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On the p-Inner Functions of ℓ^p_A

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ON THE p-INNER FUNCTIONS OF ℓ^p_A

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ABSTRACT

ON THE p-INNER FUNCTIONS OF ℓ^p_A

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Define ℓ^p_A as the space of all functions holomorphic over the unit disk whose Taylor coefficients are p-summable. Despite their classical origins and simple definition, these spaces are not as well understood as one might expect. This is particularly true when compared with the Hardy spaces, which provide a useful road map for the types of questions we might consider reasonable. In this work we examine the zero sets of ℓ^p_A, $p \in (1, \infty)$, as well as a notion of inner function that is consistent with the approach taken on numerous other function spaces. Basic properties of p-inner functions are proved. It is shown that for some values of p, there are Blaschke sequences that fail to be a zero set for ℓ^p_A. It is also shown that canonical factorization fails for ℓ^p_A.

Dedicated to Mary

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Finally, I must thank my family for their support and patience, especially my wife Mary.

NOMENCLATURE

$:=$	Equality by definition
$\approx$	Approximate equality
$\simeq$	Isometric isomorphism
$\mathbf{a}$	The sequence a_0, a_1, a_2, ...
$\langle \mathbf{a}, \mathbf{b} \rangle$	$\sum_{k=0}^{\infty} a_k b_k$
$\mathbb{R}$	The set of real numbers
$\mathbb{N}$	The set of positive integers
$\mathbb{N}_0$	The set of nonnegative integers
$\mathbb{Z}$	The set of real integers
$\mathbb{C}$	The set of complex numbers
$\mathbb{T} := \{z \in \mathbb{C} \,\vert\, \vert z\vert = 1\}$	The unit circle in $\mathbb{C}$
$\mathbb{D} := \{z \in \mathbb{C} \,\vert\, \vert z\vert < 1\}$	The open unit disk in $\mathbb{C}$
$\overline{\mathbb{D}} := \{z \in \mathbb{C} \,\vert\, \vert z\vert \leqslant 1\}$	The closed unit disk in $\mathbb{C}$
ℓ^p	$\{\mathbf{a} \,\big\vert\, \sum_{k=0}^{\infty} \vert a_k\vert^p < \infty\}$
ℓ^p_A	$\{f(z) := \sum_{k=0}^{\infty} a_k z^k \,\big\vert\, \mathbf{a} \in \ell^p\}$
$\Vert\cdot\Vert_{\mathscr{X}}$	The norm on the space $\mathscr{X}$
$\Vert\cdot\Vert_p$	The norm on ℓ^p_A
$H^p := \{f \in \mathbb{D} \,\big\vert\, \Vert f\Vert_{H^p} < \infty\}$	The Hardy space H^p supported on $\mathbb{D}$
H^∞	The set of bounded analytic functions over $\mathbb{D}$
$\perp_p$	Birkhoff-James orthogonality in ℓ^p_A
r-LWP(C), r-UWP(C)	The weak parallelogram laws
$a^{\langle s \rangle}$	$\vert a\vert^{s-1}\overline{a}$, where $a \in \mathbb{C}$
$f^{\langle r \rangle}(z)$	$\sum_{k=0}^{\infty} a_k^{\langle r \rangle} z^k$, where $f(z) = \sum_{k=0}^{\infty} a_k z^k$
q	The Hölder conjugate of p (that is, $\frac{1}{p} + \frac{1}{q} = 1$)
$\bigvee \mathscr{X}$	The closed linear span of the set $\mathscr{X}$
S	The forward shift operator
B	The backward shift operator
Q_w	The difference quotient operator
$[f] := \bigvee \{S^n f\}_{n=0}^{\infty}$	The shift-invariant subspace generated by f
$\widehat{f}$	The metric projection of f onto the subspace $[Sf]$
$\mathscr{X}^*$	The dual of the space $\mathscr{X}$

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CHAPTER 1

INTRODUCTION

1.1 OVERVIEW

The sequence space ℓ^p of p-summable sequences is among the first Banach spaces encountered by the student of functional analysis. Indeed, this space was among the earliest to be systematically studied and provided motivation for the development of Banach space theory. It seems reasonable, then, to consider the space ℓ^p_A of analytic functions whose Taylor coefficients form a sequence in ℓ^p. The natural mapping between these spaces is clearly an isomorphism, so any study of ℓ^p will reveal more about the structure of ℓ^p_A.

We know from classical theory that ℓ^p, $p \in (1, \infty)$ is a reflexive and uniformly convex Banach space with dual space ℓ^q, where $\frac{1}{p} + \frac{1}{q} = 1$. The cases $p = 1$ and $p = \infty$ also yield Banach spaces, but we lose reflexivity and uniform convexity. It is also meaningful to discuss the case $p \in (0, 1)$; however, here ℓ^p is no longer a Banach space, but rather a Fréchet space. Our focus in this work will be exclusively on $p \in (1, \infty)$. Our techniques largely rely on certain generalizations of the parallelogram law that do not apply otherwise.

Despite the natural isomorphism to ℓ^p and potential to reveal further properties of the same, surprisingly little is known about ℓ^p_A. This is particularly true when compared with other notable function spaces such as the Hardy, Bergman, or Dirichlet spaces. Some attempts were made in the mid-twentieth century to examine ℓ^p_A, largely by Russian mathematicians (see [27, 30, 31, 32, 33]), but these papers tended to be highly technical and their results underscored how ℓ^p_A may require different methods and ideas from other classical function spaces.

The classical Hardy spaces represent a sort of gold standard when it comes to function theory. Among function spaces they are particularly well understood and offer plenty of reasonable questions to explore within other spaces. Since we will be using these H^p spaces as a jumping off point in our exploration of ℓ^p_A, we begin with a brief overview of H^p and some motivating results.

There are two results in H^p theory which motivate our research, both of which are closely related to the concept of an inner function in H^p. After recounting some preliminaries related to a certain concept of inner function on ℓ^p_A, Chapter 3 will develop some properties of these so-called p-inner functions.

The first of our motivating results concerns zero sets, an area in which H^p is particularly well understood. In general, we define a zero set of a function space to be any sequence W in the domain space such that $f(w) = 0$ for all $w \in W$ for some nontrivial function f in the space. It is natural to wonder which sequences form zero sets in any given space. In the case of H^p, the answer is surprisingly simple: the zero sets are precisely those sequences $\{w_k\}$ which satisfy $\Sigma(1 - |w_k|) < \infty$. This is known as the Blaschke condition, and such sequences are called Blaschke sequences.

The question of zero sets is not so simple for ℓ^p_A. For $p \in (1, 2)$, it is not the case that every Blaschke sequence is a zero set. In Chapter 4 we construct an example that proves this. It is also the case that for $p \in (2, \infty)$ every Blaschke sequence *is* a zero set, but it is not necessary that a zero set be a Blaschke sequence. (See [31] and [15, Section 10] for an example of this.) Our construction relies on a partial characterization of the zero sets of ℓ^p_A due to Cheng, Mashreghi, and Ross.[15]

The second motivating result for our studies is the canonical factorization theorem. For any function $f \in H^p$, there is a unique inner-outer factorization (up to multiplication by a unimodular constant). This result is very closely linked to the zero sets. In particular, the inner factor caries all of the zeros of the function. In Chapter 5 we will show by counterexample that in general no such factorization exists for ℓ^p_A.

Finally, in Chapter 6 we will discuss some potential areas for further study.

CHAPTER 2

PRELIMINARIES

In this chapter, we introduce the notation and terminology for this paper, and identify the tools needed to obtain our main results. But first, we review the motivating results in the Hardy spaces.

2.1 SOME QUICK H^p THEORY

The classical Hardy spaces represent a sort of gold standard when it comes to function theory. Among function spaces they are particularly well understood and offer plenty of reasonable questions to explore within other spaces. Since we will be using these H^p spaces as a jumping off point in our exploration of ℓ^p_A, we begin with a brief overview of H^p and some motivating results.

2.1.1 DEFINITION

We first present the definition of H^p:

Definition 2.1. Let $p \in (0, \infty)$. We define H^p to be the space of functions, f, holomorphic on the unit disk $\mathbb{D}$ such that

$$\|f\|_{H^p} := \sup_{0 \leqslant r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|^p \, d\theta \right)^{1/p} \tag{1}$$

is finite. We define H^∞ to be the set of bounded holomorphic functions on $\mathbb{D}$ and set

$$\|f\|_{H^\infty} := \sup_{z \in \mathbb{D}} |f(z)|$$

When $p \in [1, \infty]$, $\|\cdot\|_{H^p}$ defines a norm and H^p is a Banach space. For the remainder of this chapter, we will assume that $p \in (1, \infty)$ unless otherwise stated.[1]

[1]While the case $p \in (0, 1)$ is meaningful, we will be restricting our discussion of ℓ^p_A to $p \in (1, \infty)$, so there is no need here to discuss H^p with $p < 1$. It is perhaps also worth mentioning that the definition of H^p can be extended to arbitrary domains within $\mathbb{C}$, however our focus for ℓ^p_A is only over $\mathbb{D}$. So for the present work, we will only consider this domain and any mention of H^p will assume $\mathbb{D}$ as the domain.

Any function $f \in H^p$ has a radial limit a.e.$[d\theta]$, and it is customary to give the radial limit function the same name, i.e.,

$$f(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta}).$$

For any $p \in (1, \infty)$, the space ℓ^p_A is defined to be the class of analytic functions on the open unit disk $\mathbb{D}$ whose Taylor coefficients belong to the sequence space ℓ^p. The space ℓ^2_A coincides with the Hardy space H^2. For other values of p, relatively little is known about ℓ^p_A compared to other classical function spaces on $\mathbb{D}$, such as the Bergman, Dirichlet and Hardy spaces. A complete description of the zero sets of ℓ^p_A when $p \neq 2$, for example, is yet to be discovered.

2.1.2 MOTIVATING RESULTS IN H^p

As mentioned in the introduction, there are two results in H^p theory which motivate our research. We recount these results here formally for context, starting with the result on zero sets.

Definition 2.2 (Blaschke sequence)**.** Let $\mathbf{a}$ be a sequence in $\mathbb{D}$. Then $\mathbf{a}$ is said to satisfy the **Blaschke condition** if

$$\sum_k (1 - |a_k|) < \infty$$

Any sequence satisfying the Blaschke condition is called a **Blaschke sequence**.

Definition 2.3 (zero set)**.** Let $\mathscr{X}$ be a function space and $\mathbf{a}$ be a sequence over $\mathbb{C}$. Then $\mathbf{a}$ is said to be a **zero set** of $\mathscr{X}$, if there exists some $f \in \mathscr{X}$ such that $f(z) = 0$ if and only if $z \in \mathbf{a}$, where the number of times a value is repeated in $\mathbf{a}$ is treated as the multiplicity of the zero.

With these definitions in hand, we may state a complete characterization of the zero sets of H^p. [24, Corollary to Theorem 2.3]

Theorem 2.4. *Let* $\mathbf{a}$ *be a sequence in* $\mathbb{D}$*, and* $p \in [1, \infty]$*. Then* $\mathbf{a}$ *is a nontrivial zero set of* H^p *if and only if* $\mathbf{a}$ *is a Blaschke sequence.*

We exclude the trivial example of the zero set of the function $f(z) = 0$ being the entire disc.

Definition 2.5 (Blaschke product)**.** Let **a** be a Blaschke sequence; then the function

$$B(z) = z^m \prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}$$

is called a **Blaschke product**.

The product for $B(z)$ converges uniformly in each disk $|z| < R < 1$. The zeros of $B(z)$ within the open unit disk $\mathbb{D}$ are precisely the elements of the sequence **a**, with multiplicities equal to the number of times they occur in the sequence, and $z = 0$ with multiplicity m. Moreover, $|B(z)| < 1$ in $\mathbb{D}$ and $|B(e^{i\theta})| = 1$ a.e.[24] Any function in H^p satisfying this last condition is called an **inner function**. As we will see shortly, there is a property of inner functions on H^2 that will provide a more useful definition for our work.

Neither of the following two definitions are important for our work, but they are quite important to our next motivating theorem. We therefore present them here for the sake of completeness.

Definition 2.6 (outer function)**.** A function G analytic in $\mathbb{D}$ is said to be an **outer function** for H^p if it is of the form

$$G(z) = e^{i\gamma} \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \psi(t)\, dt \right\}$$

where $\psi(t) \in L^p$, $\psi(t) \geqslant 0$, $\log \psi(t) \in L^1$, and $\gamma \in \mathbb{R}$.

Definition 2.7. (singular inner function) A function S is said to be a **singular inner function** if it is of the form

$$S(z) = \exp\left\{ - \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \right\}$$

where $\mu(t)$ is a bounded nondecreasing singular function ($\mu'(t) = 0$ a.e.).

Theorem 2.8 (Canonical Factorization)**.** [24, Theorem 2.8] *Let f in H^p be not identically zero. Then we may factor f as*

$$f(z) = B(z)S(z)G(z)$$

where $B(z)S(z)$ is inner with $B(z)$ a Blaschke product and $S(z)$ a singular inner function, and G(z) is outer in H^p. This factorization is unique up to multiplication by a unimodular constant. Conversely, every such product belongs to H^p.

Theorems 2.4 and 2.8 are deep and highly consequential in the theory of Hardy spaces. The characterization of zero sets reveals much about the structure of H^p functions. Not only can the zeros of an analytic function not cluster in the domain of analyticity, but the growth restrictions on a function in H^p require that their zeros must tend rapidly to the boundary. The canonical factorization theorem relates directly to Beurling's theorem and the approximation of H^p functions by polynomials.

