ for $j = 1, \dots, k$ and $n \geq 1$. For this sequence r_n let there exist a sequence $\{\tau_n\}$ in J_1^k such that $\langle r_n, \tau_n \rangle \in J_2$ and

$$\nabla (\psi_{1n}(\tau_n) + \psi_{2n}(-\langle r_n, \tau_n \rangle)) = 0 \quad (3.13)$$

and

$$\psi'_{2n}(-\langle r_n, \tau_n \rangle) > \alpha_2 \quad (3.14)$$

for some positive constant α_2 and for all $n \geq 1$.

We now state the main theorem of this section.

Theorem 3.2.1. *Let $\{T_n, n \geq 1\}$ be a sequence of random vectors in R^k . Let $\{S_n, n \geq 1\}$ be a sequence of random variables which is independent of T_n . Assume that T_n and S_n satisfy the following conditions:*

(F1) *There exist $\beta_1 < \infty$ and $\beta_2 < \infty$ such that*

$$|\psi_{1n}(z)| < \beta_1 \text{ for } z \in \Omega_1^k \quad (3.15)$$

and

$$|\psi_{2n}(z)| < \beta_2 \text{ for } z \in \Omega_2 \text{ for all } n \geq 1. \quad (3.16)$$

(F2) *Given $\delta > 0$ there exist $0 < \eta < 1$, $q > 0$ such that*

$$\limsup_n \sup_{|t| > \delta} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \right|^{1/a_n} < \eta \quad (3.17)$$

and

$$\sup_{|t| > \delta} |\psi'_{2n}(-\langle r_n, \tau_n + it \rangle)| = O(a_n^q). \quad (3.18)$$

(F3) *There exist $p, l > 0$ such that*

$$\sup_{\tau \in J_1^k} \int_{-\infty}^{\infty} \left| \frac{\phi_{1n}(\tau + it)}{\phi_{1n}(\tau)} \right|^{l/a_n} dt = O(a_n^p). \quad (3.19)$$

Then, an asymptotic expansion for the density function $\bar{g}_n$ of T_n/S_n at the point r_n is given by,

$$\bar{g}_n(r_n) = \frac{a_n^{k/2} (\psi'_{2n}(-\langle r_n, \tau_n \rangle))^k}{(2\pi)^{k/2} |d_n|^{1/2}} \quad (3.20)$$

$$\times \exp(a_n[\psi_{1n}(\tau_n) + \psi_{2n}(-\langle r_n, \tau_n \rangle)]) [1 + O(1/a_n)].$$

Remark 3.2.3. The proof of Theorem 3.2.1 is similar to that of Theorem 2.3.1. The proof has two major steps. In the first step the term I_{n1} appearing in the proof of the theorem is shown to go to zero exponentially fast and in the second step the term I_{n2} is shown to be equal to $1 + O(1/a_n)$. These steps are presented below as Lemmas 3.2.7 and 3.2.9 respectively. The proofs of these lemmas depend on lemmas 3.2.4, 3.2.8 which are also proved below.

Lemma 3.2.4. *Let ψ_{1n} and ψ_{2n} be as defined in (3.14) and (3.15). Assume that Condition (F1) of Theorem 3.2.1 holds. Let δ_1 be any real number less than $(c_1 - b_1)/2$. For $|t| < \delta_1$ let*

$$\begin{aligned} R_{1n}(\tau_n + it) &= \psi_{1n}(\tau_n + it) - \psi_{1n}(\tau_n) - i \langle t, \nabla \psi_{1n}(\tau_n) \rangle \\ &\quad + \frac{1}{2} t' \nabla^2 \psi_{1n}(\tau_n) t + i \sum_{|\alpha|=3} a_\alpha^{(n)} t^{|\alpha|}. \end{aligned} \quad (3.21)$$

Then we have the following bound :

$$\sup_n |R_{1n}(\tau_n + it)| \leq \frac{2\beta_1 t^4}{(c_1 - b_1)^4}. \quad (3.22)$$

Proof. Since $\tau_n \in J_1^k$ for all $n \geq 1$ and ψ_{1n} is holomorphic in Ω_1^k we can write for $|t| < (c_1 - b_1)/2$,

$$\begin{aligned} \psi_{1n}(\tau_n + it) &= \psi_{1n}(\tau_n) + i \langle t, \nabla \psi_{1n}(\tau_n) \rangle - \frac{1}{2} t' \nabla^2 \psi_{1n}(\tau_n) t \\ &\quad - i \sum_{|\alpha|=3} a_\alpha^{(n)} t^{|\alpha|} + \sum_{|\alpha| \geq 4} a_\alpha^{(n)} (it)^{|\alpha|}. \end{aligned} \quad (3.23)$$

By Cauchy's theorem and Condition (F1) of Theorem 3.2.1 we get the following bound for $a_\alpha^{(n)}$

$$|a_\alpha^{(n)}| \leq \frac{\beta_1}{(c_1 - b_1)^{|\alpha|}}. \quad (3.24)$$

Thus for $|t| < (c_1 - b_1)/2$ and for all $n \geq 1$,

$$\begin{aligned} |R_{1n}(\tau_n + it)| &= \left| \sum_{|\alpha| \geq 4} a_\alpha^{(n)} (it)^{|\alpha|} \right| \\ &\leq \sum_{|\alpha| \geq 4} \left[\prod_{j=1}^k |t_j|^{\alpha_j} \right] \frac{\beta_1}{(c_1 - b_1)^{|\alpha|}} \\ &\leq \frac{2 |t|^4 \beta_1}{(c_1 - b_1)^4}. \end{aligned} \quad (3.25)$$

Remark 3.2.5. Let $R_{2n}(\tau + it)$ be as defined in (2.62). As shown in Lemma 2.3.7 there exists $0 < \delta_2 < (c_2 - b_2)/2$ such that

$$\sup_{\tau \in J_2} |R_{2n}(\tau + it)| \leq \frac{2\beta_2 t^4}{(c_2 - b_2)^4}$$

for $|t| < \delta_2$ and for all $n \geq 1$.

Remark 3.2.6. Let $\sup_n |r_n| = r$. Let $\delta_3 = \delta_2/r$. Let $R(z)$ be the remainder term in the expansion of $\psi'_{2n}(-\langle r_n, z \rangle)$ as in (2.64). Proceeding as in Remark 2.3.8. we can show that for $|t| < \delta_3$,

$$|R(-\langle r_n, \tau_n + it \rangle)| \leq \frac{2\beta_2 \langle r_n, t \rangle^3}{(c_2 - b_2)^3}. \quad (3.26)$$

In the next lemma we show that the term I_{n1} defined in (3.29) goes to zero exponentially fast.

Lemma 3.2.7. Assume that the Conditions (F2) and (F3) of Theorem 3.2.1 are satisfied. Let

$$d_n = \nabla^2(\psi_{1n}(\tau_n) + \psi_{2n}(-\langle \tau_n, \tau_n \rangle)) \quad (3.27)$$

and

$$f_{1n}(z) = \psi_{1n}(z) + \psi_{2n}(-\langle \tau_n, z \rangle). \quad (3.28)$$

Then there exists $\delta > 0$ such that

$$I_{n1} = \left[\frac{a_n^k |d_n|}{(2\pi)^k} \right]^{1/2} \int_{|t| \geq \delta} \exp([a_n[f_n(\tau_n + it) - f_n(\tau_n)])] \frac{\psi'_{2n}(-\langle \tau_n, \tau_n + it \rangle)}{\psi'_{2n}(-\langle \tau_n, \tau_n \rangle)} dt \quad (3.29)$$

goes exponentially fast to zero.

Proof. Let δ be such that $\delta < \min(\delta_1, \delta_2, \delta_3)$ where δ_1 , δ_2 and δ_3 are as in Lemma 3.2.4, Remark 3.2.5 and Remark 3.2.6 respectively. Consider

$$\begin{aligned} |I_{n1}| &= \left| \left[\frac{a_n^k |d_n|}{(2\pi)^k} \right]^{1/2} \int_{|t| \geq \delta} \exp([a_n[f_n(\tau_n + it) - f_n(\tau_n)])] \frac{\psi'_{2n}(-\langle \tau_n, \tau_n + it \rangle)}{\psi'_{2n}(-\langle \tau_n, \tau_n \rangle)} dt \right| \\ &\leq \left[\frac{a_n^k |d_n|}{(2\pi)^k} \right]^{1/2} \int_{|t| \geq \delta} |\exp([a_n[f_n(\tau_n + it) - f_n(\tau_n)])]| \left| \frac{\psi'_{2n}(-\langle \tau_n, \tau_n + it \rangle)}{\psi'_{2n}(-\langle \tau_n, \tau_n \rangle)} \right| dt. \end{aligned}$$

