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Local L_p-Saturation of Positive Linear Convolution Operators

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Local L_p-saturation of positive linear convolution operators is investigated. Results are obtained for two important classes of operators previously studied by Bojanic, DeVore, Korovkin and the authors.

1. INTRODUCTION

Let $I = [0, r]$, $1 \leqslant p < \infty$, and denote by $L_p(I)$ the space of real valued pth power Lebesgue integrable functions on I. Let $\{H_n(t)\}$ be a sequence of non-negative, even and continuous functions on $[-r, r]$ such that

$$\int_{-r}^{r} H_n(t)\, dt = 1, \qquad n = 1, 2, \ldots, \tag{1.1}$$

$$\int_{-r}^{r} t^2 H_n(t)\, dt \equiv \mu_n \to 0, \qquad n \to \infty, \tag{1.2}$$

and

$$\int_{-r}^{r} t^4 H_n(t)\, dt = O(\mu_n^2), \qquad n \to \infty. \tag{1.3}$$

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For $f \in L_p(I)$ and $0 \leqslant x \leqslant r$, define the convolution operators

$$K_n(f(t), x) = \int_0^r f(t)\, H_n(t-x)\, dt, \qquad n = 1, 2,\ldots. \tag{1.4}$$

These linear operators map $L_p(I)$ into $L_p(I)$ and are positive on I. There are two important examples of (1.4).

EXAMPLE 1.1 (Korovkin operators). Let ϕ be a non-negative, even and continuous function on $[-r, r]$, decreasing on $[0, r]$ and such that $\phi(0) = 1$. Let

$$K_n(f, x) = \rho_n \int_0^r f(t)(\phi(t-x))^n\, dt, \tag{1.5}$$

where

$$\rho_n^{-1} = 2 \int_0^r (\phi(t))^n\, dt. \tag{1.6}$$

This approximating method was introduced by Korovkin [5], who used it for the uniform approximation of continuous functions. Later Bojanic and Shisha [2] showed that

$$\lim_{t \to 0^+} \frac{1-\phi(t)}{t^\lambda} = c \tag{1.7}$$

for some positive constants λ and c implies

$$\mu_n = O(n^{-2/\lambda}), \qquad n \to \infty, \tag{1.8}$$

and

$$\sup_{0 \leqslant x \leqslant r} K_n((t-x)^4, x) = O(n^{-4/\lambda}), \qquad n \to \infty. \tag{1.9}$$

Numerous important special cases of (1.5) that satisfy (1.7) are listed in [2]. Conditions (1.1)–(1.3) follow from (1.6), (1.8) and (1.9).

EXAMPLE 1.2. (Bojanic–DeVore operators). Let $\{P_n\}$ be a sequence of orthogonal polynomials on $[-1, 1]$ whose weight function w is even, non-negative and satisfies

$$0 < m \leqslant w(x) \leqslant M < \infty, \qquad x \in [-r, r], \qquad 0 < r \leqslant 1.$$

Let

$$R_n(x) = c_n \left[\frac{P_{2n}(x)}{(x^2 - \alpha_{2n}^2)(x^2 - \alpha_{2n-1}^2)} \right]^2,$$

where α_{2n}, α_{2n-1} are the smallest positive zeros of P_{2n} and $c_n > 0$ is chosen so that

$$\int_{-r}^{r} R_n(x)\, dx = 1, \qquad n = 1, 2,\dots . \tag{1.10}$$

Let

$$K_n(f, x) = \int_0^r f(t)\, R_n(t - x)\, dt. \tag{1.11}$$

Bojanic [1] and DeVore [3] have shown that

$$\mu_n = O(n^{-2}), \qquad (n \to \infty), \tag{1.12}$$

and

$$\sup_{0 \leqslant x \leqslant r} K_n((t - x)^4, x) = O(n^{-4}), \qquad n \to \infty. \tag{1.13}$$

Conditions (1.1)–(1.3) follow from (1.10), (1.12) and (1.13).