These theorems supply two of the main questions we seek to explore in this work. First, we construct a Blaschke sequence that fails to be a zero set for $p \in (1, 2)$. Second we examine whether a canonical inner-outer type factorization can exist in general in ℓ^p_A, $p \in (1, 2)$. To explore this we will need a concept of "inner" that makes sense in ℓ^p_A. The remainder of this chapter will build up the basic definitions and results we need for this.

2.2 DEFINITION OF ℓ^p_A

We now turn our attention to ℓ^p_A, beginig with a definition of the space.

Definition 2.9. For $p \in [1, \infty)$, we define

$$\ell^p_A := \left\{ f : \mathbb{D} \longmapsto \mathbb{C} \;\middle|\; f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ with } \sum_{n=0}^{\infty} |a_n|^p < \infty \right\}.$$

We further define

$$\|f\|_p := \Big(\sum_{n=0}^{\infty} |a_n|^p \Big)^{1/p},$$

That is, the function borrows the familiar ℓ^p norm of its coefficient sequence.

ℓ^p_A is a Banach space of analytic functions on the open unit disk $\mathbb{D}$.[2] We emphasize that the notation $\|\cdot\|_p$ refers to this norm, and not the norm on the Hardy space H^p. See [14, Section 2] for an exposition on the basic properties of ℓ^p_A.

Remark 2.10. In the present work, unless otherwise stated, p will always be assumed to be in the interval $(1, \infty)$, and q will denote its Hölder conjugate index. That is, p and q will satisfy $1/p + 1/q = 1$.

The dual of ℓ^p_A can be identified with ℓ^q_A by means of the usual bi-linear pairing

$$(f, g) = \sum_{k=0}^{\infty} a_k b_k, \tag{2}$$

[2] We may additionally define ℓ^∞_A by a similar analogy to the traditional ℓ^∞ sequence space and its norm; however any treatment of this space is outside the scope of our work here.

where $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \ell^p_A$, and $g(z) = \sum_{k=0}^{\infty} b_k z^k \in \ell^q_A$.[3]

2.3 BIRKHOFF-JAMES ORTHOGONALITY IN ℓ^p_A

The definition of ℓ^p_A makes sense for any $p \in (0, \infty]$. However, we focus on the range $p \in (1, \infty)$, for in this case ℓ^p_A enjoys certain geometric properties called the Weak Parallelogram Laws (see [11, 18, 19]).[4] Our methods are largely built upon these geometric foundations. Central to these developments is a general notion of orthogonality on normed linear spaces, called Birkhoff-James orthogonality ([3] contains a recent survey). This notion of orthogonality takes the following form in the context of ℓ^p_A:

Definition 2.11. (Birkhoff-James Orthogonality) For functions f and g in ℓ^p_A, we say that $f \perp_p g$ if

$$\|f + cg\|_p \geqslant \|f\|_p$$

for all scalars c.

Notice that $\perp_p$ agrees with the usual definition of orthogonality when $p = 2$. More generally, however, $\perp_p$ fails to be symmetric or linear. Birkhoff-James orthogonality arises in a natural way when considering extremal problems in Banach spaces, and the prediction theory of stochastic processes endowed with an L^p structure (see, for example, [8, 12, 17, 18, 19, 20, 26]).

In discussing this notion of orthogonality, we will find the following notation useful:

Definition 2.12. For $a \in \mathbb{C}$, and $s \in \mathbb{R}$,

$$a^{\langle s \rangle} := |a|^{s-1}\bar{a}.$$

where $0^{\langle 0 \rangle}$ is taken to be equal to 0.

Equivalently, if $a = re^{i\theta}$, with $r \geqslant 0$, we may write

$$a^{\langle s \rangle} = r^s e^{-i\theta}.$$

[3] When $p = 2$, we replace b_k with $\bar{b}_k$ in this sum; however our focus in this paper is on the non-Hilbert space case.

[4] In fact, ℓ^p_A is perfectly well defined for any $0 < p \leqslant \infty$, but fails to be a Banach space in the case $0 < p < 1$. For such values of p, $\|\cdot\|_p$ is a seminorm and ℓ^p_A is a Fréchet space.

We state some simple properties of this operation.

Lemma 2.13. *Let $p \in (1,\infty)$, $r, s \in \mathbb{R}$, and $z, w \in \mathbb{C}$. Then the following statements hold:*

1. $(zw)^{\langle s\rangle} = z^{\langle s\rangle} w^{\langle s\rangle}$
2. $|z|^s = z^{\langle s-1\rangle} z$
3. $(z^{\langle s\rangle})^r = (z^r)^{\langle s\rangle}$
4. $(a^{\langle p-1\rangle})^{\langle q-1\rangle} = a$

This last property will be of particular importance and is not particularly evident at a glance; a short proof is warranted.[5]

Proof. Let $\frac{1}{p} + \frac{1}{q} = 1$. Some simple algebra reveals that $p - 1 = \frac{p}{q}$ and $q - 1 = \frac{q}{p}$. Thus $(p-1)(q-1) = 1$. Next let $a = re^{i\theta}$, where $r \geqslant 0$.

$$\begin{aligned}
\left(a^{\langle p-1\rangle}\right)^{\langle q-1\rangle} &= \left(r^{p-1}e^{-i\theta}\right)^{\langle q-1\rangle} \\
&= r^{(p-1)(q-1)}e^{i\theta} \\
&= re^{i\theta} \\
&= a
\end{aligned}$$

□

With this notation in hand, here is an analytical criterion for $\perp_p$ [4].

Proposition 2.14. *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ are functions in ℓ^p_A. Then the condition $f \perp_p g$ is equivalent to*

$$\sum_{n=0}^{\infty} a_k^{\langle p-1\rangle} b_n = 0,$$

where any occurrence of $0^{\langle p-1\rangle}$ is interpreted as zero.[6]

---

[5] Properties 1-3 are quite simple, but proofs are still provided in Appendix B.1.

[6] Note that in the case $p = 2$, this reduces to the usual test for orthogonality in the inner product space ℓ^2_A.

From this we see that $\perp_p$ is a linear relation in its second argument; consequently, it makes sense to speak of the orthogonality of a function to a subspace of ℓ^p_A.

We will find the following notation useful:

Definition 2.15. For $f(z) = \sum_{k=0}^{\infty} a_k z^k$, let us write

$$f^{\langle r \rangle} := \sum_{k=0}^{\infty} a_k^{\langle r \rangle} z^k$$

If $f \in \ell^p_A$, it is easy to see that $f^{\langle p-1 \rangle} \in \ell^q_A$, and thus

$$f \perp_p g \Longleftrightarrow \left\langle g, f^{\langle p-1 \rangle} \right\rangle = 0. \tag{3}$$

Where $\langle \mathbf{a}, \mathbf{b} \rangle := \sum a_k b_k$ is the usual bilinear pairing.

2.3.1 PYTHAGOREAN THEOREM

There is a Pythagorean theorem for the L^p spaces (and more generally, for normed linear spaces satisfying the Weak Parallelogram Laws [6, 7, 18, 19]). It takes the form of a family of inequalities [19, Corollary 3.4].

Proposition 2.16. *Suppose that $x \perp_p y$ in L^p. If $p \in (1, 2]$, then*

$$\begin{aligned} \|x+y\|^p_{L^p} &\leqslant \|x\|^p_{L^p} + \frac{1}{2^{p-1}-1}\|y\|^p_{L^p} \\ \|x+y\|^2_{L^p} &\geqslant \|x\|^2_p + (p-1)\|y\|^2_{L^p}. \end{aligned}$$

If $p \in [2, \infty)$, then

$$\begin{aligned} \|x+y\|^p_{L^p} &\geqslant \|x\|^p_{L^p} + \frac{1}{2^{p-1}-1}\|y\|^p_{L^p} \\ \|x+y\|^2_{L^p} &\leqslant \|x\|^2_p + (p-1)\|y\|^2_{L^p}. \end{aligned}$$

When $p = 2$, the four inequalities merely simplify to the familiar Pythagorean theorem for the Hilbert space L^2. We will apply these Pythagorean inequalities in the setting of ℓ^p_A, in which the measure space is the nonnegative integers endowed with counting measure.

As a result of its uniform convexity, ℓ^p_A, enjoys the unique nearest point property. That is, given a subspace $\mathscr{M} \subseteq \ell^p_A$ and a function $f \in \ell^p_A$, there exists a unique $g \in \mathscr{M}$ such that

$$\|f - g\|_p = \inf_{h \in \mathscr{M}} \|f - h\|_p.$$

This extremal function g is called the **metric projection** of f onto $\mathscr{M}$. Furthermore, when $\mathscr{M}$ is a subspace of ℓ^p_A, we immediately have

$$f - g \ \perp_p \ \mathscr{M}.$$

2.4 OPERATORS ON ℓ^p_A

One of the things we tend to study in any function space is the bounded linear operators on it, particularly those that arise in a natural way. Our work will rely heavily on such operators, so we take a moment here to cover a few of them. The first operator we present is the forward shift:

Definition 2.17. Let $f \in \ell^p_A$. Then we define the **forward shift** operator S by

$$(Sf)(z) := zf(z), \ z \in \mathbb{D}$$

Note that S is an isometry.

The reason for calling S a shift is clear if we consider its effect on the series representation of f. Let $\mathbf{a}$ be the coefficient sequence for f.

$$\begin{aligned}(Sf)(z) &= z\sum_{k=0}^{\infty} a_k z^k \\ &= \sum_{k=0}^{\infty} a_k z^{k+1}\end{aligned}$$

Upon applying S, the coefficient a_k shifts from z^k to z^{k+1}, making the constant term zero, with every other term in $\mathbf{a}$ shifting one space to the right.

We may similarly define a backward shift operator B. This operator must first drop the constant term and then divide by z to shift the coefficient sequence to the left.

Definition 2.18. Let $f \in \ell^p_A$. Then we define the **backward shift** operator B by

$$(Bf)(z) := \frac{f(z) - f(0)}{z}$$

The forward shift will be essential to the results in this work, so for the sake of brevity, when we refer to "the shift operator" it will be assumed that we are discussing the forward shift unless otherwise stated. The backward shift is included here primarily for completeness, as well as some mild notational simplification later on.

One way in which the shift operator is essential to us is that it plays an important role in defining the p-inner functions that permeate our work. Before we can present a definition for these functions, we must first define a certain subspace of ℓ^p_A.

Definition 2.19. Let $f \in \ell^p_A$. Then we will use the following notation for the S**-invariant subspace** generated by f:

$$[f] := \bigvee \{f, Sf, S^2 f, \dots\}$$

where $\bigvee \mathscr{S}$ denotes the closed linear span of the set $\mathscr{S}$ of vectors in ℓ^p_A.

With the introduction of the S-invariant subspaces, we introduce the concept of a *cyclic vector* in ℓ^p_A.

Definition 2.20. Let $f \in \ell^p_A$. We call f **cyclic** if $[f] = \ell^p_A$.

Observe that to show that a function$f$ is cyclic, it is sufficient to show that $1 \in [f]$. This is true because any linear combination of the shifts of 1 (that is any linear combination of nonnegative integer powers of z) will then also be in $[f]$, and thus $\ell^p_A \subseteq [f]$. Since we also have $[f] \subseteq \ell^p_A$ by definition, it must be that $[f] = \ell^p_A$.

In addition, we write $\widehat{f}$ to denote the metric projection of $f \in \ell^p_A$ onto $[Sf]$. That is, $\widehat{f}$ is the unique function in $[zf]$ such that

$$\begin{aligned} \|f - \widehat{f}\|_p &= \inf_{h \in [zf]} \|f - h\|_p \\ &= \inf_{P \in \mathscr{P}(\mathbb{D})} \|f(z) - zf(z)P(z)\|_p \end{aligned} \tag{4}$$

where $\mathscr{P}(\mathbb{D})$ is the class of polynomials supported on $\mathbb{D}$.

Note that it follows immediately from Definition 2.19 that $f - \widehat{f} \perp_p [Sf]$.

We will also have need to consider the difference quotient operator:

Definition 2.21. Let $f \in \ell^p_A$ and $w \in \mathbb{D}$. Then we define the **difference quotient operator** on f at w by

$$(Q_w f)(z) := \frac{f(z) - f(w)}{z - w}$$

The mapping $f \longmapsto Q_w f$ is linear. Moreover, it is continuous on ℓ^p_A [15, Proposition 3.8].

Proposition 2.22. *Let $p \in (1, \infty)$, and let $w \in \mathbb{D}$. If $f \in \ell^p_A$, then*

$$\|Q_w f\|_p \leqslant \frac{1}{1-|w|}\|f\|_p.$$

2.5 p-INNER FUNCTIONS

Recall that an inner function in the Hardy space H^2 is a function whose radial limits are unimodular almost everywhere on the unit circle [24]. From the resulting condition

$$\int_{\partial\mathbb{D}} |f(e^{i\theta})|^2 e^{-in\theta}\frac{d\theta}{2\pi} = 0, \quad n \geqslant 1$$

we see that a unit vector $f \in H^2$ is inner if and only if

$$f \;\perp\; \bigvee\{zf(z), z^2 f(z), z^3 f(z), \ldots\}.$$

In the paper [15], a notion of inner on ℓ^p_A is introduced by analogy with this definition on H^2, consistent with the approach taken in some other settings.

Definition 2.23. A function $f \in \ell^p_A$ is p**-inner** if it is not identically zero, and

$$f \;\perp_p\; [Sf].$$

We do not require a p-inner function to be a unit vector; instead, it is convenient to adopt the normalization $J(0) = 1$ for a p-inner function J. This means of defining "inner" is consistent with the approaches taken in the Bergman space [2, 21, 22, 23] the Dirichlet space [28, 29], and other function spaces [5, 16]. It follows immediately from the definition that if $f \in \ell^p_A$ is not the zero function, then $J := f - \widehat{f}$ is p-inner; indeed, all p-inner functions arise in this manner [15, Proposition 5.4]. In addition, it must be that any zero of f is also a zero of J, multiplicities taken into account [15, Proposition 5.5].