Substituting for $f_n(\tau_n + it)$ and $f_n(\tau_n)$ (see (3.28)), we get that

$$|I_{n1}| \leq \left[\frac{a_n^k |d_n|}{(2\pi)^k} \right]^{1/2} \int_{|t| \geq \delta} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \frac{\phi_{2n}(-\langle \tau_n, \tau_n + it \rangle)}{\phi_{2n}(-\langle \tau_n, \tau_n \rangle)} \right|$$

$$\begin{aligned}
& \times \left| \frac{\psi'_{2n}(-\langle r_n, \tau_n + it \rangle)}{\psi'_{2n}(-\langle r_n, \tau_n \rangle)} \right| dt \\
& \leq \left[\frac{a_n^k |d_n|}{(2\pi)^k} \right]^{1/2} \int_{|t| \geq \delta} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \right| \left| \frac{\psi'_{2n}(-\langle r_n, \tau_n + it \rangle)}{\psi'_{2n}(-\langle r_n, \tau_n \rangle)} \right| dt \\
& \leq \left[\frac{a_n^k |d_n|}{(2\pi)^k} \right]^{1/2} \sup_{|t| \geq \delta} \left| \frac{\psi'_{2n}(-\langle r_n, \tau_n + it \rangle)}{\psi'_{2n}(-\langle r_n, \tau_n \rangle)} \right| \\
& \quad \times \int_{|t| \geq \delta} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \right|^{l/a_n} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \right|^{(1-l/a_n)} dt \\
& \leq \left[\frac{a_n^k |d_n|}{(2\pi)^k} \right]^{1/2} \sup_{|t| \geq \delta} \left| \frac{\psi'_{2n}(-\langle r_n, \tau_n + it \rangle)}{\psi'_{2n}(-\langle r_n, \tau_n \rangle)} \right| \sup_{|t| \geq \delta} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \right|^{(1-l/a_n)} \\
& \quad \times \int_{|t| \geq \delta} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \right|^{l/a_n} dt.
\end{aligned}$$

From (3.17), (3.18) and (3.19) and also noting the fact that

$\psi'_{2n}(-\langle r_n, \tau_n \rangle) > \alpha_2$ we get that

$$\begin{aligned}
|I_{n1}| & \leq \left[\frac{a_n^k |d_n|}{(2\pi)^k} \right]^{1/2} \frac{1}{\alpha_2} O(a_n^q) \eta^{(1-l/a_n)} \int_{R^k} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \right|^{l/a_n} dt \\
& \leq O(a_n^{p+q+k/2}) \exp(-\eta_1(a_n - l))
\end{aligned} \tag{3.30}$$

which goes to zero exponentially fast, since $\eta_1 = -\log \eta > 0$.

The following lemma will be used to prove Lemma 3.2.9.

Lemma 3.2.8. *Let ψ_{1n} and ψ_{2n} be as defined in (3.14) and (3.15). Let d_n be as in (3.27). Assume that Condition (F1) of Theorem 3.2.1 holds.*

Let

$$L_n(s) = \left[\exp(z_n) \frac{\psi'_{2n}(-\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle)}{\psi'_{2n}(-\langle r_n, \tau_n \rangle)} - 1 - z_n \right] \quad (3.31)$$

where

$$z_n(s) = \left[\frac{-i}{\sqrt{a_n}} \sum_{|\alpha|=3} a_\alpha^{(n)} s^\alpha + \frac{i(\langle r_n, s \rangle)^3}{6\sqrt{a_n}} \psi'''_{2n}(-\langle r_n, s \rangle) \right. \\ \left. + a_n R_{1n}(\tau_n + i \frac{s}{\sqrt{a_n}}) + a_n R_{2n}(-\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle) \right]. \quad (3.32)$$

Then, there exists $\delta > 0$ such that

$$\left[\frac{|d_n|}{(2\pi)^k} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-s' d_n s}{2}\right) L_n(s) ds = O(1/a_n). \quad (3.33)$$

Proof. Consider

$$Q_n^* = \left[\frac{|d_n|}{(2\pi)^k} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-s' d_n s}{2}\right) \\ \left[\exp(z_n) \frac{\psi'_{2n}(-\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle)}{\psi'_{2n}(-\langle r_n, \tau_n \rangle)} - 1 - z_n \right] ds \quad (3.34)$$

for $\delta > 0$. The r.h.s. of the above equation can be written as the sum of four integrals as follows:

$$Q_n^* = \left[\frac{|d_n|}{(2\pi)^k} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-s' d_n s}{2}\right) [(\exp(z_n) - 1 - z_n) \\ \times \frac{\psi'_{2n}(-\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle)}{\psi'_{2n}(-\langle r_n, \tau_n \rangle)} + \left(\frac{\psi'_{2n}(-\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle)}{\psi'_{2n}(-\langle r_n, \tau_n \rangle)} - 1 \right) \\ - (z_n) + \left(z_n \frac{\psi'_{2n}(-\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle)}{\psi'_{2n}(-\langle r_n, \tau_n \rangle)} \right)] ds \\ = I'_{n1} + I'_{n2} + I'_{n3} + I'_{n4} \text{ (say)}$$

where

$$\begin{aligned}
I'_{n1} &= \left[\frac{|d_n|}{(2\pi)^k} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-s' d_n s}{2}\right) (\exp(z_n) - 1 - z_n) \\
&\quad \frac{\psi'_{2n}(-\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle)}{\psi'_{2n}(-\langle r_n, \tau_n \rangle)} ds. \\
I'_{n2} &= \left[\frac{|d_n|}{(2\pi)^k} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-s' d_n s}{2}\right) \left(\frac{\psi'_{2n}(-\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle)}{\psi'_{2n}(-\langle r_n, \tau_n \rangle)} - 1 \right) ds. \\
I'_{n3} &= \left[\frac{|d_n|}{(2\pi)^k} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-s' d_n s}{2}\right) (-z_n) ds. \\
I'_{n4} &= \left[\frac{|d_n|}{(2\pi)^k} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-s' d_n s}{2}\right) z_n \frac{\psi'_{2n}(-\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle)}{\psi'_{2n}(-\langle r_n, \tau_n \rangle)} ds.
\end{aligned}$$

We complete the proof of this lemma by showing $I'_{ni} = O(1/a_n)$, $i = 1, 2, 3, 4$. To show that $I'_{n1} = O(1/a_n)$, we get an upperbound for $(\exp(z_n) - 1 - z_n)$.

For $|s| < \sqrt{a_n} \delta$, consider

$$\begin{aligned}
|z_n| &= \left| \frac{-i}{\sqrt{a_n}} \sum_{|\alpha|=3} a_\alpha^{(n)} s^\alpha + \frac{i(\langle r_n, s \rangle)^3}{6\sqrt{a_n}} \psi'''_{2n}(-\langle r_n, s \rangle) \right. \\
&\quad \left. + a_n R_{1n}(\tau_n + i \frac{s}{\sqrt{a_n}}) + a_n R_{2n}(-\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle) \right|.
\end{aligned} \tag{3.35}$$

The bound on $|R_{2n}(-\langle r_n, \tau_n + it \rangle)|$ given in Remark 3.2.5 yields

$$|z_n| \leq \left[\frac{1}{\sqrt{a_n}} \frac{\beta_1 |s|^3}{(c_1 - b_1)^3} + \frac{|s|^3}{\sqrt{a_n}} \frac{\beta_2 (\sum_{j=1}^k \bar{\tau}_j)^3}{(c_2 - b_2)^3} \right]$$

$$\begin{aligned}
& + \frac{2\beta_1 |s|^4}{a_n(c_1 - b_1)^4} + \frac{2|s|^4 \beta_2 (\sum_{j=1}^k \bar{r}_j)^4}{a_n(c_2 - b_2)^4} \Big]^2 \exp(M(\delta)|s|^2) \\
& \leq \frac{\beta_1 \delta |s|^2}{(c_1 - b_1)^3} + \frac{\beta_2 \delta |s|^2 (\sum_{j=1}^k \bar{r}_j)^3}{(c_2 - b_2)^3} \\
& \quad + \frac{2\beta_1 \delta^2 |s|^2}{(c_1 - b_1)^4} + \frac{2\beta_2 \delta^2 |s|^2 (\sum_{j=1}^k \bar{r}_j)^4}{(c_2 - b_2)^4} \\
& \leq \left[\frac{\beta_1 \delta}{(c_1 - b_1)^3} + \frac{\beta_2 \delta (\sum_{j=1}^k \bar{r}_j)^3}{(c_2 - b_2)^3} \right. \\
& \quad \left. + \frac{2\beta_1 \delta^2}{(c_1 - b_1)^4} + \frac{2\beta_2 \delta^2 (\sum_{j=1}^k \bar{r}_j)^4}{(c_2 - b_2)^4} \right] |s|^2
\end{aligned} \tag{3.36}$$

wherein we have used the fact that $|r_{jn}| < \bar{r}_j$. Thus we have

$$|z_n(s)| < M(\delta)|s|^2 \tag{3.38}$$

where

$$\begin{aligned}
M(\delta) = & \left[\frac{\beta_1 \delta}{(c_1 - b_1)^3} + \frac{\beta_2 \delta (\sum_{j=1}^k \bar{r}_j)^3}{(c_2 - b_2)^3} \right. \\
& \left. + \frac{2\beta_1 \delta^2}{(c_1 - b_1)^4} + \frac{2\beta_2 \delta^2 (\sum_{j=1}^k \bar{r}_j)^4}{(c_2 - b_2)^4} \right].
\end{aligned} \tag{3.39}$$