The degree of L_p-approximation with (1.4) was investigated in [10, 11]. In this work we consider local saturation of (1.4). Global saturation for the operators (1.5) is studied in [10]. L_p saturation for other types of operators has been studied by Ditzian and May [4], Maier [6], Müller and Maier [8] and Reimenschneider [9].

2. Direct Results

In the sequel, we let $e_i(x) = x^i$ for $i = 0, 1, 2,\dots$.

LEMMA 2.1. *For $n = 1, 2,\dots$, we have*

$$\int_0^r H_n(t - x)\, dt = K_n(e_0, x) \leqslant 1, \qquad 0 \leqslant x \leqslant r, \tag{2.1}$$

$$\int_0^r H_n(t - x)\, dx \leqslant 1, \qquad 0 \leqslant t \leqslant r, \tag{2.2}$$

$$|K_n((t-x),x)| = O(\mu_n), \qquad n\to\infty,\ 0<\delta\leqslant x\leqslant r-\delta<r, \tag{2.3}$$

$$|K_n(e_0,x)-1| = O(\mu_n), \qquad n\to\infty,\ 0\leqslant\delta\leqslant x\leqslant r-\delta<r, \tag{2.4}$$

$$K_n((t-x)^2,x)\leqslant\mu_n, \qquad 0\leqslant x\leqslant r, \tag{2.5}$$

$$K_n(|t-x|,x)\leqslant\mu_n^{1/2}, \qquad 0\leqslant x\leqslant r, \tag{2.6}$$

and

$$\|K_n(e_i)-e_i\|_{L_p[0,r]} = O(\mu_n^{1/2p}), \qquad n\to\infty,\ i=0,1. \tag{2.7}$$

Proof. Equations (2.1), (2.2) and (2.5)–(2.7) were established in [11]. Using (1.2), (1.3) and the method of [11, Lemma 1], it is easy to obtain (2.3) and (2.4).

Let $0\leqslant a<a_1<b_1<b\leqslant r$ and define

$$\begin{aligned}\chi(t) &= 0, \quad t\in[a,b]\\ &= 1, \quad t\notin[a,b].\end{aligned}$$

LEMMA 2.2. *If* $f\in L_p[0,r]$, *then* $\|K_n(\chi f)\|_{L_p[a_1,b_1]} = O(\mu_n)$ $(n\to\infty)$.

Proof. For $p=1$ we have

$$\begin{aligned}\|K_n(\chi f)\|_{L_1[a_1,b_1]} &= \int_{a_1}^{b_1}\left|\int_0^r \chi(t)\,f(t)\,H_n(t-x)\,dt\right|dx\\ &\leqslant \int_0^r \chi(t)\,|f(t)|\int_{a_1}^{b_1} H_n(t-x)\,dx\,dt\\ &\leqslant \|f\|_{L_1[0,r]}\left(\sup_{t\notin[a,b]}\int_{a_1}^{b_1} H_n(t-x)\,dx\right).\end{aligned}$$

If $t\notin[a,b]$ and $x\in[a_1,b_1]$, then $|t-x|\geqslant\min(a_1-a,b-b_1)>0$. Using (1.2) and (1.3) we obtain

$$\sup_{t\notin[a,b]}\int_{a_1}^{b_1} H_n(t-x)\,dx = o(\mu_n), \qquad n\to\infty.$$

Assume $1<p<\infty$, $a_1\leqslant x\leqslant b_1$, and $p+q=pq$. By (2.1) and Hölder's inequality for positive linear operators,

$$|K_n(\chi f,x)|\leqslant K_n(\chi,x)^{1/q}(K_n(\chi\,|f|^p,x)^{1/p}).$$

Hence

$$\|K_n(\chi f, x)\|_{L_p[a_1,b_1]} \leqslant \Big(\sup_{a_1 \leqslant x \leqslant b_1} (K_n(\chi, x))^{1/q}\Big) \cdot \left(\int_0^r \int_{a_1}^{b_1} \chi(t)\,|f(t)|^p H_n(t-x)\,dx\,dt\right)^{1/p}$$

$$\leqslant \Big(\sup_{a_1 \leqslant x \leqslant b_1} (K_n(\chi, x))^{1/q}\Big)\,\|f\|_{L_p[0,r]} \times \left(\sup_{t \notin [a,b]} \int_{a_1}^{b_1} H_n(t-x)\,dx\right)^{1/p}.$$