Note that by virtue of the unique nearest point property, there is some choice $f(z)R(z)$ for which $\widehat{f}(z) = zf(z)R(z)$, though R itself need not be in ℓ^p_A.

In the case of a linear polynomial $f(z) = 1 - z/w$, with $w \in \mathbb{D} \setminus \{0\}$, the corresponding p-inner function $J = f - \widehat{f}$ takes the form

$$J(z) = \frac{1 - z/w}{1 - w^{\langle q-1\rangle} z}.$$

Indeed, when $p = 2$, this function is a Blaschke factor, apart from a multiplicative constant.[13, Proposition 8.3.6]

More generally, we have the following characterization of the p-inner function J associated with a polynomial with specified roots [15, Proposition 5.9].

Proposition 2.24. *Fix $p \in (1, \infty)$. Suppose that s_1, s_2,..., s_d are distinct nonzero elements of $\mathbb{D}$, and let n_1, n_2,..., n_d be positive integers. Let f be the polynomial*

$$f(z) = \Big(1 - \frac{z}{s_1}\Big)^{n_1} \Big(1 - \frac{z}{s_2}\Big)^{n_2} \cdots \Big(1 - \frac{z}{s_d}\Big)^{n_d}.$$

Then $J = f - \widehat{f}$ is of the form

$$J(z) = 1 + \sum_{k=1}^{\infty} \big(\sum_{m=1}^{d} \sum_{j=0}^{n_m - 1} C_{j,m} k^j s_m^k \big)^{\langle q-1 \rangle} z^k, \tag{5}$$

and the constants $C_{j,m}$ are uniquely determined by the conditions $J^{(m)}(s_k) = 0$ for all k, $1 \leqslant k \leqslant d$ and all m, $0 \leqslant m < n_k$, where $J^{(m)}$ stands for the mth derivative of J.

Notice that such J is analytic in a neighborhood of the closed disk $\overline{\mathbb{D}}$. Indeed, if $R = \max\{|s_1|, |s_2|, \ldots, |s_d|\}$, then the radius of convergence of the Taylor series for J is $1/R^{q-1}$. For additional examples and further developments, see [15].

We rely on the following partial characterization of zero sets of ℓ^p_A ([15, Theorem 2.2]).

Theorem 2.25. *Let $p \in (1, \infty)$ and suppose that $W = (w_1, w_2, \ldots) \subseteq \mathbb{D} \setminus \{0\}$. Define, for each $n = 1, 2, 3, \ldots$,*

$$f_n(z) := \Big(1 - \frac{z}{w_1}\Big)\Big(1 - \frac{z}{w_2}\Big) \cdots \Big(1 - \frac{z}{w_n}\Big)$$

and

$$J_n := f_n - \widehat{f_n}.$$

Then

1. *J_n is p-inner for all $n = 1, 2, 3, \ldots$;*

2. *$\|J_n\|_p$ is monotone increasing with n;*

3. *$f(W) = \{0\}$ for some nontrivial $f \in \ell^p_A$ if and only if*

$$\sup_{n \geqslant 1} \|J_n\|_p < \infty.$$

In this case, J_n converges in the norm of ℓ^p_A to a p-inner function $J \in \ell^p_A$ such that $J(W) = \{0\}$.

Note that this characterization is limited in the sense that both f and the p-inner function J could vanish at points outside of W (or at points of W with higher multiplicity). This problem of "extra zeros" will be addressed in Chapter 5 of this work.

The theorem ensures that for a given sequence W in $\mathbb{D}$, we may determine whether W is contained in a nontrivial zero set for ℓ^p_A by examining an associated sequence of p-inner functions with finitely many prescribed zeros. If the resulting norms tend toward infinity, then W fails to be a zero set.

To apply this criterion it is helpful to have a means to estimate the associated norms from below. For this purpose, we rely on the *point evaluation functionals* $k^{(n)}_w$ for $f^{(n)}(z)$, defined by

$$k^{(n)}_w(z) = \sum_{j=n}^{\infty} \frac{j!}{(j-n)!} w^{j-n} z^j = \frac{n! z}{(1-wz)^{n+1}} \tag{6}$$

for each $w \in \mathbb{D}$.[7] Then $k^{(n)}_w \in \ell^q_A$, where $1/p + 1/q = 1$.

Now for any $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \ell^p_A$, under the usual pairing we have

$$\langle f, k^{(n)}_w \rangle = \sum_{j=n}^{\infty} \frac{a_j j!}{(j-n)!} w^{j-n} = f^{(n)}(w).$$

The following is established in [16] by means of a duality argument.

Theorem 2.26. *Let $w_1, w_2, \ldots, w_M$ be a collection of distinct nonzero points of $\mathbb{D}$, and let $J := f - \widehat{f}$, where*

$$f(z) := \left(1 - \frac{z}{w_1}\right)\left(1 - \frac{z}{w_2}\right) \cdots \left(1 - \frac{z}{w_M}\right).$$

Then

$$\|J\|_p = \left[\inf \|1 + b_1 k^{(0)}_{w_1} + b_2 k^{(0)}_{w_2} + \cdots + b_M k^{(0)}_{w_M}\|_q\right]^{-1},$$

where the infimum is over the coefficients $b_1, b_2, \ldots, b_M$.

Thus any particular selection of the constants $b_1, b_2, \ldots, b_M$ gives rise to an upper bound for the infimum – and hence a lower bound for the norm of J. In this work, we will use G to notate the argument of the above infimum for a given p-inner function J. Such G exists since ℓ^q_A is uniformly convex, and G is the nearest point from a subspace of ℓ^q_A to the constant

[7] See Appendix B.2 for a proof that these are in fact the point evaluation functionals.

function 1. That is, for p-inner J with finite zero set $W = \{w_i\}_{i=1}^{M}$ with $w_i \neq w_j$ whenever $i \neq j$,

$$G(z) := 1 + b_1 k_{w_1}^{(0)} + b_2 k_{w_2}^{(0)} + \cdots + b_M k_{w_M}^{(0)} \tag{7}$$

Note: if the roots are not distinct, then a similar claim holds involving the appropriate kernel functions for the derivatives. Specifically, we would include terms with each $\{k_{w_j}^{(i)}\}_{i=0}^{m_j-1}$ for each root w_j with multiplicity m_j.

CHAPTER 3

PROPERTIES OF p-INNER FUNCTIONS

3.1 THE NORM OF J

We begin this chapter with a look at the norms of p-inner functions, J, corresponding to finite zero sets. In particular, we establish a bound on such norms as well as explore the norming functional of J. In our exploration of the norming functional, we also establish more about the relationship between J and its dual extremal function G.[8]

3.1.1 A BOUND ON $\|J\|_p$

Proposition 3.1. *Let $p \in (1, \infty)$, and $\varepsilon > 0$. Fix a positive integer d. There exists $R \in (0, 1)$, such that if f is a polynomial of degree d with all its roots in the annulus $\{R < |z| < 1\}$ and $f(0) = 1$, then*

$$1 \leqslant \|J\|_p \leqslant 1 + \varepsilon$$

where $J = f - \widehat{f}$.

Proof. For any positive integer n, and $1 \leqslant k \leqslant d$, define

$$u_k(z) := 1 - \frac{1}{n}\Big[\frac{z}{w_k} + \frac{z^2}{w_k^2} + \cdots + \frac{z^n}{w_k^n}\Big].$$

Obviously $u_k(w_k) = 0$, and $u_k(0) = 1$. Thus by the extremal property of J,

$$\|J\|_p \leqslant \|u_1 u_2 \cdots u_d\|_p.$$

We further establish that

$$\begin{aligned} \|u_k - 1\|_p^p &= \frac{1}{n^p|w_k|^p} + \frac{1}{n^p|w_k|^{2p}} + \cdots + \frac{1}{n^p|w_k|^{np}} \\ &\leqslant \frac{1}{n^p|w_k|^{np}} + \frac{1}{n^p|w_k|^{np}} + \cdots + \frac{1}{n^p|w_k|^{np}} \end{aligned}$$

[8] See equation (7).

$$\begin{aligned} &= \frac{1}{n^{p-1}|w_k|^{np}} \\ \|u_k - 1\|_p &\leqslant \frac{1}{n^{1/p'}|w_k|^n} \quad \text{and} \\ \|u_k\|_1 &= 1 + \frac{1}{n|w_k|} + \frac{1}{n|w_k|^2} + \cdots + \frac{1}{n|w_k|^n} \\ &\leqslant 1 + \frac{1}{|w_k|^n}. \end{aligned}$$

From

$$\begin{aligned} \|J\|_p^p - 1 &\leqslant \|u_1 u_2 \cdots u_d\|_p^p - 1 \\ &= \|1 - u_1 u_2 \cdots u_d\|_p^p \end{aligned}$$

it follows that we'll be done if we can show that $\|1 - u_1 u_2 \cdots u_d\|_p$ can be made arbitrarily small by choosing n sufficiently large. With that in mind,

$$\begin{aligned} \|1 - u_1 u_2 \cdots u_d\|_p &= \|(1-u_1) + u_1(1-u_2) + u_1u_2(1-u_3) + \cdots + u_1 u_2 \cdots u_{d-1}(1-u_d)\|_p \\ &\leqslant \|1-u_1\|_p + \|u_1(1-u_2)\|_p + \cdots + \|u_1u_2\cdots u_{d-1}(1-u_d)\|_p \\ &\leqslant \|1-u_1\|_p + \|u_1\|_1\|1-u_2\|_p + \cdots + \|u_1\|_1\|u_2\|_1\cdots\|u_{d-1}\|_1\|1-u_d\|_p \end{aligned}$$

Now if all of the roots $w_1, w_2, \ldots, w_d$ lie inside the annulus $\{R < |z| < 1\}$ for some $R \in (0,1)$, then the last quantity is bounded above by

$$\frac{d}{n^{1/p'}R^n}\left(1 + \frac{1}{R}\right)^d.$$

This can indeed be made arbitrarily small by choosing n sufficiently large, and R sufficiently close to 1. □

This extends to polynomials with a mix of zeros being perturbed toward the boundary or within the disk.

3.1.2 THE NORMING FUNCTIONAL OF J

As a result of the Hahn-Banach theorem, we know that any nonzero element of a smooth Banach space[9] $\mathscr{X}$ must have a unique norming functional λ. That is, given $x \neq 0 \in \mathscr{X}$,

[9] For $p \in (1,\infty)$, ℓ^p_A is, in fact, uniformly smooth. A Banach space is uniformly smooth if and only if its continuous dual is uniformly convex [25, Proposition 1.e.2, p. 61]. Since, as previously stated, ℓ^q_A is uniformly convex for any $q \in (1,\infty)$, it follows that ℓ^p_A, $p \in (1,\infty)$ is uniformly smooth.

there exists $\lambda \in \mathscr{X}^*$ with $\|\lambda\|_{\mathscr{X}^*} = 1$, such that $\lambda x = \|x\|_{\mathscr{X}}$. Here we consider the norming functional of a p-inner function corresponding to a finite zero set.

Proposition 3.2. *Let J be the p-inner function corresponding to a finite zero set of ℓ^p_A, $p \in (1, \infty)$, and G its associated dual extremal function as defined in* (7)*. Then the norming functional λ for J can be expressed in terms of either J or G as follows:*

$$\lambda = \frac{J^{\langle p-1\rangle}}{\|J\|_p^{p-1}} = \frac{G}{\|G\|_q}$$

That is

$$\left\langle J, \frac{J^{\langle p-1\rangle}}{\|J\|_p^{p-1}} \right\rangle = \left\langle J, \frac{G}{\|G\|_q} \right\rangle = \|J\|_p$$

Proof. We already know that $\|J\|_p = \|G\|_q^{-1}$.

Let $J = \sum_k a_k z^k$. We check by inspection that the function

$$\frac{J^{\langle p-1\rangle}}{\|J\|_p^{p-1}}$$

in ℓ^q_A is a norming functional for J:

$$\begin{aligned}
\left\langle J, \frac{J^{\langle p-1\rangle}}{\|J\|_p^{p-1}} \right\rangle &= \frac{1}{\|J\|_p^{p-1}} \left\langle \sum_k a_k z^k, \sum_k a_k^{\langle p-1\rangle} z^k \right\rangle \\
&= \frac{1}{\|J\|_p^{p-1}} \sum_k a_k^{\langle p-1\rangle} a_k \\
&= \frac{1}{\|J\|_p^{p-1}} \sum_k |a_k|^p \\
&= \frac{1}{\|J\|_p^{p-1}} \|J\|_p^p \\
&= \|J\|_p
\end{aligned}$$

Let's show that another norming functional for J is

$$\frac{G}{\|G\|_q}.$$

It is obviously norm one. Furthermore,

$$\langle J, G\rangle = \langle J, 1 + b_1\lambda_{w_1} + \cdots + b_N\lambda_{w_N}\rangle$$

$$
\begin{aligned}
&= J(0) + b_1 J(w_1) + \cdots + b_N J(w_N) \\
&= J(0) \\
&= 1.
\end{aligned}
$$

Therefore $\langle J, G/\|G\|_q \rangle = 1/\|G\|_q = \|J\|_p$. By uniqueness of norming functionals, this forces

$$
\frac{J^{\langle p-1 \rangle}}{\|J\|_p^{p-1}} = \frac{G}{\|G\|_q}.
$$

□

3.2 ROOTS OF UNIT MODULUS

Here we establish that removing a linear factor from a function does not have an effect on the shift invariant subspace generated if the root of that factor has unit modulus. Consequently any calculation relying on such subspaces can ignore any such factors of the generating function.

