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Stephan Olariu
Old Dominion University

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Note

No Antitwins in Minimal Imperfect Graphs

STEPHAN OLARIU

Department of Computer Science, Old Dominion University, Norfolk, Virginia 23508

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It is customary to call vertices x and y twins if every vertex distinct from x and y is adjacent either to both of them or to neither of them. By analogy, we shall call vertices x and y antitwins if every vertex distinct from x and y is adjacent to precisely one of them. Lovász proved that no minimal imperfect graph has twins. The purpose of this note is to prove the analogous statement for antitwins.

Claude Berge proposed to call a graph G *perfect* if for every induced subgraph H of G, the chromatic number of H equals the largest number of pairwise adjacent vertices in H. At the same time he conjectured that a graph is perfect if and only if its complement is perfect. Lovász [2] proved this conjecture which is known as the Perfect Graph Theorem.

It is customary to call vertices x and y *twins* if every vertex distinct from x and y is adjacent either to both of them or to neither of them. By analogy, we shall call x and y *antitwins* if every vertex distinct from x and y is adjacent to precisely one of them.

A graph G is *minimal imperfect* if G itself is imperfect but every proper induced subgraph of G is perfect. Lovász [2] proved that no minimal imperfect graph has twins. The purpose of this note is to prove the analogous statement for antitwins.

THEOREM. *No minimal imperfect graph contains antitwins.*

Proof. Assume the statement false: some minimal imperfect graph G contains antitwins u and v. Let A denote the set of all neighbours of u other than v, and let B denote the set of neighbours of v other than u.

As usual, a *clique* is a set of pairwise adjacent vertices, and a *stable set* is

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a set of pairwise non-adjacent vertices; we let α and ω denote the largest size of a stable set and a clique, respectively, in G. We claim that

B contains a clique of size $\omega - 1$ that extends into no clique of size ω in $A \cup B$. (1)

To justify this claim, colour $G - v$ by ω colours and let S be the colour class that includes u. Since $G - S$ cannot be coloured by $\omega - 1$ colours, it contains a clique of size ω; since $G - S - v$ is coloured by $\omega - 1$ colours, we must have $v \in C$. Hence $C - v$ is a clique in B of size $\omega - 1$. If a vertex x extends $C - v$ into a clique of size ω then $x \notin A$ (since $A \cap S = \varnothing$ and $G - S - v$ is coloured by $\omega - 1$ colours) and $x \notin B$ (since otherwise x would extend C into a clique of size $\omega + 1$). Thus (1) is justified.

The Perfect Graph Theorem guarantees that the complement of G is minimal imperfect; hence (1) implies that

A contains a stable set of size $\alpha - 1$ that extends into no stable set of size α in $A \cup B$. (2)

Now let C be the clique featured in (1), and let S be the stable set featured in (2); let x be a vertex in C that has the smallest number of neighbours in S. By (2), x has a neighbour z in S; by (1), z is non-adjacent to some y in C. Since y has at least as many neighbours in S as x, it must have a neighbour w in S that is non-adjacent to x. Now u, z, x, y, w induce in G a chordless cycle. Since this cycle is imperfect, G is not minimal imperfect, a contradiction.

Chvátal *et al.* [1] call a graph G an (α, ω)-*graph* if it satisfies the following conditions:

(i) G contains exactly $\alpha\omega + 1$ vertices.

(ii) For every vertex w of G, the vertex-set of $G - w$ can be partitioned into α disjoint cliques of size ω and into ω disjoint stable sets of size α.

(iii) Each vertex of G is included in precisely α stable sets of size α and in precisely ω cliques of size ω.

(iv) Each stable set of size α is disjoint from precisely one clique of size ω and each clique of size ω is disjoint from precisely one stable set of size α.

Padberg [3] proved that every minimal imperfect graph is an (α, ω)-graph. However, there exist (α, ω)-graphs that contain antitwins. One such graph is featured in Fig. 1: the vertices 0 and 5 are antitwins.

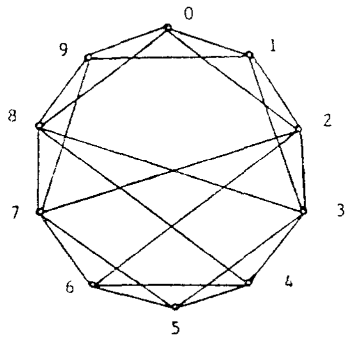

FIGURE 1

ACKNOWLEDGMENT

The author is indebted to Vašek Chvátal for the example in Fig. 1 and for many inspiring ideas.

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