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All minimal prime extensions of hereditary classes of graphs

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Abstract

The substitution composition of two disjoint graphs G_1 and G_2 is obtained by first removing a vertex x from G_2 and then making every vertex in G_1 adjacent to all neighbours of x in G_2. Let $\mathcal{F}$ be a family of graphs defined by a set $\mathcal{Z}$ of forbidden configurations. Giakoumakis [V. Giakoumakis, On the closure of graphs under substitution, Discrete Mathematics 177 (1997) 83–97] proved that $\mathcal{F}^*$, the closure under substitution of F, can be characterized by a set $\mathcal{Z}^*$ of forbidden configurations — the *minimal prime extensions* of $\mathcal{Z}$. He also showed that $\mathcal{Z}^*$ is not necessarily a finite set. Since substitution preserves many of the properties of the composed graphs, an important problem is the following: find necessary and sufficient conditions for the finiteness of $\mathcal{Z}^*$. Giakoumakis [V. Giakoumakis, On the closure of graphs under substitution, Discrete Mathematics 177 (1997) 83–97] presented a sufficient condition for the finiteness of $\mathcal{Z}^*$ and a simple method for enumerating all its elements. Since then, many other researchers have studied various classes of graphs for which the substitution closure can be characterized by a finite set of forbidden configurations.

The main contribution of this paper is to completely solve the above problem by characterizing all classes of graphs having a finite number of minimal prime extensions. We then go on to point out a simple way for generating an infinite number of minimal prime extensions for all the other classes of $\mathcal{F}^*$.

Keywords: Graph theory; Graph decompositions; Modular decomposition

1. Motivation

The substitution composition of graphs has been widely used by researchers in the study of both theoretical as well as practical problems; we refer the interested reader to Brandstädt et al. [4] for a comprehensive discussion. The appeal of the substitution composition is, most certainly, due to the fact that it preserves many of the properties of the composed graphs. For example, Lovász [12] relied on the well known fact that substitution preserves perfection[1] in order to prove that a graph is *perfect* if and only if its complement is. We recall that the famous Strong Perfect Graph Conjecture (SPGC, for short) introduced by Berge [1] in 1961 was recently answered in the affirmative becoming the

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[1] For the definition of perfect graphs see Berge [2] or Brandstädt et al. [4].

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Strong Perfect Graph Theorem. We refer the reader to Chvàtal's web page [8] for a detailed survey and to Chudnovsky et al. [7] for the proof of the Strong Perfect Graph Theorem.

Let $\mathcal{F}$ be a family of graphs defined by a set $\mathcal{Z}$ of forbidden configurations and let $\mathcal{F}^*$ be the closure of $\mathcal{F}$ under substitution.

Problem 1. Find a forbidden induced subgraph characterization of $\mathcal{F}^*$.

Giakoumakis [10] proved that:

1. $\mathcal{F}^*$ can be characterized by a set $\mathcal{Z}^*$of forbidden configurations;
2. $\mathcal{Z}^*$ is not necessarily a finite set;
3. If no graph Z of $\mathcal{Z}$ contains a *module* of more than two vertices then $\mathcal{Z}^*$ is finite.

Problem 2. Find necessary and sufficient conditions for $\mathcal{Z}^*$ to be finite.

Various researchers investigated Problem 2 and many sufficient conditions have been presented [5,6,10,11,13,16,17]. These, and other similar papers, give forbidden subgraph characterizations of the closure under substitution of various classes of graphs. It is worth noting that such characterizations are very likely to lead to efficient graph optimization algorithms. Indeed, for optimization problems including finding the weighted stability number (see [3] and [9]) and the domination problem (see [15]), efficient solutions can be found when dealing with hereditary classes of graphs.

The main contribution of this paper is to offer a complete answer to Problem 2 by characterizing all classes of graphs defined by a finite set of forbidden configurations, whose closure under substitution can also be defined by a finite number of forbidden subgraphs. For all the other classes of $\mathcal{F}^*$ we give a simple way for generating an infinite number of minimal prime extensions.

2. Notation and previous results

The main goal of this section is to establish notation and terminology and to review a number of known results that will be needed in the subsequent sections of the paper.

2.1. Notation and terminology

For terms not defined here the reader is referred to [2] and [4]. All the graphs in this work are finite, with no loops nor multiple edges. Given a graph $G = (V, E)$, the set V of its vertices will also be denoted by $V(G)$; similarly, the set E of its edges will be denoted by $E(G)$. We also write $n = |V|$ and $m = |E|$ to denote the cardinality of V and E.

Let x be a vertex of graph G. The neighborhood of x will be denoted by $N(x)$; we let $degree(x)$ stand for $| N(x) |$. The subgraph of G induced by $V(G) - \{x\}$ will be denoted by $G \setminus x$.

If $N(x) = V(G) - \{x\}$, x is said to be a *universal* vertex of G. The graph induced by a set $X \subseteq V$ will be denoted by $[X]$; $[X]$ is a *proper* induced subgraph of G if X is strictly contained in $V(G)$. We shall let $N_G(X)$ stand for the set of vertices in $V(G) - X$ adjacent to at least one vertex of X.

The set X is said to be *stable* (resp. *complete*) if the graph $[X]$ is edgeless (resp. fully connected). A stable (resp. complete) set of r vertices is denoted by S_r (resp. K_r). The edgeless graph of r vertices will be denoted by O_r. The graph induced by $V(G) - X$ is also written as $G \setminus X$ and the graph induced by $V(G) - \{x\}$ where x a vertex of G, will be written as $G \setminus x$. A vertex x is *total, indifferent* or *partial* with respect to X if it is, respectively, adjacent to all, to none or to some but not all of the vertices of X. A set of vertices Y is total (or universal) with respect to X if every vertex of Y is adjacent to all the vertices of X; Y is indifferent with respect to X if no vertex of Y is adjacent to a vertex of X; finally, Y is partial with respect to X if at least one of the vertices of Y is partial with respect to X.

We shall write P_k (resp. C_k) to denote a chordless path (resp. cycle) on k vertices. The complementary graph of a chordless path is referred to as a *co-path*. A $2K_2$ is the complementary graph of a C_4. When no confusion is possible, we shall use the notation P_k to design also the set of vertices of the chordless chain P_k.

The notation $G_1 \sim G_2$ signifies that the graph G_1 is isomorphic to the graph G_2.

Let $\mathcal{Z}$ be a set of graphs. A graph G is said to be $\mathcal{Z}$-free if G contains no induced subgraph isomorphic to a graph of $\mathcal{Z}$. A set of graphs $\mathcal{F}$ is $\mathcal{Z}$-free if every graph of $\mathcal{F}$ is $\mathcal{Z}$-free.

A set $M \subseteq V(G)$ is called a *module* if every vertex of G outside M is adjacent to all vertices of M or to none of them. The empty set, $V(G)$ and the singletons are *trivial* modules. A graph G that contains only trivial modules

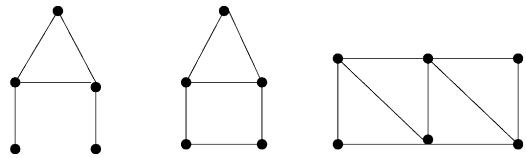

Fig. 1. The minimal prime extensions of a C_3.

is termed *prime* or *indecomposable*. A module M that is a strict subset of $V(G)$ and contains at least two vertices is said to be *non-trivial* or a *homogeneous set*.[2] A graph that contains a non-trivial module is said to be *substitution-decomposable* or, simply, *decomposable*.

Let M be a module of a graph G. M is said to be a *strong* module if for every non-trivial module M' of G either $M \cap M' = \emptyset$ or one of M and M' is included in the other. The decomposition of a graph into its modules was discovered independently by researchers in many seemingly unrelated areas. We refer the reader to Brandstädt et al. [4] for a comprehensive discussion and further references.

The *modular decomposition* of a graph G is a form of decomposition that associates with G a unique decomposition tree $T(G)$. The set of leaves of $T(G)$ is the set $V(G)$. The set of leaves associated with a subtree of $T(G)$ rooted at a node f of $T(G)$ is *leaves*(f). It is well known that for each internal node f of $T(G)$ different from its root, *leaves*(f) forms a strong module of G and that $\{leaves(f)\}$ is the set of all strong modules of G. An internal node f is labeled P, S, or N to denote respectively, *parallel*, *series* or *neighbourhood* modules. The subgraph induced in G by a parallel module is disconnected, the one induced by a series module is connected and has a disconnected complement; and, finally, the one induced by a neighbourhood module is connected both in the graph and the complement.

Let $f_1, \ldots, f_k$ be the set of children of f in $T(G)$ and let H be the subgraph of G whose vertex-set consists of one vertex from each module *leaves*(f_i), $i = 1, \ldots, k$. Clearly, H is an edgeless graph whenever f is a P-node, a complete graph whenever f is an S-node, and a prime graph whenever f is an N-node.

Due to its vast array of practical applications the problem of finding efficient algorithms (both sequentially and parallel) for the modular decomposition and for the construction of the corresponding decomposition tree has received a great deal of attention in the recent literature. We refer the reader to the excellent web page [18] for a very informative synopsis of research in this area.

Definition 2.1. Let G be a graph. The graph G' is a *minimal prime extension* of G if the following conditions are satisfied:

- G' is prime,
- G' contains an induced subgraph isomorphic to G, and
- G' is minimal with respect to set inclusion and primality.

In other words, if G' is a minimal prime extension of G, no proper prime induced subgraph of G' contains an induced subgraph isomorphic to G. Observe that if G itself is prime then G' coincides with G.

Notation 2.2. Let G be a graph and let $\text{Ext}(G)$ denote the set of minimal prime extensions of G.

Let $\mathcal{F}$ be a family of graphs defined by a set $\mathcal{Z}$ of forbidden configurations. Giakoumakis [10] proved the following result:

Lemma 2.3 (*[10]*). *The closure $\mathcal{F}^*$ of $\mathcal{F}$ under substitution is defined by a set $\mathcal{Z}^*$ of forbidden configurations which is the union of the sets* $\text{Ext}(Z)$ *where Z is a graph of $\mathcal{Z}$.*

2.2. Known results

We begin by recalling two results concerning minimal prime extensions of various classes of graphs that we shall need in the following sections.

Theorem 2.4 (*[13]*). *The substitution composition of C_3-free graphs is defined by the three forbidden configurations depicted in Fig. 1.*

[2] We warn the reader that in the sequel of this paper 'homogeneous set' and 'non-trivial module' will be regarded as synonyms and used interchangeably.

Theorem 2.5 (*[16]*). *If every nontrivial module of a graph G induces a subgraph of a P_4 then the set of all minimal prime extensions of G is finite.*

Notation 2.6. A graph whose every nontrivial module induces a subgraph of a P_4 is called *P_4-homogeneous*. We prefer this terminology to that of a *simple* graph used in [16], in order to avoid any possible confusion with the meaning of the term 'simple' used in other contexts in graph theory.

An interesting procedure proposed by Zverovich [14] generating prime extensions of a graph G is the *Reducing Pseudopath Method*. We recall its definition using the notation of [14].