Proposition 3.3. *Let $p \in (1, \infty)$. If $f \in \ell^p_A$, and $|w| = 1$, then $[f(z)] = [f(z)(z - w)]$.*

Proof. We treat the case $w = 1$, the others being similar. It is clear that $[f(z)(1 - z)] \subseteq [f(z)]$. Thus, we will be done if we show that the expression

$$
\Phi_r(z) := f(z)(1 - z) + rzf(z)(1 - z) + r^2 z^2 f(z)(1 - z) + \ldots
$$

converges in ℓ^p_A to f.

Observe that

$$
\begin{aligned}
\Phi_r(z) &= f(z)(1 - z) + rzf(z)(1 - z) + r^2 z^2 f(z)(1 - z) + \ldots \\
&= f(z) - zf(z) + rzf(z) - rz^2 f(z) + r^2 z^2 f(z) - r^2 z^3 f(z) + \cdots \\
&= f(z) - (1 - r)zf(z) - (1 - r)rz^2 f(z) - (1 - r)r^2 z^3 f(z) - \cdots .
\end{aligned}
$$

This series converges absolutely in ℓ^p_A, because of the geometrically decaying factor r^k. Indeed, by the Triangle Inequality,

$$
\|(1 - r)zf(z) + (1 - r)rz^2 f(z) + (1 - r)r^2 z^3 f(z) + \cdots \|_p
$$

$$\begin{aligned}
&\leqslant (1-r)\|zf(z)\|_p + (1-r)r\|z^2 f(z)\|_p + (1-r)r^2\|z^3 f(z)\|_p + \cdots \\
&= \|f\|_p(1-r)(1+r+r^2+\cdots) \\
&= \|f\|_p.
\end{aligned}$$

Also, the kth term of the expression

$$f(z) - \Phi_r(z) = (1-r)zf(z) + (1-r)rz^2 f(z) + (1-r)r^2 z^3 f(z) + \cdots \tag{8}$$

is

$$(1-r)(c_0 r^{k-1} + c_1 r^{k-2} + \cdots + c_{k-1}).$$

By Hölder's Inequality this term is bounded as follows:

$$\begin{aligned}
&|(1-r)(f_0 r^{k-1} + f_1 r^{k-2} + \cdots + f_{k-1})|^p \\
\leqslant\ & (1-r)^p \|f\|_p^p \frac{1}{(1-r^q)^{p/q}}.
\end{aligned}$$

We can apply L'Hôpital's Rule to $(1-r)^q/(1-r^q)$ to see that

$$\frac{(1-r)^p}{(1-r^q)^{p/q}} = \left[\frac{(1-r)^q}{(1-r^q)}\right]^{p/q}$$

tends to the value zero as r increases to 1. That is, the ℓ_A^p norm of the expression $f(z)-\Phi_r(z)$, viewed as a function of its summation index, converges pointwise to zero.

With a view to applying the Dominated Convergence Theorem, we bound the kth term in the summation for $\|f(z) - \Phi_r(z)\|_p^p$ as follows:

$$\begin{aligned}
&\left|f_0 r^{k-1} + f_1 r^{k-2} + \cdots + f_{k-1}\right|^p \\
&\leqslant \left(|f_0| r^{k-1} + |f_1| r^{k-2} + \cdots + |f_{k-1}|\right)^p \\
&\leqslant \left(r^{k-1} + r^{k-2} + \cdots + 1\right)^{p-1}\left(|f_0|^p r^{k-1} + |f_1|^p r^{k-2} + \cdots + |f_{k-1}|^p\right) \\
&< \frac{1}{(1-r)^{p-1}}\left(|f_0|^p r^{k-1} + |f_1|^p r^{k-2} + \cdots + |f_{k-1}|^p\right),
\end{aligned}$$

where we have applied Jensen's Inequality. Thus we have

$$\|f(z) - \Phi_r(z)\|_p^p = \sum_{k=1}^{\infty} (1-r)^p \left|f_0 r^{k-1} + f_1 r^{k-2} + \cdots + f_{k-1}\right|^p$$

$$\leqslant \frac{(1-r)^p}{(1-r)^{p-1}} \sum_{k=1}^{\infty} \left(|f_0|^p r^{k-1} + |f_1|^p r^{k-2} + \cdots + |f_{k-1}|^p\right)$$
$$\leqslant (1-r) \sum_{k=1}^{\infty} \left(|f_0|^p r^{k-1} + |f_1|^p r^{k-2} + \cdots + |f_{k-1}|^p\right).$$

For each r, $0 < r < 1$, the series is summable in k, and indeed the sum is bounded above by

$$(1-r)\big(|f_0|^p + |f_1|^p + |f_2|^p \cdots\big)\big(1 + r + r^2 + \cdots\big) = \|f\|_p^p.$$

Thus, the Dominated Convergence Theorem indeed applies, with the expression

$$(1-r)\big(|f_0|^p r^{k-1} + |f_1|^p r^{k-2} + \cdots + |f_{k-1}|^p\big),$$

viewed as a function of the index k, serving as a suitable dominating function. The conclusion is that

$$\lim_{r\uparrow 1} \|f(z) - \Phi_r(z)\|_p^p = 0.$$

$\square$

Of course, if $|w| > 1$, then the polynomial $f(z) := 1 - z/w$ is cyclic. This is because

$$1 = (1 - z/w)(1 - z/w)^{-1} = f(z)\Big(1 + \frac{z}{w} + \frac{z^2}{w^2} + \cdots\Big),$$

and partial sums converge geometrically in norm. This shows that 1 lies in the span of $\{f(z), zf(z), z^2 f(z), \ldots\}$, and hence the invariant subspace generated by f is all of ℓ^p_A.

3.3 p-INNER PART

The following result illustrates that p-inner functions connect to the invariant subspaces for ℓ^p_A. Indeed, a stronger connection is established for H^p by Beurling's celebrated theorem and its corollaries. [24, Theorem 7.4] Unfortunately, in Chapter 5 of the present work we see that Beurling's theorem fails to extend to ℓ^p_A, $p \neq 2$.[10]

Proposition 3.4. *If $[f] = [g]$ in ℓ^p_A, and $f(0) = g(0) = 1$, then $f - \widehat{f} = g - \widehat{g}$.*

Proof.

$$\|g - \widehat{g}\|_p \leqslant \|g(z) + zg(z)\{(1 + zQ(z))P(z) + Q(z)\}\|_p$$

[10] See Corollary 5.2 for specifics.

$$
\begin{aligned}
&\leqslant \|g(z)[1+zQ(z)]+zg(z)(1+zQ(z))P(z)\|_p \\
&\leqslant \|f(z)+zf(z)P(z)\|_p+\|f-g(z)(1+zQ(z))\|_p \\
&\qquad +\|P\|_1\|zf(z)-zg(z)(1+zQ(z))\|_p \\
&\leqslant \|f-\widehat{f}\|_p+2\epsilon+\|P\|_1\epsilon.
\end{aligned}
$$

where we first choose a polynomial P, to get $\|f(z)+zf(z)P(z)\|_p$ close to $\|f(z)-\widehat{f}\|_p$; then choose Q, so that $\|f-g(z)[1+zQ(z)]\|_p$ is sufficiently small.

The argument reverses, so we've shown that $\|f-\widehat{f}\|_p=\|g-\widehat{g}\|_p$.

Next, choose polynomials Q_n so that $g(z)+zg(z)Q_n(z)\longrightarrow f$. Then, since $\widehat{f}\in[zg(z)]$,

$$
\begin{aligned}
\|f-\widehat{f}\|_p &= \lim_{n\to\infty}\|g(z)+zg(z)Q_n(z)-\widehat{f}\|_p \\
&= \|g-\widehat{g}\|_p.
\end{aligned}
$$

But the metric projection of g onto $[zg(z)]$ is unique, so this forces

$$
\lim_{n\to\infty}(zg(z)Q_n(z)-\widehat{f})=-\widehat{g}.
$$

Finally,

$$
\begin{aligned}
f-\widehat{f} &= \lim_{n\to\infty}\big(g(z)+zg(z)Q_n(z)\big)-\widehat{f} \\
&= g(z)+\lim_{n\to\infty}\big(zg(z)Q_n(z)-\widehat{f}\big) \\
&= g-\widehat{g}.
\end{aligned}
$$

□

We picked up the following:

Corollary 3.5. *If $g\in[f]$, and $f(0)=g(0)=1$, then $\|f-\widehat{f}\|_p\leqslant\|g-\widehat{g}\|_p$.*

3.4 CONTINUITY OF J

We now present the main result of this chapter, on the continuity of p-inner functions with respect to their zero sets.

Theorem 3.6. *Let $p\in(1,\infty)$, and J be p-inner. Then J is continuous with respect to perturbing finitely many zeros.*

Our proof of this result will require the following lemma:

Lemma 3.7. *Given $g \in \ell_A^\infty$, let $\Psi_g : \mathbb{D} \times \ell_A^p \mapsto \ell_A^p$ be defined as follows:*

$$\Psi_g(w, P) := \left(1 - \frac{z}{w}\right) g(z) - z\left(1 - \frac{z}{w}\right) g(z)P(z)$$

If $w \neq 0$, then Ψ is continuous in both arguments.

The proof of lemma requires a particular generalization of Hölder's inequality [34]. Specifically, if $r \in (0, \infty]$, and $p_1, \ldots, p_n \in (0, \infty]$ such that $\sum_{k=1}^n \frac{1}{p_k} = \frac{1}{r}$, then for all measurable functions $f_1, \ldots, f_n$ in the space,

$$\left\| \prod_{k=1}^n f_k \right\|_r \leqslant \prod_{k=1}^n \|f_k\|_{p_k}. \tag{9}$$

In particular, we will choose $n = 2$, $r = p_1 = p$, and $p_2 = \infty$ to get $\|fg\|_p \leqslant \|g\|_\infty \|f\|_p$, assuming all norms exist. With this in hand, we now prove lemma 3.4.

Proof. Fix $w \in \mathbb{D}\backslash\{0\}$ and $P \in \ell_A^p$ and choose $\delta \in (0, |w|/2)$. Suppose $\widetilde{w}$ and $\widetilde{P}$ are such that

$$|w - \widetilde{w}| < \delta$$

and

$$\|P - \widetilde{P}\|_p < \delta$$

First observe that $\frac{1}{2}|w| \leqslant |\widetilde{w}| \leqslant \frac{3}{2}|w|$.

$$\begin{aligned} |\widetilde{w}| &= |\widetilde{w} - w + w| \\ &\leqslant |\widetilde{w} - w| + |w| \\ &\leqslant \delta + |w| \\ &\leqslant \frac{|w|}{2} + |w| \\ &= \frac{3}{2}|w| \end{aligned}$$

$$\begin{aligned} |w| &= |w - \widetilde{w} + \widetilde{w}| \\ &\leqslant |w - \widetilde{w}| + |\widetilde{w}| \\ &\leqslant \delta + |\widetilde{w}| \end{aligned}$$

$$\leqslant \frac{|w|}{2} + |\widetilde{w}|$$
$$\frac{1}{2}|w| \leqslant |\widetilde{w}|$$

From this we may deduce that $\frac{1}{2}|w|^2 \leqslant |w\widetilde{w}| \leqslant \frac{3}{2}|w|^2$.

We now have the following:

$$\begin{aligned}
\|\Psi(w,P) - \Psi(\widetilde{w},\widetilde{P})\|_p &= \left\| f + zfP - \widetilde{f} - z\widetilde{f}\widetilde{P} \right\|_p \\
&= \left\| \left[1 - \frac{z}{w} + \left(z - \frac{z^2}{w} \right) P - 1 + \frac{z}{\widetilde{w}} - \left(z - \frac{z^2}{w} \right) \widetilde{P} \right] g(z) \right\|_p \\
&= \left\| \left[\left(\frac{1}{\widetilde{w}} - \frac{1}{w} \right) + (P - \widetilde{P}) + z \left(\frac{1}{\widetilde{w}}\widetilde{P} - \frac{1}{w}P \right) \right] zg(z) \right\|_p \\
&\leqslant \|zg(z)\|_\infty \left\| \left(\frac{1}{\widetilde{w}} - \frac{1}{w} \right) + (P - \widetilde{P}) + z \left(\frac{1}{\widetilde{w}}\widetilde{P} - \frac{1}{w}P \right) \right\|_p \qquad (10) \\
&\leqslant \|g(z)\|_\infty \left[\left| \frac{1}{\widetilde{w}} - \frac{1}{w} \right| + \left\| P - \widetilde{P} \right\|_p + \left\| \frac{1}{\widetilde{w}}\widetilde{P} - \frac{1}{\widetilde{w}}P + \frac{1}{\widetilde{w}}P - \frac{1}{w}P \right\|_p \right] \\
&\leqslant \|g(z)\|_\infty \left[\left| \frac{w - \widetilde{w}}{w\widetilde{w}} \right| + \delta + \left\| \frac{1}{\widetilde{w}}\widetilde{P} - \frac{1}{\widetilde{w}}P \right\|_p + \left\| \frac{1}{\widetilde{w}}P - \frac{1}{w}P \right\|_p \right] \\
&= \|g(z)\|_\infty \left(\frac{\delta}{|w\widetilde{w}|} + \delta + \frac{1}{|\widetilde{w}|} \left\| \widetilde{P} - P \right\|_p + \left| \frac{1}{\widetilde{w}} - \frac{1}{w} \right| \|P\|_p \right) \\
&\leqslant \|g(z)\|_\infty \left(\frac{\delta}{|w\widetilde{w}|} + \delta + \frac{1}{|\widetilde{w}|}\delta + \frac{\delta}{|w\widetilde{w}|}\|P\|_p \right) \\
&= \delta\|g(z)\|_\infty \left(\frac{1}{|w\widetilde{w}|} + 1 + \frac{1}{|\widetilde{w}|} + \frac{1}{|w\widetilde{w}|}\|P\|_p \right) \\
&= \delta\|g(z)\|_\infty \frac{1 + |w\widetilde{w}| + |w| + \|P\|_p}{|w\widetilde{w}|} \\
&\leqslant \delta\|g(z)\|_\infty \frac{1 + \frac{3}{2}|w|^2 + |w| + \|P\|_p}{\frac{1}{2}|w|^2}
\end{aligned}$$

Where for the step (10) we have used the generalization of Hölder's inequality (9).