For $a_1 \leqslant x \leqslant b_1$ and $\delta = \min(a_1 - a, b - b_1)$ we have

$$K_n(\chi, x) = \int_0^r \chi(t)\,H_n(t-x)\,dt$$

$$\leqslant \frac{1}{\delta^4} \int_0^r (t-x)^4 H_n(t-x)\,dt$$

$$= O(\mu_n^2), \qquad n \to \infty.$$

For $t \notin [a, b]$,

$$\int_{a_1}^{b_1} H_n(t-x)\,dx \leqslant \frac{1}{\delta^4} \int_0^r (t-x)^4 H_n(t-x)\,dx$$

$$= O(\mu_n^2), \qquad n \to \infty.$$

Thus

$$\|K_n(\chi f, x)\|_{L_p[a_1,b_1]} \leqslant \|f\|_{L_p[0,r]} \cdot O(\mu_n^{2/q}) \cdot O(\mu_n^{2/p})$$

$$= \|f\|_{L_p[0,r]} \cdot O(\mu_n^2)$$

$$= o(\mu_n), \qquad n \to \infty.$$

This completes the proof of Lemma 2.2.

For $1 < p < \infty$ and $0 \leqslant a < b \leqslant r$, we denote by $L_p^2[a, b]$ the space of those functions f such that $f \in L_p[0, r]$, f' is absolutely continuous on $[a, b]$ and $f'' \in L_p[a, b]$. $BV[a, b]$ denotes the space of functions of bounded variation on $[a, b]$.

THEOREM 2.3. *Let $f \in L_p[0, r]$, $1 \leqslant p < \infty$ and $0 < a < a_1 < b_1 < b < r$. Then*

(i) *if* $1 < p < \infty$, *then* $f \in L_p^2[a,b]$ *implies* $\|K_n(f) - f\|_{L_p[a_1,b_1]} = O(\mu_n)$, $n \to \infty$;

(ii) *if* $p = 1$, *then* $f' \in BV[a,b]$ *implies* $\|K_n(f) - f\|_{L_p[a_1,b_1]} = O(\mu_n)$, $n \to \infty$;

(iii) *if* $1 \leqslant p < \infty$, *then* f *linear on* $[a,b]$ *implies* $\|K_n(f) - f\|_{L_p[a_1,b_1]} = o(\mu_n)$, $n \to \infty$.

Proof. (i) Let χ be defined as in Lemma 2.2 and let $\chi_1(t) = 1 - \chi(t)$. Then

$$\|K_n(f) - f\|_{L_p[a_1,b_1]} \leqslant \|K_n(\chi f)\|_{L_p[a_1,b_1]} + \|K_n(\chi_1 f) - (\chi_1 f)\|_{L_p[a_1,b_1]}.$$

By Lemma 2.2,

$$\|K_n(\chi f)\|_{L_p[a_1,b_1]} = O(\mu_n), \qquad n \to \infty.$$

Assume $1 < p < \infty$ and $f \in L_p^2[a,b]$. Then

$$\begin{aligned}
&\|K_n((\chi_1 f)(t) - (\chi_1 f)(x), x)\|_{L_p[a_1,b_1]} \\
&\quad = \left(\int_{a_1}^{b_1} \left|\int_0^r (\chi_1(t) f(t) - f(x))\, H_n(t-x)\, dt\right|^p dx\right)^{1/p} \\
&\quad \leqslant \left(\int_{a_1}^{b_1} \left|\int_a^b (f(t) - f(x))\, H_n(t-x)\, dt\right| dx\right)^{1/p} \\
&\qquad + \left(\int_{a_1}^{b_1} |f(x)|^p \left(\int_0^a H_n(t-x)\, dt\right)^p dx\right)^{1/p} \\
&\qquad + \left(\int_{a_1}^{b_1} |f(x)|^p \left(\int_b^r H_n(t-x)\, dt\right)^p dx\right)^{1/p}.
\end{aligned}$$