Definition 2.7. Let G be an induced subgraph of a graph H and let W be a homogeneous set of G. We define a reducing W-pseudopath in H as a sequence $R = (u_1, u_2, \ldots, u_t)$, with $t \geq 1$, of pairwise distinct vertices of $V(H) \setminus V(G)$ satisfying the following conditions:

1. u_1 is partial with respect to W;
2. $\forall\, i = 2, \ldots, t$, either u_i is adjacent to u_{i-1} and indifferent with respect to $W \cup \{u_1, \ldots, u_{i-2}\}$ or u_i is total with respect to $W \cup \{u_1, \ldots, u_{i-2}\}$ and non-adjacent to u_{i-1} (when $i = 2$, $\{u_1, u_2, \ldots, u_{i-2}\} = \emptyset$);
3. $\forall\, i = 1, \ldots, t-1$, vertex u_i is total with respect to $N(W)$ in G and indifferent with respect to $V(G) - N(W) - W$ and either u_t is non-adjacent to a vertex of $N(W)$ or u_t is adjacent to a vertex of $V(G) - N_G(W)$.

We refer the reader to Fig. 2 in Section 3.2 for an illustration of a reducing W-pseudopath.

Theorem 2.8 (*[14]*). *Let H be a prime extension of its induced subgraph G and let W be a homogeneous set of G. Then there exists a reducing W-pseudopath with respect to every induced copy of G in H.*

The remainder of the paper is organized as follows: in Section 3 we give two methods for constructing a minimal prime extension of a graph G. In Section 4 we discuss necessary and sufficient conditions for finiteness of $\mathrm{Ext}(G)$. Finally, Section 5 offers concluding remarks and ideas for possible extensions of the results presented in this paper.

3. Two basic constructions

The main goal of this section is to introduce two basic constructions that provide the framework for our main result given in Section 4. Both these constructions build minimal prime extensions of a decomposable graph.

3.1. Constructing the basic extension of a decomposable graph

Let $G = (V, E)$ be a connected graph and let $T(G)$ be the corresponding modular decomposition tree.

Notation 3.1. Let $\pi(G) = \{H_1, \ldots, H_l\}$ be the partition of V obtained by the following equivalence relation R on V: for vertices x and y in V we write xRy if and only if x and y have the same parent in $T(G)$.

Throughout the remainder of this section we shall assume that G is not prime.

Remark 3.2. If G is decomposable, at least one of the H_i's in $\pi(G)$ is a non-trivial module of G.

Notation 3.3. Let $\rho(G) = \{M_1, \ldots, M_k\}$ be the subset of $\pi(G)$ consisting of all the non-trivial modules in G. Thus, for $i = 1, 2, \ldots, k$, every M_i is a non-trivial module in G .

Remark 3.4. Let $f(M_i)$ be the parent in $T(G)$ of the vertices of $M_i \in \rho(G)$. Obviously if M_i is stable then $f(M_i)$ is a P-node, if M_i is a complete set then $f(M_i)$ is an S-node and if M_i induces a prime graph then $f(M_i)$ is an N-node. Furthermore, if $M_i = leaves(f(M_i))$ then M_i is a strong nontrivial module of G, minimal with respect to set inclusion. In particular, this case occurs whenever $f(M_i)$ is an N-node (i.e. if $f(M_i)$ is an N-node then $M_i = leaves(f(M_i))$). It is worth noting that since G is connected, every vertex of module $M_i \in \rho(G)$ has a neighbour in $V - M_i$.

Let us associate with every module M_i of $\rho(G)$ a set V_i' of new vertices (i.e. $V_i' \cap V = \emptyset$, $V_i' \cap V_j' = \emptyset$, $i, j = 1, \ldots, k$, $i \neq j$) and a set E_i' of edges connecting the vertices of M_i with the vertices of V_i' in the following manner:

1. if M_i is a stable set or a complete set $\{x_1, \ldots, x_r\}$ then $V_i' = \{y_1, \ldots, y_{r-1}\}$ and E_i' is the set of edges $x_j y_j$, $j = 1, \ldots, r-1$.
2. If M_i induces in G a prime graph then V_i' is a singleton $\{y\}$ and E_i' is the edge yx where x is a vertex of M_i.

Let G' be the graph whose vertex set is $V \cup V'$, where $V' = V'_1 \cup \cdots \cup V'_k$ and whose edge set is $E \cup E'$, where $E' = E'_1 \cup \cdots \cup E'_k$. Clearly V' is a stable set in G' and each vertex of this set has exactly one neighbour in G', this neighbour being its own *'private'* neighbour.

We propose to show that G' is a minimal prime extension of G. For this purpose, however, we need the following result:

Lemma 3.5. *Let H be a connected graph, let $x \in V(H)$ be a vertex of degree 1, and let M be a non-trivial module of the graph $H \setminus x$ containing the unique neighbour, say y, of x in H. If x is contained in a nontrivial module Q of H then y is a universal vertex of H.*

Proof. Suppose not. Since H is connected and since $degree(x) = 1$, the neighbourhood of any vertex of module Q outside this module must be a singleton and, thus, $N(Q) = \{y\}$. Let Q_1 be the set $N(y) - Q$ and let Q_2 be the set of the remaining vertices of H. Then, no vertex of Q can be adjacent to a vertex of $Q_1 \cup Q_2$. Since, by assumption, $Q_2 \neq \emptyset$, the connectedness of H implies that $Q_1 \neq \emptyset$. Let z be a vertex of M different from y. If $z \in Q \setminus x$, every vertex of Q_1 would be adjacent to z and if $z \in Q_1 \cup Q_2$ any vertex of $Q \setminus x$ would be adjacent to z, a contradiction. □

Proposition 3.6. *The graph G' is a prime graph.*

Proof. Assume to the contrary that there exists a nontrivial module M in G'.

Claim 1. *M contains no vertices of V'.*

Proof. If M contains a vertex x of V' then by Lemma 3.5 the unique neighbour y of x in G' is a universal vertex of G'. Since every vertex of V' has his own private neighbour in G', it follows that V' contains only the vertex x. Let y be the private neighbour of x and let f be the parent of y in $T(G)$. Clearly, f is neither a P-node nor a N-node, for otherwise y would not be universal in G, a contradiction. Consequently, f must be a S-node.

Assume first that f is the root of $T(G)$. If f has a child that is an internal node or if f has more than two children distinct from y that are leaves, then V' contains more than one vertex, a contradiction. It follows that f contains exactly two children y and z, implying that G is prime, since it is isomorphic to a K_2. It follows that $V' = \emptyset$, a contradiction.

Thus, f cannot be the root of $T(G)$. Since f is a S-node, the parent of f in $T(G)$ is either a P-node or a S-node, implying that y is not a universal vertex of G, a contradiction. □

Claim 1 guarantees that M is a nontrivial module of G. Let f be the least common ancestor in $T(G)$ of all vertices of M and let $U = \{f_1, \ldots, f_r\}$ be the set of children of f in $T(G)$.

Claim 2. *If $leaves(f_i) \cap M \neq \emptyset$ then $leaves(f_i) \subset M$, $f_i \in U$.*

Proof. Indeed, if $f_i \in U$ is a leaf then we are done and if not, $leaves(f_i)$ is a strong module of G and since by definition M is not entirely contained into the set $leaves(f_i)$, M strictly contains $leaves(f_i)$ as claimed. □

Claim 3. *No vertex of V' is adjacent to a vertex of M.*

Proof. The result follows from the fact that every vertex of V' is of degree 1. □

Let $U' \subseteq U$ be the set of $f_i \in U$ such that $leaves(f_i) \subseteq M$. If $f_i \in U'$ is an internal node of $T(G)$, then by construction there must exist a vertex of V' adjacent to $leaves(f_i)$, contradicting Claim 3. Hence, every $f_i \in U'$ is a leaf of $T(G)$ that is, M contains only vertices whose least common ancestor f in $T(G)$ is a parent of all of them and, thus, M is entirely contained into a module of $\rho(G)$. Hence, there must exist a vertex of V' which distinguishes the vertices of M, a contradiction. It follows that G' is a prime graph as claimed. □

Proposition 3.7. *Every proper subgraph G'' of G' that contains G as induced subgraph is not prime.*

Proof. Suppose not and consider an arbitrary vertex x of $V(G') - V(G'')$. Clearly x is a vertex of V'. Let M be the module of $\rho(G)$ containing the unique neighbour, say y, of x in G. If M induces a prime graph in G, then x is the unique vertex in G' that is partial for M and consequently M is a module in G'', a contradiction.

If M is a stable or a complete set in G then the vertex y together with the vertex of M that has no neighbour in V' forms a nontrivial module in G'', a contradiction. □

Proposition 3.8. *If G' contains a subgraph $H \neq G$ isomorphic to G by an isomorphism σ, then $V(G) - V(H)$ is a stable set whose vertices have degree 1 in G' and have a private neighbour in $V(G) \cap V(H)$.*

Proof. Observe first that $V(H)$ cannot be a subset of V' since $| V' | < | V(H) |$ and, additionally, V' is a stable set while H is connected. Thus, it must be that $V(H) \cap V(G) \neq \emptyset$ and $V(H) \cap V' \neq \emptyset$. Let G_1 be the graph induced by $V(H) \cap V(G)$. Write $X = V(G) - V(H)$, $Y = V(H) - V(G)$ and $Z = V' - Y$. It is obvious that G_1 is not the empty graph, $X, Y \neq \emptyset$,$| X | = | Y |$ and $Y \subseteq V'$. Let x be a vertex of Y and y its private neighbor in G_1. By the definition of G' there must be a nontrivial module M in G containing y. Moreover M is either a stable set or a complete set or it induces a prime graph in G. We recall that if M is a stable or a complete set there is a vertex z of M having no neighbour in V' while all the others vertices of M have their private neighbor in V'. Let M'_0 be a submodule of M formed as follows:

1. if M is a stable or a complete set then M'_0 contains z and all vertices of M having a neighbour in Y
2. if M induces a prime graph in G then $M'_0 = M$.

Let Y_0 be the subset of Y which is the neighborhood of M'_0 in Y. Let M_1 be the nontrivial module of H isomorphic to M'_0 by σ. It is clear that no vertex of M_1 can belong to Y and consequently M_1 is entirely contained in G_1. Now, if M_1 is not an homogeneous set in G there must be a set of vertices X'_1 outside M_1 that distinguishes the vertices of M_1. Since M_1 is a non- trivial module of H, X'_1 must be a subset of X. Let M'_1 be a maximal submodule of M_1 in H. Let $\mu = (M_1, M'_1), \ldots, (M_l, M'_l)$ be the longest sequence of pair of sets in G_1 such that for $0 \leq i \leq l$ and $l \geq 1$ we have:

1. $M_i = \sigma(M'_{i-1})$
2. M_i is a non trivial module of H
3. M'_i is a maximal submodule of M_i which is a non trivial module of G
4. M_l is not a nontrivial module of G.

Let X_i be the set of vertices of X that distinguishes the vertices of M_i $i \in [1, l]$. It is clear that if $X_i = \emptyset$ then $M_i = M'_i$. Since $M'_0 \neq \emptyset$ (M'_0 contains the vertex y) and M'_0 is not an homogeneous set of H we deduce that $\forall i, j \in [0, l]$, $i \neq j$ $M_i \cap M_j = \emptyset$ and that $l < | V(G_1) |$. It is easy to see also that if M_i is a stable or a complete set then the number of edges between X_i and M_i is at least $| M_i - M'_i |$. Since $| Y_0 | = | M'_0 | -1$ whenever M'_0 is a stable or a complete set and $| Y_0 | = 1$ whenever M'_0 induces a prime graph, we can easily verify that the number of edges between $X_1 \cup \cdots \cup X_l$ and G_1 is at least $| Y_0 |$. Now, we proceed in an analogous way by considering the set $Y_1 = Y - Y_0$, then the set $Y_2 = Y - Y_1$ and so on until obtaining $Y_r = Y - Y_{r-1} = \emptyset$. Let X' be a minimal with respect to set inclusion subset of X such that each vertex of X' 'breaks' a module of G_1 during the previous process. We can easily see that the number of edges between X' and G_1 is at least $| Y |$.