This last expression clearly vanishes as δ approaches 0. Thus Ψ is continuous in both w and P. □

3.4.1 PROOF OF THE THEOREM

We now present our proof of Theorem 3.6.

Proof. Fix $p \in (1, \infty)$ and let W be a zero set for ℓ^p_A with corresponding p-inner function J_W. Furthermore, let $\mathbf{w}$ be a sequence in $\mathbb{D}$ that converges to some $w \in \mathbb{D}$, not equal to 0. Finally, let J_n be the p-inner function corresponding to the zero set $W \cup \{w_n\}$, and J the p-inner function corresponding to $W \cup \{w\}$.

First note that by the extremal property of p-inner functions, J_n has the minimal norm of all functions in ℓ^p_A with zero set $W \cup \{w_n\}$. This together with Young's convolution inequality, gives the following bound.

$$\begin{aligned} \|J_n\|_p &\leqslant \left\| \left(1 - \frac{z}{w_n}\right) J_W(z) \right\|_p \\ &\leqslant \left\| 1 - \frac{z}{w_n} \right\|_1 \|J_W(z)\|_p \\ &= \left(1 + \frac{1}{|w_n|}\right) \|J_W\|_p \end{aligned} \tag{11}$$

We may similarly claim that $\|J\|_p \leqslant \left(1 + \frac{1}{|w|}\right) \|J_W\|_p$.

Recall that

$$\|J_n\|_p = \inf_{P(z)} \left\| \left(1 - \frac{z}{w_n}\right) J_W(z) + z\left(1 - \frac{z}{w_n}\right) J_W(z)P(z) \right\|_p$$

where the infimum is attained for some $J_W P \in \ell^p_A$. For simplicity in notation, let R_n replace P for this extremal function. Note that R_n need not be a function, but $J_W R_n$ is well defined. Similarly, let R replace P for the extremal function corresponding to $\|J\|$.

Since $w_n \to w \neq 0$, we may assume that $\mathbf{w}$ is bounded away from zero. Then by (11), there is some positive constant M such that $\|J_n\|_p \leqslant M\|J_W\|_p$.

Furthermore,

$$\begin{aligned} \|J_W R_n\|_p &\leqslant \|J_W(z) + zJ_W(z)R_n(Z)\|_p + \|J_W(z)\|_p \\ &\leqslant \frac{|w_n|}{1 - |w_n|} \left\| \left(1 - \frac{z}{w_n}\right) J_W(z) + z\left(1 - \frac{z}{w_n}\right) J_W(z)R_n(z) \right\|_p + \|J_W(z)\|_p \\ &\leqslant \frac{|w_n|}{1 - |w_n|} \|J_n\|_p + \|J_W\|_p \\ &\leqslant \left(\frac{M|w_n|}{1 - |w_n|} + 1\right) \|J_W\|_p \end{aligned} \tag{12}$$

Since $\mathbf{w}$ converges inside $\mathbb{D}$, this is a uniform bound.

Now we may calculate

$$
\begin{aligned}
\|J_W\|_p &= \inf_{P(z)} \left\| \left(1-\frac{z}{w}\right) J_W(z) + z\left(1-\frac{z}{w}\right) J_W(z)P(z) \right\|_p \\
&\leqslant \left\| \left(1-\frac{z}{w}\right) J_W(z) + z\left(1-\frac{z}{w}\right) J_W(z)R_n(z) \right\|_p \\
&\leqslant \left\| \left(1-\frac{z}{w_n}\right) J_W(z) + z\left(1-\frac{z}{w_n}\right) J_W(z)R_n(z) \right\|_p \\
&\quad + \left\| \left(\frac{z}{w_n}-\frac{z}{w}\right) J_W(z) + z\left(\frac{z}{w_n}-\frac{z}{w}\right) J_W(z)R_n(z) \right\|_p \\
&\leqslant \|J_n(z)\|_p + \left|\frac{z}{w_n}-\frac{z}{w}\right| \cdot \|J_W(z) + zJ_W(z)R_n(z)\|_p \\
&\leqslant \|J_n\|_p + \left|\frac{z}{w_n}-\frac{z}{w}\right| \cdot \left(\frac{M|w_n|}{1-|w_n|}+2\right) \|J_W\|_p
\end{aligned}
$$

which implies

$$
\|J_W\|_p \leqslant \liminf_{n\to\infty} \|J_n\|_p \tag{13}
$$

Similarly,

$$
\begin{aligned}
\|J_n\|_p &= \inf_{P(z)} \left\| \left(1-\frac{z}{w_n}\right) J_W(z) + z\left(1-\frac{z}{w_n}\right) J_W(z)P(z) \right\|_p \\
&\leqslant \left\| \left(1-\frac{z}{w_n}\right) J_W(z) + z\left(1-\frac{z}{w_n}\right) J_W(z)R(z) \right\|_p \\
&\leqslant \left\| \left(1-\frac{z}{w}\right) J_W(z) + z\left(1-\frac{z}{w}\right) J_W(z)R(z) \right\|_p \\
&\quad + \left\| \left(\frac{z}{w_n}-\frac{z}{w}\right) J_W(z) + z\left(\frac{z}{w_n}-\frac{z}{w}\right) J_W(z)R(z) \right\|_p \\
&\leqslant \|J(z)\|_p + \left|\frac{z}{w_n}-\frac{z}{w}\right| \cdot \|J_W(z) + zJ_W(z)R(z)\|_p \\
&\leqslant \|J\|_p + \left|\frac{z}{w_n}-\frac{z}{w}\right| \cdot \left(\frac{M|w_n|}{1-|w_n|}+2\right) \|J_W\|_p
\end{aligned}
$$

which implies

$$
\limsup_{n\to\infty} \|J_n\|_p \leqslant \|J_W\|_p \tag{14}
$$

Combining (13) and (14) gives

$$
\lim_{n\to\infty} \|J_n\|_p = \|J_W\|_p \tag{15}
$$

By Alaoglu's theorem, there is a subsequence $\{J_{n_k}\}$ which converges weakly to some

$L \in \ell^p_A$.[11] Together with (15), this shows we have J_{n_k} converges in norm to L.

Since $J_n(z) = \left(1 - \frac{z}{w_n}\right) J_W(z) + z\left(1 - \frac{z}{w_n}\right) J_W(z)R_n(z)$, we have

$$J_W R_n = -w_n Q_{w_n} B\left[J_n - \left(1 - \frac{z}{w_n}\right) J_W\right]$$

where B is the backward shift operator (that is, division by z), and Q_{w_n} is the difference quotient operator. This implies $J_W R_{n_k}$ converges in ℓ^p_A to $J_W\tilde{R}$ for some $J_W\tilde{R}$.

Next we consider the function

$$\Psi : \mathbb{C} \times [J_W] \to \ell^p_A$$

defined by

$$\Psi(\zeta, P) := \left(1 - \frac{z}{\zeta}\right) J_W(z) + z\left(1 - \frac{z}{\zeta}\right) J_W(z)P(z)$$

By Lemma 3.4, Ψ is continuous for ζ bounded away from the origin, and for all $[J_W]$. Continuity confers the property that

$$\Psi(w_{n_k}, R_{n_k}) \to \Psi(w, \tilde{R}).$$

By definition of R as the extremal case of P, we also have

$$\|\Psi(w, \tilde{R})\|_p \geqslant \|\Psi(w, R)\|_p,$$

with strict inequality if $\tilde{R} \neq R$.

Since $\Psi(w_{n_k}, R) \to \Psi(w, R)$, we get $\|\Psi(w_{n_k}, R)\|_p \to \|\Psi(w, R)\|_p$.

But $\|\Psi(w_{n_k}, R)\|_p \geqslant \|\Psi(w_{n_k}, R_{n_k})\|_p$ for all k. By taking limits, we see that

$$\|\Psi(w, R)\|_p \geqslant \|\Psi(w, \tilde{R})\|_p$$

forcing $\tilde{R} = R$.

Because any subsequence of $\mathbf{w}$ itself has a subsequence for which the corresponding $J_W R_{n_k}$ converges in norm to $J_W R$, we have that $J_W R_n$ converges to $J_W R$.

Thus $J_n \to J$ in norm, and there is continuity with respect to perturbing finitely many zeros, completing the proof. $\square$

[11] For reflexive spaces such as ℓ^p_A, $p \in (1, \infty)$, the weak and weak-$*$ topologies coincide.

CHAPTER 4

A BLASCHKE SEQUENCE THAT IS NOT A ZERO SET

4.1 INTRODUCTION

From the Hausdorff-Young Inequality [24, Theorem 6.1], we know that when $2 < p < \infty$, the Hardy class H^q is contained in ℓ^p_A, where $1/p + 1/q = 1$. Consequently, all Blaschke sequences are zero sets for ℓ^p_A, when $p \in (2, \infty)$. Vinogradov [31] showed by example that the containment is proper. Another example, based on a different approach, is furnished in [15, Section 10].

On the other hand, when $p \in (1, 2)$, the space ℓ^p_A is a subset of H^q. As a result, all nontrivial zero sets for ℓ^p_A in this instance must be Blaschke sequences. Our present aim is to show by construction that the containment is proper. The main result is the following — it has been published in [9].

Theorem 4.1. *If $p \in (1, 2)$, then there exists a Blaschke sequence W of points in $\mathbb{D}$ such that any function $f \in \ell^p_A$ vanishing on W must vanish identically.*

The proof relies on the partial characterization of the zero sets of ℓ^p_A given by Theorem 2.25.

Although the result is not surprising, the underlying construction is new, and the ideas and methods surrounding it can shed further light on the space ℓ^p_A. Indeed, inner functions and zero set properties make contact with canonical factorization, invariant subspaces, interpolation, and other important unsolved problems about ℓ^p_A.

4.2 PROOF OF THE THEOREM

We now present the proof of Theorem 4.1.

Proof. Let $p \in (1, 2)$, and let $1/p + 1/q = 1$. To prove Theorem 4.1 it will suffice to exhibit a Blaschke sequence that fails to be a zero set for ℓ^p_A. This sequence will comprise the 2^nth roots of unity, multiplied by some common radius r_n, for each $n = 1, 2, 3, \ldots$. We can choose $\{r_n\}_{n=1}^{\infty}$ so that the resulting sequence of points satisfies the Blaschke condition. With this selection of points, a lower bound for the norms of associated sequence of p-inner functions can be computed, by use of Theorem 2.26. It is then shown that these lower bounds diverge to infinity. The proof is then completed by invoking Theorem 2.25.

We begin with an observation. Let n be an integer greater than 1, and consider the nth roots of unity

$$e^{2\pi i\cdot 0/n},\ e^{2\pi i\cdot 1/n},\ e^{2\pi i\cdot 2/n},\ldots,\ e^{2\pi i\cdot(n-1)/n}.$$

They form a group $\mathscr{G}$ under multiplication. For any non-negative integer j, their respective powers

$$e^{2\pi ij\cdot 0/n},\ e^{2\pi ij\cdot 1/n},\ e^{2\pi ij\cdot 2/n},\ldots,\ e^{2\pi ij\cdot(n-1)/n}.$$

constitute a subgroup of $\mathscr{G}$. It is the trivial subgroup precisely if $\mathrm{GCD}(j,n)$ is equal to n. In this case, the sum of these powers is just n.

Let us denote the sum of these powers by $\Theta(n,j)$:

$$\Theta(n,j) := e^{2\pi ij\cdot 0/n} + e^{2\pi ij\cdot 1/n} + e^{2\pi ij\cdot 2/n} + \cdots + e^{2\pi ij\cdot(n-1)/n}.$$

When $\mathrm{GCD}(j,n) < n$, $\Theta(n,j)$ is equal to zero, due to the symmetric placement of terms around the origin. Indeed, if the subgroup contains r members, where r divides n, then exactly n/r elements of $\mathscr{G}$ map to each element of the subgroup. These properties will enable the subsequent calculation of norm estimates to be tractable.

Next, suppose that

$$0 < r_1 < r_2 < r_3 < \ldots < 1$$

and consider the finite collection S_N of points in $\mathbb{D}$ consisting of

$$\begin{aligned}
&r_1 e^{2\pi i\cdot 0/2^1},\ r_1 e^{2\pi i\cdot 1/2^1},\\
&r_2 e^{2\pi i\cdot 0/2^2},\ r_2 e^{2\pi i\cdot 1/2^2},\ r_2 e^{2\pi i\cdot 2/2^2},\ r_2 e^{2\pi i\cdot 3/2^2},\\
&\ldots\\
&r_N e^{2\pi i\cdot 0/2^N},\ r_N e^{2\pi i\cdot 1/2^N},\ r_N e^{2\pi i\cdot 2/2^2},\ldots,\ r_N e^{2\pi i\cdot(2^N-1)/2^N},
\end{aligned}$$

for $N = 1,2,3,\ldots$. Note that $S_1 \subseteq S_2 \subseteq S_3 \subseteq \cdots$. By choice of the radii r_n, we will ensure that the union W of these sets S_N will serve as the Blaschke sequence that fails to be a zero set.