Let us choose our δ such that $M(\delta) < \alpha_1/2k$. Using the simple inequality,

$$\begin{aligned}
|\exp(z_n) - 1 - z_n| & \leq |z_n|^2 \exp(|z_n|), \text{ we obtain,} \\
|\exp(z_n) - 1 - z_n| & \leq \left[\frac{1}{\sqrt{a_n}} \frac{\beta_1 |s|^3}{(c_1 - b_1)^3} + \frac{|s|^3}{\sqrt{a_n}} \frac{\beta_2 (\sum_{j=1}^k \bar{r}_j)^3}{(c_2 - b_2)^3} \right. \\
& \quad \left. + \frac{2\beta_1 |s|^4}{a_n(c_1 - b_1)^4} + \frac{2|s|^4 \beta_2 (\sum_{j=1}^k \bar{r}_j)^4}{a_n(c_2 - b_2)^4} \right]^2 \exp(M(\delta)|s|^2).
\end{aligned} \tag{3.40}$$

Consider

$$|I'_{n1}| = \left| \left[\frac{|d_n|}{(2\pi)^k} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-s' d_n s}{2}\right) (\exp(z_n) - 1 - z_n) \right. \\ \left. \times \frac{\psi'_{2n}(-\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle)}{\psi'_{2n}(-\langle r_n, \tau_n \rangle)} ds \right|.$$

Substituting the bound in (4.46) in the above equation, we get

$$|I'_{n1}| \leq \left[\frac{|d_n|}{(2\pi)^k} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-s' d_n s}{2}\right) \exp(M(\delta)|s|^2) \\ \times \left[\frac{1}{\sqrt{a_n}} \frac{\beta_1 |s|^3}{(c_1 - b_1)^3} + \frac{|s|^3}{\sqrt{a_n}} \frac{\beta_2 (\sum_{j=1}^k \bar{r}_j)^3}{(c_2 - b_2)^3} + \frac{2\beta_1 |s|^4}{a_n (c_1 - b_1)^4} \right. \\ \left. + \frac{2|s|^4 \beta_2 (\sum_{j=1}^k \bar{r}_j)^4}{a_n (c_2 - b_2)^4} \right]^2 \left| \frac{\psi'_{2n}(-\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle)}{\psi'_{2n}(-\langle r_n, \tau_n \rangle)} \right| ds \\ \leq \frac{1}{a_n} \left[\frac{|d_n|}{(2\pi)^k} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-\alpha_1 s' s}{2}\right) \exp(M(\delta)|s|^2) \\ \times \left[\frac{\beta_1 |s|^3}{(c_1 - b_1)^3} + \frac{|s|^3 \beta_2 (\sum_{j=1}^k \bar{r}_j)^3}{(c_2 - b_2)^3} + \frac{2\beta_1 |s|^4}{\sqrt{a_n} (c_1 - b_1)^4} \right. \\ \left. + \frac{2|s|^4 \beta_2 (\sum_{j=1}^k \bar{r}_j)^4}{\sqrt{a_n} (c_2 - b_2)^4} \right]^2 \left| \frac{\psi'_{2n}(-\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle)}{\psi'_{2n}(-\langle r_n, \tau_n \rangle)} \right| ds. \quad (3.41)$$

Since $k(s's) \geq |s|^2$, we have

$$|I'_{n1}| \leq \frac{1}{a_n} \left[\frac{|d_n|}{(2\pi)^k} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(-\alpha_1 \frac{s' s}{2} + M(\delta)k(s's)\right) \\ \times \left[\frac{\beta_1 |s|^3}{(c_1 - b_1)^3} + \frac{|s|^3 \beta_2 (\sum_{j=1}^k \bar{r}_j)^3}{(c_2 - b_2)^3} + \frac{2\beta_1 |s|^4}{\sqrt{a_n} (c_1 - b_1)^4} \right.$$

$$\begin{aligned}
& + \frac{2|s|^4 \beta_2 (\sum_{j=1}^k \bar{r}_j)^4}{\sqrt{a_n} (c_2 - b_2)^4} \Big]^2 \left| \frac{\psi'_{2n}(-\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle)}{\psi'_{2n}(-\langle r_n, \tau_n \rangle)} \right| ds \quad (3.42) \\
& \leq \frac{1}{a_n} \left[\frac{|d_n|}{(2\pi)^k} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp \left(-\frac{s' s}{2} (\alpha_1 - 2kM(\delta)) \right) \\
& \quad \times \left[\frac{\beta_1 |s|^3}{(c_1 - b_1)^3} + \frac{|s|^3 \beta_2 (\sum_{j=1}^k \bar{r}_j)^3}{(c_2 - b_2)^3} + \frac{2\beta_1 |s|^4}{\sqrt{a_n} (c_1 - b_1)^4} \right. \\
& \quad \left. + \frac{2|s|^4 \beta_2 (\sum_{j=1}^k \bar{r}_j)^4}{\sqrt{a_n} (c_2 - b_2)^4} \right]^2 \left| \frac{\psi'_{2n}(-\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle)}{\psi'_{2n}(-\langle r_n, \tau_n \rangle)} \right| ds. \quad (3.43)
\end{aligned}$$

Proceeding as in the univariate case we can show that

$$|\psi'_{2n}(-\langle r_n, \tau_n + it \rangle) / \psi'_{2n}(-\langle r_n, \tau_n \rangle)|$$

is bounded in n . Since $M(\delta) < \alpha_1/2k$, the inequality (3.45) shows that $I'_{n1} = O(1/a_n)$. Next consider,

$$\begin{aligned}
|I'_{n2}| &= \left[\frac{|d_n|}{(2\pi)^k} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp \left(\frac{-s' d_n s}{2} \right) \left(\frac{\psi'_{2n}(-\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle)}{\psi'_{2n}(-\langle r_n, \tau_n \rangle)} - 1 \right) ds \\
&= \left[\frac{|d_n|}{(2\pi)^k} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp \left(\frac{-s' d_n s}{2} \right) \left[-\frac{i \langle r_n, s \rangle}{a_n} \frac{\psi''_{2n}(-\langle r_n, \tau_n \rangle)}{\psi'_{2n}(-\langle r_n, \tau_n \rangle)} \right. \\
& \quad \left. - \frac{\langle r_n, s \rangle^2}{a_n} \frac{\psi'''_{2n}(-\langle r_n, \tau_n \rangle)}{\psi'_{2n}(-\langle r_n, \tau_n \rangle)} + \frac{R(\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle)}{\psi'_{2n}(-\langle r_n, \tau_n \rangle)} \right] ds. \quad (3.44)
\end{aligned}$$

It is easy to see that the first integral on the right hand side of I'_{n2} is zero, second integral is $O(1/a_n)$. We can use the bound on $|R(\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle)|$ (see Remark 3.2.6) to show that the third integral is $O(1/a_n)$.

Therefore

$$I'_{n2} = O(1/a_n). \quad (3.45)$$

Now, consider

$$\begin{aligned}
|I'_{n3}| &= \left[\frac{|d_n|}{(2\pi)^k} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-s' d_n s}{2}\right) (-z_n) ds \\
&= (-1) \left[\frac{|d_n|}{(2\pi)^k} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-s' d_n s}{2}\right) \\
&\quad \left[\frac{-i}{\sqrt{a_n}} \sum_{|\alpha|=3} a_\alpha^{(n)} s^\alpha + \frac{i \langle r_n, s \rangle^3}{6 \sqrt{a_n}} \psi_{2n}'''(-\langle r_n, \tau_n \rangle) \right. \\
&\quad \left. + a_n R_{1n}(\tau_n + i \frac{s}{\sqrt{a_n}}) + a_n R_{2n}(-\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle) \right] ds.
\end{aligned} \tag{3.46}$$

The first two integrals on the right hand side of I'_{n3} are zero. We use the bounds on $R_{1n}(\tau + it)$ and $R_{2n}(\tau + it)$, obtained in Lemma 3.2.4 and Remark 3.2.5, to show that the third and fourth integrals are $O(1/a_n)$.

