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J. J. Swetits  
*Old Dominion University*

S. E. Weinstein  
*Old Dominion University*

Yuesheng Xu  
*Old Dominion University*

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Best $L_p$ Approximation
with Multiple Constraints for $1 \leq p < \infty$

J. J. Swetits, S. E. Weinstein, and Yuesheng Xu

Department of Mathematics and Statistics, Old Dominion University,
Norfolk, Virginia 23529, U.S.A.

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The problem considered in this paper is best $L_p$ approximation with multiple constraints for $1 \leq p < \infty$. Characterizations of best $L_p$ approximations from multiple $n$-convex splines and functions are established and the relationship between them is investigated. Applications to best monotone convex approximation are studied.

1. INTRODUCTION

In this paper, we consider best $L_p$ approximation with multiple constraints for $1 \leq p < \infty$. The classes of approximating functions are the class of multiple $n$-convex splines and the class of multiple $n$-convex functions, which are defined below.

A real-valued function $g$ is said to be $n$-convex in $(0, 1)$ if for any $n + 1$ distinct points $x_0, x_1, \ldots, x_n$ in $(0, 1)$, the $n$th order divided difference is nonnegative, i.e.,

$$[x_0, x_1, \ldots, x_n] g \geq 0.$$

The set of $n$-convex functions is a convex cone. Note that 1-convex functions are nondecreasing and 2-convex functions are convex in the usual sense.

It is known (e.g., [2]) that if $g$ is an $n$-convex function on $(0, 1)$ then $g^{(n-2)}$ exists and is convex on $(0, 1)$. Hence, $g^{(n-2)}$ is absolutely continuous on any closed subinterval of $(0, 1)$, the $(n-1)$st left-derivative $g^{(n-1)}$ exists and is left-continuous and nondecreasing in $(0, 1)$, the $(n-1)$st right-derivative $g^{(n-1)}$ exists and is right-continuous and nondecreasing in $(0, 1)$, $g^{(n-1)}$ exists a.e. in $(0, 1)$, and $g^{(n-1)} = g^{(n-1)} = g^{(n-1)}$ a.e. in $(0, 1)$. If $g \in C^n[0, 1]$, then $g$ is $n$-convex if and only if $g^{(n)} \geq 0$. The set of $n$-convex functions is a convex cone.
functions contains the subspace of polynomials of degree $n - 1$. Some additional properties of $n$-convex functions can be found in [2, 11, 16, 19].

Given $0 \leq m \leq n$, $g$ is said to be $(m, n)$-convex if $(-1)^i g$ is $(m + i)$-convex for $i = 0, 1, \ldots, n - m$. Note that for $n > m$, $(m, n)$-convex functions are functions with multiple constraints. Let $K_{m,n}$ denote the set of $(m, n)$-convex functions. Then clearly $K_{m,n}$ is the finite intersection of $k$-convexity cones. The finite intersections of generalized convexity cones with respect to an ECT-system were defined in [20, 21]. Clearly, $K_{m,n}$ is a finite intersection of the convexity cone with respect to the ECT-system $\{1, x, x^2, \ldots, x^{n-1}\}$.

From the above definition, $(n, n)$-convex functions are $n$-convex functions and $(0, n)$-convex functions are $n$-time monotone functions. For some applications of $n$-time monotone functions, see [18] and other references therein. In addition, $(0, \infty)$-convex functions are completely monotone functions (see [17]). More generally, we define $(m, n)_\sigma$-convexity. Let $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_{n-m})$, where each $\sigma_i$ is 1 or $-1$. A function $g$ is said to be $(m, n)_\sigma$-convex if $\sigma_i (-1)^i g$ is $(m + i)$-convex, for $i = 0, 1, \ldots, n - m$. In this paper, for the sake of simplicity we restrict ourselves to $(m, n)$-convex functions. All results we obtain here can be extended to the setting with arbitrary $\sigma$ without any difficulty.

Let $K_{m,n}^\rho$ denote the intersection of $K_{m,n}$ and $L_\rho = L_\rho[0, 1]$. Then $K_{m,n}^\rho$ is a closed convex cone in $L_\rho$. Given a partition $\mathcal{A}$ of $[0, 1]$, with $\mathcal{A}: 0 = x_0 < x_1 < \cdots < x_k , i = 1$, let $S_{m,n}^k(\mathcal{A})$ denote the space of polynomial splines of degree $n - 1$ with $k$ simple knots at $x_1, \ldots, x_k$, i.e.,

$$ S_{m,n}^k(\mathcal{A}) = \text{span}\{(1 - x)^{i-1}, i = 1, 2, \ldots, n, (x_j - x)^{n-1}, j = 1, 2, \ldots, k\}. $$

Define

$$ S_{m,n}^k(\mathcal{A}) = S_{m,n}^k(\mathcal{A}) \cap K_{m,n}. $$

Since polynomials of degree $n - 1$ are contained in both $S_{m,n}^k(\mathcal{A})$ and $K_{m,n}$, $S_{m,n}^k(\mathcal{A})$ is a nonempty convex cone. In particular, $S_{m,n}^0(\mathcal{A})$ is the set of $(m, n)$-convex polynomials of degree $n - 1$.

Given $f \in L_\rho[0, 1]$, $s^* \in K_{m,n}^\rho$ (resp., $S_{m,n}^k(\mathcal{A})$) is called a best $(m, n)$-convex (resp., $(m, n)$-convex spline) $L_\rho$ approximation to $f$ if

$$ \|f - s^*\|_\rho = \inf\{\|f - s\|_\rho : s \in K_{m,n}^\rho (\text{resp., } S_{m,n}^k(\mathcal{A}))\}. $$

The existence of a best $n$-convex $L_1$ approximation was proved in [7] and [16] independently, and uniqueness is proved under some additional restrictions in [22]. The characterizations of best 1-convex (nondecreasing) $L_\rho$ approximations for $1 \leq p < \infty$ were established in [12, 13]. A partial characterization of a best $n$-convex $L_1$ approximation was proved in [22]. The complete characterization of a best $n$-convex $L_\rho$ approximation, for $1 \leq p < \infty$, is considered in [14, 15, 19]. Existence of a best $n$-convex
uniform approximation was proved in [3, 24]. Burchard [4] and Brown [1] have characterized best uniform $n$-convex approximation. Some additional properties of best uniform $n$-convex approximation are considered in [23].

For $1 \leq p < \infty$, the existence of a best approximation to $f \in L_p[0, 1]$ from $S_{m,n}^k(A)$ follows from the fact that $S_{m,n}^k(A)$ is a finite dimensional, closed subset of $L_p$. For $1 < p < \infty$, unicity follows from the fact that $L_p$ is strictly convex. For $p = 1$, unicity was proved by Pence in [9]. In Section 2, the characterizations of best $(m, n)$-convex spline $L_p$ approximations for $1 \leq p < \infty$ are established. As consequences, we also consider best $L_p$ approximation by $n$-convex splines of degree $n - 1$.