Since G and H are isomorphic we have that $| E(G_1) | + | Y | = | E(G_1) | + | X |$. Putting together the facts that G is connected, $| X | = | Y |$ and the number of edges between X' and G_1 is at least $| Y |$, we deduce that $X' = X$ and that X is a stable set whose every vertex is of degree 1 in G. It follows that since X' is a minimal subset of X whose every vertex 'breaks' a non trivial module in G_1, every vertex of X is of degree 1 in G' and has a private neighbour in G_1, as claimed. □

Theorem 3.9. *The graph G' is a minimal prime extension of G.*

Proof. If the only subgraph of G' which is isomorphic to G is the graph G itself, then by Proposition 3.7 we deduce that G' is a minimal prime extension of G, as claimed. Assume then that there exists a subgraph $H \neq G$ of G' which is isomorphic to G and let Q be a subgraph of G' which is prime and contains H. Write $X = V(G) - V(H)$, $Y = V(H) - V(G)$ and $Z = V' - Y$. Let $\rho_1 = \{M_1, \ldots, M_s\}$ and $\rho_2 = \{M'_1, \ldots, M'_t\}$ be a bipartition of $\rho(H)$ such that every module of ρ_1 is a stable or a complete set and every module of ρ_2 induces a prime graph in H. Clearly, the set $\rho(H)$ is isomorphic to the set $\rho(G)$.

Letting R stand for the set $V(Q) - V(H)$ we have $R \subseteq X \cup Z$. Consider an arbitrary M in $\rho(H)$. If $M \in \rho_1$ then since M is either a stable or a complete set, every pair of vertices in M forms a non-trivial module in H.

Consequently, since by Proposition 3.8 every vertex of M has at most one neighbour in R and this neighbour is private, at least $| M | -1$ vertices of R are needed for 'breaking' every submodule of two vertices of M. If $M \in \rho_2$ then since M induces a prime graph in H, at least one vertex which is partial for M is needed for 'breaking' the module M in

G'. It follows that $| R | \geq \sum_{i=1}^{i=s}(| M_i | -1)+t$. But since $| X |=| Y |$ and $| Y | + | Z |=| V' | = \sum_{i=1}^{i=s}(| M_i | -1)+t$ we deduce that Q is precisely the graph G' and the result follows. □

Definition 3.10. The minimal prime extension G' of G described in this section, will be called henceforth the *basic extension* of G and will be noted $basic(G)$.

3.2. The path extension of a decomposable graph

The main goal of this subsection is to present a method for constructing an infinite number of minimal prime extensions of a connected decomposable graph G satisfying the following condition: there exists a nontrivial module M of size at least three in G such that $[M]$, the subgraph of G induced by M, is connected and non-isomorphic to a chordless path $P_k,\ k \geq 3$. Importantly, this construction constitutes the framework for characterizing all cases where a graph possesses an infinite number of minimal prime extensions.

Let us now recall the following result of Giakoumakis [10].

Proposition 3.11 (*[10]*). *Q is a minimal prime extension of a graph G if, and only if, $\overline{Q}$ is a minimal prime extension of $\overline{G}$.*

Corollary 3.12. *A graph G has an infinite number of minimal prime extensions if, and only if, $\overline{G}$ has an infinite number of minimal prime extensions.*

Corollary 3.12 allows us to restrict ourselves to the case where G is a connected graph.

Assume now that G contains a nontrivial module M of at least three vertices such that $[M]$ is connected and distinct from a $P_k,\ k \geq 3$. We may assume without loss of generality that M is maximal with respect to set inclusion, connectivity and that $[M]$ is not isomorphic to a chordless path. Let A be the neighbourhood of M in G and let B stand for its neighbourhood in the complement of G. Since G is connected, it follows that A $\neq \emptyset$.

Consider the basic extension $G' = basic(G)$ of G and denote by Q the set of vertices of $V(basic(G)) - V(G)$ such that $N(Q) \subset M$ and denote by D the vertices of $V(basic(G)) - V(G)$ such that $N(D) \subset (A \cup B)$. Put differently, Q is the set of new vertices that *break* the module M in G and any nontrivial module of $[M]$, while D is the set of new vertices that *break* any nontrivial module of $[A \cup B]$ in G. It is easy to see that $Q \cup D$ is stable and that every vertex in $Q \cup D$ has exactly one neighbour in G and this neighbour is private in the sense defined above.

Finally, write $D_A = N(A) \cap D$ and $D_B = N(B) \cap D$.

Lemma 3.13. $[M \cup Q]$ *is a prime graph.*

Proof. Assume not and let M' be a nontrivial module of $[M \cup Q]$. Observe that M' is neither entirely contained in Q (because every vertex of Q has exactly one neighbour in $basic(G)$ which is in M and it is private) nor entirely contained in M, for otherwise M' would be a module in $basic(G)$, a contradiction.

It follows that M' contains vertices from both M and Q. Let x be a vertex of $M' \cap Q$ and y its neighbour in M (this neighbour is unique in $[M \cup Q]$). Because $basic(G)$ is prime, y belongs to a non-trivial module of $[M \cup Q] \setminus x$, a contradiction. Notice that by Lemma 3.5, y must be a universal vertex of $[M \cup Q]$ and, consequently, Q must be a singleton.

Let Q' be the neighbourhood of y in $[M]$. Clearly, Q' must be a singleton for otherwise Q' would be a non-trivial module in G'. It follows that $[M]$, which by assumption is different from a chordless chain, is isomorphic to a K_2, a contradiction. Thus, $[M \cup Q]$ is a prime graph, as claimed. □

Notation 3.14. Let G^+ be the graph obtained from $basic(G)$ in the following way: $V(G^+) = V(basic(G))$ and $E(G^+) = E(basic(G)) \cup \{\{x, y\} \mid x \in Q, y \in A\}$. In other words, every vertex of Q is adjacent in G^+ to every vertex of A, which implies that $M \cup Q$ is a non-trivial module of G^+.

Lemma 3.15. *If $M \cup Q$ is not the unique nontrivial module of G^+ then G^+ contains exactly a second nontrivial module formed involving vertices of $M \cup Q \cup \{w\}$, where w is a vertex of B.*

Proof. Consider the subgraph H of G^+ induced by $(V(G^+) \setminus (M \cup Q)) \cup \{h\}$, where h is a vertex of the module $M \cup Q$. In other words, H is obtained by 'contracting' $M \cup Q$ to a single vertex. Observe that H is also a proper subgraph of $basic(G)$. If H is prime then there is nothing to prove since in this case the only non-trivial module in G^+ is the set $M \cup Q$.

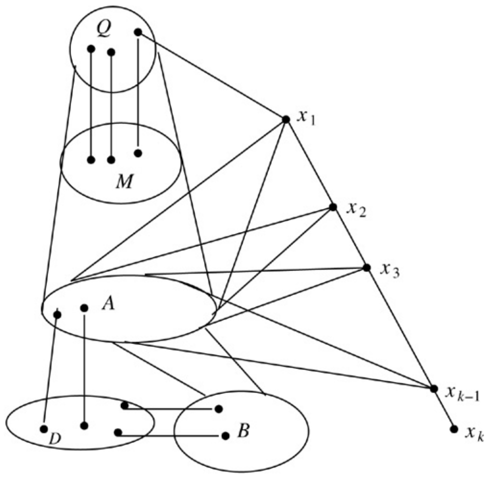

Fig. 2. The path extension of a graph $G = [M \cup A \cup B]$.

If H contains a nontrivial module $\{w, h\}$ then, again, there is nothing to prove since w cannot be adjacent to h, for otherwise it would be total for M and, consequently, M would not be maximal with respect to set inclusion, a contradiction.

Finally, assume that H contains a module M' such that $M' - \{h\}$ contains at least two vertices. Then, since no vertex of $M \cup Q$ can distinguish the vertices of $M' - \{h\}$ neither in G^+ nor in *basic*(G), this set would be also a nontrivial module of *basic*(G), a contradiction. □

Corollary 3.16. *The subgraph H of basic(G) induced by $\{x, y\} \cup A \cup B \cup D$ where $x \in M$ and y is the private neighbour of x in Q, is a prime graph.*

Proof. Let H' be the subgraph of G^+ such that $V(H') = V(H)$. Lemma 3.15 guarantees that H' contains at most two non-trivial modules $\{x, y\}$ and $\{x, y, w\}$ where w is a vertex of B nonadjacent to $\{x, y\}$. Since in H the vertex y has exactly one neighbour, namely x, the result follows. □

Notation 3.17. Let $G \otimes P_k$ be the graph obtained from G^+ in the following way: $V(G \otimes P_k) - V((G)^+)$ induces a chordless chain $P_k = x_1, \ldots, x_k$ such that

- x_1 is adjacent to exactly one vertex of Q,
- every vertex of $\{x_1, \ldots, x_{k-1}\}$ is total with respect to A and adjacent to no vertices of $M \cup B \cup D$,
- no vertex in $\{x_2, \ldots, x_{k-1}\}$ is adjacent to a vertex of Q and,
- x_k is adjacent to no vertices of G^+.

The structure of $G \otimes P_k$ is illustrated in Fig. 2.

Proposition 3.18. *The graph $G \otimes P_k$ is prime.*

Proof. Since $[M \cup Q]$ is prime (see Lemma 3.13) and since x_1 is partial with respect to $M \cup Q$, this set cannot be a module in $G \otimes P_k$ and this is the case as well whenever there exists the module $M \cup Q \cup \{w\}$ described in Lemma 3.15. It is easy to verify that the addition of the chain P_k to G^+ does not create any nontrivial modules and, hence, the resulting graph $G \otimes P_k$ must be prime, as claimed. □

Notation 3.19.

1. G^* is a minimal prime extension of G contained, as an induced subgraph, in $G \otimes P_k$;
2. G^* is said to be of type 1 if it contains P_k and of type 2 otherwise;
3. G_1 is an induced subgraph of G^* isomorphic to G by an isomorphism σ;
4. (M_1, A_1, B_1) denotes the partition of $V(G_1)$ for which M_1, A_1, B_1 are isomorphic by σ to M, A, B, respectively.

Lemma 3.20. $M_1 \cap D = \emptyset$.

Proof. Indeed, since the degree in G of every vertex in M is at least two and the degree in $G \otimes P_k$ of every vertex of D is exactly one, no vertex of D can belong to M_1. □

Proposition 3.21. *$M_1 \cup A_1$ is not entirely contained in $P_k \cup Q \cup M$.*

Proof. Assume not; since $[M_1]$ is connected and distinct from a chordless chain, M_1 is not entirely contained in $P_k \cup Q$ and hence $M_1 \cap M \neq \emptyset$. Since every vertex of M is total with respect to A and indifferent with respect to $B \cup D$, the connectedness of G_1 implies that $B_1 \cap (B \cup D) = \emptyset$ and, consequently, $V(G_1)$ is entirely contained in $P_k \cup Q \cup M$.