Thus

$$I'_{n3} = O(1/a_n). \tag{3.47}$$

Lastly, consider

$$\begin{aligned}
I'_{n4} &= \left[\frac{|d_n|}{(2\pi)^k} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-s' d_n s}{2}\right) z_n \frac{\psi_{2n}'(-\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle)}{\psi_{2n}'(-\langle r_n, \tau_n \rangle)} ds \\
&= \left[\frac{|d_n|}{(2\pi)^k} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp\left(\frac{-s' d_n s}{2}\right) \left[\frac{-i}{\sqrt{a_n}} \sum_{|\alpha|=3} a_\alpha^{(n)} s^\alpha \right. \\
&\quad \left. + \frac{i \langle r_n, s \rangle^3}{6 \sqrt{a_n}} \psi_{2n}'''(-\langle r_n, \tau_n \rangle) + a_n R_{1n}(\tau_n + i \frac{s}{\sqrt{a_n}}) \right. \\
&\quad \left. + a_n R_{2n}(-\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle) \right]
\end{aligned}$$

$$\left[1 - \frac{i \langle r_n, s \rangle}{a_n} \frac{\psi_{2n}''(-\langle r_n, \tau_n \rangle)}{\psi_{2n}'(-\langle r_n, \tau_n \rangle)} - \frac{\langle r_n, s \rangle^2}{a_n} \frac{\psi_{2n}'''(-\langle r_n, \tau_n \rangle)}{\psi_{2n}'(-\langle r_n, \tau_n \rangle)} + \frac{R(\langle r_n, \tau_n + i \frac{s}{a_n} \rangle)}{\psi_{2n}'(-\langle r_n, \tau_n \rangle)} \right] ds.$$

Using the argument similar to the univariate case the term I_{n4}' can be easily verified to be equal to $O(1/a_n)$. This completes the proof of Lemma 3.2.6.

The next lemma shows that the term I_{n2} appearing in the proof of the main theorem is $1 + O(1/a_n)$.

Lemma 3.2.9 *Let $f_n(t)$ be as defined in (3.31). Let $\delta > 0$ as in Lemma 3.2.8. Then*

$$\begin{aligned} I_{n2} &= \left[\frac{a_n^k |d_n|}{(2\pi)^k} \right]^{1/2} \int_{|t| < \delta} \exp([a_n(f_n(\tau_n + it) - f_n(\tau_n))]) \frac{\psi_{2n}'(-\langle r_n, \tau_n + it \rangle)}{\psi_{2n}'(-\langle r_n, \tau_n \rangle)} dt \\ &= 1 + O(1/a_n). \end{aligned} \quad (3.48)$$

Proof. Consider

$$\begin{aligned} I_{n2} &= \left[\frac{a_n^k |d_n|}{(2\pi)^k} \right]^{1/2} \int_{|t| < \delta} \exp(a_n[\psi_{1n}(\tau_n + it) - \psi_{1n}(\tau_n) \\ &\quad + \psi_{2n}(-\langle r_n, \tau_n + it \rangle) - \psi_{2n}(-\langle r_n, \tau_n \rangle)]) \frac{\psi_{2n}'(-\langle r_n, \tau_n + it \rangle)}{\psi_{2n}'(-\langle r_n, \tau_n \rangle)} dt. \end{aligned} \quad (3.49)$$

By making a change of variable $t = s/\sqrt{a_n}$, we have,

$$\begin{aligned} I_{n2} &= \left[\frac{|d_n|}{(2\pi)^k} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp \left(a_n \left[\psi_{1n} \left(\tau_n + i \frac{s}{\sqrt{a_n}} \right) - \psi_{1n}(\tau_n) \right. \right. \\ &\quad \left. \left. + \psi_{2n}(-\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle) - \psi_{2n}(-\langle r_n, \tau_n \rangle) \right] \right) \frac{\psi_{2n}'(-\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle)}{\psi_{2n}'(-\langle r_n, \tau_n \rangle)} ds. \end{aligned} \quad (3.50)$$

For $|s| < \sqrt{a_n} \delta$, we use Taylor's series for ψ_{1n} and ψ_{2n} to get

$$\begin{aligned}
I_{n2} = & \left[\frac{|d_n|}{(2\pi)^k} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp \left(-a_n \left[\frac{1}{2a_n} s' \nabla^2 \psi_{1n}(\tau_n) s \right. \right. \\
& + \frac{\langle r_n, s \rangle^2}{2a_n} \psi_{2n}''(-\langle r_n, \tau_n \rangle) - \frac{i}{a_n \sqrt{a_n}} \sum_{|\alpha|=3} a_\alpha^{(n)} s^\alpha \\
& - \frac{i \langle r_n, s \rangle^3}{6a_n \sqrt{a_n}} \psi_{2n}'''(-\langle r_n, \tau_n \rangle) - R_{1n}(\tau_n + i \frac{s}{\sqrt{a_n}}) \\
& \left. \left. - R_{2n}(-\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle) \right] \right) \frac{\psi_{2n}'(-\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle)}{\psi_{2n}'(-\langle r_n, \tau_n \rangle)} ds.
\end{aligned} \tag{3.51}$$

We note that $s' \nabla^2 \psi_{1n}(\tau_n) s + \langle r_n, s \rangle^2 \psi_{2n}''(-\langle r_n, \tau_n \rangle) = s' d_n s$. Thus we have,

$$\begin{aligned}
|I_{n2}| = & \left[\frac{|d_n|}{(2\pi)^k} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp \left(\frac{-s' d_n s}{2} \right) \exp \left(-\frac{i}{\sqrt{a_n}} \sum_{|\alpha|=3} a_\alpha^{(n)} s^\alpha \right. \\
& - \frac{i \langle r_n, s \rangle^3}{6 \sqrt{a_n}} \psi_{2n}'''(-\langle r_n, \tau_n \rangle) + a_n R_{1n}(\tau_n + i \frac{s}{\sqrt{a_n}}) \\
& \left. + a_n R_{2n}(-\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle) \right) \frac{\psi_{2n}'(-\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle)}{\psi_{2n}'(-\langle r_n, \tau_n \rangle)} ds \\
= & \left[\frac{|d_n|}{(2\pi)^k} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp \left(\frac{-s' d_n s}{2} \right) \left[1 - \frac{i}{\sqrt{a_n}} \sum_{|\alpha|=3} a_\alpha^{(n)} s^\alpha \right. \\
& - \frac{i \langle r_n, s \rangle^3}{6 \sqrt{a_n}} \psi_{2n}'''(-\langle r_n, \tau_n \rangle) + a_n R_{1n}(\tau_n + i \frac{s}{\sqrt{a_n}}) \\
& \left. + a_n R_{2n}(-\langle r_n, \tau_n + i \frac{s}{\sqrt{a_n}} \rangle) + L_n(s) \right] ds
\end{aligned} \tag{3.52}$$

where d_n , $L_n(s)$ and $z_n(s)$ are as defined in (3.30), (3.35) and (3.36) respectively. The right hand side of (3.56) involves six integrals. The 1st integral is $1 + o(1/a_n)$. Trivially, 2nd and 3rd integrals are equal to zero. In order to show that the 4th integral is $1 + O(1/a_n)$, we use the bound on $R_{1n}(\tau_n + i s/\sqrt{a_n})$ given in Lemma 3.2.4. Thus

$$\begin{aligned}
& \left| \left[\frac{|d_n|}{(2\pi)^k} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} a_n R_{1n}(\tau_n + i \frac{s}{\sqrt{a_n}}) \exp(\frac{-s' d_n s}{2}) ds \right| \\
& \leq a_n \left[\frac{|d_n|}{(2\pi)^k} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} \exp(\frac{-s' d_n s}{2}) \left| R_{1n}(\tau_n + i \frac{s}{\sqrt{a_n}}) \right| ds \\
& \leq \frac{1}{a_n} \left[\frac{|d_n|}{(2\pi)^k} \right]^{1/2} \frac{2\beta_1}{(c_1 - b_1)^4} \int_{|s| < \sqrt{a_n} \delta} |s|^4 \exp(\frac{-s' d_n s}{2}) ds \\
& = O(1/a_n). \tag{3.53}
\end{aligned}$$

Similarly, the 5th integral can be shown to be equal to $O(1/a_n)$. The 6th integral, $\left[|d_n|/(2\pi)^k \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} L_n(s) \exp(-s' d_n s/2) ds$ is Q_n^* defined in (3.37). Therefore, it follows from Lemma 3.2.8 that

$$\left[\frac{|d_n|}{(2\pi)^k} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta} L_n(s) \exp(\frac{-s' d_n s}{2}) ds = O(1/a_n). \tag{3.54}$$

Thus

$$I_{n2} = 1 + O(1/a_n).$$

We now proceed with the proof of main Theorem.

Proof. Let G_n be the distribution function of $R_n = T_n/S_n$. Then, for $r = (r_1, r_2, \dots, r_k)$

$$G_n(r) = Pr \left[\frac{T_{1n}}{S_n} \leq r_1, \dots, \frac{T_{kn}}{S_n} \leq r_k \right]$$

$$\begin{aligned}
&= \int_0^\infty Pr [T_{1n} \leq r_1 y, \dots, T_{kn} \leq r_k y] dF_{2n}(y) \\
&= \int_0^\infty F_{1n}(ry) dF_{2n}(y). \tag{3.55}
\end{aligned}$$

The density function g_n of G_n is given by

$$g_n(r) = \int_0^\infty y^k f_{1n}(ry) dF_{2n}(y) \tag{3.56}$$

where f_{1n} is the p.d.f. of T_n .