For $1 < p < \infty$, the existence of a unique best $L_p$ approximation from $K_{m,n}^p$ follows from the facts that $L_{m,n}^p$ is closed and convex in the reflexive Banach space $L_p$ and that the $L_p$ norm is uniformly convex. In Section 3, we prove the existence of a best $L_1$ approximation of $f \in L_1[0, 1]$ from $K_{m,n}^1$ and characterize best $L_p$ approximation to a function $f$ in $L_p[0, 1]$ from $K_{m,n}^p$ for $1 \leq p < \infty$. An interesting relationship between best $L_p$ approximations to $f \in C[0, 1]$ from $S_{m,n}^k(A)$ and $K_{m,n}^p$ is investigated in Section 4. In Section 5, best monotone convex $L_p$ approximations are studied and best convex $L_p$ approximation is characterized in terms of best monotone convex $L_p$ approximations.

2. BEST $L_p$ APPROXIMATION FROM $S_{m,n}^k(A)$

By a corollary of the Hahn-Banach Theorem (see [5, 6]), if $K_p$ is a convex cone in $L_p[0, 1]$ for $1 \leq p < \infty$, then

(i) for $1 < p < \infty$, $s_p^* \in K_p$ is a best $L_p$ approximation to $f \in L_p[0, 1]$ from $K_p$ if and only if

\[
\int_0^1 s_p^* \phi_p = 0, \tag{2.1}
\]

and

\[
\int_0^1 s \phi_p \leq 0, \quad \text{for all } s \in K_p, \tag{2.2}
\]

where $\phi_p = \text{sign}(f - s_p^*) |f - s_p^*|^{p-1}$; and

(ii) for $p = 1$, $s_1^* \in K_1$ is a best $L_1$ approximation to $f \in L_1[0, 1]$ from $K_1$ if and only if there exists a $\phi_1 \in L_\infty$ with $\|\phi_1\|_\infty = 1$ and $\int_0^1 \phi_1(f - s_1^*) = \|f - s_1^*\|_1$, satisfying (2.1) and (2.2) with $p = 1$.

The above result shall be referred to as the duality theorem.
Let $d\mu$ be a signed measure of bounded variation on $(0, 1)$. The dual cone to a cone $K$ of functions is the set of signed measures $d\mu$ such that

$$
\int_0^1 g(x) \, d\mu(x) \geq 0 \quad \text{for all} \quad g \in K.
$$

With this definition, the above duality theorem can be restated as follows: For $1 < p < \infty$ $s^*_p \in K_p$ is a best $L_p$ approximation to $f \in L_p$ from $K_p$ if and only if $\phi_p$ is orthogonal to $s^*_p$ and $-\phi_p(x) \, dx$ is in the dual cone to $K_p$. For $p = 1$, we can similarly restate the duality theorem. The dual cone to a finite intersection of generalized convexity cones with respect to an ECT-system was characterized by Ziegler in [20, 21].

By applying the duality theorem, we have the following characterization of best $L_p$ approximation to $f \in L_p$ from $S^k_{m,n}(A)$ for $1 \leq p < \infty$. Let $N_{m,n} = \{m+1, \ldots, n\}$ and $N_n = N_{0,n}$.

**THEOREM 2.1 (Characterization).** For $1 \leq p < \infty$, let $f \in L_p[0, 1]$ and let $s^*_p \in S^k_{m,n}(A)$.

(a) For $1 < p < \infty$, let $\phi_p = \text{sign}(f - s^*_p) |f - s^*_p|^{p-1}$, and

$$
H_{p,i}(x) = \left\{ \frac{1}{(i-1)!} \right\} \int_0^x (x-t)^i \phi_p(t) \, dt, \quad x \in [0, 1], \ i \in N_n. \quad (2.3)
$$

Then $s^*_p$ is the best $L_p$ approximation to $f$ from $S^k_{m,n}(A)$ if and only if

(i) $H_{p,i}(1) = 0$, $i \in N_m$;

(ii) $(-1)^m H_{p,i}(1) \leq 0$, $i \in N_{m,n}$;

(iii) $(-1)^m H_{p,j}(x_j) \leq 0$, $j \in N_k$;

(iv) if $(-1)^m H_{p,i}(1) < 0$ for some $i \in N_{m,n}$, then $s^*_p|^{i-1}(1) = 0$;

(v) if $(-1)^m H_{p,n}(x_j) < 0$ for some $j \in N_k$, then $s^*_p|^{n-1}(x_j) = s^*_p|^{n-1}(x_j^*)$.

(b) For $p = 1$, $s^*_p$ is a best $L_1$ approximation from $S^k_{m,n}(A)$ to $f$ if and only if there exists a $\phi_1 \in L_\infty$ with $\|\phi_1\|_\infty = 1$ and $\int_0^1 \phi_1(f - s^*_1) = \|f - s^*_1\|_1$ satisfying (i)-(v) of part (a) with $p = 1$. We call $\phi_1$ an associated functional of $s^*_p$.

**Proof.** (a) This proof will depend on the above duality theorem. Since $S^k_{m,n}(A)$ is a closed convex cone in $L_p[0, 1]$, by the duality, $s^*_p$ is the best approximation to $f$ from $S^k_{m,n}(A)$ if and only if

$$
\int_0^1 s^*_p \phi_p = 0, \quad (2.4)
$$

(b) This proof will depend on the above duality theorem. Since $S^k_{m,n}(A)$ is a closed convex cone in $L_p[0, 1]$, by the duality, $s^*_p$ is the best approximation to $f$ from $S^k_{m,n}(A)$ if and only if

$$
\int_0^1 s^*_p \phi_p = 0, \quad (2.4)
$$
and

$$\int_0^1 s\phi_p \leq 0 \quad \text{for all } s \in S_{m,n}^k(\Delta). \quad (2.5)$$

(Necessity) First, note that \((1 - x)^{-1/(i - 1)!}/(i - 1)! \in S_{m,n}^k(\Delta)\) for \(i = 1, 2, \ldots, m\). By substituting these functions into inequality (2.5), we find

$$\int_0^1 \{(1 - x)^{-1/(i - 1)!}\} \phi_p(x) \, dx = 0, \quad i = 1, 2, \ldots, m.$$ 

This proves (i).