Since no vertex of P_k is adjacent to a vertex of M, no vertex of A_1 can be in P_k and consequently A_1 is entirely contained in $Q \cup M$.

Observe that in the graph $[P_k \cup Q \cup M]$ every vertex $x \in Q$ has either degree 1 or degree 2 precisely when x is the unique vertex of Q adjacent to a vertex of P_k. Now, since the degree of every vertex of A in G is at least $| M |$ and since M contains, by assumption, at least three vertices, no vertex of A_1 can be in Q. It follows that A_1 is entirely contained in M. Then, since M_1 is total with respect to A_1 and no vertex of P_k is adjacent to M we have that $M_1 \cap P_k = \emptyset$. Therefore since $M_1 \cap M \neq \emptyset$, M_1 contains vertices from M and Q. Furthermore, the connectedness of $[M_1]$ together with the fact that M_1 is not entirely contained in M (otherwise, $A_1 = \emptyset$) implies that $M_1 \cap Q \neq \emptyset$ and that the unique neighbour of every vertex $x \in M_1 \cap Q$ belongs to M_1. It follows that M_1 cannot be total for A_1 in G_1, a contradiction. □

Proposition 3.22. *$V(G^*) \cap A \neq \emptyset$ and $V(G^*) \cap P_k \neq \emptyset$.*

Proof. Assume first that $V(G^*) \cap A = \emptyset$. Since no vertex in $M \cup Q \cup P_k$ is adjacent to a vertex of $B \cup D$ and since M_1 is total with respect to A_1, the connectedness of G_1 and Proposition 3.21, combined, imply that $V(G_1)$ is entirely contained in $B \cup D_B$. G_1 is proper subgraph of G^* and since by assumption $V(G^*) \cap A = \emptyset$, the connectedness of G^* implies that $V(G^*)$ is contained in $B \cup D_B$, which is in contradiction with the fact that *basic*(G) is a minimal prime extension of G. It follows that $V(G^*) \cap A \neq \emptyset$, as claimed.

Assume next that $V(G^*) \cap P_k = \emptyset$. Since $V(G^*) \cap A \neq \emptyset$, the set $V(G^*) \cap (M \cup Q)$ must contain at most one vertex; otherwise G^* would contain a homogeneous set, a contradiction. It follows that G^* is isomorphic to a proper subgraph of *basic*(G), a contradiction. □

Lemma 3.23. *If G^* is of type 2 then the set $P_k \cap V(G^*)$ is*

1. *either a subchain $P' = x_i, x_{i+1}, \ldots, x_{k-1}, x_k$ with $1 < i < k$ of P_k*
2. *or $P' \cup \{x_j\}$ with $1 \leq j < i - 1$.*

Proof. The conclusion follows immediately from Proposition 3.22 and the fact that the graph G^* is prime. □

Proposition 3.24. *If G^* is of type 2 then $T = V(G^*) \cap (M \cup Q)$ contains at most one vertex.*

Proof. Assume to the contrary that $| T |> 1$. Since the vertices of T have the same neighbourhood in $V(G^*) \cap A$ and since G^* is a prime graph, there must exist a set $T' \subseteq V(G^*)$ containing T which is not a homogeneous set of G^*. It is easy to see that T' and, consequently, G^* must contain the whole chain P_k, a contradiction. □

In Lemma 3.15 we proved that in addition to the module $M \cup Q$, the graph G^+ may also contain the module $M \cup Q \cup \{w\}$ where w is a vertex of B (and hence nonadjacent to M), whose neighbourhood in G is the set A.

To simplify the notation, we shall let $M \cup Q \cup \{w\}$ refer to the set $M \cup Q$ when w does not exist. Now, Notation 3.14, Lemma 3.23 and Proposition 3.24, combined, suggest the following result.

Corollary 3.25. *The set $V(G^*) \cap (P_k \cup M \cup Q \cup \{w\})$ equals either P', or $P' \cup \{x_j\}$, or $P' \cup \{w\}$, or $P' \cup \{h\}$ where h is a vertex of $M \cup Q$.*

Proof. The conclusion follows directly by observing that we cannot have both w and h or both w and x_j or both h and x_j in $V(G^*) \cap (P_k \cup M \cup Q)$, for otherwise the set of these two vertices would be a homogeneous set in G^*, a contradiction. □

Notation 3.26. To simplify the notation, the set $V(G^*) \cap (P_k \cup M \cup Q \cup \{w\})$ will be denoted by P^*.

At this point it is easy to verify the result below which turns out to be a valuable tool in some of the proofs in the sequel of this section.

Lemma 3.27. *Let X be a subset of $G \otimes P_k$ such that $[X]$ is connected and distinct from a chordless chain. Let X_1 be the set $X \cap P_k$ and let T be the set of vertices of P_k that are total with respect to X. If $X_1 \neq \emptyset$ then $\mid T \mid \leq 2$ and, moreover:*

1. $\mid T \mid = 2$ *implies that* $[T \cup X_1]$ *is isomorphic to a* $P_3 = abc$ *with* $a, c \in T$
2. $\mid T \mid = 1$ *implies that* $\mid X_1 \mid = 1$ *or that* X_1 *contains exactly two nonadjacent vertices.*

Proposition 3.28. *If G^* is of type 2 then $M_1 \cap P^* = \emptyset$.*

Proof. Assume not. Since the graph $[M_1]$ is connected and non-isomorphic to a chordless chain, it cannot be entirely contained in P^*. Write $X_1 = M_1 \cap P^*$, $X_2 = M_1 - X_1$, $Y_1 = A_1 \cap P^*$ and $Y_2 = A_1 - Y_1$.

Assume first that $X_2 \cap B \neq \emptyset$. Since the graph induced by M_1 is connected, we have $X_2 \cap A \neq \emptyset$. Since $P^* - \{x_k\}$ is total with respect to A, x_k is indifferent with respect to A, P^* is indifferent for $B \cup D$ and M_1 is indifferent for B_1, it follows that $A_1 \cap P^* = \emptyset$ and $A_1 \cap D = \emptyset$. Consequently, A_1 must be entirely contained in $A - \{X_2 \cap A\}$, a contradiction.

Thus, X_2 must be entirely contained in A. Since no vertex of P^* is adjacent to a vertex of $B \cup D$, it must be the case that $Y_2 \cap (B \cup D) = \emptyset$ and consequently Y_2 must be entirely contained in A. This implies that $Y_1 \neq \emptyset$. Since, by assumption, $X_1 \neq \emptyset$, Lemma 3.27 guarantees that $\mid Y_1 \mid \leq 2$.

Assume that Y_1 contains two vertices, say x and y. Now, Lemma 3.27 guarantees that these vertices are nonadjacent and X_1 is a singleton. Since no vertex of P^* is adjacent to a vertex of $B \cup D$ and since $X_1 \neq \emptyset$, it must be that $Y_2 \subset A$. Since $[M_1]$ is not isomorphic to a chordless chain, X_2 contains exactly two adjacent vertices, say z and t, that is, $[M_1]$ is isomorphic to a C_3.

Let θ stand for the number of edges of the graph induced by Y_2 and let θ_1 denote the number of edges of the graph induced by A_1. We have that $\theta_1 = \theta + \mid Y_2 \mid + \mid Y_2 \mid$ (x and y are total with respect to Y_2). But the number of edges of $[A]$ is at least $\theta + \mid Y_2 \mid + \mid Y_2 \mid + 1$ (z and t are total with respect to Y_2 and $\{z, t\}$ is an edge of $[X_2]$), a contradiction. Hence, Y_1 is a singleton and either X_1 is a singleton or it contains two non-adjacent vertices. In either case, the set X_2 must contain at least two vertices and hence $\mid A_1 \mid < \mid A \mid$, a contradiction. □

Proposition 3.29. *If G^* is of type 2 then either M_1 is entirely contained in A or M_1 is entirely contained in B.*

Proof. By Lemma 3.20 and Proposition 3.28, it must be the case that M_1 is entirely contained in $A \cup B$. Assume for the sake of contradiction that $X_1, X_2 \neq \emptyset$ where $X_1 = M_1 \cap A$ and $X_2 = M_1 \cap B$. Since every vertex of P^* is total with respect to A and indifferent for $B \cup D$, no vertex of $A_1 \cup B_1$ can be in P^* and, as a consequence, $A_1 \cup B_1$ is entirely contained in $A \cup B \cup D$.

Consider the proper subgraph H of *basic*(G) induced by $\{x, y\} \cup A \cup B \cup D$ where x is a vertex of M and y is the private neighbour of x in Q. Since G_1 is a subgraph of H and since by Corollary 3.16 H is prime, H must contain an extension of G_1, contradicting the fact that H is a proper subgraph of *basic*(G). □

Theorem 3.30. *G^* is of type* 1.

Proof. Assume not. Clearly, we can write $V(G^*) \subseteq A \cup B \cup D \cup P^*$. Let R be the set of vertices of G that are partial with respect to M_1 in $V(G) - V(G_1)$. Proposition 3.29 guarantees that every vertex of M is either total or indifferent with respect to M_1. Thus, it must be that $R \cap M = \emptyset$ and, consequently, $R \subset A \cup B$.

Let $S \subset A \cup B$ be the set of vertices that are total or indifferent with respect to M_1 in the graph induced by $V(G) - V(G_1)$. Let μ be a sequence of vertices $\mu = x_0 x_1 \cdots x_s$ such that:

1. x_0 is a vertex of M or a vertex of $R \cup S$;
2. Every vertex of $\mu \setminus x_0$ belongs to $A \cup B$;
3. $x_i = \sigma(x_{i-1})$, $1 \leq i \leq s$;
4. μ is as long as possible with the above properties.

We shall call the path μ a *special path*. In this context, we denote by *init*(μ) the vertex x_0 and by *term*(μ) the vertex x_s. If *init*(μ) is a vertex of M then μ will be a special path of type 1 and if *init*(μ) is a vertex of $R \cup S$ it will be a special path of type 2. Let Γ be the set of special paths in G. We denote by Γ^1 the set of special paths of type 1 and by Γ^2 the set of special paths of type 2. Finally, we shall let $\mu(x)$ denote the special path to which vertex x belongs.

Let μ be a path of G. It is easy to see that:

1. $\sigma(term(\mu))$ belongs to $D \cup P^*$;
2. No two special paths μ_1 and μ_2 share common vertices and

$$\sigma(term(\mu_1)) \neq \sigma(term(\mu_2)).$$

Claim 1. *Let x be a vertex of a special path μ distinct from $term(\mu)$ such that the neighbourhood of x in G is not a stable set. Then neither the neighbourhood of $\sigma(x)$ in G nor the neighbourhood of $\sigma(term(\mu))$ in G_1 can be stable sets.*

Proof. If $\sigma(x)$ belongs to A the result is obvious since M belongs to G. Assume then that $\sigma(x)$ belongs to B and consider two adjacent vertices a and b of G which are adjacent to x. Since a and b are both in G, none of $\sigma(a)$ and $\sigma(b)$ can be a vertex of D. Since none of the vertices of P^* can be adjacent to $\sigma(x) \in B$, it follows that $\sigma(a)\sigma(b)$ is an edge of $[A \cup B]$ which proves that the neighbourhood of $\sigma(x)$ in G is not a stable set, as claimed.