Using the conjugate distribution technique, we get,

$$f_{1n}(ry) = \left[\frac{1}{2\pi} \right]^k \int_{R^k} \phi_{1n}(\tau + it) \exp(-\langle ry, \tau + it \rangle) dt. \tag{3.57}$$

Therefore,

$$\begin{aligned}
g_n(r) &= \int_0^\infty y^k \left[\frac{1}{2\pi} \right]^k \int_{R^k} \phi_{1n}(\tau + it) \exp(-\langle ry, \tau + it \rangle) dt dF_{2n}(y) \\
&= \left[\frac{1}{2\pi} \right]^k \int_{R^k} \phi_{1n}(\tau + it) \left[\int_0^\infty y^k \exp(-\langle ry, \tau + it \rangle) dF_{2n}(y) \right] dt \\
&= \left[\frac{1}{2\pi} \right]^k \int_{R^k} \phi_{1n}(\tau + it) \left[\int_0^\infty y^k \exp(-\langle ry, \tau + it \rangle) dF_{2n}(y) \right] dt \\
&= \left[\frac{1}{2\pi} \right]^k \int_{R^k} \phi_{1n}(\tau + it) \phi'_{2n}(-\langle r, \tau + it \rangle) dt \\
&= \left[\frac{a_n}{2\pi} \right]^k \int_{R^k} \exp(a_n [\psi_{1n}(\tau + it) + \psi_{2n}(-\langle r, \tau + it \rangle)]) \psi'_{2n}(-\langle r, \tau + it \rangle) dt.
\end{aligned}$$

We now replace r by r_n and τ by τ_n to get

$$g_n(r_n) = \left[\frac{a_n}{2\pi} \right]^k \int_{R^k} \exp(a_n [\psi_{1n}(\tau_n + it) + \psi_{2n}(-\langle r_n, \tau_n + it \rangle)])$$

$$\begin{aligned}
& \times \psi'_{2n}(-\langle r_n, \tau_n + it \rangle) dt \\
& = \left[\frac{a_n}{2\pi} \right]^{k/2} \frac{\psi'_{2n}(-\langle r_n, \tau_n \rangle)}{|d_n|^{1/2}} \exp(a_n[\psi_{1n}(\tau_n) + \psi_{2n}(-\langle r_n, \tau_n \rangle)]) I_n
\end{aligned} \tag{3.58}$$

where

$$\begin{aligned}
I_n &= \left[\frac{a_n^k |d_n|}{(2\pi)^k} \right]^{1/2} \int_{R^k} \exp(a_n[\psi_{1n}(\tau_n + it) - \psi_{1n}(\tau_n) \\
& \quad + \psi_{2n}(-\langle r_n, \tau_n + it \rangle) - \psi_{2n}(-\langle r_n, \tau_n \rangle)]) \frac{\psi'_{2n}(-\langle r_n, \tau_n + it \rangle)}{\psi'_{2n}(-\langle r_n, \tau_n \rangle)} dt.
\end{aligned} \tag{3.59}$$

Writing $\psi_{1n}(t)$ and $\psi_{2n}(t)$ in terms of $f_n(t)$ (see (3.31)) we get,

$$\begin{aligned}
I_n &= \left[\frac{a_n^k |d_n|}{(2\pi)^k} \right]^{1/2} \int_{R^k} \exp(a_n[f_n(\tau_n + it) - f_n(\tau_n)]) \frac{\psi'_{2n}(-\langle r_n, \tau_n + it \rangle)}{\psi'_{2n}(-\langle r_n, \tau_n \rangle)} dt \\
&= \left[\frac{a_n^k |d_n|}{(2\pi)^k} \right]^{1/2} \left[\int_{|t| \geq \delta} \exp(a_n[f_n(\tau_n + it) - f_n(\tau_n)]) \frac{\psi'_{2n}(-\langle r_n, \tau_n + it \rangle)}{\psi'_{2n}(-\langle r_n, \tau_n \rangle)} dt \right. \\
& \quad \left. + \int_{|t| < \delta} \exp(a_n[f_n(\tau_n + it) - f_n(\tau_n)]) \frac{\psi'_{2n}(-\langle r_n, \tau_n + it \rangle)}{\psi'_{2n}(-\langle r_n, \tau_n \rangle)} dt \right]
\end{aligned} \tag{3.60}$$

where δ is chosen such that $\delta = \min(\delta_1, \delta_2, \delta_3)$ and $M(\delta) < \alpha_1/2k$ for $M(\delta)$ in (3.45). Thus

$$I_n = I_{n1} + I_{n2} \tag{3.61}$$

where

$$I_{n1} = \left[\frac{a_n^k |d_n|}{2\pi^k} \right]^{1/2} \int_{|t| \geq \delta} \exp(a_n[f_n(\tau_n + it) - f_n(\tau_n)]) \frac{\psi'_{2n}(-\langle r_n, \tau_n + it \rangle)}{\psi'_{2n}(-\langle r_n, \tau_n \rangle)} dt \tag{3.62}$$

and

$$I_{n2} = \left[\frac{a_n^k |d_n|}{2\pi^k} \right]^{1/2} \int_{|t| < \delta} \exp(a_n[f_n(\tau_n + it) - f_n(\tau_n)]) \frac{\psi'_{2n}(-\langle \tau_n, \tau_n + it \rangle)}{\psi_{2n}(-\langle \tau_n, \tau_n \rangle)} dt. \quad (3.63)$$

Lemma 3.2.7 shows that the term I_{n1} goes exponentially fast to zero and Lemma 3.2.9 shows that $I_{n2} = 1 + O(1/a_n)$. Thus

$$I_n = 1 + O(1/a_n). \quad (3.64)$$

This completes the proof of the theorem.

Theorem 3.2.10 stated below is analogous to Theorem 3.2.1 except that in Theorem 3.2.10 we take the components of T_n and S_n to be sums of n i.i.d. random variables, particular case of Theorem 3.2.1.

Theorem 3.2.10 *Let $\{X_n, n \geq 1\}$ be a sequence of non-lattice i.i.d. random vectors in R^k with distribution function F_1 and m.g.f. ϕ_1 . Let $\{Y_n, n \geq 1\}$ be another sequence of non-lattice, positive random variables with distribution function F_2 and m.g.f. ϕ_2 . We assume that Y_n 's are independent of the components of X_n 's. Let ϕ_1 and ϕ_2 be analytic and non-vanishing in Ω_1^k and Ω_2 respectively, where $\Omega_i = \{z \in \mathbb{C} : |z| < c_i\}$ for $c_i > 0$, $i = 1, 2$. Let*

$$\psi_i(z) = \frac{1}{n} \log \phi_i(z),$$

for $i = 1, 2$. Let $J_1 = (-b_1, b_1)$, $0 < b_1 < c_1$ and $J_2 = (-b_2, b_2)$, $0 < b_2 < c_2$. Let $\{a_n\}$ be a sequence of real numbers such that $a_n \rightarrow \infty$.

Let $\nabla \psi_1(z) = (D_1 \psi_1(z), D_2 \psi_1(z), \dots, D_k \psi_1(z))$ be the vector of the first order partial derivatives and $\nabla^2 \psi_1(z)$ denote the matrix of second order partial derivatives, that is,

$$\nabla^2 \psi_1(z) = (D_{ij} \psi_1(z)). \quad (3.65)$$

The determinant of the matrix $\nabla^2 \psi_1(z)$ is denoted by $|\nabla^2 \psi_1(z)|$.

Let $d_n = \nabla^2 (\psi_1(\tau) + \psi_2(-\langle \tau, \tau \rangle))$ be positive definite. Further, let the eigen values of d_n be bounded below by $\alpha_1 > 0$, for all $\tau \in J_1^k$ and $n \geq 1$. Let $\{r_n\}$ be a sequence of vectors in R^k such that $|r_{jn}| < r_j^*$ for $j = 1, \dots, k$ and $n \geq 1$. For this sequence r_n let there exist a sequence $\{\tau_n\}$ in J_1^k such that $\langle r_n, \tau_n \rangle \in J_2$

$$\nabla (\psi_1(\tau_n) + \psi_2(-\langle r_n, \tau_n \rangle)) = 0 \quad (3.66)$$

and

$$\psi_2'(-\langle r_n, \tau_n \rangle) > \alpha_2 \quad (3.67)$$

where $\alpha_2 > 0$ and $n \geq 1$.

Let us assume the following conditions on the moment generating functions ϕ_1 and ϕ_2 .