By (1.2) and (1.3), the last two terms are $o(\mu_n)$, $n \to \infty$, since $|t - x| \geqslant \delta = \min(a_1 - a, b - b_1) > 0$. The first term is dominated by

$$\begin{aligned}
&\left(\int_{a_1}^{b_1} \left|\int_a^b f'(x)(t-x)\, H_n(t-x)\, dt\right|^p dx\right)^{1/p} \\
&\quad + \left(\int_{a_1}^{b_1} \left|\int_a^b \left(\int_x^t (t-u)\, f''(u)\, du\right) H_n(t-x)\, dt\right|^p dx\right)^{1/p},
\end{aligned}$$

since $f(t) - f(x) = f'(x)(t-x) + \int_x^t (t-u) f''(u)\, du$. Using (1.2) and (1.3), it can be shown that the first term is dominated by

$$\sup_{a_1 \leqslant x \leqslant b_1} |f'(x)| \cdot \|K_n(t-x, x)\|_{L_p[a_1,b_1]} + o(\mu_n) \qquad = o(\mu_n), \qquad n \to \infty.$$

Now let

$$\theta(f'', x) = \sup_{\substack{a \leqslant t \leqslant b \\ t \neq x}} \frac{1}{(t-x)} \int_x^t |(f''(u)| \, du, \qquad a_1 \leqslant x \leqslant b_1,$$

denote the Hardy–Littlewood majorant of f'' at x. Since $f'' \in L_p[a, b]$ and $p > 1$,

$$\| \theta(f'', x)\|_{L_p[a,b]} \leqslant A_p \|f''\|_{L_p[a,b]},$$

where $A_p > 0$ depends only on p [12, Theorem 13.15]. Hence

$$\left(\int_{a_1}^{b_1} \left| \int_a^b \left(\int_x^t (t-u) f''(u) \, du \right) H_n(t-x) \, dt \right|^p dx \right)^{1/p}$$

$$\leqslant \left(\int_{a_1}^{b_1} |\theta(f'', x)|^p \left(\int_a^b (t-x)^2 H_n(t-x) \, dt \right)^p dx \right)^{1/p}$$

$$= O(\mu_n), \qquad n \to \infty.$$

Finally,

$$\|K_n(\chi_1 f) - (\chi_1 f)\|_{L_p[a_1,b_1]} \leqslant \|K_n((\chi_1 f)(t) - (\chi_1 f)(x), x)\|_{L_p[a_1,b_1]} + \|(\chi_1 f)(x)(K_n(e_0, x) - 1)\|_{L_p[a_1,b_1]}$$

and part (i) follows from (2.4) and the above calculations.

(ii) Now assume $p = 1$ and $f' \in BV[a, b]$. Then we have, for x, $t \in [a, b]$,

$$f(t) - f(x) = f'(x)(t-x) + \int_x^t (t-u) \, df'(u).$$

From the proof of (i), we need only show that

$$\left\| K_n \left(\chi_1(t) \int_x^t (t-u) \, df'(u), x \right) \right\|_{L_1[a_1,b_1]} = O(\mu_n), \qquad n \to \infty.$$