Toward the goal of applying Theorem 2.25, we define J_N to be the p-inner function with unit constant term that vanishes on S_N, $N = 1,2,3,\ldots$. (It may happen to vanish at other points as well—this would have no effect on the construction). That is, $J_N := f_N - \widehat{f_N}$, where

$$f_N(z) := \prod_{w \in S_N} \Big(1 - \frac{z}{w}\Big).$$

We would like to estimate the norm in ℓ^p_A of J_N, using the formula from Theorem 2.26. Accordingly we enumerate the points of S_N as

$$S_N = \{w_1, w_2, w_3, \dots, w_M\}$$

(thus $M = 2^1 + 2^2 + \cdots + 2^N$), and consider

$$\|J_N\|_p = \Big[\inf \|1 + B_1 k_{w_1} + B_2 k_{w_2} + \cdots + B_M k_{w_M}\|_q\Big]^{-1}, \tag{16}$$

where the infimum is over the complex coefficients $B_1, B_2, \dots, B_M$.

We may, at the permissible cost of exceeding the infimum, assume that the points of S_N with the same radius r_j share a common coefficient b_j in (16). Then

$$\begin{aligned}
&1 + B_1 k_{w_1} + B_2 k_{w_2} + \cdots + B_M k_{w_M} \\
&= (1 + 2^1 b_1 + 2^2 b_2 + \cdots + 2^N b_N) \\
&\quad + \big[b_1 r_1 \Theta(2^1, 1) + b_2 r_2 \Theta(2^2, 1) + \cdots + b_N r_N \Theta(2^N, 1)\big] z^1 \\
&\quad + \big[b_1 r_1^2 \Theta(2^1, 2) + b_2 r_2^2 \Theta(2^2, 2) + \cdots + b_N r_N^2 \Theta(2^N, 2)\big] z^2 \\
&\quad + \cdots \\
&\quad + \big[b_1 r_1^j \Theta(2^1, j) + b_2 r_2^j \Theta(2^2, j) + \cdots + b_N r_N^j \Theta(2^N, j)\big] z^j \\
&\quad + \cdots .
\end{aligned}$$

Writing

$$\Delta_N := \inf \|1 + B_1 k_{w_1} + B_2 k_{w_2} + \cdots + B_M k_{w_M}\|_q^q,$$

we now have the bound

$$\begin{aligned}
\Delta_N \leqslant\ & \big|1 + 2^1 b_1 + 2^2 b_2 + \cdots + 2^N b_N\big|^q \\
& + \big|b_1 r_1 \Theta(2^1, 1) + b_2 r_2 \Theta(2^2, 1) + \cdots + b_N r_N \Theta(2^N, 1)\big|^q \\
& + \big|b_1 r_1^2 \Theta(2^1, 2) + b_2 r_2^2 \Theta(2^2, 2) + \cdots + b_N r_N^2 \Theta(2^N, 2)\big|^q \\
& + \cdots \\
& + \big|b_1 r_1^j \Theta(2^1, j) + b_2 r_2^j \Theta(2^2, j) + \cdots + b_N r_N^j \Theta(2^N, j)\big|^q
\end{aligned}$$

$$+\cdots. \tag{17}$$

Next, use the fact that if j is a multiple of 2^n, then perforce it must be a multiple of 2, 2^2,..., 2^{n-1} as well. In this situation $\Theta(2^n, j) = 2^n$ for all n, $1 \leqslant n \leqslant j$. Therefore, the above sum over j could be grouped into separate sums over odd multiples of 2, odd multiples of 2^2, odd multiples of 2^3, and so on. When we reach the last layer of 2^N roots, we'll have to sum over all the multiples (not merely the odd multiples), so as to account for all j. We are discarding the terms with odd values of j, since all of the corresponding $\Theta(2^n, j)$ are zero.

Writing $\mathscr{O}$ for the set of odd positive integers, and substituting the numerical values of each $\Theta(2^n, j)$, we obtain

$$\begin{aligned}
\Delta_N \leqslant& \left|1 + 2^1 b_1 + 2^2 b_2 + \cdots + 2^N b_N\right|^q \\
&+ \sum_{j\in 2\cdot\mathscr{O}} \left|b_1 r_1^j \Theta(2^1, j) + b_2 r_2^j \Theta(2^2, j) + \cdots + b_N r_N^j \Theta(2^N, j)\right|^q \\
&+ \sum_{j\in 2^2\cdot\mathscr{O}} \left|b_1 r_1^j \Theta(2^1, j) + b_2 r_2^j \Theta(2^2, j) + \cdots + b_N r_N^j \Theta(2^N, j)\right|^q \\
&+ \sum_{j\in 2^3\cdot\mathscr{O}} \left|b_1 r_1^j \Theta(2^1, j) + b_2 r_2^j \Theta(2^2, j) + \cdots + b_N r_N^j \Theta(2^N, j)\right|^q \\
&+ \cdots \\
&+ \sum_{j\in 2^{N-1}\cdot\mathscr{O}} \left|b_1 r_1^j \Theta(2^1, j) + b_2 r_2^j \Theta(2^2, j) + \cdots + b_N r_N^j \Theta(2^N, j)\right|^q \\
&+ \sum_{j\in 2^N\cdot\mathbb{N}} \left|b_1 r_1^j \Theta(2^1, j) + b_2 r_2^j \Theta(2^2, j) + \cdots + b_N r_N^j \Theta(2^N, j)\right|^q \\
=& \left|1 + 2^1 b_1 + 2^2 b_2 + \cdots + 2^N b_N\right|^q \qquad (18) \\
&+ \sum_{j\in 2\cdot\mathscr{O}} \left|b_1 r_1^j \cdot 2^1\right|^q \\
&+ \sum_{j\in 2^2\cdot\mathscr{O}} \left|b_1 r_1^j \cdot 2^1 + b_2 r_2^j \cdot 2^2\right|^q \\
&+ \sum_{j\in 2^3\cdot\mathscr{O}} \left|b_1 r_1^j \cdot 2^1 + b_2 r_2^j \cdot 2^2 + b_3 r_3^j \cdot 2^3\right|^q \\
&+ \cdots \\
&+ \sum_{j\in 2^{N-1}\cdot\mathscr{O}} \left|b_1 r_1^j \cdot 2^1 + b_2 r_2^j \cdot 2^2 + \cdots + b_{N-1} r_{N-1}^j \cdot 2^{N-1}\right|^q \\
&+ \sum_{j\in 2^N\cdot\mathbb{N}} \left|b_1 r_1^j \cdot 2^1 + b_2 r_2^j \cdot 2^2 + \cdots + b_N r_N^j \cdot 2^N\right|^q \qquad (19)
\end{aligned}$$

We will be done if we can find constants $b_1, b_2, \ldots, b_N$ (which can depend on N), and a sequence of radii $\{r_n\}_{n=1}^{\infty}$, such that the expression on the right hand side tends to zero as $N \longrightarrow 0$, and the resulting sequence W satisfies the Blaschke condition. With that goal in mind, let us take

$$\begin{aligned} b_n &= -1/(N \cdot 2^n), \;\; 1 \leqslant n \leqslant N; \\ r_n &= e^{-1/(2^n n^{3-p})}, \;\; n \geqslant 1. \end{aligned}$$

Already the first term (18) of the final expression is zero, so we need only be concerned about the remaining sums:

$$\left|1 + 2^1 b_1 + 2^2 b_2 + \cdots + 2^N b_N\right|^q = |1 - (1/N) - (1/N) - \cdots - (1/N)|^q = 0.$$

Notice that

$$e^{-x} \geqslant 1 - x$$

for all $x \geqslant 0$. Hence we have

$$\begin{aligned} \sum_{w \in W} (1 - |w|) &= \sum_{n=1}^{\infty} 2^n (1 - r_n) \\ &= \sum_{n=1}^{\infty} 2^n (1 - e^{-1/[2^n n^{3-p}]}) \\ &\leqslant \sum_{n=1}^{\infty} \frac{2^n}{2^n n^{3-p}} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{3-p}} \\ &< \infty. \end{aligned}$$

That is, the prescribed collection of points $W = \bigcup_{N=1}^{\infty} S_N$ in $\mathbb{D}$ is a Blaschke sequence.

Moving on, the final sum in (19) can be expressed via re-indexing as follows:

$$\begin{aligned} &\sum_{j \in 2^N \cdot \mathbb{N}} \left|(1/N) r_1^j + (1/N) r_2^j + \cdots + (1/N) r_N^j\right|^q \qquad (20) \\ &= \frac{1}{N^q} \sum_{j \in 2^N \cdot \mathbb{N}} \left|r_1^j + r_2^j + \cdots + r_N^j\right|^q \end{aligned}$$

$$= \frac{1}{N^q} \sum_{j=1}^{\infty} \left| e^{-2^N j/(2^1 1^{3-p})} + e^{-2^N j/(2^2 2^{3-p})} + \cdots + e^{-2^N j/(2^N N^{3-p})} \right|^q$$

Among the terms inside the absolute values,

$$e^{-2^N j/(2^1 1^{3-p})} + e^{-2^N j/(2^2 2^{3-p})} + \cdots + e^{-2^N j/(2^N N^{3-p})}, \tag{21}$$

there are some exceeding than the quantity e^{-Nj}. For such terms, indexed by s,

$$\begin{aligned} \frac{-2^N j}{2^s s^{3-p}} &\geqslant -Nj \\ 2^N &\leqslant 2^s s^{3-p} N \\ N &\leqslant s + (3-p)\log_2 s + \log_2 N \\ N - (4-p)\log_2 N &\leqslant s. \end{aligned}$$

Since we already have $s \leqslant N$, there can be at most $(4-p)\log_2 N$ such terms. They are bounded above by the largest of them, namely $e^{-j/N^{3-p}}$.

Among the remaining terms in (21), there are no more than N of them, and they are each bounded above by e^{-Nj}. Thus, combining these two estimates by means of the elementary bound

$$|x+y|^q \leqslant 2^{q-1}(|x|^q + |y|^q),$$

we have

$$\begin{aligned} &\left| e^{-2^N j/(2^1 1^{3-p})} + e^{-2^N j/(2^2 2^{3-p})} + \cdots + e^{-2^N j/(2^N N^{3-p})} \right|^q \\ &\leqslant 2^{q-1} N^q e^{-Njq} + 2^{q-1}(4-p)^q [\log_2 N]^q e^{-jq/N^{3-p}}. \end{aligned}$$

Now sum over $j \geqslant 1$, and divide by N^q, to get

$$\frac{2^{q-1} e^{-Nq}}{1 - e^{-Nq}} + \frac{2^{q-1}(4-p)^q [\log_2 N]^q e^{-q/N^{3-p}}}{N^q (1 - e^{-q/N^{3-p}})} \tag{22}$$

as an upper bound for the final sum (20).

The first term in the bound (22) tends to zero as N increases without bound. For N

large, the second term behaves like

$$\frac{2^{q-1}(4-p)^q[\log_2 N]^q e^{-q/N^{3-p}}}{qN^{p+q-3}},$$

which also tends to zero as $N \to \infty$. Thus we have the expression in (20) under control.

We now lay this aside, and turn to the mth of the sums in (19), $1 \leqslant m < N$, which is

$$\frac{1}{N^q} \sum_{j \in 2^m \cdot \mathscr{O}} \left| r_1^j + r_2^j + \cdots + r_m^j \right|^q.$$

By re-indexing, we find that this is equivalent to

$$\frac{1}{N^q} \sum_{j \in \cdot \mathscr{O}} \left| r_1^{j2^m} + r_2^{j2^m} + \cdots + r_m^{j2^m} \right|^q. \tag{23}$$

Focusing on the summand of (23), we have the expression

$$e^{-2^m j/(2^1 1^{3-p})} + e^{-2^m j/(2^2 2^{3-p})} + e^{-2^m j/(2^3 3^{3-p})} + \cdots + e^{-2^m j/(2^m m^{3-p})}$$

inside the absolute value signs.

Again, there are terms which exceed the quantity e^{-Nj}, in which case (with the terms indexed by t)

$$\begin{aligned} e^{-2^m j/(2^t t^{3-p})} &\geqslant e^{-Nj} \\ 2^m &\leqslant 2^t t^{3-p} N \\ m - (3-p)\log_2 m - \log_2 N &\leqslant t. \end{aligned}$$

Since $t \leqslant m$, there can be at most $(3-p)\log_2 m + \log_2 N$ of these terms, which we can estimate with the largest of them, namely $e^{-j/m^{3-p}}$. As before, the sum of the remaining terms is bounded above by Ne^{-Nj}. Thus the expression (23) is bounded above by

$$\frac{2^{q-1}N^q e^{-Nq}}{N^q(1-e^{-2Nq})} + \frac{2^{q-1}[(3-p)\log_2 m + \log_2 N]^q e^{-q/m^{3-p}}}{N^q(1-e^{-2q/m^{3-p}})}. \tag{24}$$

If we perform the sum of the quantity (24) from $m = 1$ to $N-1$, and take $N \to \infty$, the contribution from the first term behaves like

$$\frac{(N-1)\cdot 2^{q-1}e^{-Nq}}{(1-e^{-2Nq})} \longrightarrow 0.$$

Next, note that for real x near zero, $1/(1-e^{-x})$ is comparable to $1/x$. Also, there exists a constant C, depending only on q, such that

$$(\log_2 2)^q + (\log_2 3)^q + (\log_2 4)^q + \cdots + (\log_2 N)^q \leqslant CN(\log_2 N)^q$$

for all $N \geqslant 2$. Furthermore, it is elementary to check that if $p \neq 2$, then $p+q > 4$. Hence, the contribution from the second term of (24) behaves as

$$\frac{2^{q-1}(3-p)^q N(\log_2 N)^q}{N^{p+q-3}} = \frac{2^{q-1}(3-p)^q(\log_2 N)^q}{N^{p+q-4}} \longrightarrow 0.$$

We have shown that $\Delta_N \to 0$ as $N \to \infty$. This implies that $\|J_N\|_p$ diverges to infinity with increasing N. By Theorem 2.25, the Blaschke sequence W fails to be a zero set for ℓ^p_A. The proof is complete. $\square$

Note that when $2 \leqslant p < 3$, this construction yields a sequence W that fails to be a zero set for ℓ^p_A, and fails to be a Blaschke sequence as well.