Next, since \((-1)^m(1 - x)^{-1/(i - 1)!} \in S_{m,n}^k(\Delta), i = m + 1, \ldots, n\), by using (2.5) once again, we obtain (ii). Similarly, in (2.5), let \(s = (-1)^m(x_j - x)^{n-1/(n - 1)!}, j = 1, 2, \ldots, k\), and we have

$$\int_0^{x_j} \{(-1)^m(x_j - x)^{n-1/(n - 1)!}\} \phi_p(x) \, dx \leq 0, \quad j = 1, 2, \ldots, k$$

Now, by integrating by parts and by using (i),

$$\int_0^1 s_p^*(x) \phi_p(x) \, dx = \int_0^1 (-1)^m H_{p,m}(x) \cdot s_p^*(m)(x) \, dx$$

$$= \sum_{i = m}^{n} (-1)^i H_{p,i+1}(1) \cdot s_p^*(i)(1)$$

$$+ \sum_{j = 1}^{k} (-1)^n H_{p,n}(x_j) [s_p^*(n-1)(x_j^+) - s_p^*(n-1)(x_j^-)].$$

where the last equality holds because \(s_p^*\) is a polynomial of degree \(n - 1\) on each subinterval \((x_j, x_{j+1})\). Combining the above equation with (2.4) gives

$$\sum_{i = m}^{n} (-1)^i H_{p,i+1}(1) \cdot s_p^*(i)(1)$$

$$+ \sum_{j = 1}^{k} (-1)^n H_{p,n}(x_j) [s_p^*(n-1)(x_j^+) - s_p^*(n-1)(x_j^-)] = 0. \quad (2.6)$$

Since \(s_p^* \in K_{m,n}\), \((-1)^i m s_p^*(i)(1) \geq 0\) and \((-1)^n m [s_p^*(n-1)(x_j^+) - s_p^*(n-1)(x_j^-)] \geq 0\). It follows from (ii) and (iii) that each term in (2.6) is nonpositive. Hence,

$$(-1)^m H_{p,i+1}(1) \cdot s_p^*(i)(1) = 0, \quad i = m, m + 1, \ldots, n - 1. \quad (2.7)$$
and
\[( -1)^m H_{p,n}(x_j)[s_p^{*\int x_j^s}(x_j^s - s_p^{*\int x_j^s})] = 0, \quad j = 1, 2, \ldots, k. \tag{2.8}\]

Then (2.7) implies (iv) and (2.8) implies (v).

(Sufficiency) If \( s_p^* \in S_{m,n}^k(A) \) satisfying conditions (i)-(v), then by integration by parts, it is easy to verify that (2.4) and (2.5) hold. Therefore, \( s_p^* \) is the best approximation to \( f \) from \( S_{m,n}^k(A) \).

(b) The proof is similar to (a). Thus, we omit the details. This proves Theorem 2.1.

We remark that since \( H_{p,n}^\prime(x) = H_{p,n}^\prime (x) \) conditions (i)-(v) of Theorem 2.1 can be restated in terms of \( H_{p,n}^\prime \) and its derivatives. For example, conditions (i) and (ii) are equivalent to \( H_{p,n}^{n-1}(1) = 0, i \in N_m, \) and \( ( -1)^m H_{p,n}^{n-1}(1) \leq 0, i \in N_{m,n} \), respectively.

In order to derive some structural properties of a best approximation, we introduce some additional notation. Let \( 1 \leq p < \infty \) and \( \phi_p \in (L_p)^* \), the dual space of \( L_p \). Define \( H_{p,i} \) as in (2.3),
\[ I(\phi_p) = \{ i \in N_m : ( -1)^m H_{p,i}(1) < 0 \}, \tag{2.9}\]
and
\[ J(\phi_p) = \{ j \in N_k : ( -1)^m H_{p,n}(x_j) < 0 \}. \tag{2.10}\]

We define a subspace of \( S_k^*(A) \) by
\[ S_{k^*}(A, \phi_p) = \{ s \in S_{n,k}^*(A) : s^{(i)}(1) = 0, i \in I(\phi_p); \]
\[ s^{(n-1)}(x_j^s) = s^{(n-1)}(x_j^s), j \in J(\phi_p) \}. \tag{2.11}\]

It is easily proved that \( S_{k^*}^*(A, \phi_p) \) has a basis
\[ \{ (1-x)^{i-1}, i \in N_n - I(\phi_p), (x_j-x)^{i-1}, j \in N_k - J(\phi_p) \}. \tag{2.12}\]

The next theorem gives an alternate characterization of best \( L_p \) approximation from \( S_{m,n}^k(A) \), which indicates that best \( L_p \) approximation from the convex set \( S_{m,n}^k(A) \) is equivalent to best \( L_p \) approximation from the subspace \( S_{k^*}^*(A, \phi_p) \).

**Theorem 2.2.** Let \( 1 \leq p < \infty \) and let \( f \in L_p[0, 1] \).

(a) For \( 1 < p < \infty \), \( s_p^* \in S_{m,n}^k(A) \) is the best \( L_p \) approximation to \( f \) from \( S_{m,n}^k(A) \) if and only if \( s_p^* \) is the best \( L_p \) approximation from \( S_{k^*}^*(A, \phi_p) \), where
\[ \phi_p = \text{sign}(f - s_p^*) |f - s_p^*|^\frac{p-1}{p}. \]
(b) For $p = 1$, $s^*_p \in S^k_{m,n}(\Delta)$ is a best $L_1$ approximation to $f$ from $S^k_{m,n}(\Delta)$ if and only if there is a $\phi \in L_\infty$ with $\|\phi\|_\infty = 1$ and 
$$\int_0^1 \phi_1(f - s^*_p) = \|f - s^*_p\|_1,$$ 

such that $s^*_p$ is a best $L_1$ approximation to $f$ from $S^k_{n}(*)(\Delta, \phi_p)$.

**Proof.** Let $1 < p < \infty$. By Theorem 2.1, $s^*_p$ is the best $L_p$ approximation to $f$ from $S^k_{m,n}(\Delta)$ if and only if conditions (i)–(v) are satisfied. It follows from the definitions of $I(\phi_p)$ and $J(\phi_p)$ that conditions (i)–(v) are equivalent to the conditions

$$\int_0^1 (1 - t)^{i-1} \phi_p(t) \, dt = 0, \quad i \in N_n, \quad I(\phi_p), \quad (2.13)$$

and

$$\int_0^1 (x_j - t)^{n-1} \phi_p(t) \, dt = 0, \quad j \in N_k \setminus J(\phi_p). \quad (2.14)$$

This is equivalent to the statement that $s^*_p$ is the best $L_p$ approximation to $f$ from $S^k_{n}(*)(\Delta, \phi_p)$, since $S^k_{n}(*)(\Delta, \phi_p)$ is a finite dimensional subspace of $S^k_{n}(*)(\Delta)$ and $S^k_{n}(*)(\Delta, \phi_p)$ has the basis (2.12).