Fix $\delta > 0$. We have

$$\left| \int_a^b \int_x^t (t-u) \, df'(u) \, H_n(t-x) \, dt \right|$$

$$\leqslant \int_a^b |t-x| \, H_n(t-x) \left| \int_x^t |df'(u)| \, dt \right.$$

$$= \int_a^b |t-x| \, H_n(t-x) \left| \int_0^{|t-x|} |df'(x+y)| \right| dt$$

$$\leqslant \sum_{j=0}^{[(b-a)/\delta]} I_{nj}(x),$$

where

$$I_{nj}(x)=\int_{j\delta\leqslant|t-x|\leqslant(j+1)\delta}|t-x|\,H_n(t-x)\int_0^{|t-x|}|df'(x+y)|\,dt.$$

Clearly,

$$I_{nj}(x)\leqslant S_{nj}(\delta,x)\int_0^{(j+1)\delta}|df'(x+y)|,$$

where

$$S_{nj}(\delta,x)=\int_{j\delta\leqslant|t-x|<(j+1)\delta}|t-x|\,H_n(t-x)\,dt.$$

Next, we estimate $S_{nj}(\delta,x)$ for $j=0$ and $1\leqslant j\leqslant[(b-a)/\delta]$ separately. We have

$$\begin{aligned}S_{nO}(\delta,x)&=\int_{0\leqslant|t-x|<\delta}|t-x|\,H_n(t-x)\,dt\\&<\delta\int_{-r}^{r}H_n(t)\,dt\\&=\delta.\end{aligned}$$

For $1\leqslant j$,

$$\begin{aligned}S_{nj}(\delta,x)&\leqslant\frac{1}{(j\delta)^3}\int_{j\delta\leqslant|t-x|<(j+1)\delta}(t-x)^4H_n(t-x)\,dt\\&\leqslant\frac{1}{(j\delta)^3}\int_{-r}^{r}t^4H_n(t)\,dt\\&=\frac{1}{(j\delta)^3}\cdot O(\mu_n^2)\end{aligned}$$

by (1.3).

It follows that

$$\sum_{j=0}^{[(b-a)/\delta]}I_{nj}(x)\leqslant\delta\int_0^{\delta}|df'(x+y)|+\left(\sum_{j=1}^{[(b-a)/\delta]}\frac{1}{(j\delta)^3}\int_0^{(j+1)\delta}|df'(x+y)|\right)\cdot O(\mu_n^2).$$

Integrating this last inequality with respect to x, and taking into account that

$$\int_{a_1}^{b_1}\int_0^{(j+1)\delta} |df'(x+y)|\,dx \leqslant (j+1)\delta\,\|f'\|_{BV[a,b]}$$

we obtain

$$\left\|K_n\left(\chi_1(t)\int_x^t (t-u)\,df'(u)\right), x\right\|_{L_1[a_1,b_1]}$$
$$\leqslant \delta^2\,\|f'\|_{BV[a,b]} + \frac{\|f'\|_{BV[a,b]}}{\delta^2}\sum_{j=1}^{\infty}\frac{j+1}{j^3}\cdot O(\mu_n^2).$$

If we choose $\delta = \mu_n^{1/2}$, the last term is $O(\mu_n)$, $n \to \infty$.

(iii) If f is linear on $[a, b]$, the above calculations, along with (2.3), (2.4) and Lemma 2.2, imply the conclusion of (iii).

3. A Local Saturation Theorem

The following theorem provides a local inverse to Theorem 2.3 and shows that the convolution operators (1.4) are locally saturated with order $O(\mu_n)$.

THEOREM 3.1. *If $0 < a < b < r$ and $f \in L_p[0, r]$, we have*

(i) *if* $1 < p < \infty$, *then* $\|K_n(f) - f\|_{L_p[a,b]} = O(\mu_n)$, $n \to \infty$, *implies* $f \in L_p^2[a, b]$;

(ii) *if* $p = 1$, *then* $\|K_n(f) - f\|_{L_p[a,b]} = O(\mu_n)$, $n \to \infty$, *implies* $f' \in BV[a, b]$;

(iii) *if* $1 \leqslant p < \infty$, *then* $\|K_n(f) - f\|_{L_p[a,b]} = o(\mu_n)$, $n \to \infty$, *implies f is linear on* $[a, b]$.