CHAPTER 5

FAILURE OF THE CANONICAL FACTORIZATION

5.1 INTRODUCTION

Proceeding by analogy with H^p, it is natural to ask whether every nontrivial function $f \in \ell^p_A$ has a factorization

$$f = Jg, \tag{25}$$

where J is p-inner and g is analytic and nonvanishing in $\mathbb{D}$ (it would have been desirable to attach an "outer" condition to g; however, even analyticity is too much to ask). It is shown that for certain values of p, there exist polynomials f for which this factorization fails; in this situation, every p-inner function J that vanishes at the roots of f, with multiplicities taken into account, must also vanish at another point of $\mathbb{D}$. That is, the p-inner function J associated with a given f may have "extra zeros." It follows that there are polynomials f such that the shift-invariant subspace $[f]$ of ℓ^p_A cannot be generated by a p-inner function. This furnishes negative answers to some fundamental questions about ℓ^p_A.

Using ideas from the geometry of Banach spaces developed in [11, 19], we also obtain bounds for extra zeros of p-inner functions. It is shown that any extra zeros must lie near the boundary of the unit disk. By generalizing these methods to certain weighted ℓ^p_A spaces, we derive a sufficient condition for a polynomial f to have a factorization (25). The present results have been published in [10].

5.2 EXISTENCE OF AN EXTRA ZERO FOR ℓ^p_A

Let us now derive a means to exhibit the existence of an extra zero for some values of p. We will establish rigorously that an extra zero exists when $p = 4/3$. More generally, our numerical studies are able to show that extra zeros exist for p in the range $1.025 < p < 1.8$.

The intuition underlying our construction is as follows. For $r \in (0, 1)$ and a positive integer n, the function

$$B(z) := \frac{1 - z^n/r^n}{1 - r^{n(q-1)} z^n}$$

is p-inner. To see this, recall from equation (5) that $B(z)$ is p-inner when $n = 1$; for other values of n, replacing z by z^n merely spaces out the Taylor coefficients by n steps, and thus preserves the orthogonality of B to its forward shifts. Then $B(z)$ is the p-inner function corresponding to the zero set

$$r, re^{2\pi i/n}, re^{2\pi i\cdot 2/n}, re^{2\pi i\cdot 3/n}, \ldots, re^{2\pi i\cdot(n-1)/n}.$$

Plainly $B(z)$ does not have any extra zeros. Now, if we remove the single point r from the zero set, we could expect the resulting p-inner function to be "close" to $B(z)$. Indeed, by manipulating our choices of r and n, we might hope to coax an extra zero into the unit disk.

With that in mind, let $p \in (1, \infty)$, let $0 < r < 1$ be fixed, let n be a positive integer, and define the roots

$$w_j := re^{2\pi ij/n},$$

for all $j = 0, 1, 2, \ldots, n-1$. Suppose that f is the polynomial

$$f(z) := \frac{1 - z^n/r^n}{1 - z/r}.$$

Thus f has all of these roots w_j except for $w_0 = r$.

As ever, define $J := f - \widehat{f}$ in ℓ^p_A, the p-inner function arising from f. We know that J must be of the form

$$\begin{aligned}
J(z) &= 1 + \sum_{k=1}^{\infty} \left(C_1 w_1^k + C_2 w_2^k + \cdots C_{n-1} w_{n-1}^k\right)^{\langle q-1\rangle} z^k \\
&= 1 + \sum_{k=1}^{\infty} \left(C_1 r^k e^{2\pi ik/n} + C_2 r^k e^{2\pi ik\cdot 2/n} + \cdots C_{n-1} r^k e^{2\pi ik\cdot(n-1)/n}\right)^{\langle q-1\rangle} z^k \\
&= 1 + \sum_{k=1}^{\infty} \left(C_1 e^{2\pi ik/n} + C_2 e^{2\pi ik\cdot 2/n} + \cdots C_{n-1} e^{2\pi ik\cdot(n-1)/n}\right)^{\langle q-1\rangle} r^{k(q-1)} z^k.
\end{aligned}$$

Since all of the Taylor coefficients of f are real, it must be that each Taylor coefficient of J is real as well (or else $\overline{J(\bar{z})}$ would also be a p-inner function corresponding to f, in violation of the uniqueness of metric projections). Thus the complex power $\langle q-1\rangle$ is simply a signed power.

The expression

$$D_k := (C_1 e^{2\pi ik/n} + C_2 e^{2\pi ik\cdot 2/n} + \cdots C_{n-1} e^{2\pi ik\cdot(n-1)/n})^{\langle q-1\rangle}, \tag{26}$$

as k varies over the positive integers, takes at most n distinct values, repeating in cycles of

n. We may therefore express J in the form

$$
\begin{aligned}
J(z) &= 1 + \sum_{j=1}^{\infty} \bigl(D_1 r^{(jn+1)(q-1)} z^{jn+1} + D_2 r^{(jn+2)(q-1)} z^{jn+2} + \cdots + D_n r^{(jn+n)(q-1)} z^{jn+n}\bigr) \\
&= 1 + \frac{D_1 r^{(q-1)} z}{1 - r^{n(q-1)} z^n} + \frac{D_2 r^{2(q-1)} z^2}{1 - r^{n(q-1)} z^n} + \cdots + \frac{D_n r^{n(q-1)} z^n}{1 - r^{n(q-1)} z^n}. \qquad (27)
\end{aligned}
$$

It is easy to rearrange this last expression into the ratio of two polynomials in z, each of degree exactly equal to n. The constant coefficient of the numerator is 1, while the leading coefficient of the numerator is given by

$$(D_n - 1) r^{n(q-1)}.$$

This quantity is the negative reciprocal of the product of the n roots of the numerator polynomial, which coincide with the zeros of J. We know $n-1$ of the these zeros (they are the roots of f), and their product is r^{n-1}. Therefore the remaining zero must be real, and its value is

$$w = \frac{1}{r^{n-1} \cdot r^{n(q-1)} \bigl(1 - D_n\bigr)}.$$

We can calculate D_n in the following manner. From the conditions $C_1 J(w_1) = C_2 J(w_2) = \cdots = C_{n-1} J(w_{n-1}) = 0$, we see that

$$
\begin{aligned}
0 &= C_1 + \sum_{k=1}^{\infty} \bigl(C_1 w_1^k + C_2 w_2^k + \cdots C_{n-1} w_{n-1}^k\bigr)^{\langle q-1\rangle} C_1 w_1^k \\
0 &= C_2 + \sum_{k=1}^{\infty} \bigl(C_1 w_1^k + C_2 w_2^k + \cdots C_{n-1} w_{n-1}^k\bigr)^{\langle q-1\rangle} C_2 w_2^k \\
&\cdots \\
0 &= C_{n-1} + \sum_{k=1}^{\infty} \bigl(C_1 w_1^k + C_2 w_2^k + \cdots C_{n-1} w_{n-1}^k\bigr)^{\langle q-1\rangle} C_{n-1} w_{n-1}^k.
\end{aligned}
$$

Adding all of these equations together yields

$$
\begin{aligned}
C_1 + C_2 + \cdots + C_{n-1} &= -\sum_{k=1}^{\infty} \bigl|C_1 w_1^k + C_2 w_2^k + \cdots C_{n-1} w_{n-1}^k\bigr|^q \\
&= -\sum_{k=1}^{\infty} \bigl|(C_1 w_1^k + C_2 w_2^k + \cdots C_{n-1} w_{n-1}^k)^{\langle q-1\rangle}\bigr|^p \\
&= -\|J\|_p^p + 1,
\end{aligned}
$$

where we have used $(q-1)p = (1-1/q)qp = (1/p)qp = q$.

By definition, $D_n = (C_1 + C_2 + \cdots + C_{n-1})^{\langle q-1\rangle}$. But then, this quantity takes the value

$$D_n = -(\|J\|_p^p - 1)^{q-1}.$$

Thus we have

$$w = \frac{1}{r^{n-1} \cdot r^{n(q-1)}\left[1 + (\|J\|_p^p - 1)^{q-1}\right]}. \tag{28}$$

If we can come up with a close estimate for $\|J\|_p$ from below, and the resulting upper bound for w turns out to be less than unity, then we will be done.

To obtain a lower bound for $\|J\|_p$, let us use the formula

$$\|J\|_p = \left[\inf \left\|1 + b_1 k_1 + b_2 k_2 + \cdots + b_{n-1}k_{n-1}\right\|_q\right]^{-1}, \tag{29}$$

from Theorem 2.26, where k_j is the reproducing kernel function

$$k_j(z) = \frac{1}{1 - w_j z}, \ \ 1 \leqslant j \leqslant n-1,$$

and the infimum is over the coefficients $b_1, b_2, \ldots, b_{n-1}$. Let us call the expression inside the infimum norm $G(z)$, and notice

$$\begin{aligned}
\|G(z)\|_q^q &= \left\|1 + b_1 k_1 + b_2 k_2 + \cdots + b_{n-1}k_{n-1}\right\|_q^q \\
&= \left\|(1 + b_1 + b_2 + \cdots + b_{n-1}) + \sum_{j=1}^{\infty}(b_1 w_1^j + b_2 w_2^j + \cdots + b_{n-1}w_{n-1}^j) z^j\right\|_q^q \\
&= \left|1 + b_1 + b_2 + \cdots + b_{n-1}\right|^q + \sum_{j=1}^{\infty}\left|b_1 w_1^j + b_2 w_2^j + \cdots + b_{n-1}w_{n-1}^j\right|^q \\
&= \left|1 + b_1 + b_2 + \cdots + b_{n-1}\right|^q + \sum_{j\equiv 1}\left|b_1 e^{2\pi i j/n} + b_2 e^{2\pi i j\cdot 2/n} + \cdots b_{n-1}e^{2\pi i j\cdot(n-1)/n}\right|^q r^{jq} \\
&\qquad + \sum_{j\equiv 2}\left|b_1 e^{2\pi i j/n} + b_2 e^{2\pi i j\cdot 2/n} + \cdots b_{n-1}e^{2\pi i j\cdot(n-1)/n}\right|^q r^{jq} + \ldots \\
&\qquad + \sum_{j\equiv n}\left|b_1 e^{2\pi i j/n} + b_2 e^{2\pi i j\cdot 2/n} + \cdots b_{n-1}e^{2\pi i j\cdot(n-1)/n}\right|^q r^{jq},
\end{aligned}$$

where the congruences are modulo n, and $j \geqslant 1$. The calculation continues

$$= \left|1 + b_1 + b_2 + \cdots + b_{n-1}\right|^q + \left|b_1 e^{2\pi i/n} + b_2 e^{2\pi i\cdot 2/n} + \cdots b_{n-1}e^{2\pi i\cdot(n-1)/n}\right|^q \frac{r^q}{1 - r^{nq}}$$

$$+\left|b_1 e^{2\cdot 2\pi i/n} + b_2 e^{2\cdot 2\pi i\cdot 2/n} + \cdots b_{n-1} e^{2\cdot 2\pi i\cdot (n-1)/n}\right|^q \frac{r^{2q}}{1-r^{nq}} + \cdots$$
$$+\left|b_1 e^{n\cdot 2\pi i/n} + b_2 e^{n\cdot 2\pi i\cdot 2/n} + \cdots b_{n-1} e^{n\cdot 2\pi i\cdot (n-1)/n}\right|^q \frac{r^{nq}}{1-r^{nq}}.$$

Any selection of the constants b_j results in a valid lower bound for $\|J\|_p$. We can now proceed by selectively evaluating this quantity to obtain an estimate for $\|J\|_p$, and then using that in turn to see if the nth zero w lies in $\mathbb{D}$.

Let us carry this out this estimation procedure when $p = 4/3$, using $n = 4$ and $r = 0.9$. In this specific situation, the roots are ir, $-r$ and $-ir$, and $q = 4$. Thus we need to find the norm in ℓ^q_A of

$$G(z) = 1 + \frac{A}{1-irz} + \frac{B}{1+irz} + \frac{C}{1+rz}$$

for some choice of parameters A, B and C. By symmetry we may assume that C is real and that $\bar{B} = A$. Now

$$\begin{aligned}
\|G\|_4^4 &= \left\|1 + \frac{A}{1-irz} + \frac{\bar{A}}{1+irz} + \frac{C}{1+rz}\right\|^4 \\
&= \left|1 + A + \bar{A} + C\right|^4 + \sum_{m=1}^{\infty} \left|\big(Ai^m + \bar{A}(-i)^m\big) + C(-1)^m\right|^4 r^{4m} \\
&= \left|1 + A + \bar{A} + C\right|^4 + \sum_{m\equiv 1} \left|(A-\bar{A})i - C\right|^4 r^{4m} \\
&\quad + \sum_{m\equiv 2} \left|-\big(A+\bar{A}\big) + C\right|^4 r^{4m} + \sum_{m\equiv 3} \left|-\big(A-\bar{A}\big)i - C\right|^4 r^{4m} \\
&\quad + \sum_{m\equiv 4} \left|\big(A+\bar{A}\big) + C\right|^4 r^{4m}
\end{aligned}$$

where in the sums, the equivalences are modulo 4, and $m \geqslant 1$. Since we want to minimize this quantity, we are aided by the assumption that A is real, so that the contributions from the expression $(A - \bar{A})i$ are zero. We expand the geometric series to get

$$\|G\|_4^4 = \left|1 + 2A + C\right|^4 + \frac{C^4 r^4}{1-r^{16}} + \frac{(-2A+C)^4 r^8}{1-r^{16}} + \frac{C^4 r^{12}}{1-r^{16}} + \frac{(2A+C)^4 r^{16}}{1-r^{16}}.$$

Again, *any* choice of A and C will result in $\|G\|_4^{-1}$ being a lower bound for $\|J\|_{4/3}$. If we choose the parameter values $A = -0.205683$ and $C = -0.202725$, then the resulting value of $\|G\|_4^4$ is approximately 0.0574716. This provides the following bound for the norm of J:

$$\|J\|_{4/3} \geqslant 2.042381.$$

Substituting this into (28) gives us the bound for the extra zero w

$$w \leqslant 0.965699.$$

We already know that $w > 0$, and hence w is an extra zero for J in $\mathbb{D}$. We have proved the following.