The proof of (b) is similar to that of (a). This completes the proof.

In the rest of this section we apply the general results that we just obtained to best $L_p$ approximations from $S^k_{n,n}(\Delta)$, the set of $n$-convex splines of degree $n - 1$, for $1 \leq p < \infty$.

**COROLLARY 2.1.** For $1 \leq p < \infty$, let $f \in L_p[0, 1]$ and let $s^*_p \in S^k_{n,n}(\Delta)$.

(a) For $1 < p < \infty$, the following statements are equivalent:

1. $s^*_p$ is the best $L_p$ approximation to $f$ from $S^k_{n,n}(\Delta)$;
2. $s^*_p$ satisfies three conditions
   (i) $H_{\rho,i}(1) = 0$, $i = 1, 2, \ldots, n$,
   (ii) $(-1)^n H_{\rho,n}(x_j) \leq 0$, $j = 1, 2, \ldots, k$,
   (iii) if $(-1)^n H_{\rho,n}(x_j) < 0$ for some $j \in N_k$, then $s^*_{p(n-1)}(x_j^{-}) = s^*_{p(n-1)}(x_j^{+})$.
3. $s^*_p$ is the best $L_p$ approximation to $f$ from $S^k_{n,*}(*)(\Delta, \phi_p)$ defined by
   $$S^k_{n,*}(*)(\Delta, \phi_p) = \{ s \in S^k_{n}(\Delta) : s^{(n-1)}(x_j) = s^{(n-1)}(x_j^{+}), j \in J(\phi_p) \},$$
   where $J(\phi_p) = \{ j \in N_k : (-1)^n H_{\rho,n}(x_j) < 0 \}$. 


(b) For \( p = 1 \), \( s^*_1 \) is a best \( L_1 \) approximation to \( f \) from \( S^*_{n,n}(\Delta) \) if and only if there exists a \( \phi_1 \in L_{\infty} \) with \( \| \phi_1 \|_{\infty} = 1 \) and \( \int_0^1 \phi_1(f - s^*_1) = \| f - s^*_1 \|_1 \), satisfying the conditions (i)-(iii) of part (a) with \( p = 1 \), and if and only if there exists a \( \phi_1 \) as above such that \( s^*_1 \) is the best \( L_p \) approximation to \( f \) from \( S^*_{n,n}(\Delta, \phi_1) \).

3. Best \( L_p \) Approximation from \( K^p_{m,n} \)

In this section, we consider best \( L_p \) approximation to \( f \in L_p \) from \( K^p_{m,n} \) for \( 1 \leq p < \infty \).

First of all, we study the existence of a best \((m, n)\)-convex \( L_1 \) approximation. It will be proved to be a consequence of an existence theorem in a recent paper [16] by Ubhaya. We first state a definition and a theorem that appear in [16]. Let \( H \) be the set of all extended real-valued functions on \([0, 1]\). We say \( P \subset H \) is sequentially closed if it is closed under pointwise convergence of sequences of functions. We denote by \( \overline{P} \) the smallest superset of \( P \) which is sequentially closed.

**Theorem 3.1** [16]. Let \( P \) be a nonempty set in \( H \). Assume the following two conditions are satisfied:

1. \( P \cap L_p = \overline{P} \cap L_p \);
2. There exists a positive integer \( z \) which depends only upon \( P \), and the following holds: If \( k \in P \), there exist an integer \( 1 \leq r \leq z \) and points \( \{x_i : i = 0, 1, \ldots, r\} \) with \( 0 = x_0 < x_1 < \cdots < x_r = 1 \) so that \( k \) is monotone on each interval \((x_i, x_{i+1})\).

Then a best approximation to \( f \) in \( L_p \) from \( P \cap L_p \) exists for \( 1 \leq p < \infty \).

The following theorem is a consequence of Theorem 3.1.

**Theorem 3.2.** Let \( f \in L_1[0, 1] \). Then there exists a best \((m, n)\)-convex \( L_1 \) approximation to \( f \).

**Proof.** Let

\[
K_i = \{ g \in H : (-1)^i g \text{ is } (m + i)\text{-convex} \}.
\]

Then \( K_{m,n} = \bigcap_{i=0}^{m-n} K_i \). Thus,

\[
K_{m,n} \cap L_1 = \left( \bigcap_{i=0}^{n-m} K_i \right) \cap L_1 = \bigcap_{i=0}^{n-m} \{ K_i \cap L_1 \}.
\]
By Proposition 3.4 of [16], we have 
\[ K_i \cap L_1 = \bigcap_{i=0}^{n-m} (K_i \cap L_1) \]
Hence,
\[ \bigcap_{i=0}^{n-m} (K_i \cap L_1) = \bigcap_{i=0}^{n-m} \bigcap_{j=i}^{n-m} K_j \cap L_1. \]

Therefore, condition (1) in Theorem 3.1 is satisfied. In addition, since an
\((m, n)\)-convex function is \(m\)-convex, by a property of \(m\)-convex functions
(see [15, 16]), condition (2) is also satisfied. It follows from Theorem 3.1
that there exists a best \(L_1\) approximation to \(f\) from \(K_{m,n}^1\). This completes
the proof.

Next we establish a characterization of best \(L_p\) approximation by \((m, n)\)-
convex functions, for \(1 \leq p < \infty\). To do this, we first prove the following:

**Lemma 3.1.** Let \(g\) be \((m, n)\)-convex on \([0, 1]\). Then \(g^{(n-1)}(1^-)\) and
\(g^{(m+i)}(1^-), i = 0, 1, \ldots, n-m-2\), are finite.

**Proof.** Since \(g\) is \((m, n)\)-convex and \((-g)\) is \((m+1)\)-convex. We then
find that \(g^{(m)}\) is nonincreasing and \(g^{(m)}(x) \geq 0\) for all \(x \in (0, 1)\). Hence, for
an arbitrarily small \(\varepsilon\) with \(0 < \varepsilon < \frac{1}{2}\),
\[ 0 \leq g^{(m)}(1 - \varepsilon) \leq g^{(m)}(\frac{1}{2}). \]

However, \(g^{(m)}(\frac{1}{2}) < +\infty\). It follows that \(g^{(m)}(1^-)\) is finite. This proof can be
completed by induction on \(i\).

We are now ready to state our main theorem in this section.