Finally, it is clear that the neighbourhood in G_1 of $\sigma(term(\mu))$ is not a stable set if and only if the neighbourhood of x in G is not a stable set. □

In Corollary 3.25 it was shown that P^* is formed by the subchain $P' = x_i, \ldots, x_k$ of P_k such that $1 < i < k$ and possibly by a vertex of $\{x_j, w, h\}$ with $x_j \in P_k$, $j < i - 1$, $w \in B$ and $h \in M \cup Q$. In the following, we shall assume that P^* is formed by the vertices of $P' \cup \{h\}$ where h is a vertex of Q. It is an easy task to verify that the claimed result of this theorem holds when considering all the other possibilities concerning P^*.

Let $\mathcal{U} = \{M_0, \ldots, M_i, \ldots, M_q\}$ be the set satisfying the following conditions:

1. $M_0 = M$,
2. $M_i = \sigma(M_{i-1}), i = 1, \ldots, q$,
3. $M_i \subseteq A \cup B, i = 1, \ldots, q$,
4. $\mathcal{U}$ is the largest set with respect to the above properties.

Property of $\mathcal{U}$. Since $M_0 \cap V(G^*) = \emptyset$, it is easy to verify that for every choice of M_i and M_j, $1 \le i \neq j \le q$, in $\mathcal{U}$, $M_i \cap M_j = \emptyset$.

Claim 2. *Let x be a vertex of $M_i \in \mathcal{U}$, $i = 0, \ldots, q$ and let $\mu \in \Gamma^1$ be the special path containing x. Then $\sigma(term(\mu)) \in P^*$.*

Proof. The maximality of μ implies that $\sigma(term(\mu)) \in P^* \cup D$ and since the neighbourhood of any vertex of M_0 is not a stable set, the result follows from Claim 1. □

Claim 3. *$M_{q+1} = \sigma(M_q)$ contains vertices from P^* and $A \cup B$.*

Proof. If not, by the previous claim $\sigma(M_q)$ would be entirely contained in P^* and, consequently, M_q would induce a graph isomorphic to a chordless chain, a contradiction. □

The reader can easily verify the two following claims.

Claim 4. *Let M_i be a set of $\mathcal{U}$ and X a set of vertices of G outside M_i. Then $\sigma(X)$ is partial (resp. total, indifferent) with respect to $\sigma(M_i)$ if and only if X is partial (resp. total, indifferent) with respect to M_i.*

Claim 5. *The number of partial (resp. total, indifferent) vertices of $M_i \in \mathcal{U}$ in G is equal to the number of partial (resp. total, indifferent) vertices of $M_{i+1} \in \mathcal{U}$ in G_1, $i = 1, \ldots, q - 1$.*

Let $\mathcal{H} = \{H_0, \ldots, H_i, \ldots, H_r\}$ be a set satisfying the following properties.

1. H_0 is a module of M,
2. $H_i = \sigma(H_{i-1}), i = 1, \ldots, r$,
3. $H_i \subseteq A \cup B, i = 1, \ldots, r$,
4. $| H_i | > 1$, and
5. $\mathcal{H}$ is the largest set with respect to the above properties.

Claim 6. *Let H_i, $1 \leq i \leq r$, be a set in $\mathcal{H}$ and let x be a vertex of G which is partial with respect to H_i. The following two conditions are satisfied:*

1. *H_i is either entirely contained in A or else entirely contained in B;*
2. *x belongs to a path μ of type 2, i.e. $\mu \in \Gamma^2$.*

Proof. The proof is by induction on i. By Proposition 3.29, M_1 is entirely contained in A or is entirely contained in B and, hence, every vertex of M is either total or indifferent with respect to M_1 and, consequently, with respect to H_1. It follows that every partial vertex with respect to H_1 in G belongs to R and hence the result holds for $i = 1$.

Assume that the result holds for H_t, $1 \leq t < r$, and consider H_{t+1}. If H_{t+1} contains vertices from both A and B, then every vertex of P^* is partial with respect to this set in G_1. By the induction hypothesis the set, say J, of partial vertices with respect to H_t consists of vertices belonging to special paths in Γ^2. Since $\sigma(J)$ belongs to $A \cup B \cup D \cup P^*$, it follows that every vertex y of P^* must satisfy $\sigma^{-1}(y) \in J$, contradicting Claim 2. Hence H_{t+1} is entirely contained in A or entirely contained in B, as claimed.

Consider now a vertex x of G which is partial for H_{t+1}. If $x \in V(G_1) \cap V(G)$ then $x \in \Gamma^2$ for otherwise $\sigma^{-1}(x) \notin \Gamma^2$, contradicting the induction hypothesis. If $x \in V(G) - V(G_1)$ which equals $M \cup R \cup S$ then since M is total or indifferent in G for M_{t+1} and consequently for H_{t+1}, x must be a vertex of $R \cup S$ which belongs to Γ^2 as claimed. □

Observation. Since the set H_0 of $\mathcal{H}$ is not necessarily a nontrivial module of $M_0 = M$, H_0 can be M itself and, consequently, Claim 6 holds for every set M_i of $\mathcal{U}$.

Denote by Ω the set $\Omega = \{\sigma(term(\mu(x))) \mid x \in M\}$. Observe that by virtue of Claim 2, Ω is entirely contained in P^*. Let Y be the set $M_{q+1} \cap P^*$, let T be the set of vertices of P^* which are total with respect to Y and let Ω' be the set $\Omega - (T \cup Y)$.

Assume first that M contains at least four vertices. By Lemma 3.27, if $T \neq \emptyset$, $T \cup Y$ contains at most three vertices and, consequently, $\Omega' \neq \emptyset$. Let J be the set of partial vertices of M_q in G. By Claim 6, for $x \in J$ we have $\sigma(term(\mu(x))) \notin \Omega$ and the number of partial vertices of M_{q+1} in G_1 is at least $\mid J \mid + \mid \Omega' \mid$, a contradiction.

Assume now that M contains fewer than four vertices. Since $[M]$ is connected and distinct from a chordless path, $[M]$ must be isomorphic to a C_3, say abc, induced, in the obvious way, by vertices a, b, and c. If Y contains two adjacent vertices then $T = \emptyset$ and, consequently, $\mid \Omega' \mid = 1$ which implies that the number of partial vertices of M_{q+1} in G_1 is at least $\mid J \mid +1$, a contradiction.

Hence Y contains exactly one vertex, say, a. Clearly if $J = \emptyset$ or if $\sigma(J) \subseteq P^*$ then $\{b, c\}$ is a module in G_1 and, consequently, in G^* since there is no vertex of $V(G_1)$ that "breaks" the module $\{b, c\}$, a contradiction. Let b^* and c^* be, respectively, the vertices $b^* = \sigma(term(\mu(b)))$ and $c^* = \sigma(term(\mu(c)))$ which by Claim 2 belong to P^*. If at least one of these vertices is not total with respect to abc, the number of partial vertices of abc in G_1 is at least $\mid J \mid +1$, a contradiction. Hence $Y = \{b^*, c^*\}$.

Let $\theta = b_0c_0, b_1c_1, \ldots, b_rc_r$ be the longest sequence of edges in G such that $b_0 = b$, $c_0 = c$ and $b_i = \sigma(b_{i-1})$, $c_i = \sigma(c_{i-1})$, $i = 1, \ldots, r$. In other words $b_i \in \mu(b_o)$ and $c_i \in \mu(c_o)$, $i = 1, \ldots, r$. Since by Claim 6 the set of partial vertices of b_rc_r in G belong to paths in Γ^2, and b^* is nonadjacent to c^* then either $\sigma(b_r) = b^*$ or $\sigma(c_r) = c^*$ but not both.

Assume, without loss of generality, that $\sigma(b_r) = b^*$ and write $c_{r+1} = \sigma(c_r)$. Since b_rc_r is an edge of G, it follows that b^*c_{r+1} is an edge of G_1 and so c_{r+1} belongs to A. Thus, c^* distinguishes in G_1 the vertices of $\{b^*, c_{r+1}\}$. Let I denote the set of partial vertices with respect to $\{b_r, c_r\}$ in G. By Claim 6 every vertex of I belongs to a path of Γ^2. By Claim 2, $c^* \notin \sigma(I)$, a contradiction, since the number of partial vertices in G_1 of $\{b^*, c_{r+1}\}$ is larger than the number of partial vertices of $\{b_r, c_r\}$ in G. □

We are now in a position to state the main result of this section.

Theorem 3.31. *Let G be a connected graph containing a maximal nontrivial module M such that $[M]$ induces a connected graph with at least three vertices non-isomorphic to a P_k, $k \geq 3$. Then, the set of minimal prime extensions* $\mathrm{Ext}(G)$ *of G is infinite.*

Proof. The result follows from the fact that every prime extension of G obtained by the path construction $G \otimes P_k$ is of type 1 and the fact that the chain P_k is of arbitrary length. □

4. All minimal prime extensions: The finite and infinite cases

The main goal of this section is to characterize all classes of graphs whose closure under substitution closure can be defined by a finite set of forbidden subgraphs. Our result will be obtained by an exhaustive examination of the structure of non-trivial modules of a connected graph G. The cases that may arise are illustrated in the Fig. 3.

Theorem 4.1. *Let G be a connected graph which is not P_4-homogeneous and such that every module of G that induces a connected graph is isomorphic to a chordless chain. If $\overline{G}$ is disconnected then it contains exactly two connected components.*

Proof. Suppose not. If $\overline{G}$ contains at least four components, say F_1, F_2, F_3, and F_4 then $F_1 \cup F_2 \cup F_3$ is a module in G whose induced graph is not isomorphic to a chordless chain since it contains a C_3, a contradiction.

Assume, next, that $\overline{G}$ contains three connected components F_1, F_2 and F_3. If two of these components, say F_1 and F_2, are not single vertices, then the module $F_1 \cup F_2$ of G contains a C_4, a contradiction. Hence two of these components, say F_2 and F_3, are single vertices, while $| F_1 | > 2$, for otherwise G would be P_4-homogeneous, a contradiction. It is easy to see now that the nontrivial module $F_1 \cup F_2$ of G cannot be isomorphic to a chordless chain, a contradiction. □

4.1. $2P_4$-homogeneous graphs

We shall now present a class of graphs whose set of minimal prime extensions is finite.

Definition 4.2. Let G be a connected graph which is not P_4-homogeneous and having a universal vertex u. We shall call G a *pseudo-gem* if $G' = G \setminus u$ is a P_4- homogeneous graph which is isomorphic to a subgraph of a chordless chain.

The following result clarifies the structure of a pseudo-gem.