(F11) There exist $\beta_1 < \infty$ and $\beta_2 < \infty$ such that

$$|\psi_1(z)| < \beta_1 \text{ for } z \in \Omega_1^k \quad (3.68)$$

and

$$|\psi_2(z)| < \beta_2 \text{ for } z \in \Omega_2. \quad (3.69)$$

(F12) Given $\delta > 0$, there exists $q > 0$ such that

$$\sup_{|t| > \delta} |\psi_2'(-\langle r_n, \tau_n + it \rangle)| = O(n^q). \quad (3.70)$$

(F13) There exists $B_1 < \infty$ such that

$$\sup_{\tau \in J_1^k} \int_{-\infty}^{\infty} \left| \frac{\phi_1(\tau + it)}{\phi_1(\tau)} \right|^{l/a_n} dt < B_1. \quad (3.71)$$

(F12) Given $\delta > 0$, there exists $q > 0$ such that

$$\sup_{|t| > \delta} |\psi'_2(-\langle r_n, \tau_n + it \rangle)| = O(n^q). \quad (3.70)$$

(F13) There exists $B_1 < \infty$ such that

$$\sup_{\tau \in J_1^k} \int_{-\infty}^{\infty} \left| \frac{\phi_1(\tau + it)}{\phi_1(\tau)} \right|^{l/a_n} dt < B_1. \quad (3.71)$$

Let $T_n = X_1 + \cdots + X_n$, $S_n = Y_1 + \cdots + Y_n$. Then an asymptotic expression for the p.d.f. of $R_n = T_n/S_n$ at the point r_n is given by,

$$\bar{g}_1(r_n) = \frac{n^{k/2} (\psi'_2(-\langle r_n, \tau_n \rangle))^k}{(2\pi)^{k/2} |d_n|^{1/2}} \quad (3.72)$$

$$\times \exp(n[\psi_1(\tau_n) + \psi_2(-\langle r_n, \tau_n \rangle)]) [1 + O(1/n)].$$

3.3 Large deviation local limit theorems for lattice random vectors

Now we proceed to obtain similar local limit theorems for ratio statistic $R_n = T_n/S_n$ where the random vector T_n and the random variable S_n both are lattice. We continue to use the same notations introduced in Section 3.2.

Before we state and prove the results of this section we give below a few definitions and results for lattice random vectors.

Definition 3.3.1. Consider R^k as a group under vector addition. A subgroup L is discrete if every ball in R^k has only a finite number of points of L in it.

Definition 3.3.2. A discrete subgroup L is called a lattice if there exists k linearly independent vectors $\{\xi_1, \dots, \xi_k\}$ such that

$$L = [\xi: \xi = m_1\xi_1 + \dots + m_k\xi_k; m_i \text{ integer}, \forall i = 1, \dots, k].$$

Definition 3.3.3. A random vector X in R^k is a lattice random vector if there exists $x_0 \in R^k$ and a lattice such that $Pr(X \in x_0 + L) = 1$.

Definition 3.3.4. A random vector X in R^k is degenerate if there exists a hyperplane $H = \{x: \langle a, x \rangle = c\}$, where $a \neq 0$ and c is a constant such that $Pr(X \in H) = 1$, otherwise we say that X is non-degenerate.

Lemma 3.3.5. *Let X be a nondegenerate lattice random vector. Then there exists a unique lattice L_0 such that the following two properties hold:*

- (i) $Pr(X \in x + L_0) = 1$ for all x such that $Pr(X = x) > 0$.
(ii) If M is any closed subgroup such that $Pr(X \in y_0 + M) = 1$ for some $y_0 \in R^k$, then $L_0 \subset M$.

Proof: See Bhattacharya and Ranga Rao (1976), pp. 226-227.

The lattice L_0 is called the minimal lattice for X . In what follows we shall consider only the minimal lattice. Let $\phi(t) = Ee^{i\langle t, x \rangle}$, $t \in R^k$ be the characteristic function of X . A vector $t_0 \in R^k$ is said to be a period of $|\phi|$ if $|\phi(t + t_0)| = |\phi(t)|$ for all $t \in R^k$.

Lemma 3.3.6. *Let X be a lattice random vector with characteristic function ϕ . The set L_1 of periods of $|\phi|$ is a lattice if and only if X is nondegenerate. Further, let $L_1 = \{t : \langle t, \xi \rangle \in 2\pi m \forall \xi \in L, m \text{ integer}\}$. Let $\{n_1, \dots, n_k\}$ be a dual basis, that is, $\langle \xi_j, n_{j'} \rangle = 0$ if $j \neq j'$ and 1 if $j = j'$. Then $L_1 = \{2\pi t : t = m_1 n_1 + \dots + m_k n_k, m_i \text{ integer}\}$.*

Proof. See Bhattacharya and Ranga Rao (1976), pp. 227-228.

This completes the preliminaries. We now proceed with Theorem 3.3.7 which gives an asymptotic expression for the density of the ratio statistic $R_n = T_n/S_n$ where $\{T_n, n \geq 1\}$ is a sequence of lattice random vectors and $\{S_n, n \geq 1\}$ is a sequence of lattice valued random variables.

Theorem 3.3.7. *Let L_n be the lattice $\{\xi : \xi = h_n(m_1 e_1 + \dots + m_k e_k), m_i \text{ integer}\}$, where $\{h_n, n \geq 1\}$ is a sequence of real numbers and $\{e_1, \dots, e_k\}$ is an orthonormal basis of R^k . Let $\{T_n, n \geq 1\}$ be a sequence of nondegenerate*

lattice random vectors defined on L_n with distribution functions F_{1n} . Let $\{S_n, n \geq 1\}$ be a sequence of lattice random variables taking values in the set $S' = \{a'_n + kh'_n : a'_n, h'_n \text{ are such that } a'_n + kh'_n > 0\}$. Let S_n be independent of the components of the vector T_n for all $n \geq 1$. Let $r_n = (r_{1n}, \dots, r_{kn})$ be a sequence of vectors in R^k such that $|r_{jn}| < r$ for $j = 1, \dots, k$ and $n \geq 1$ and $r > 0$.

Assume that the following conditions are satisfied.

(H1) There exist $\beta_1 < \infty$ and $\beta_2 < \infty$ such that

$$|\psi_{1n}(z)| < \beta_1 \text{ for } z \in \Omega_1^k \quad (3.73)$$

and

$$|\psi_{2n}(z)| < \beta_2 \text{ for } z \in \Omega_2 \quad (3.74)$$

for $n \geq 1$.

(H2) Given $\delta > 0$, there exists $0 < \eta < 1$ such that

$$\limsup_n \sup_{\delta \leq |t| \leq \pi/|h_n|} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \right|^{1/a_n} < \eta. \quad (3.75)$$

(H3) There exist $p, l > 0$ such that

$$\sup_{\tau \in J_1^k} \int_{-\pi/|h_n|}^{\pi/|h_n|} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \right|^{1/a_n} = O(a_n^p). \quad (3.76)$$

Let $P_n(r_n) = P(T_n = r_n S_n)$. Then

$$\begin{aligned} P_n(r_n) &= \left[\frac{|h_n|^k}{(2\pi)^k a_n^k |\nabla^2(\psi_{1n}(\tau_n) + \psi_{2n}(-\langle r_n, \tau_n \rangle))|} \right]^{1/2} \\ &\times \exp(a_n[\psi_{1n}(\tau_n) + \psi_{2n}(-\langle r_n, \tau_n \rangle)] [1 + O(1/a_n)]). \end{aligned} \quad (3.77)$$

Proof: Consider,

$$P(T_n = r_n S_n) = \sum_{y \in S', ry \in L_n} P(T_n = ry) P(S_n = y). \quad (3.78)$$

We use the conjugate distribution technique to obtain the probability $P(T_n = ry)$ and it is given by

$$P(T_n = ry) = \frac{|h_n|^k}{(2\pi)^k} \int_{E^n} \phi_{1n}(\tau + it) e^{-\langle ry, \tau + it \rangle} dt \quad (3.79)$$

where $E^n = \{(t_1, \dots, t_k) : |t_j| < \pi/|h_n|\}$. In view of (3.82), (3.81) becomes

$$\begin{aligned} P(T_n = r S_n) &= \sum_{y \in S', ry \in L_n} \left[\frac{|h_n|^k}{(2\pi)^k} \int_{E^n} \phi_{1n}(\tau + it) e^{-\langle ry, \tau + it \rangle} dt \right] P(S_n = y) \\ &= \left[\frac{|h_n|^k}{(2\pi)^k} \right] \int_{E^n} \phi_{1n}(\tau + it) \sum_{y \in S', ry \in L_n} e^{-\langle ry, \tau + it \rangle} P(S_n = y) \\ &= \left[\frac{|h_n|^k}{(2\pi)^k} \right] \int_{E^n} \phi_{1n}(\tau + it) \phi_{2n}(-\langle ry, \tau + it \rangle) dt \\ &= \left[\frac{|h_n|^k}{(2\pi)^k} \right] \int_{E^n} \exp(a_n[\psi_{1n}(\tau + it) + \psi_{2n}(-\langle ry, \tau + it \rangle)]) dt. \end{aligned} \quad (3.80)$$

The saddle point allows us to replace τ by τ_n when r is replaced by r_n .