**Theorem 3.3 (Characterization).** For \(1 \leq p < \infty\), let \(f \in L_p[0, 1]\) and
let \(g_p^* \in K_{m,n}^p\).

(a) For \(1 < p < \infty\), let \(\phi_p = \text{sign}(f - g_p^*)|f - g_p^*|^{p-1}\), and define \(H_{p,i}\)
as (2.3). Then \(g_p^*\) is the best \(L_p\) approximation to \(f\) from \(K_{m,n}^p\) if and only if

(i) \(H_{p,i}(1) = 0, \ i = 1, 2, \ldots, m;\)

(ii) \((-1)^m H_{p,i}(1) \leq 0, \ i = m + 1, \ldots, n;\)

(iii) \((-1)^m H_{p,n}(x) \leq 0, \ x \in [0, 1];\)
(iv) if \((-1)^m H_{p,i}(1) < 0\) for some \(i \in \{m+1, \ldots, n\}\), then \(g_p^{(i-1)}(1) = 0\);

(v) if \((-1)^m H_{p,n}(x) < 0\) for some \(x \in (0, 1)\), then \(g_p^*\) is a polynomial of degree \(n - 1\) in a neighborhood of \(x\).

(b) For \(p = 1\), \(g_1^*\) is a best \(L_1\) approximation to \(f\) from \(K_{m,n}^p\) if and only if there exists \(\phi_1 \in L_\infty\) with \(\|\phi_1\|_\infty = 1\) and \(\int_0^1 \phi_1(f - g_1^*) = \|f - g_1^*\|_1\), satisfying the conditions (i)–(v) of part (a) with \(p = 1\).

**Proof:** (a) This proof depends on the duality theorem, as the proof of Theorem 23.1.

(Necessity) The proof for (i)–(iii) is similar to the proof for (i)–(iii) in Theorem 2.1. To prove (iv) and (v), we establish the following integration by parts:

\[
\int_0^1 g_p^* \phi_p = \sum_{i=1}^m (-1)^i H_{p,i+1}(1) g_p^{(i)}(1^-) + (-1)^m \int_0^1 H_{p,n} d(g_p^{(m-1)}). \tag{3.1}
\]

A similar reasoning as in the proof of Theorem 1 of [15] gives

\[
\int_0^1 g_p^* \phi_p = (-1)^m \int_0^1 H_{p,m} g_p^{(m)},
\]

and \(H_{p,m} g_p^{(m)} \in L_1[0,1]\). By Lemma 3.1, \(g_p^{(m)}(1^-)\) is finite, and thus, for an arbitrarily small \(\varepsilon > 0\), \(H_{p,m+1} g_p^{(m+1)} \in L_1[\varepsilon, 1]\). Hence, integration by parts yields

\[
\int_0^1 H_{p,m} g_p^{(m)} = H_{p,m+1}(1) g_p^{(m)}(1^-) - H_{p,m+1}(\varepsilon) g_p^{(m)}(\varepsilon)
- \int_\varepsilon^1 H_{p,m+1} g_p^{(m+1)}. \tag{3.2}
\]

If \(g_p^{(m)}(0^+)\) is finite, then by letting \(\varepsilon \to 0\) in (3.2), we obtain

\[
\int_0^1 H_{p,m} g_p^{(m)} = H_{p,m+1}(1) g_p^{(m)}(1^-) - \int_0^1 H_{p,m+1} g_p^{(m+1)}, \tag{3.3}
\]

and \(H_{p,m+1} g_p^{(m+1)} \in L_1[0,1]\). Otherwise, we must have \(g_p^{(m)}(0^+) = +\infty\). Since \(g_p^{(m)}\) is nonincreasing, there exists a \(t \in (0, 1)\) such that \(g_p^{(m)}\) is nonincreasing on \((0, t)\). Whenever \(0 < \varepsilon < t\),

\[
|H_{p,m+1}(\varepsilon) g_p^{(m)}(\varepsilon)| < |g_p^{(m)}(\varepsilon)| < |g_p^{(m)}(\varepsilon)| \int_0^\varepsilon |H_{p,m}| \leq \int_0^\varepsilon |g_p^{(m)}| H_{p,m}|
\]
Since \( H_{p,m} g_p^{(m)} \in L_1[0, 1] \), we have \( \lim_{\epsilon \to 0} \int_0^\epsilon |g_p^{(m)}(\epsilon) H_{p,m}| = 0 \). By the above inequality,

\[
\lim_{\epsilon \to 0} \int_0^\epsilon |g_p^{(m)}(\epsilon) H_{p,m+1}(\epsilon)| = 0.
\]

Letting \( \epsilon \to 0 \), we also come out with (3.3) and \( H_{p,m+1} g_p^{(m+1)} \in L_1[0, 1] \).

This procedure can be repeated to obtain (3.1).

Combining the duality theorem and (3.1) yields

\[
(-1)^i H_{p,i+1}(1) g_p^{(i)}(1) + (-1)^n \int_0^1 H_{p,n} d(g_p^{(n-1)}) = 0.
\]

The definition of an \((m, n)\)-convex function together with (ii) and (iii) implies that

\[
(-1)^i H_{p,i+1}(1) g_p^{(i)}(1) = 0, \quad i = m, ..., n - 1,
\]

and

\[
\int_0^1 H_{p,n} d(g_p^{(n-1)}) = 0.
\]

Equations (3.4) and (3.5) give (iv) and (v), respectively.

(Sufficiency) Assume \( g_p^* \in K_{m,n}^p \) and it satisfies conditions (i)–(v).

Then by (3.1), (2.1) holds. Also, (3.1) is true if we replace \( g_p^* \) by any \( g \in K_{m,n}' \). Hence, (2.2) holds by using conditions (i)–(v). Consequently, \( g_p^* \) is a best \( L_p \) approximation to \( f \) from \( K_{m,n}' \), since \( K_{m,n}' \) is a convex cone.

(b) Since the proof for \( p = 1 \) is similar, we omit the details. This completes the proof.

This theorem can be extended to characterize a best \( L_p \) approximation from \((m, n)_c\)-convex functions.

4. A Relationship between Best Approximations from \( S_{m,n}^k(A) \) and \( K_{m,n}^p \)

We assume throughout this section that \( f \in C[0, 1] \) and \( 1 \leq p < \infty \). To establish a relationship between best \( L_p \) approximations to \( f \) from \( S_{m,n}^k(A) \) and \( K_{m,n}^p \), we need the following theorem.

**Theorem 4.1.** For \( 1 \leq p < \infty \), let \( f \in C[0, 1] \) and \( g_p^* \in K_{m,n}^p \) be given.
Assume that \( f \neq g_p^* \) a.e. in \([0, 1]\) and that \( f - g_p^* \) has a finite number of sign changes in \((0, 1)\). Let

\[
\phi_p = \begin{cases} 
\text{sign}(f - g_p^*) |f - g_p^*|^{p-1} & 1 < p < \infty \\
\text{sign}(f - g_p^*) & p = 1 
\end{cases}
\]

and define \( H_{p,n}(x) \) as (2.3). Then \( g_p^* \) is a best \( L_p \) approximation to \( f \) from \( K_{m,n}^p \) if and only if (i)-(iv) (of Theorem 3.3) hold with \( 1 < p < \infty \), and

(v) \( g_p^* \) is a spline of degree \( n - 1 \) with simple knots \( \xi_1, \xi_2, \ldots, \xi_r \), the distinct zeros of \( H_{p,n} \) in \((0, 1)\).