Proof. To prove (i) and (ii), let $f \in L_p[0, r]$ and choose $\psi \in C^2[a, b]$ with $\psi(a) = \psi'(a) = \psi''(a) = \psi(b) = \psi'(b) = \psi''(b) = 0$. Let $\psi(x) = 0$ for $x \notin [a, b]$, so that $\psi \in C^2[0, r]$ and $\psi(x) = \psi'(x) = 0$ for $x \notin [a, b]$. We utilize the bilinear functional (compare [6, 8])

$$A_n(f, \psi) = \frac{1}{\mu_n}\int_a^b K_n(f(t), x) - f(x))\,\psi(x)\,dx.$$

Fix ψ as above. We first demonstrate that $\{A_n(\cdot, \psi)\}$ is uniformly bounded on $L_p[0, r]$. We have

$$\int_a^b K_n(f(t), x)\, \psi(x)\, dx = \int_0^r \psi(x) \int_0^r f(t)\, H_n(t-x)\, dt\, dx$$

$$= \int_0^r f(t) \int_0^r \psi(x)\, H_n(t-x)\, dx\, dt.$$

For x, $t \in [0, r]$ we can write

$$\psi(x) = \psi(t) + \psi'(t)(x+t) + \frac{\psi''(\eta)}{2}(x-t)^2$$

for some η between x and t. Hence

$$\int_0^r \int_0^r f(t)\, \psi(x)\, H_n(t-x)\, dx\, dt$$

$$= \int_0^r f(t)\, \psi(t) \int_0^r H_n(t-x)\, dx\, dt$$

$$+ \int_0^r f(t)\, \psi'(t) \int_0^r (x-t)\, H_n(t-x)\, dx\, dt$$

$$+ \frac{1}{2} \int_0^r f(t) \int_0^r \psi''(\eta)(x-t)^2 H_n(t-x)\, dx\, dt$$

$$= I_1 + I_2 + I_3.$$

Using (1.1) we have

$$I_1 = \int_0^r f(t)\, \psi(t) \int_{t-r}^t H_n(u)\, du\, dt$$

$$= \int_0^r f(t)\, \psi(t)\, dt - \int_0^r f(t)\, \psi(t) \int_{-r}^{t-r} H_n(u)\, du\, dt$$

$$- \int_0^r f(t)\, \psi(t) \int_t^r H_n(u)\, du\, dt.$$

Since $\psi(t) = 0$ for $t \notin [a, b]$, (1.2) and (1.3) yield

$$\left| \int_0^r f(t)\, \psi(t) \int_t^r H_n(u)\, du\, dt \right|$$

$$\leqslant \frac{1}{a^4} \int_0^r |f(t)|\, |\psi(t)| \int_t^r u^4 H_n(u)\, du\, dt$$

$$\leqslant \sup_{0 \leqslant t \leqslant r} |\psi(t)| \left(\frac{1}{a^4} \int_{-r}^r u^4 H_n(u)\, du \right) \|f\|_{L_1[0,r]}$$

$$= o(\mu_n)\, \|f\|_{L_p[0,r]}, \qquad n \to \infty.$$

Similarly

$$\left| \int_0^r f(t)\, \psi(t) \int_{-r}^{t-r} H_n(u)\, du\, dt \right| = o(\mu_n)\, \|f\|_{L_p[0,r]}, \qquad n \to \infty.$$

Hence

$$I_1 = \int_0^r f(t)\, \psi(t)\, dt + o(\mu_n)\, \|f\|_{L_p[0,r]}, \qquad n \to \infty.$$

Similar calculations and the fact that $\psi'(t) = 0$ for $t \notin [a, b]$ yield $I_2 = o(\mu_n)\, \|f\|_{L_p[0,r]}$, $n \to \infty$. Finally

$$|I_3| \leqslant \tfrac{1}{2} \sup_{0 \leqslant x \leqslant r} |\psi''(x)|\, \mu_n\, \|f\|_{L_1[0,r]},$$

and it follows that $\{A_n(\cdot, \psi)\}$ is uniformly bounded on $L_p[0, r]$.

