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Note

A New Conjecture about Minimal Imperfect Graphs

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H. Meyniel proved that in every minimal imperfect graph, every pair of vertices is joined by a chordless path containing an odd number of edges. We conjectured that in every minimal imperfect graph, every pair of vertices is joined by a path containing an even number of edges. We give an equivalent version of this new conjecture. © 1989 Academic Press, Inc.

Claude Berge [1] proposed to call a graph G *perfect* if for every induced subgraph H of G the chromatic number of H equals the largest number of pairwise adjacent vertices in H.

We shall let $\omega(G)$ and $\alpha(G)$ stand for the largest number of pairwise adjacent vertices in G and in its complement $\bar{G}$, respectively.

A graph G is *minimal imperfect* if G itself is imperfect but every proper induced subgraph of G is perfect.

The only known minimal imperfect graphs are the odd chordless cycles of length at least 5 (also called *odd holes*) and their complements.

Recently, one of us (see Meyniel [2]) proved that in a minimal imperfect graph, every pair of vertices is joined by a chordless path containing an odd number of edges.

Call vertices x, y of a graph G an *odd pair* if every path in G joining x and y and containing an even number of edges has a chord distinct from the edge xy.

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Both authors, independently, proposed the following conjecture:
No minimal imperfect graph contains an odd pair.

The purpose of this note is to present an equivalent version of this conjecture. For this purpose, call an edge xy of a graph G *deficient* if x and y have no common neighbours in G.

THEOREM. *The following two statements are equivalent:*

(i) *A minimal imperfect graph contains a deficient edge if and only if it is an odd hole.*

(ii) *No minimal imperfect graph contains an odd pair.*

Proof. To prove the implication (i) → (ii), consider a minimal imperfect graph G. We only need prove that G contains no odd pair. This is trivially true if $\omega(G)=2$.

Now, $\omega(G) \geqslant 3$. Let x, y be arbitrary vertices in G. We only need prove that x, y are not an odd pair. If x and y have a common neighbour, then we are done. We shall assume, therefore, that x and y have no common neighbours.

If x and y are adjacent, then the edge xy is deficient and, by (i), $\omega(G)=2$, a contradiction.

Now we may assume that x and y are non-adjacent. Let G' be the graph obtained from G by adding the edge xy. By assumption, $\omega(G)=\omega(G')$. We note that G' is imperfect, for otherwise any colouring of G' using $\omega(G)$ colours is also a colouring of G, a contradiction.

Let H induce a minimal imperfect subgraph G'_H of G', and let G_H be the subgraph of G induced by H. We note that both x and y belong to H. (This is true if $G_H=G$; if G_H is a proper induced subgraph of G, then since G_H is perfect and G'_H is minimal imperfect, we must have $x, y \in H$, as claimed.)

If $\omega(G'_H) \geqslant 3$, then, since the edge xy is deficient we contradict (i).

Hence, $\omega(G'_H)=2$. It follows that G'_H is an odd hole, and so H induces a chordless path with an even number of edges in G. Hence, x and y are not an odd pair.

To prove the implication (ii) → (i), let G be a minimal imperfect graph that is a counterexample to our claim: G contains a deficient edge xy and yet $\omega(G) \geqslant 3$.

Since (ii) is assumed to be true, there exists a path

$$x = w_0, w_1, ..., w_{2p} = y$$

in G joining x and y, containing an even number of edges, whose only chord is the edge xy.

However, since xy is a deficient edge, $\{w_0, w_1, ..., w_{2p}\}$ induces an odd hole, contradicting the minimality of G.

This completes the proof of the theorem.

Note. A graph G is called an (α, ω)-*graph* if it satisfies the following conditions:

(i) G contains exactly $\alpha\omega + 1$ vertices.

(ii) For every vertex w of G, the vertex-set of $G - w$ can be partitioned into α disjoint cliques of size ω and into ω disjoint stable sets of size α.

(iii) Each vertex of G is included in precisely α stable sets of size α and in precisely ω cliques of size ω.

(iv) Each stable set of size α is disjoint from precisely one clique of size ω and each clique of size ω is disjoint from precisely one stable set of size α.

It is easy to see that every minimal imperfect graph is an (α, ω)-graph. Recently, Bruce Reed [3] proved that in an (α, ω)-graph, every pair of vertices is joined by a chordless path containing an odd number of edges, thus generalizing Meyniel's result [2]. We note, however, that there exist (α, ω)-graphs which contain an odd pair (see Fig. 1). Thus any proof of the conjecture stated in this note must rely on properties that distinguish minimal imperfect graphs from (α, ω)-graphs.

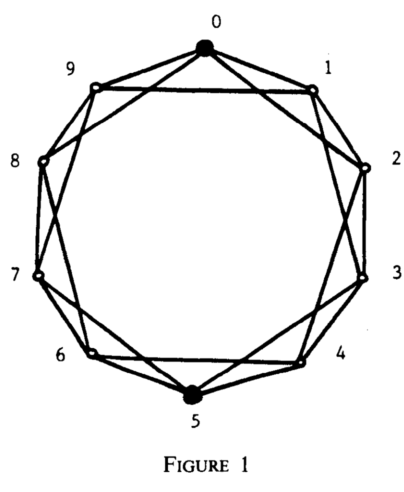

FIGURE 1

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