**Proof.** Let \( g_p^* \in K_{m,n}^p \) be a best \((m, n)\)-convex \( L_p \) approximation to \( f \). By the hypothesis, \( f - g_p^* \) has a finite number of sign changes in \((0, 1)\). Assume that the number of sign changes of \( f - g_p^* \) in \((0, 1)\) is \( N \). By the definition of \( \phi_p \) for \( 1 \leq p < \infty \), \( \phi_p \) has \( N \) sign changes in \((0, 1)\). Since \( \phi_p = H_{p,n}^{(n)} \), by Rolle's Theorem, \( H_{p,n} \) has at most \( N + n \) zeros in \((0, 1)\), computing multiplicities. Let \( \xi_1 < \xi_2 < \cdots < \xi_r \) be the distinct zeros of \( H_{p,n} \) in \((0, 1)\), where \( r \leq N + n \). Let \( \xi_0 = 0 \) and \( \xi_{r+1} = 1 \). Note that \((-1)^m H_{p,n}(x) \leq 0, x \in [0, 1] \).

Hence,

\[
(-1)^m H_{p,n}(x) < 0 \quad \text{for } x \in (\xi_i, \xi_{i+1}), \quad i = 0, 1, \ldots, r.
\]

Thus, by (v) of Theorem 3.3, \( g_p^* \) is a polynomial of degree \( n - 1 \) on each subinterval \((\xi_i, \xi_{i+1})\). Since \( g_p^* \in C^{n-2}(0, 1) \), \( g_p^* \) is a spline of degree \( n - 1 \) with simple knots \( \xi_1, \xi_2, \ldots, \xi_r \).

Conversely, let \( g_p^* \) satisfy the assumptions and conditions (i)-(iv) and (v)'. If

\[
(-1)^m H_{p,n}(x_0) < 0 \quad \text{for some } x_0 \in (0, 1),
\]

then \( x_0 \notin \{\xi_1, \xi_2, \ldots, \xi_r\} \). Hence, \( x_0 \in (\xi_j, \xi_{j+1}) \) for some index \( j \in \{0, 1, \ldots, r\} \). By (v)' \( g_p^* \) is a polynomial of degree \( \leq n - 1 \) on \((\xi_j, \xi_{j+1})\), which is a neighborhood of \( x_0 \). Thus, by Theorem 3.3, \( g_p^* \) is a best \( L_p \) approximation to \( f \) from \( K_{m,n}^p \). This completes the proof.

By Theorem 2.1, 3.3 and the above theorem, the following theorem is readily proved, which establishes a relationship between best approximations to \( f \in C[0, 1] \) from \( S_{m,n}(\Delta') \) and \( K_{m,n}^p \), where \( \Delta' = \{\xi_i\}_{i=1}^r \).

**Theorem 4.2.** For \( 1 \leq p < \infty \), let \( g_p^* \) be a best \( L_p \) approximation to \( f \in C[0, 1] \) from \( K_{m,n}^p \) with all assumptions in Theorem 4.1 satisfied. Let \( \Delta' = \{\xi_i\}_{i=1}^r \) be the distinct zeros of \( H_{p,n} \) in \((0, 1)\). Then

(i) \( g_p^* \) is the best approximation to \( f \) from \( S_{m,n}(\Delta') \);
(ii) \( g_p^* \) is the best \( L_p \) approximation to \( f \) from the subspace
\[
S_n^{\star \star \star}(\Delta', \phi_p) = \{ s \in S_n'(\Delta'): s^{(i)}(1) = 0, \ i \in I(\phi_p) \};
\]

(iii) \( g_p^* \) is the unique best \( L_1 \) approximation to \( f \) from \( K_{m,n}^p \).

\textbf{Proof.} (i) It follows directly from Theorem 2.1, 3.3 and 4.1 that \( g_p^* \) is the best \( L_p \) approximation to \( f \) from \( S_n^{\star \star \star}(\Delta') \).

(ii) By (i) and Theorem 2.2, \( g_p^* \) is the best \( L_p \) approximation to \( f \) from \( S_n^{\star \star \star}(\Delta', \phi_p) \). Note \( \{ \zeta_i \}_{i=1}^r \) are the distinct zeros of \( H_{p,n} \) in \((0, 1)\). \( J(\phi_p) \) is an empty set. Hence, (ii) follows.

(iii) The uniqueness follows from (i) and the fact that there is a unique best \( L_1 \) approximation to \( f \in C[0, 1] \) from \( S_n^{\star \star \star}(\Delta') \) (see [9]).

\textbf{THEOREM 4.3.} For \( 1 < p < \infty \), let \( s_p^* \in S_{m,n}^k(\Delta) \) be the best \( L_p \) approximation to \( f \in L_p[0, 1] \) from \( S_{m,n}^k(\Delta) \). Assume that each knot \( x_j \) in \( \Delta \) is a non-trivial knot of \( s_p^* \). Then \( s_p^* \) is a best \( L_p \) approximation to \( f \) from \( K_{m,n}^p \) if and only if

\[
(-1)^m \int_{x_j}^{x} (x-t)^{n-1} \phi(t) \, dt \leq 0, \quad x \in [x_j, x_{j+1}], \ j = 0, 1, \ldots, n. \tag{4.2}
\]

\textbf{Proof.} Let \( s_p^* \) be a best \( L_p \) approximation to \( f \) from \( K_{m,n}^p \). By Theorem 3.3, we have \((-1)^m H_{p,n}(x) \leq 0\) for \( x \in [0, 1] \). Since \( x_j \) is a non-trivial knot of \( s_p^* \), \((-1)^m H_{p,n}(x_j) = 0\) for \( j \in N_k \). Hence,

\[
(-1)^m \int_{x_j}^{x} (x-t)^{n-1} \phi(t) \, dt = (-1)^m H_{m,n}(x) - (-1)^m H_{m,n}(x_j)
= (-1)^m H_{m,m}(x), \tag{4.3}
\]

and thus, (4.2) holds.

Conversely, let (4.2) hold. By (4.3), we have \(-1)^m H_{p,n}(x) \leq 0\) for \( x \in [0, 1] \). Conditions (i), (ii), (iv), and (v) of Theorem 2.1, and the above inequality imply conditions (i)–(v) of Theorem 3.3. Hence, \( s_p^* \) is a best \( L_p \) approximation to \( f \) from \( K_{m,n}^p \). This completes the proof.