Lemma 4.3. *Let G be a pseudo-gem and u a universal vertex of G, then $G' = G \setminus u$ is one of the following types of graphs:*

1. *G' is isomorphic to a chordless chain P_l, $l \geq 5$;*
2. *G' is the disjoint union of two chordless chains P_l and P_t such that*
   - *$l = 1$ and $3 \leq t \leq 4$, or*
   - *$l = 2$ and $2 \leq t \leq 4$, or*
   - *$l = 3$ and $1 \leq t \leq 4$, or*
   - *$l = 4$ and $1 \leq t \leq 4$;*
3. *G' is isomorphic to an O_3 or is the union of an O_2 and a P_2.*

Proof. Indeed, since u is a universal vertex of G and G is not a P_4-homogeneous graph, G' cannot be a subgraph of a P_4. Also, since G' is a P_4-homogeneous graph isomorphic to a subgraph of a chordless chain, then either G' is prime and, consequently, isomorphic to a chordless chain P_l, $l \geq 5$ (Case 1 above) or every module of G' induces a subgraph of a P_4. The conclusion follows. □

Definition 4.4. Let G be a connected graph which is not P_4-homogeneous such that $\overline{G}$ contains exactly two connected components C_1 and C_2. Then, G and $\overline{G}$ are said to be $2P_4$*-homogeneous* graphs if $[C_1]$ and $[C_2]$ are subgraphs of a chordless chain and one of the following conditions holds:

- G is isomorphic to a pseudo gem
- $[C_2]$ is isomorphic to a subgraph of a P_4 and $[C_1]$ is a P_4-homogeneous graph which is $2K_2$-free.

The following result clarifies the structure of a $2P_4$-homogeneous graph.

Lemma 4.5. *Let G be a connected $2P_4$-homogeneous graph which is not isomorphic to a pseudo gem. Let C_1 and C_2 be the two connected components of $\overline{G}$ with $[C_2]$ a subgraph of a P_4. Then*

1. *$[C_2]$ is isomorphic to a P_4, $\overline{P}_3$ or a $\overline{P}_2$*
2. *$[C_1]$ is isomorphic to a $P_1 \cup P_3$ or to a $P_1 \cup P_4$ or to an O_3 or to an $O_2 \cup P_2$.*

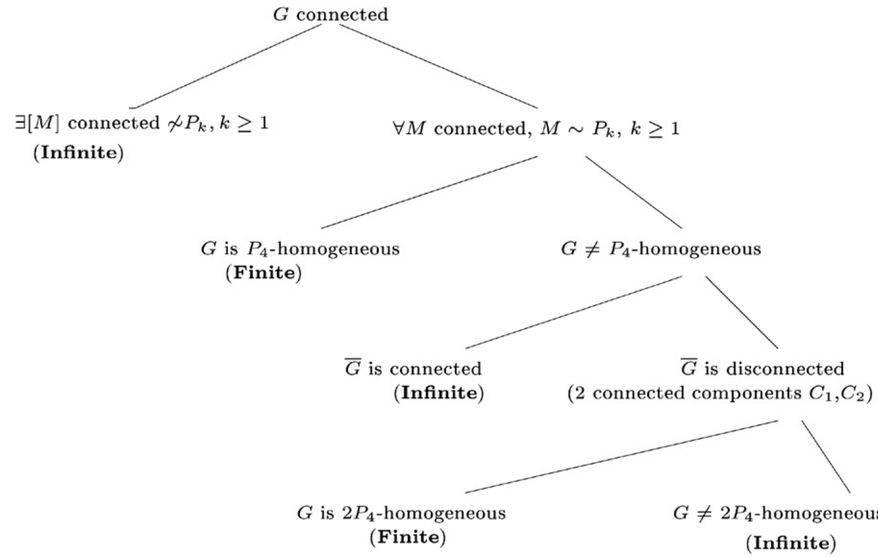

Fig. 3. Illustration of the various cases that may occur for graph G and module M. For each case we indicate whether Ext(G) is finite or infinite.

Proof. Since G is not isomorphic to a pseudo gem, C_2 is not a singleton and since $\overline{[C_2]}$ is connected and isomorphic to a subgraph of a P_4, $[C_2]$ must be isomorphic to a P_4 or to a $\overline{P}_3$ or to a $\overline{P}_2$ as claimed. Let us prove now the second assertion of the lemma. Indeed, since $[C_1]$ is $2K_2$-free, $[C_1]$ cannot contain a subgraph isomorphic to either a chordless chain $P_l, l \geq 5$ or to the disjoint union of two chordless chains which are both distinct from a P_1. Since also G is not a P_4-homogeneous graph, $[C_1]$ cannot be isomorphic to a chordless chain having at most 4 vertices. Finally, since $[C_1]$ is a P_4-homogeneous graph which is a subgraph of a chordless chain, $[C_1]$ is isomorphic to a $P_1 \cup P_3$ or to a $P_1 \cup P_4$ or to an $O_2 \cup P_2$ to an O_3 as claimed. □

We shall prove now that the set of minimal prime extensions of a $2P_4$-homogeneous graph is finite. Before this we need some preliminaries results.

Theorem 4.6 ([16]). *If W is a nontrivial module of a graph G and W induces a subgraph of a P_4 then in any minimal prime extension of G there exists a W-pseudopath P having at most two vertices. Moreover*

1. *if $[W]$ is not isomorphic to a P_4, P has exactly one vertex.*
2. *If W is isomorphic to a P_4 $abcd$ and P has two vertices, the first vertex of P is adjacent to the two middle vertices b and c of $[W]$ and misses the two other vertices a and c.*

Notation 4.7. Let Q be a minimal prime extension of its induced connected subgraph G and let W be a non trivial module of G. A W-pseudopath $P_k = (x_1, \ldots, x_k)$ in Q will be called a *strong* pseudopath if there is no homogeneous set $W' \subseteq W$ in the graph $G \cup P_k$. In other words the vertex x_1 'breaks' any nontrivial module of $[W]$.

We can derive from Theorem 4.6 the following result:

Proposition 4.8. *Let Q be a minimal prime extension of its induced connected subgraph G and let W be a nontrivial module of G such that W induces a subgraph of a P_4. Then there exists in Q a W-strong pseudopath P having at most two vertices.*

Proof. If W contains two or four vertices, the result is directly obtained from Theorem 4.6. Assume then that $[W]$ is isomorphic to a P_3 or to a $\overline{P_3}$. Let x_1 be the first vertex of P which as we recall, is partial for W. If there exists an homogeneous set $W' \subseteq W$ in the graph induced by $G \cup P$, then W' has exactly two vertices. Consequently, by Theorem 4.6 there must be a W'-pseudopath P' in Q. Clearly P' is a strong pseudopath and is also a W-pseudopath. We can easily verify that no subset of W is an homogeneous set in the graph $G \cup P'$ and hence we are done. □

Proposition 4.9. *Let Q be a minimal prime extension of its induced connected subgraph G and let W be a maximal homogeneous set of G such that W induces a subgraph of a P_4. If W is the unique maximal homogeneous set of G then $\mid V(Q) \mid < \mid V(G) + 2 \mid$.*

Proof. Indeed, by Proposition 4.8 there exists in Q a W-strong pseudopath P having at most two vertices. Clearly since M is the unique maximal homogeneous set in G the graph $G \cup P$ is prime. Since $Q \in \text{Ext}(G)$, the result follows. □

We are now in position to present the main theorem of this subsection.

Theorem 4.10. *If G is a $2P_4$-homogeneous graph, then* $\text{Ext}(G)$ *is a finite set.*

Proof. Assume that G is connected and let $\mathcal{E}$ be the set of minimal prime extensions of G. Let C_1 and C_2 be the two connected components of $\overline{G}$ and assume w.l.o.g. that $[C_2]$ is a subgraph of a P_4.

Consider the bi-partition of $\mathcal{E}$ into the following sets:

1. E_1 is the set of graphs belonging to $\text{Ext}([C_1]) \cap \mathcal{E}$;
2. E_2 is the set of graphs in $\mathcal{E} - E_1$.

Claim 1. *E_1 is finite.*

Proof. Since, by assumption, $[C_1]$ is a P_4-homogeneous graph, Theorem 2.5 guarantees that $\text{Ext}([C_1])$ is finite and, consequently, the same must hold for E_1. □

Our next task is to prove that E_2 is finite. We shall distinguish the two complementary cases: C_2 is a singleton (i.e. G is a pseudo gem) and $C_2 \neq$ singleton.

Case 1 *C_2 is a singleton*

Let Q be an arbitrary graph in E_2. Let H be a subgraph of Q isomorphic to G and $H' = H \setminus v$ where v is a universal vertex of H. Since Q is prime there must be a subgraph Q' in Q containing H' as induced subgraph such that $Q' \in \text{Ext}(H')$.

Claim 2. $|\ V(Q)\ | \leq |\ V(Q')\ | + |\ V(G)\ |$.

Proof. Assume first that the vertex v is not adjacent to all vertices of Q'. Then since v is adjacent to all vertices of H', v is partial with respect to $V(Q')$. Since Q' is a prime graph, the graph formed by Q' and the vertex v is also prime and consequently this graph is the graph Q. Assume now that the vertex v is total with respect to $V(Q')$ that is, $V(Q')$ is a nontrivial module of the graph induced by $V(Q') \cup \{v\}$. Then by Theorem 2.8 there must be in Q a $V(Q')$-pseudopath $P = y_1, \ldots, y_r$. Let $P' = y_1, \ldots, y_s$, $1 \leq s \leq r$ be the longest sequence of vertices of P inducing a chordless chain. We show now that $s <|\ V(H')\ | + 1$. Assume the contrary, then since H' is isomorphic to a subgraph of a chordless chain, P would contain a subgraph isomorphic to H'. It follows that P together with the vertex v would form a prime graph containing a minimal prime extension of H strictly contained in Q, a contradiction. We shall show now that P' is exactly P. Assume the contrary and consider the graph Q'' induced by $V(Q') \cup P' \cup y_{s+1}$. By the definition of P', y_{s+1} is adjacent to all vertices of Q' and all vertices of $P' \setminus y_s$. We can easily verify that the subgraph of Q'' formed by the vertices of H' and the vertex y_{s+1}, is isomorphic to G. Consequently, since Q'' is a prime graph it contains a minimal prime extension of G. Since Q'' is a proper subgraph of Q (Q contains also the vertex v) we obtain a contradiction. □

Since H' is a P_4-homogeneous graph, by Theorem 2.8 $\text{Ext}(H')$ is a finite set. Therefore since each minimal prime extension of H is obtained from a graph of $\text{Ext}(H')$ by adding at most $s <|\ V(G)\ | + 1$ vertices, we deduce that whenever G is isomorphic to a pseudo-gem, E_2 is a finite set.

Case 2 *$C_2 \neq$ singleton*

Let Q be an arbitrary graph in $\text{Ext}(G)$ and G' be a subgraph of Q isomorphic to G. Denote by $[C_1']$ and $[C_2']$ the subgraphs of G' isomorphic respectively to $[C_1]$ and $[C_2]$. Clearly, there is a subgraph H_1 in Q containing $[C_1']$ as proper subgraph such that $H_1 \in \text{Ext}[C_1']$.

In [17] all the minimal prime extensions for $[C_1']$ are given. The reader can easily verify that there are 10 minimal prime extensions whenever $[C_1'] \sim O_2 \cup P_2$, 3 minimal prime extensions whenever $[C_1'] \sim P_1 \cup P_3$ or $[C_1'] \sim O_3$ and 9 minimal prime extensions whenever $[C_1'] \sim P_1 \cup P_4$. Furthermore, there is exactly one of these minimal prime extensions extensions whose number of vertices is $|\ C_1'\ | + 3$ while the number of vertices of all others is $|\ C_1'\ | + 2$. Finally, there are exactly three of them having a universal vertex x with respect to $[C_1']$. In Fig. 4 of this paper we give these three minimal prime extensions using the same notations as in [17].