Next we observe that, for $f \in C^2[0, r]$,

$$\lim_{n \to \infty} A_n(f, \psi) = \frac{1}{2} \int_a^b f(x)\, \psi''(x)\, dx. \tag{3.1}$$

This follows from the definition of ψ and

$$\lim_{n \to \infty} \frac{1}{\mu_n} (K_n(f, x) - f(x)) = \frac{1}{2} f''(x), \tag{3.2}$$

uniformly on $[a, b]$. Equation (3.2) is a consequence of Mamedov's asymptotic theorem [7], (1.2), (1.3) and (2.3)–(2.5).

Since $\{A_n(\cdot, \psi)\}$is uniformly bounded on $L_p[0, r]$, and $C^2[0, r]$ is dense in $L_p[0, r]$, (3.1) yields

$$\lim_{n \to \infty} A_n(f, \psi) = \frac{1}{2} \int_a^b f(x)\, \psi''(x)\, dx \tag{3.3}$$

for any $f \in L_p[0, r]$.

Fix $f \in L_p[0, r]$ and consider the sequence of linear functionals $\{A_n(f, \cdot)\}$. Since $\|K_n(f) - f\|_{L_p[a,b]} = O(\mu_n)$, $n \to \infty$, there exists $h \in L_p[a, b]$ $(p > 1)$ and $h \in BV[a, b]$ $(p = 1)$ and a subsequence $\{A_{n_i}(f, \cdot)\}$ such that

$$\begin{aligned} \lim_{i \to \infty} A_{n_i}(f, \psi) &= \int_a^b h(x)\, \psi(x)\, dx, \qquad p > 1 \\ &= \int_a^b \psi(x)\, dh(x), \qquad p = 1. \end{aligned} \tag{3.4}$$

Equating (3.3) and (3.4) we obtain

$$\frac{1}{2}\int_a^b f(x)\,\psi''(x)\,dx = \int_a^b h(x)\,\psi(x)\,dx, \qquad p>1$$
$$= \int_a^b \psi(x)\,dh(x), \qquad p=1. \tag{3.5}$$

A particular solution to (3.5) is

$$\frac{1}{2} f(x) = \int_\eta^x \int_\eta^\xi h(\mu)\,d\mu\,d\xi, \qquad p>1$$
$$= \int_\eta^x \int_\eta^\xi dh(\mu)\,d\xi, \qquad p=1.$$

The homogeneous problem

$$\int_a^b f(x)\,\psi''(x)\,dx = 0$$

has the general solution $f(x) = C_1 x + C_2$ for $a \leqslant x \leqslant b$, since $\psi \in C^2[a,b]$, $\psi(a) = \psi'(a) = \psi(b) = \psi'(b) = 0$ is arbitrary. Hence, if $1 < p < \infty$, $f \in L_p^{(2)}[a,b]$, and if $p=1$ then $f' \in BV[a,b]$.

If $\|K_n(f) - f\|_{L_p[a,b]} = o(\mu_n)$, $n \to \infty$, then

$$|A_n(f,\psi)| \leqslant \frac{1}{\mu_n}\int_a^b |K_n(f,x) - f(x)| \cdot |\psi(x)|\,dx$$
$$\leqslant \big(\sup_{a \leqslant x \leqslant b} |\psi(x)|\big)\frac{A_p}{\mu_n}\|K_n(f) - f\|_{L_p[a,b]},$$

where $A_p > 0$ is independent of n. Hence

$$\lim_{n\to\infty} A_n(f,\psi) = 0. \tag{3.6}$$

Comparing (3.3) and (3.6) we obtain

$$\frac{1}{2}\int_a^b f(x)\,\psi''(x)\,dx = 0$$

and, consequently, f is linear on $[a,b]$.

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