As an application, let us establish an interesting relationship between best \( n \)-convex \( L_p \) approximation and best \( L_p \) approximation by \( n \)-convex splines of degree \( n-1 \). Let \( K_{n,p} \) denote the set of \( n \)-convex functions in \( L_p[0, 1] \).

\textbf{COROLLARY 4.1.} Let \( f \in C[0, 1] \). For \( 1 \leq p < \infty \), let \( g_p^* \in K_{n,p} \) such that \( f \neq g_p^* \) a.e. in \([0, 1]\) and \( f - g_p^* \) has a finite number of sign changes in \((0, 1)\). Define \( \phi_p \) as (4.1) and \( H_{p,n} \) as before. If \( g_p^* \) is a best \( L_p \) approximation to
f from $K_{n,p}$, then $g^*_p$ is a best $L_p$ approximation to $f$ from $S^{*}_{n,n}(A')$, where $A' : 0 < \xi_1 < \cdots < \xi_r < 1$, and the $\xi_i$'s are the distinct zeros of $H_{p,n}$ in $(0, 1)$.

5. **Best Monotone Convex $L_p$ Approximation**

As applications of the results in Section 3, we consider best $L_p$ approximation by monotone convex functions, and the relationship between best convex $L_p$ approximation and best monotone convex $L_p$ approximation.

For $1 \leq p < \infty$, let $M_\Omega(a, b)$ (resp., $M_\iota(a, b)$) be the set of nonincreasing (resp., nondecreasing) convex functions on $(a, b)$. Thus, $g(x) \in M_\Omega(a, b)$ if and only if $G(x) = g(-x) \in M_\iota(-b, -a)$. In addition, $g^*(x)$ is a best $L_p$ approximation to $f$ from $M_\Omega(a, b)$ if and only if $G^*(x) \equiv g^*(-x)$ is a best $L_p$ approximation to $F(x) = f(-x)$ from $M_\iota(-b, -a)$.

Since a nondecreasing convex function is $(1, 2)_\sigma$-convex with $\sigma = (1, -1)$ and a nonincreasing function is $(1, 2)_\rho$-convex with $\rho = (-1, -1)$, a similar reasoning to the proof of Theorem 3.3 gives the following two corollaries of Theorem 3.3:

**Corollary 5.1.** (a) For $1 < p < \infty$, $g^* \in M_\Omega(a, b)$ (resp., $M_\iota(a, b)$) is the best nonincreasing (resp., nondecreasing) convex $L_p$ approximation to $f \in L_p[a, b]$ if and only if

(i) $\int_a^b \phi_p(x) \, dx = 0$;

(ii) $\int_a^t (x - t) \phi_p(x) \, dx \leq 0$ (resp., $\int_a^t (t - x) \phi_p(x) \, dx \leq 0$) for all $t \in [a, b]$;

(iii) if $\int_a^b x \phi_p(x) \, dx > 0$ (resp., $\int_a^b x \phi_p(x) \, dx < 0$), then $g^*_p (b) = 0$ (resp., $g^*_p (a^-) = 0$);

(iv) if $\int_a^{t_0} (t_0 - x) \, dx < 0$ (resp., $\int_a^{t_0} (x - t_0) \phi_p(x) \, dx < 0$) for some $t_0 \in (a, b)$, then $g^*_p$ is a linear polynomial in a neighborhood of $t_0$.

(b) For $p = 1$, $g^*_\iota \in M_\Omega(a, b)$ (resp., $M_\iota(a, b)$) is a best nonincreasing (resp., nondecreasing) convex $L_1$ approximation to $f \in L_1[a, b]$ if and only if there exists a $\phi_1 \in L_\iota[a, b]$ with $\|\phi_1\|_\iota = 1$, $\int_a^b \phi_1(f - g^*_\iota) = \|f - g^*_\iota\|_1$, satisfying conditions (i)-(iv) in (a) with $p = 1$.

The next three theorems establish some relationships between best convex $L_p$ approximation and best monotone convex $L_p$ approximation.

**Theorem 5.1.** Let $g^*_p$ be a best convex $L_p$ approximation to $f \in L_p[0, 1]$, for $1 \leq p < \infty$. Then, there exists a $t \in [0, 1]$ such that $g^*_p$ is
both a best nonincreasing convex $L_p$ approximation to $f$ on $[0, t]$ and a best nondecreasing convex $L_p$ approximation to $f$ on $[t, 1]$.

**Proof.** If $g_p^*$ is nonincreasing (nondecreasing) on $(0, 1)$, then let $t = 1$ ($t = 0$). Assume that $g_p^*$ is a nonmonotone convex function. Let

$$m = \inf \{ g_p^*(x) : x \in [0, 1] \}.$$

Then the set $A = \{ x \in [0, 1] : g_p^*(x) = m \}$ is a nonempty and closed interval contained in $(0, 1)$. Define $t = \inf A$. Then, $g_p^*$ is nonincreasing on $(0, t)$ and nondecreasing on $(t, 1)$. By the definition of $t$, $g_p^*$ cannot be a linear polynomial in any neighborhood of $t$ which contains $t$ as an interior point. The characterization of best convex approximation implies $\int_0^t (t - x) \phi_p(x) \, dx = 0$. Thus, $g_p^*$ is a best approximation to $f$ on both $[0, t]$ and $[t, 1]$ (see [15, 19]). Since the set of nonincreasing convex functions in $L_p[0, t]$ is contained in the set of convex functions in $L_p[0, t]$, $g_p^*$ is also a best nonincreasing convex approximation to $f$ on $[0, t]$. Similarly, $g_p^*$ is a best nondecreasing convex approximation to $f$ on $[t, 1]$.

**Theorem 5.2.** For $1 < p < \infty$ let $f \in L_p[0, 1]$. Assume $t \in (0, 1)$. Let $g_p \in M_{\phi}(0, t)$ (resp., $g_p \in M_{\phi}(t, 1)$) be the best nonincreasing (resp., nondecreasing) convex $L_p$ approximation to $f$ on $[0, t]$ (resp., on $[t, 1]$). Define

$$\phi_{p, D}(x) = \text{sign} [f(x) - g_D(x)] |f(x) - g_D(x)|^{\frac{1}{p} - 1}, \quad \text{for} \quad x \in [0, t],$$

$$\phi_{p, 1}(x) = \text{sign} [f(x) - g_1(x)] |f(x) - g_1(x)|^{\frac{1}{p} - 1}, \quad \text{for} \quad x \in [t, 1],$$

and

$$g(x) = \begin{cases} g_D(x), & x \in [0, t] \\ g_1(x), & x \in (t, 1]. \end{cases}$$

Then, $g$ is the best convex $L_p$ approximation to $f$ on $[0, 1]$ if and only if

(i) $g_D(t) = g_1(t),$

(ii) $\int_0^t (t - x) \phi_{p, D}(x) \, dx = \int_t^1 (x - t) \phi_{p, 1}(x) \, dx.$

**Proof.** Let

$$\phi_p(x) = \begin{cases} \phi_{p, D}(x), & x \in [0, t] \\ \phi_{p, 1}(x), & x \in (t, 1]. \end{cases}$$

Assume $g$ is the best convex $L_p$ approximation to $f$ on $[0, 1]$. Then $g$ is continuous on $(0, 1)$ and thus $g_D(t) = g_1(t)$. In addition, by the charac-
terization of best convex $L_p$ approximation, we have $\int_0^1 \phi_p = 0$, and $\int_0^1 x\phi_p(x) \, dx = 0$. Hence,

$$\int_0^1 (t - x) \phi_p(x) \, dx = 0 \quad \text{for all} \quad t \in (0, 1).$$

It follows from this equation that (ii) holds.