Thus, we have,

$$\begin{aligned} P(T_n = r_n S_n) &= \left[\frac{|h_n|^k}{(2\pi)^k} \right] \int_{E^n} \exp(a_n[\psi_{1n}(\tau_n + it) + \psi_{2n}(-\langle r_n y, \tau_n + it \rangle)]) dt \\ &= \left[\frac{|h_n|^k}{a_n^{k/2} (2\pi)^{k/2} |d_n|^{1/2}} \right] \int_{E^n} \exp(a_n[\psi_{1n}(\tau_n) + \psi_{2n}(-\langle r_n, \tau_n \rangle)]) I_n \end{aligned} \quad (3.81)$$

where

$$I_n = \left[\frac{a_n^{k/2} |d_n|^{1/2}}{2\pi^{k/2}} \right] \int_{E^n} \exp(a_n[\psi_{1n}(\tau_n + it) - \psi_{1n}(\tau_n) + \psi_{2n}(-\langle r_n, \tau_n + it \rangle)])$$

$$-\psi_{2n}(-\langle r_n, \tau_n \rangle)) dt. \quad (3.82)$$

Using the techniques similar to the techniques in Theorem 3.3.7, we can easily show that $I_n = 1 + O(1/a_n)$. Thus

$$\begin{aligned} P(T_n = r_n S_n) &= \left[\frac{|h_n|^k}{a_n^{k/2} (2\pi)^{k/2} |d_n|^{1/2}} \right] \\ &\times \exp(a_n[\psi_{1n}(\tau_n) + \psi_{2n}(-\langle r_n, \tau_n \rangle)]) [1 + O(1/a_n)]. \end{aligned} \quad (3.83)$$

The following Theorem 3.3.8 is a particular case of Theorem 3.3.7 proved above. Here T_n is the sum of n i.i.d. lattice valued random vectors defined on the lattice L_n and S_n is the sum of n i.i.d. lattice valued random variables Y_n . This theorem gives an asymptotic expression for the density of the ratio of two statistics.

Theorem 3.3.8. *Let L_n be the lattice $\{\xi : \xi = h_n(m_1 e_1 + \dots + m_k e_k), m_i \text{ integer}\}$, where $\{h_n, n \geq 1\}$ is a sequence of real numbers and $\{e_1, \dots, e_k\}$ is an orthonormal basis of R^k . Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. lattice valued random vectors defined on lattice L_n . Let $\{Y_n, n \geq 1\}$ be an i.i.d. sequence of lattice random variables taking values in the set $S' = \{a'_n + kh'_n : a'_n, h'_n \text{ are such that } a'_n + kh'_n > 0\}$. Let Y_n be independent of the components of the random vectors X_n for all $n \geq 1$. Let $\tau_n = (\tau_{1n}, \dots, \tau_{kn})$ be a sequence of vectors in R^k such that $|r_{jn}| < r$ for $j = 1, \dots, k$ and $n \geq 1$ and $r > 0$. For this sequence $\{\tau_n\}$, let there exist τ'_n s in J_1^k such that*

$$\psi'_1(\tau_n) - r_n \psi'_2(-\langle r_n, \tau_n \rangle) = 0 \text{ and } \langle r_n, \tau_n \rangle \in J_2.$$

Let $v_n = \nabla^2(\psi_1(\tau) + \psi_2(-\langle r, \tau \rangle))$ for $\tau \in J_1^k$ and $\langle r, \tau \rangle \in J_2$. Let the eigen values of v_n be bounded below by $\alpha_1 > 0$. Assume that the following conditions are satisfied.

(H11) There exist $\beta_1 < \infty$ and $\beta_2 < \infty$ such that

$$|\psi_1(z)| < \beta_1 \text{ for } z \in \Omega_1^k \quad (3.84)$$

and

$$|\psi_2(z)| < \beta_2 \text{ for } z \in \Omega_2. \quad (3.85)$$

(H13) There exists B_3 such that

$$\sup_{\tau \in J_1^k} \int_{-\pi/|h|}^{\pi/|h|} \left| \frac{\phi_1(\tau + it)}{\phi_1(\tau)} \right|^n < B_3. \quad (3.86)$$

Let $T_n = X_1 + \cdots + X_n$ and $S_n = Y_1 + \cdots + Y_n$. Let $P(T_n = r_n S_n) = P_n(r_n)$. Then,

$$\begin{aligned} P_n(r_n) &= \left[\frac{|h|^k}{(2\pi)^k n^k |\nabla^2(\psi_1(\tau) + \psi_2(-\langle r, \tau \rangle))|} \right]^{1/2} \\ &\times \exp(a[\psi_1(\tau) + \psi_2(-\langle r, \tau \rangle)] [1 + O(1/n)]). \end{aligned} \quad (3.87)$$

Chapter 4

Strong large deviation results for ratio statistics

4.1 Introduction

Let $\{T_n, n \geq 1\}$ be a sequence of random variables such that $T_n/a_n \rightarrow 0$ for some sequence of real numbers $a_n \rightarrow \infty$. In most examples the probability of the event $\{T_n \geq x_n\}$ goes to zero exponentially fast whenever $x_n = O(a_n)$ and x_n is positive. The event $\{T_n \geq x_n\}$ for x_n positive is known in the literature as a large deviation event. Numerous authors including Cramér (1938), Chernoff (1952), Ellis (1984), Varadhan (1984) have obtained asymptotic expressions for $\log P(T_n \geq x_n)$ under some conditions on the moment generating function of T_n . Bahadur and Ranga Rao (1960) obtained asymptotic expression for $P(T_n \geq x_n)$ when T_n is the sum of n i.i.d. random variables and $x_n = O(n)$. Chaganty- Sethuraman coined the

term Weak large deviation results for the results of the former type and Strong large deviation results for the results of the latter type. In a recent paper, Chaganty-Sethuraman extended the results of Bahadur and Ranga Rao (1960) to an arbitrary sequence of random variables T_n , not necessarily sums of i.i.d. random variables. We state these results precisely in Theorems 4.2.2 and 4.3.2.

The purpose of this chapter is to obtain strong large deviation result for the ratio statistic $R_n = T_n/S_n$, where T_n and S_n are arbitrary sequences of random variables. In Theorem 4.2.3, we show that Conditions (Q1) and (Q2) on the m.g.f.'s of T_n and S_n imply Conditions (P1) and (P2) of theorem 4.2.2 for the random variable T'_n . Thus Conditions (P1) and (P2) of Theorem 4.2.2 are satisfied for arbitrary random variables $T'_n = T_n - r_n S_n$ giving an asymptotic expression for $P(T_n/S_n \geq r_n)$ i.e. for $P(T_n - r_n S_n \geq 0)$. The case of arbitrary lattice random variables T_n and S_n is treated in Theorem 4.3.3.

4.2 Non-lattice case

This section contains strong large deviation limit theorems for non-lattice random variables.

Strong large deviation theorem for an arbitrary sequence of non-lattice random variables is stated below in Theorem 4.2.2.

Theorem 4.2.1 *Let $\{W_n, n \geq 1\}$ be a sequence of nonlattice valued random variables with m.g.f. $\phi_n(z)$. Let $\phi_n(z)$ be analytic and nonvanishing in*

$\Omega = \{z \in \mathbb{C} : |z| < c\}$, where $c > 0$. Let

$$\psi_n(z) = \frac{1}{a_n} \phi_n(z) \quad (4.1)$$

for $z \in \Omega$, $n \geq 1$. Also, let

$$\gamma_n(u) = \sup_{|s| < a} [us - \psi_n(s)], \text{ for real } u. \quad (4.2)$$

Let $\{m_n, n \geq 1\}$ be a sequence of real numbers such that there exists a sequence τ_n satisfying $\psi'_n(\tau_n) = m_n$ and $d < \tau_n < c < c_1$ for some positive numbers c and d and for all $n \geq 1$. Assume the following conditions for T_n .

(P1) There exists $\beta < \infty$ such that $|\psi_n(z)| < \beta$ for all $n \geq 1$, $z \in \Omega$.

(P2) There exists $\delta_1 > 0$ such that

$$\sup_{|t| \geq \delta} \left| \frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)} \right| = o\left(\frac{1}{\sqrt{a_n}}\right) \quad (4.3)$$

for all $0 < \delta < \delta_1$.

(P3) There exists $\alpha > 0$ such that $\psi''_n(\tau_n) \geq \alpha$ for all $n \geq 1$.

Then

$$P\left(\frac{W_n}{a_n} \geq m_n\right) \sim \frac{\exp(-a_n \gamma_n(m_n))}{\sqrt{2\pi\tau_n} \sqrt{a_n \psi''_n(\tau_n)}}. \quad (4.4)$$

The following Theorem 4.2.3 generalizes Theorem 4.2.2 stated above to ratio statistic.

Theorem 4.2.2 Let $\{T_n, n \geq 1\}$ be an arbitrary sequence of nonlattice random variables with m.g.f. $\phi_{1n}(z) = E[\exp(zT_n)]$ and $\{S_n, n \geq 1\}$ be another sequence of nonlattice positive random variables with m.g.f. $\phi_{2n}(z) =$

$E[\exp(zT_n)]$. Assume that the two sequences are independent of each other. Let ϕ_{1n} and ϕ_{2n} be nonvanishing and analytic in the region $\Omega = \{z \in \mathbb{C} : |z| < c\}$, where $c > 0$ and $\mathbb{C}$ is the set of all complex numbers. Let $\{a_n\}$ be a sequence of real numbers such that $a_n \rightarrow \infty$. Let

$$\psi_{in}(z) = \frac{1}{a_n} \log \phi_{in}(z), \text{ for } z \in \Omega, i = 1, 2. \quad (4.5)$$

Let $\{r_n\}$ be a positive bounded sequence of real numbers such that there exists τ_n satisfying $0 < d < \tau_n < c$ and

$$\psi'_{1n}(\tau_n) - r_n \psi'_{2n}(-r_n \tau_n) = 0. \quad (4.6)$$

Assume the following conditions for T_n and S_n :

(Q1) There exist $\beta_i < \infty$ such that $|\psi_i(z)| < \beta_i$ for $z \in \Omega$, for $i = 1, 2$.