It is easy to see now that we have the following result:

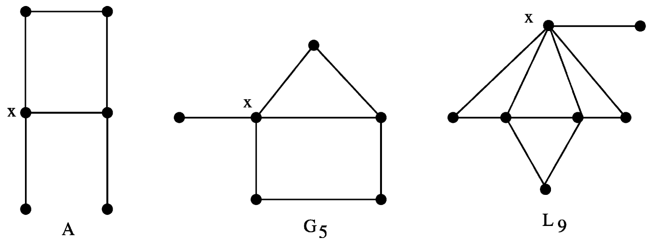

Fig. 4. Three minimal prime extensions: $A \in \text{Ext}(O_3) = \text{Ext}(P_1 \cup P_3)$, $G_5 \in \text{Ext}(O_2 \cup P_2)$, $L_9 \in \text{Ext}(P_1 \cup P_4)$.

Claim 3. $\mid V(H_1) \mid \leq \mid C'_1 \mid +3$ *and there is at most one vertex of* H_1 *which is total with respect to* C'_1 *and not adjacent to any vertex of* $V(H_1) - C'_1$.

Claim 4. *If* $V(H_1)$ *is not an homogeneous set of the graph* $H_1 \cup [C'_2]$, *then* $\mid V(Q) \mid < \mid V(G) \mid +5$.

Proof. Let M be a maximal nontrivial module of $H_1 \cup [C'_2]$. If M is entirely contained in C'_2, then since H_1 is a prime graph, M will be the unique maximal homogeneous set of $H_1 \cup [C'_2]$. The result follows by Proposition 4.9 and Claim 3. Assume then that M is not entirely included in C'_2. Then since H_1 is a prime graph, there is exactly one vertex of M, say x belonging to $V(H_1)$. Consequently $M' = M - \{x\}$ is entirely contained in C'_2. Since every vertex of C'_2 is total with respect to C'_1 we deduce that x is adjacent to every vertex of C'_1 which implies that H_1 is isomorphic to one of the graphs A, G_5, L_9 depicted in Fig. 4. Since the vertex x is not adjacent to any vertex of $V(H_1) - C'_1$, the same holds for any vertex of M'. Consequently the graph H'_1 induced by $\{V(H_1) \cup \{y\}\} - \{x\}$, with y a vertex of M', is isomorphic to H_1. It follows that $H'_1 \cup C'_2$ contains a unique maximal homogeneous set inducing a subgraph of a P_4 and the result follows by Proposition 4.9 and Claim 3. □

Assume now that $V(H_1)$ is a nontrivial module of $[V(H_1) \cup C'_2]$ which is the last case to be examined. By Proposition 4.8 there is a C'_2- strong pseudopath P in Q, having at most two vertices. If the last vertex of P, say z, is partial with respect to $V(H_1)$ then the graph $[V(H_1) \cup C'_2] \cup P$ is prime and consequently by Proposition 4.8 and Claim 3 we have that $\mid V(Q) \mid < \mid V(G) \mid +5$. Assume then that z is not partial with respect to $V(H_1)$ then since the nonneighbourhood of C'_2 in $[V(H_1) \cup C'_2]$ is the empty set, z cannot be total with respect to the set $V(H_1)$. It follows that z is indifferent with respect to $V(H_1)$. Consequently, $V(H_1)$ is the unique maximal homogeneous set of $[V(H_1) \cup C'_2] \cup P$. By Theorem 2.8 there must be in Q a $V(H_1)$- pseudopath $R_k = (x_1, \ldots, x_k)$. Since $V(H_1)$ is the unique maximal homogeneous set of $[V(H_1) \cup C'_2] \cup P$, we deduce that the graph $[V(H_1) \cup C'_2] \cup P \cup R_k$ is prime.

Our next task is to prove that $k \leq c$, where c is a constant.

Let $\mathcal{A} = \{A_1, \ldots, A_l\}$ be the largest set of chordless chains obtained from R_k in the following manner:

1. A_1 is the longest chordless chain formed by consecutive vertices of R_k and having as first vertex x_1.
2. A_i is the longest chordless chain formed by consecutive vertices of $R_k \setminus A_1 \cup \cdots \cup A_{i-1}$ and whose first vertex is the first vertex of $R_k \setminus A_1 \cup \cdots \cup A_{i-1}$, $1 < i \leq l$.

Let a_i be the first vertex and b_i be the last vertex of A_i, $i = 1, \ldots, l$. Clearly if $A_i \sim P_1$, $a_i = b_i$. If $A_i \sim P_r$ such that $r > 1$ we shall note c_i the last but one vertex of A_i, $i = 1, \ldots, l$.

Claim 5. A_i, $i = 1, \ldots, l$ *contains at most six vertices.*

Proof. Assume on the contrary that there exists $A_i \in \mathcal{A}$ having more than six vertices, then A_i contains an induced graph $[C''_1]$ isomorphic to $[C'_1]$. Since every vertex of $R_k \setminus x_k$ is total with respect to C'_2, the graph $[C''_1] \cup [C'_2]$ is isomorphic to G. But since the graph $A_i \cup \cdots \cup A_l \cup P \cup [C'_2]$ is prime, it contains a subgraph isomorphic to G as induced subgraph and is strictly contained in Q, we obtain a contradiction. □

We shall show now that $l < 10$. Assume the contrary and consider the set $\mathcal{A}' = \mathcal{A} \setminus A_1 \cup A_l$.

Claim 6. *There are not in* $\mathcal{A}'$ *three chains* A_i, A_{i+1}, A_{i+2}, $1 < i < l - 3$ *such that each one is isomorphic to a* P_1.

Proof. If not, A_i, A_{i+1}, A_{i+2} together with the first vertex of A_{i+3} would induce a copath containing a subgraph isomorphic to $[C'_2]$. Since every vertex of this copath is total with respect to H_1, the graph $H_1 \cup R_k$ which is prime and is strictly contained in Q, would contain an induced subgraph isomorphic to G, a contradiction. □

Since by assumption $l > 9$, Claim 6 implies that there exists three chordless chains in $\mathcal{A}'$, A_r,A_s and A_t, $r < s < t$ such that none of them is isomorphic to a P_1. Assume w.l.o.g. that s is as small as possible that is, there is no A_i, $r < i < s$ which is not isomorphic to a P_1. If $s = r + 1$ then the set of vertices $\{c_r, b_r, c_s, b_s, b_t\}$ induces a subgraph isomorphic to a $P_1 \cup P_4$. If $s > r + 1$ then since by assumption s is as small as possible, A_{s-1} is isomorphic to a P_1. It follows that the set of vertices $\{a_r, a_{s-1}, c_s, b_s, b_t\}$ induces a subgraph isomorphic to a $P_1 \cup P_4$. Consequently in both cases there exists in R_k a subgraph isomorphic to $[C_1']$. It follows that the graph $R_k \cup [C_2'] \cup P$ which is prime and is strictly contained in Q, contains an induced subgraph isomorphic to G a contradiction.

Since $| V(H_1) | \leq | C_1' | +3$ (Claim 3), P has at most 2 vertices (Proposition 4.9) and $l < 10$, it is easy to see now that every minimal prime extension of G has $| V(G) | +c$ vertices, where c is a constant and this completes the proof of the theorem. □

4.2. The main theorem

Theorem 4.11. *Given a decomposable graph G, $\mathrm{Ext}(G)$ is finite if and only if G is P_4-homogeneous or a $2P_4$-homogeneous graph.*

Proof. The 'if' part follows from Theorem 2.5 and Theorem 4.10, combined.

We shall now turn to the 'only if' part. For this purpose, assume that G is connected and non-isomorphic to a P_4-homogeneous or to a $2P_4$-homogeneous graph. Our goal is to show that $\mathrm{Ext}(G)$ is an infinite set for the different cases illustrated in Fig. 3.

If there exists a module M in G such that $[M]$ is connected and nonisomorphic to a chordless chain P_k, $k \geq 3$, the conclusion follows from Theorem 3.31.

Assume, next, that every module M of G that induces a connected graph is isomorphic to a chordless chain P_k, $k \geq 1$. If $\overline{G}$ is connected, then since $\overline{G}$ is not P_4-homogeneous, it must contain a module M maximal with respect to set inclusion and non-isomorphic to a chordless chain P_r with $r > 2$. By Theorem 3.31, $\mathrm{Ext}(\overline{G})$ is infinite and by Proposition 3.11 this must also be the case for $\mathrm{Ext}(G)$.

Therefore, in the remainder of the proof we assume that

$$\overline{G} \text{ is disconnected.}$$

Recall that by Theorem 4.1, $\overline{G}$ contains exactly two connected components C_1 and C_2. We shall distinguish the two complementary cases:

1. Neither C_1 nor C_2 induce a chordless chain in $\overline{G}$.
2. At least one of C_1 or C_2 induces a chordless chain in $\overline{G}$.

Case 1 *Neither C_1 nor C_2 induce a chordless chain in $\overline{G}$.*

Let Q_1 (resp. Q_2) be the set of new vertices that need to be added to $\overline{[C_1]}$ (resp. to $\overline{[C_2]}$) in order to obtain its basic extension U_1 (resp. U_2).

We construct a connected graph H by joining U_1 with U_2 with a chordless chain $P_r = x_1x_2\ldots x_r$ with $r > 2 | V(G) |$, and such that

- x_1 is adjacent to all but one vertex of U_1,
- x_r is adjacent to all but one vertices of U_2, and
- no vertex of $\{x_2, \ldots, x_{r-1}]$ is adjacent to any vertex of $V(U_1) \cup V(U_2)$.

Clearly the graph H constructed above is prime and, therefore, it contains a minimal prime extension H' of $\overline{G}$. We claim that:

$$H' \text{ contains the whole chain } P_r. \tag{1}$$

In order to argue for (1) observe that neither U_1 nor U_2 can be an extension of $\overline{G}$. Indeed, assume that one of U_1 or U_2, say U_1, is an extension of $\overline{G}$ and let F_2, F_1 be two vertex-disjoint subgraphs of U_1 isomorphic, respectively, to $\overline{[C_2]}$ and to $\overline{[C_1]}$. Since $\overline{[C_2]}$ is connected its vertex set cannot be entirely included in the stable set Q_1. Therefore, since there is no edge between $\overline{[C_2]}$ and $\overline{[C_1]}$ in $\overline{G}$, we can easily deduce that $F_1 \neq C_1$. Let F_2' be the set $V(F_2) \cap C_1$ and let F_1' be the set $V(F_1) \cap C_1$. By Proposition 3.8 F_2' is a stable set and every vertex of this set has his private neighbour in F_1' which contradicts the fact that there is no edge between F_1 and F_2. Since neither $\overline{[C_1]}$ nor $\overline{[C_2]}$ is isomorphic to a chordless chain, the vertex-set of any induced copy of $\overline{[C_1]}$ in H' is formed by a subset of C_1 and some of the vertices of the subchain $P_1 = x_1 \dots x_t$, $t < r/2$ and the vertex set of any induced copy of $\overline{[C_2]}$ in H' is formed by a subset of C_2 and some of the vertices of the subchain $P_2 = x_z \dots x_r$ with $z > r/2$. Since H' must be connected, it contains the whole chain P_r. Since P_r has arbitrary length larger than $| V(G) |$ the proof of (1) is complete.

Case 2 *At least one of C_1 or C_2 induces a chordless chain in $\overline{G}$.*

Assume, without loss of generality, that C_2 induces a chordless chain in $\overline{G}$. Clearly, $[C_2]$ is isomorphic to a P_1 or to a $\overline{P_2}$ or to a $\overline{P_3}$ or to a P_4.