Condition (i) with the facts that $g_D$ is nonincreasing convex on $[0, r]$ and $g_1$ is nondecreasing convex on $[r, 1]$ implies that $g$ is convex on $[0, 1]$. By the assumptions, we find

$$\int_0^1 \phi_p(x) \, dx = \int_0^r \phi_{p,D}(x) \, dx + \int_r^1 \phi_{p,1}(x) \, dx = 0$$

and

$$\int_0^1 x\phi_p(x) \, dx - \int_0^r (t - x) \phi_{p,D}(x) \, dx + \int_r^1 (t - x) \phi_{p,1}(x) \, dx = 0.$$

For $x \in [0, r],

$$\int_0^x (x - u) \phi_p(u) \, du = \int_0^x (x - u) \phi_{p,D}(u) \, du \leq 0,$$

and for $x \in (t, 1]$, by condition (ii),

$$\int_0^x (x - u) \phi_p(u) \, du$$

$$= \int_0^r (x - u) \phi_{p,D}(u) \, du + \int_r^x (x - u) \phi_{p,1}(u) \, du$$

$$= \int_0^r (t - u) \phi_{p,D}(u) \, du + \int_t^x (x - t) \phi_{p,1}(u) \, du + \int_t^x (t - u) \phi_{p,1}(u) \, du$$

$$= \int_t^1 (u - t) \phi_{p,1}(u) \, du - \int_t^x (u - t) \phi_{p,1}(u) \, du + \int_t^x (x - t) \phi_{p,1}(u) \, du$$

$$= \int_x^1 (u - t) \phi_{p,1}(u) \, du - \int_x^1 (x - t) \phi_{p,1}(u) \, du$$

Assume that for some $x_0 \in (0, 1), \int_0^{x_0} (x_0 - u) \phi_p(u) \, du < 0$. If $x_0 \in (0, t)$, then $g_D$ is a linear polynomial in a neighborhood of $x_0$ and so is $g$. If $x_0 \in (t, 1)$,
then by the above reasoning, we have \[ \int_{x_0}^{x} (u - x_0) \phi_{p,1}(u) \, du < 0. \] Thus, \( g_1 \) is a linear polynomial in a neighborhood of \( x_0 \) and so is \( g \). If \( x_0 = t \), in view of the continuity of \( \int_{0}^{x} (x - u) \phi_{p}(u) \, du \) for \( x \in [0, 1] \),

\[
\int_{0}^{x} (x - u) \phi_{p}(u) \, du < 0, \quad x \in (t - \delta_1, t], \text{ for some } \delta_1 > 0.
\]

By the characterization of best nonincreasing convex \( L_p \) approximation, we find that \( g'_-(t^-) = g'_{D-}(t^-) = 0 \) and \( g \) is a linear polynomial on \( (t - \delta_1, t] \).

In addition, since (ii) holds, \[ \int_{t}^{1} (x-t) \phi_{p,1}(x) \, dx < 0. \] Similarly, \( g'_+(t^-) = g'_{D+}(t^+) - 0 \), and \( g \) is a linear polynomial on \([t, t + \delta_2] \) for some \( \delta_2 > 0 \).

Hence,

\[ 0 = g'_-(t^-) \leq g'(t) \leq g'_+(t^+) = 0. \]

Thus \( g'(t) \) exists and vanishes. Therefore \( g \) is a constant on \( (t - \delta_1, t + \delta_2) \).

The conditions that we verify guarantee that \( g \) is the best convex \( L_p \) approximation to \( f \) on \([0, 1]\).

For \( p = 1 \), we have the following similar result:

**Theorem 5.3.** Let \( f \in L_1[0, 1] \) and \( t \in (0, 1) \). Assume \( g_D \in M_D(0, t) \) (resp., \( g_1 \in M_1(t, 1) \)) is a best nonincreasing (resp., nondecreasing) convex \( L_1 \) approximation to \( f \) on \([0, t] \) (resp., on \([t, 1]\)). Define

\[
g(x) = \begin{cases} 
g_D(x) & x \in [0, t] 
g_1(x) & x \in (t, 1] \end{cases},
\]

Let \( \Phi(g_D) \) be the set of \( \phi \in L_\infty[0, 1] \) with \( \|\phi\|_\infty = 1 \) and \( \int_{0}^{t} \phi(f - g_D) = \|f - g_D\|_1 \), satisfying conditions (i)-(v) of Corollary 5.1. Let \( \Phi(g_1) \) be the set of \( \phi \in L_\infty[t, 1] \) with \( \|\phi\|_\infty = 1 \) and \( \int_{t}^{1} \phi(f - g_1) = \|f - g_1\|_1 \), satisfying conditions (i)-(v) of Corollary 5.1. Then, \( g \) is a best convex \( L_1 \) approximation to \( f \) on \([0, 1]\) if and only if

(i) \( g_D(t) = g_1(t) \),

(ii) there exist \( \phi_D \in \Phi(g_D) \) and \( \phi_1 \in \Phi(g_1) \) such that

\[
\int_{0}^{t} (t-x) \phi_D(x) \, dx = \int_{t}^{1} (x-t) \phi_1(x) \, dx.
\]

**Proof.** Let

\[
\phi(x) = \begin{cases} 
\phi_D(x) & x \in [0, t] 
\phi_1(x) & x \in (t, 1] \end{cases}.
\]
Then, $\|\phi\|_\infty = 1$ and
\[
\int_0^1 \phi(f - g) = \int_0^1 \phi_D(f - g_D) + \int_1^1 \phi_D(f - g_1) = \|f - g\|_1.
\]

The rest of this proof is similar to the proof of Theorem 5.2. This completes the proof of Theorem 5.3.

**Remark.** All results in this paper could be generalized to Tchebycheffian splines and to functions generalized convex with respect to an $ECT$-system.

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