(Q2) There exists $\delta_1 > 0$ such that

$$\sup_{|t| \geq \delta} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \right| = o(1/a_n) \quad (4.7)$$

for any $0 < \delta < \delta_1$.

(Q3) Let α_1 be a positive real number such that $\psi''_{1n}(\tau) > \alpha_1$.

Then

$$P\left(\frac{T_n}{S_n} \geq r_n\right) \sim \frac{\exp(a_n[\psi_{1n}(\tau_n) + \psi_{2n}(-r_n \tau_n)])}{\sqrt{2\pi\tau_n} \sqrt{a_n[\psi''_{1n}(\tau_n) + r_n^2 \psi''_{2n}(-r_n \tau_n)]}}. \quad (4.8)$$

Proof: Consider

$$\begin{aligned} P\left(\frac{T_n}{S_n} \geq r_n\right) &= P(T_n - r_n S_n \geq 0) \\ &= P\left(\frac{T_n^*}{a_n} \geq 0\right) \end{aligned} \quad (4.9)$$

where $T_n^* = T_n - r_n S_n$. Using the independence of T_n and S_n we get the following relationship between the m.g.f.'s of T_n , S_n and T_n^*

$$\phi_{0n}(\tau) = \phi_{1n}(\tau) \phi_{2n}(-r\tau) \text{ for } \tau \in J_0 \quad (4.10)$$

which implies that

$$\psi_{0n}(z) = \psi_{1n}(z) + \psi_{2n}(-rz) \quad (4.11)$$

where ϕ_{0n} is the m.g.f. of T_n^* , $\psi_{0n}(z) = \frac{1}{a_n} \log \phi_{0n}(z)$. Let $\gamma_n^*(u) = \sup_{|s| < c} [us - \psi_{0n}(s)]$. From Condition (Q1) and (4.11), it follows that

$$|\psi_{0n}(z)| < \beta_1 + \beta_2. \quad (4.12)$$

Thus, there exists $\beta_0 = \beta_1 + \beta_2 < \infty$ such that $|\psi_{0n}(z)| < \beta_0$. Hence Condition (P1) of Theorem 4.2.2 is satisfied.

We have for any $0 < \delta < \delta_1$

$$\begin{aligned} \sup_{|t| \geq \delta} \left| \frac{\phi_{0n}(\tau_n + it)}{\phi_{0n}(\tau_n)} \right| &= \sup_{|t| \geq \delta} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \frac{\phi_{2n}(\tau_n + it)}{\phi_{2n}(\tau_n)} \right| \\ &\leq \sup_{|t| \geq \delta} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{0n}(\tau_n)} \right|. \end{aligned}$$

From Condition (Q2) it follows that

$$\sup_{|t| \geq \delta} \left| \frac{\phi_{0n}(\tau_n + it)}{\phi_{0n}(\tau_n)} \right| = o(1/a_n). \quad (4.13)$$

Hence Condition (P2) of Theorem 4.2.2 is also satisfied. In view of $\psi_{1n}''(\tau) > \alpha_1$ and (4.11) Condition (P3) is easily satisfied. Thus an arbitrary sequence $\{T_n^*, n \geq 1\}$ satisfies all the conditions of Theorem 4.2.2 and hence the conclusion of the Theorem 4.2.2 yields

$$P\left(\frac{T_n^*}{a_n} \geq 0\right) \sim \frac{\exp(-a_n \gamma_n^*(0))}{\sqrt{2\pi\tau_n} \sqrt{a_n \psi_{0n}''(\tau_n)}}. \quad (4.14)$$

which in turn gives Strong large deviation result for T_n/S_n , namely,

$$P\left(\frac{T_n}{S_n} \geq r_n\right) \sim \frac{\exp(a_n[\psi_{1n}(\tau_n) + \psi_{2n}(-r_n\tau_n)])}{\sqrt{2\pi\tau_n}\sqrt{a_n[\psi''_{1n}(\tau_n) + r_n^2\psi''_{2n}(-r_n\tau_n)]}}. \quad (4.15)$$

4.3 The Lattice case

In this section we state strong large deviation theorem for lattice valued random variables. These theorems are analogous to the theorems for lattice valued random variables of the previous section.

The next theorem provides an estimate of $P(T_n/a_n \geq m_n)$ where $\{T_n, n \geq 1\}$ is a sequence of lattice valued random variables and m_n is in the large deviation of T_n/a_n . We first introduce few notations.

Let $\{T_n, n \geq 1\}$ be an arbitrary sequence of lattice valued random variables taking values in the lattice $\{t_n + kp_n : k = 0, \pm 1, \pm 2, \dots\}$, $p_n > 0$. Let the c.f. of T_n , $\phi_n(z)$, be analytic and nonvanishing in the region

$\Omega = \{z \in \mathcal{C} : |z| < c_1\}$. Let $\{a_n\}$ be a sequence of real numbers such that $a_n \rightarrow \infty$ and $p_n = o(\sqrt{a_n})$. Let

$$\psi_n(z) = \frac{1}{a_n} \log \phi_n(z), \quad (4.16)$$

be a well defined analytic function on Ω . Let $\{m_n\}$ be a sequence of real numbers such that there exists $0 < \tau_n < b_1 < c_1$ satisfying $\psi'_n(\tau_n) = m_n$ and $\tau_n\sqrt{a_n} \rightarrow \infty$ as $n \rightarrow \infty$. We now state Theorem 4.3.2.

Theorem 4.3.1 *Assume that T_n satisfies Conditions (P1) and (P3) of Theorem 4.2.2 and the following Condition (P2'):*

(P2') There exists $\delta_1 > 0$, such that for $0 < \delta < \delta_1$,

$$\sup_{\delta \leq |t| < \pi/h_n} \left| \frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)} \right|. \quad (4.17)$$

Then

$$P\left(\frac{T_n}{a_n} \geq m_n\right) \sim \frac{p_n}{\sqrt{2\pi} \sqrt{a_n \psi_n''(\tau_n)}} \frac{\exp(-a_n \gamma_n(m_n))}{(1 - \exp(-p_n \tau_n))} \quad (4.18)$$

where $\gamma_n(m_n) = m_n \tau_n - \psi_n(\tau_n)$.

Next, we present Theorem 4.3.3 which provides an estimate of the large deviation probability for the ratio T_n/S_n , where $\{T_n, n \geq 1\}$ and $\{S_n > 0, n \geq 1\}$ denote the arbitrary sequences of random variables. First we begin with few preliminaries.

Let $\{T_n, n \geq 1\}$ be a sequence of lattice random variables with distribution functions F_{1n} . Let T_n take values in the set $S_1 = \{a_n + kh_n : a_n \text{ and } h_n \text{ are real numbers and } k \text{ is an integer}\}$. Let $\{S_n > 0, n \geq 1\}$ be a sequence of positive lattice random variables taking values in the set $S_2 = \{a'_n + kh'_n : a'_n \text{ and } h'_n \text{ are such that } a'_n + kh'_n > 0\}$. Let the m.g.f.'s of T_n and S_n , ϕ_{1n} and ϕ_{2n} , be analytic and non-vanishing in $\Omega = \{z \in \mathbb{C} : |z| < b\}$. Let r_n be a positive bounded sequence of real numbers such that there exist τ_n such that $0 < \tau_n < b_1 < b$ and

$$\psi'_{1n}(\tau_n) - r_n \psi'_{2n}(-r_n \tau_n) = 0. \quad (4.19)$$

for all $n \geq 1$.

Theorem 4.3.2 Assume that these two sequences meet the following conditions: (R1) There exist $\beta_1 < \infty$ and $\beta_2 < \infty$ such that

$$|\psi_{jn}(z)| < \beta_j \text{ for } z \text{ with } |z| < b \text{ for } j = 1, 2 \text{ and } n \geq 1. \quad (4.20)$$

(R2) There exists $\delta_1 > 0$ such that

$$\sup_{\delta \leq |t| < \pi/h_n} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \right| = o(1/a_n) \quad (4.21)$$

for any $0 < \delta < \delta_1$.

(R3) Let α_1 be positive real numbers such that

$$\psi''_{1n}(\tau) > \alpha_1. \quad (4.22)$$

Then

$$\begin{aligned} P\left(\frac{T_n}{S_n} \geq r_n\right) &\sim \frac{h_n}{\sqrt{2\pi} \sqrt{a_n[\psi''_{1n}(\tau_n) + r_n^2 \psi''_{2n}(-r_n \tau_n)]}} \\ &\times \frac{\exp(a_n[\psi_{1n}(\tau_n) + \psi_{2n}(-r_n \tau_n)])}{(1 - \exp(-h_n \tau_n))}. \end{aligned} \quad (4.23)$$

Proof. The proof of this theorem is similar to the proof of theorem and hence we omit it.

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