Claim 1. *If $[C_1]$ is not a P_4-homogeneous graph then* $\mathrm{Ext}(G)$ *is an infinite set.*

Proof. We shall prove that $\mathrm{Ext}(\overline{G})$ is an infinite set. Since by assumption $[C_1]$ is not a P_4 - homogeneous graph, $[C_1]$ is a disjoint union of a set of chordless chains $R_1, R_2, \dots, R_l$, $l > 1$. Assume w.l.o.g. that $length(R_{i-1}) \geq length(R_i)$, $1 < i \leq l$. Consider now the graph $R_1 \cup \dots \cup R_{l-1}$; clearly the set of vertices of this graph forms a nontrivial module M inducing in $[C_1]$ a subgraph different from a subgraph of a P_4. More precisely $[M]$ is either isomorphic to a chordless chain P_r, $r > 4$ or is the disjoint union of a set of chordless chains. It follows that M induces in $\overline{[C_1]}$ a connected graph different from a chordless chain and hence we can use our construction in Section 3.2 for obtaining a graph H isomorphic to a path extension $\overline{[C_1]} \otimes P_k$ where $P_k = x_1 \dots x_k$ is a chordless path of k vertices. We assume w.l.o.g. that P_k contains at least 4 vertices. Let H^* be a minimal prime extension of $\overline{[C_1]}$ contained in H.

Fact 1. *If H contains a subgraph isomorphic to $\overline{G}$, then* $\mathrm{Ext}(G)$ *is an infinite set.*

Proof. Indeed, by Proposition 3.18 H is a prime graph and consequently H contains a minimal prime extension F of $\overline{G}$. We claim that F contains the whole chain P_k. Indeed, in Theorem 3.30 we proved that any minimal prime extension of $\overline{[C_1]}$ contained in H, contains the whole chain P_k. Since F contains a minimal prime extension of $\overline{[C_1]}$ as induced subgraph, F contains the whole chain P_k, as claimed. Now, since P_k is of arbitrary length, the result follows. □

Fact 2. *If C_2 is a singleton then* $\mathrm{Ext}(G)$ *is an infinite set.*

Proof. Since in H the vertex x_k is not adjacent to any vertex of $\overline{[C_1]}$, H contains an induced subgraph isomorphic to $\overline{G}$. The result follows from Fact 1. □

We assume then in the following that H does not contain a subgraph isomorphic to $\overline{G}$ and that $\overline{C_2}$ is not isomorphic to a P_1. Let y be one of the extremities of the chordless chain $\overline{[C_2]}$ and let Q be the graph whose vertex set is $V(H^*) \cup C_2 \cup \{v\}$, where v is a new vertex; the edge set of Q is $E(Q) = E(H^*) \cup E(\overline{[C_2]}) \cup \{vz\}$, with $z \neq x_k, y$. In other words Q is obtained by adding edges between a new vertex v with all vertices of H^* except the vertex x_k and all vertices of the chain $\overline{[C_2]}$ except one of its extremities y.

It is easy to see that Q is a prime graph containing a subgraph isomorphic to $\overline{G}$.

Let Q' be a prime extension of $\overline{G}$ contained in Q. Let G' a subgraph of Q' isomorphic to G, and let U_1 and U_2 be two subgraphs of G' isomorphic to $\overline{[C_1]}$ and respectively to $\overline{[C_2]}$. Since we assumed that H does not contain a subgraph isomorphic to $\overline{G}$, G' is not a subgraph of H^*.

Fact 3. *G' does not contain the vertex v.*

Proof. Assume first that v belongs to U_2 then since U_1 contains more than two vertices and v misses at most two vertices in Q, there would be an edge between U_1 and U_2, a contradiction. Assume now that v belongs to U_1, then if C_2 contains more than two vertices there would be an edge between U_1 and U_2, a contradiction. Assume now that C_2 contains exactly two vertices. Since there is no edge between U_1 and U_2, U_2 would be formed by the two non adjacent vertices x_k and y which contradicts the connectedness of U_2. Since we assumed that U_2 is not isomorphic to a P_1, we deduce that G' does not contain the vertex v, as claimed. □

Fact 3 implies that since U_1 is not isomorphic to a subgraph of a P_4 and U_1 is connected, U_1 is entirely contained in H^*.

Fact 4. *Q' contains H^* as induced subgraph.*

Proof. Assume the contrary, then since any subgraph of Q' isomorphic to $\overline{[C_1]}$ is contained in H^*, there must be a subgraph H_1 of Q' strictly contained in H^* which is connected, it contains a subgraph isomorphic to $\overline{[C_1]}$ and is maximal with respect to set inclusion and the above properties. The maximality of H_1 implies that it contains the vertex x_k for otherwise the neighbourhood of H_1 in Q' would be the vertex v and consequently $V(H_1)$ would be a nontrivial module of Q', a contradiction. Also, H_1 can not be a prime graph for otherwise H^* would not be a minimal prime extension of $\overline{[C_1]}$, a contradiction. Let W be a maximal nontrivial module of H_1, then if W does not contain the vertex x_k, W would be a non trivial module of Q', a contradiction. But since x_k has a unique neighbour in H^* which is the vertex x_{k-1}, W can not contain the vertex x_{k-1}, for otherwise a neighbour of W in H_1 would not be adjacent to x_k, a contradiction. It follows that the neighbourhood of W in H_1 is the vertex x_{k-1}. Now, W must be formed by x_k and exactly one other vertex, for otherwise $W \setminus \{x_k\}$ would be a non trivial module in Q', a contradiction. Let z be the second vertex of W. Since x_{k-1} is the neighbourhood of W in H_1, z is either the vertex x_{k-2} or a vertex belonging to the neighbourhood of M in $\overline{[C_1]}$. Assume first that z is the vertex x_{k-2}, then the vertex x_{k-3} which distinguishes the vertices x_{k-2} and x_k does not belong to H_1. It follows that since H_1 contains more than three vertices ($\overline{[C_1]}$ contains at least four vertices), there must be a neighbour, say u, of x_{k-1} belonging to H_1. The vertex u must belong to $V(R_l)$ which is the neighbourhood of M in $\overline{[C_1]}$ and we obtain a contradiction since x_{k-2} would be also adjacent to u. Consequently, z belongs to R_l. It follows that no vertex of $M \cup \{x_1, \ldots, x_{k-2}\}$ belongs to H_1 since each one of this vertices is adjacent to z and not adjacent to x_k. It is easy to see now that R_l cannot be isomorphic to a P_1 for otherwise the graph H_1 would be isomorphic to a P_3, a contradiction. Consequently, R_l contains at least two vertices and hence since H_1 is assumed to be connected, the only vertices of H that can belong to H_1 are x_k, x_{k-1}, the vertices of R_l and the vertex w which is the only vertex needed in the basic extension of $\overline{[C_1]}$ for 'breaking' the nontrivial module $V(R_l)$ of $\overline{[C_1]}$. But since R_l contains at least two vertices, it is easy to see that M must contain at least four vertices and we obtain a contradiction. □

Since H^* contains the whole chain P_k which is of arbitrary length, we deduce that Ext(G) is an infinite set, as claimed. □

Assume now that $[C_1]$ is a P_4-homogeneous graph then since G is not a $2P_4$-homogeneous graph, $[C_2]$ cannot be a singleton, for otherwise G would be a pseudo-gem, a contradiction. It follows that $[C_1]$ cannot be $2K_2$-free.

Let L be the shortest chordless chain containing $[C_1]$ as induced subgraph and G_1 be the graph obtained by adding all missing edges between L and $[C_2]$. Let H be a prime graph containing G obtained by adding to G_1 an L-pseudopath $P_k = x_1, x_2, \ldots, x_k$, $k \geq 4$ and k even, that satisfies:

1. for $i = 1, \ldots, k-1$, if i is odd then x_i, x_{i+1} is an edge of H and if i is even then x_i, x_{i+1} is a non edge of H
2. x_1 is adjacent to all but one vertices of $[C_1]$
3. the vertex x_k is adjacent to $[C_2]$ as follows:
   (a) If $[C_2]$ is isomorphic to a $\overline{P_2} = ab$ then x_k is adjacent to a and not adjacent to b
   (b) If $[C_2]$ is isomorphic to a $\overline{P_3} = abc$ with bc the unique edge of $[C_2]$, x_k is adjacent to a and b and not adjacent to c
   (c) If $[C_2]$ is isomorphic to a $P_4 = abcd$, x_k is adjacent to the middle vertices b and c of $[C_2]$ and not adjacent to a and d.

Clearly, since P_k is a L-pseudopath, every vertex of $P \setminus x_k$ is adjacent to every vertex of $L \cup [C_2]$, x_{2i+1} is adjacent to any vertex x_j, $j < 2i$, and x_{2i} is not adjacent to any vertex x_j, $j < 2i - 1$, $i = 1, \ldots, \frac{k}{2}$.

Claim 2. *P_k is $2K_2$-free and there are not two nonadjacent vertices in P_k that are both adjacent to a vertex of $[C_1]$.*

Proof. Indeed assume first that there exists a $2K_2$ in P_k. Let $x_r x_s$ and $x_t x_v$ be the two edges of this $2K_2$ then one of r, s, say r and one of t, v, say t is odd and we obtain a contradiction since x_r is adjacent to x_t. Let x_j and x_l be two nonadjacent vertices of P_k, then at least one of j, l, say j is even and consequently x_j is not adjacent to any vertex of P_k, a contradiction. □

Since H is a prime graph it contains a minimal prime extension H' of G'. We claim that H' contains entirely the L-pseudopath P_k. Indeed, since $[C_1]$ contains a $2K_2$ by Claim 2 there cannot be a subgraph of P_k isomorphic to $[C_1]$ and hence $[C_1]$ is not entirely contained into P_k. From the other hand since $[C_2]$ contains two nonadjacent vertices, Claim 2 implies that $[C_2]$ cannot be a subgraph of P_k. The reader can easily verify now that H' must contain the whole pseudopath P_k. Since this pseudopath is of arbitrary length, we deduce the claimed result. □

5. Concluding remarks

First of all we may observe that the proofs given in the previous section suggest a general method for enumerating in the finite case all minimal prime extensions of a graph G. Consider for example the case where G has two connected components, one being an isolated vertex and the second inducing a P_4 (i.e. G is the complementary graph of a *Gem*). It is easy to see that from the different cases examined in the proof of the Lemma 4.3, we can derive all extensions of G.

It must be pointed out here that, since no general result had been available concerning the set of minimal prime extensions in the finite case, it was necessary for obtaining this set to examine separately each particular case of the graphs under consideration — see, for example, [6,17]. Hence, now it becomes interesting to enumerate by a systematic way derived from the results given in this paper, all the minimal prime extensions in the finite case. If the number of minimal prime extensions is large, instead of exhibiting all these extensions we could propose a simple algorithm for it. In this way for instance, we could characterize all the new classes of perfect graphs which are the substitution-composite of subclasses of P_4-homogeneous and $2P_4$-homogeneous graphs already been showed to be perfect.

It would also be interesting to search for different methods generating infinite sets of extensions which could be for instance beneficial to a better understanding of the structure of prime graphs that, to this day, are not well understood. Both of these directions are for us an exciting area for further work.

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