Theorem 5.1. *Let $p = 4/3$. There exists a polynomial f such that the p-inner function $J := f - \widehat{f}$ has an extra zero.*

A similar program can be pursued for other values of p. By careful choice of r and n in the above construction, one can obtain extra zeros when $1.025 < p < 1.8$. However, this approach does not appear to yield extra zeros when p is closer to 1 or 2[12], or when $p > 2$. The existence of extra zeros in these cases remains open. Other mechanisms for conjuring an extra zero may need to be developed.

With the choice $p = 4/3$, $r = 0.9$ and $n = 4$, the complex power $a^{\langle q-1\rangle}$, applied to a real number a, is just a^3. In this situation we can calculate J numerically, by solving for the coefficients C_1, C_2 and C_3 in

$$J(z) = 1 + \sum_{k=1}^{\infty} (C_1 w_1 + C_2 w_2 + C_3 w_3)^3 z^k$$

that satisfy $J(w_1) = J(w_2) = J(w_3) = 0$. This approach gives the estimate

$$J(z) \;\approx\; 1 + \frac{(0.075587)z + (0.0839856)z^2 + (0.0933173)z^3 - (1.13804)z^4}{1 - (0.28243)z^4},$$

and J has an extra zero at

$$w \approx 0.965694580489323.$$

This confirms our findings above.

[12]At least, the calculations become too taxing for our code to handle. As p approaches either 1 or 2, the number of calculations increases considerably, causing rounding errors to compound and making the numerical estimates less reliable.

Corollary 5.2. *Let $p = 4/3$. There exists a polynomial $f \in \ell^p_A$ such that $[f] \neq [K]$ for any p-inner function K.*

Proof. Let f and g satisfy $[f] = [g]$ in ℓ^p_A. Without loss of generality, we may assume that $f(0) = 1$ and $g(0) = 1$, by dividing out a common power of z if necessary. By hypothesis, given any polynomial P, and any $\epsilon > 0$, there exists a polynomial Q such that

$$\big\|[f(z) + zf(z)P(z)] - [g(z) + zg(z)Q(z)]\big\| < \epsilon.$$

Likewise, given Q and ϵ, there exists a P such that the above holds. It follows

$$\inf_P \|f(z) + zf(z)P(z)\|_p = \inf_Q \|g(z) + zg(z)Q(z)\|_p.$$

Consequently, $f - \widehat{f} = g - \widehat{g}$. In particular, if $[f] = [K]$ for some p-inner function K, then necessarily $K = f - \widehat{f}$. This is impossible if $f - \widehat{f}$ has an extra zero. □

This shows that Beurling's theorem, which characterizes the shift-invariant subspaces of H^2, does not carry over to $\ell^{4/3}_A$. (We add that for $p > 2$, it has been established that ℓ^p_A has shift-invariant subspaces of arbitrary index [1]; this constitutes another means by which Beurling's theorem has no counterpart in ℓ^p_A.)

5.3 BOUNDS FOR EXTRA ZEROS

We have established that extra zeros can exist in principle; we now turn to the question of their location. The construction from the previous section suggests that any extra zeros must lie close to the boundary of the unit disk. Indeed that turns out to be the case.

Theorem 5.3. *Suppose that f is a nonconstant function in ℓ^p_A, with $f(0) = 1$. Let $J = f - \widehat{f}$ be the associated p-inner function. Let $w \in \mathbb{D}$ be a zero of J that is not a zero of f (multiplicities taken into account). If $p \in (1, 2]$, then $|w| \geqslant \frac{p}{2}$; if $p \in [2, \infty)$, then $|w|^p - (1 - |w|)^p \geqslant 1/(2^{p-1} - 1)$.*

The proof of Theorem 5.3 relies on the following lemma, which allows for removing certain zeros of functions in the shift-invariant subspace $[f]$.

Lemma 5.4. *Let $p \in (1, \infty)$, and suppose that $f \in \ell^p_A$. If $(z - w)U(z) \in [f]$ for some $w \in \mathbb{D}$ such that $f(w) \neq 0$, then $U \in [f]$.*

Proof. By hypothesis there are polynomials φ_n, $n = 1, 2, 3, \ldots$, such that $\varphi_n f \to (z-w)U(z)$ in ℓ^p_A. By continuity of the difference-quotient operation, we have

$$\begin{aligned}\frac{\varphi_n(z)f(z) - \varphi_n(w)f(w)}{z-w} &\to \frac{(z-w)U(z) - (w-w)U(w)}{z-w} \\ &= U(z) \\ \frac{\varphi_n(z) - \varphi_n(w)}{z-w} f(z) + \frac{f(z) - f(w)}{z-w}\varphi_n(w) &\to U(z).\end{aligned}$$

Since convergence in ℓ^p_A implies convergence pointwise in the disk, we have

$$\varphi_n(w)f(w) \to (w-w)U(w) = 0;$$

so, with $f(w) \neq 0$, it must be that $\varphi_n(w) \to 0$. Hence the second term on the left side above vanishes in the limit. Therefore,

$$\frac{\varphi_n(z) - \varphi_n(w)}{z-w} f(z) \to U(z).$$

The difference-quotient of φ_n is itself a polynomial, and in conclusion $U \in [f]$. □

From this we see that if J has an extra zero w, then $J(z)/(z-w) \in [f]$. This is utilized below.

We now verify Theorem 5.3.

Proof. By Lemma 5.4, the function $J(z)/(1-z/w)$ belongs to $[f]$. We can split this function into two terms

$$\begin{aligned}\frac{J(z)}{1-\frac{z}{w}} &= \frac{J(z)}{1-\frac{z}{w}}\Big(1 - \frac{z}{w} + \frac{z}{w}\Big) \\ &= J(z) + \frac{z}{w}\frac{J(z)}{1-\frac{z}{w}}.\end{aligned}$$

Notice that on the right side the first term is $\perp_p$ to the second term, which belongs to $S[f]$. Therefore the Pythagorean inequality of Theorem 2.16 applies, with the result

$$\left\|\frac{J(z)}{1-\frac{z}{w}}\right\|_p^r \geqslant \|J(z)\|_p^r + \frac{K}{|w|^r}\left\|\frac{zJ(z)}{1-\frac{z}{w}}\right\|_p^r,$$

where r and K are the Pythagorean parameters appropriate to p. In the last term, multiplying by z does not alter the norm. So we can transpose this term to get

$$\left(1 - \frac{K}{|w|^r}\right) \left\| \frac{J(z)}{1 - \frac{z}{w}} \right\|_p^r \geqslant \|J(z)\|_p^r. \tag{30}$$

Our last step is to use

$$\begin{aligned} \frac{1}{|w|} \left\| \frac{J(z)}{1 - \frac{z}{w}} \right\|_p &= \left\| \frac{J(z) - 0}{z - w} \right\|_p \\ &= \left\| \frac{J(z) - J(w)}{z - w} \right\|_p \\ &\leqslant \frac{1}{1 - |w|} \|J(z)\|_p, \end{aligned}$$

where we have used the norm of the difference-quotient operator from Proposition 2.22. Combining the last two estimates gives us

$$\left(1 - \frac{K}{|w|^r}\right) \left\| \frac{J(z)}{1 - \frac{z}{w}} \right\|_p^r \geqslant \left(\frac{1 - |w|}{|w|}\right)^r \left\| \frac{J(z)}{1 - \frac{z}{w}} \right\|_p^r.$$

Our assumptions on f assure that $J(0) = 1$; hence the above bound simplifies to

$$1 - \frac{K}{|w|^r} \geqslant \left(\frac{1 - |w|}{|w|}\right)^r.$$

When $p \in (1, 2]$, the Pythagorean parameters are $r = 2$ and $K = p - 1$. The condition on $|w|$ then simplifies to

$$|w| \geqslant \frac{p}{2}.$$

When $p \in [2, \infty)$, the Pythagorean parameters are $r = p$ and $K = 1/(2^{p-1} - 1)$. The condition then reads

$$|w|^p - (1 - |w|)^p \geqslant \frac{1}{2^{p-1} - 1}.$$

□

Our second main result in this section is a sharper bound on the extra zeros of the p-inner function $J = f - \widehat{f}$, in the special case that f is a polynomial. It exploits the fact that there is a positive distance between the roots of f in $\mathbb{D}$ and the boundary of $\mathbb{D}$.

Theorem 5.5. *Let f be a polynomial, and let $J = f - \widehat{f}$ in ℓ^p_A. Suppose that all of the roots of f lie inside the disk $\{z : |z| \leqslant R\}$ for some $R \leqslant 1$, and J has an extra zero $w \in \mathbb{D}$. If*

$p \in (1,2]$, then

$$|w| \geqslant \frac{p-1}{2R} + \frac{1}{2R^{q-1}}.$$

If $p \in [2,\infty)$, then

$$|w|^p - \Big(\frac{1}{R^{q-1}} - |w|\Big)^p \geqslant \frac{1}{R^q(2^{p-1}-1)}.$$

We see that any extra zeros resulting from a polynomial must lie even closer to the boundary.

The proof of Theorem 5.5 rests on some properties of weighted ℓ^p spaces and their associated function spaces.

Definition 5.6. Let $v > 0$ and $p \in (1,\infty)$. We define the space $\ell^p_A(v)$ to be the linear space of functions

$$f(z) = \sum_{k=0}^{\infty} f_k z^k$$

of the complex variable z, $|z| < v^{1/p}$, endowed with the norm

$$\|f\|_{\ell^p_A(v)} := \Big(\sum_{k=0}^{\infty} |f_k|^p v^k\Big)^{1/p}.$$

If $|z| < v^{1/p}$, then indeed the series for $f(z)$ converges absolutely:

$$\begin{aligned}\sum_{k=0}^{\infty} |f_k z^k| &\leqslant \sum_{k=0}^{\infty} |f_k| v^{k/p}(1/v^{k/p})|z|^k \\ &\leqslant \Big(\sum_{k=0}^{\infty} |f_k|^p v^k\Big)^{1/p} \Big(\sum_{k=0}^{\infty} (1/v^{kq/p})|z|^{kq}\Big)^{1/q} \\ &= \|f\|_{\ell^p(v)_A} \Big[\frac{1}{1-(|z|/v^{1/p})^q}\Big]^{1/q}.\end{aligned}$$

Then $a(z) := \sum_{k=0}^{\infty} a_k z^k$ belongs to $\ell^p_A(v)$ precisely when $a(zv^{1/p})$ belongs to ℓ^p_A.

Let the relation $\perp_{\ell^p_A(v)}$ be defined as in Definition 2.11. It is simple to check that $a \perp_{\ell^p_A(v)} b$ precisely when

$$\sum_{k=0}^{\infty} a_k^{\langle p-1 \rangle} v^k b_k = 0,$$

and this is equivalent to

$$0 = \sum_{k=0}^{\infty} a_k^{\langle p-1 \rangle} v^{k/q} v^{k/p} b_k = \sum_{k=0}^{\infty} (a_k v^{k/p})^{\langle p-1 \rangle}(v^{k/p} b_k),$$

or

$$a(v^{1/p}z) \perp_p b(v^{1/p}z).$$

This could also be deduced from the fact that the mapping

$$a(z) \longmapsto a(v^{1/p}z)$$

is an invertible (linear) isometry from $\ell^p_A(v)$ onto ℓ^p_A. Under this mapping, the subspace $[a]$ of $\ell^p_A(v)$ corresponds to the span of $\{a(v^{1/p}z), za(v^{1/p}z), z^2a(v^{1/p}z), \ldots\}$ in ℓ^p_A. Furthermore, since an isometry preserves Birkhoff-James orthogonality, it is easy to see that if $J = f - \widehat{f}$ in $\ell^p_A(v)$, then $J(v^{1/p}z) = f(v^{1/p}z) - \widehat{f(v^{1/p}z)}$ in ℓ^p_A.

Let $c > 0$ and consider the weight $v = c^p$. Taking a difference-quotient is a bounded operation on $\ell^p_A(c^p)$.

Lemma 5.7. *Suppose that $p \in (1, \infty)$ and $c > 0$. If $f \in \ell^p_A(c^p)$, and $|w| < c$, then*

$$\|Q_w f\|_{\ell^p_A(c^p)} \leqslant \frac{1}{c - |w|}\|f\|_{\ell^p_A(c^p)}.$$

Proof. The function $f(cz)$ belongs to ℓ^p_A. If $|w| < c$, then $w/c \in \mathbb{D}$. On that basis, using Proposition 2.22, we can say that

$$\left\|\frac{f(cz) - f(c[w/c])}{z - w/c}\right\|_p \leqslant \frac{1}{1 - |w/c|}\|f(cz)\|_p.$$

But the mapping $f(cz) \longmapsto f(z)$ is an isometry of ℓ^p_A to $\ell^p_A(c^p)$. Thus, the above could be written

$$\left\|\frac{f(z) - f(w)}{z/c - w/c}\right\|_{\ell^p_A(c^p)} \leqslant \frac{1}{1 - |w/c|}\|f\|_{\ell^p_A(c^p)},$$

or

$$\left\|\frac{f(z) - f(w)}{z - w}\right\|_{\ell^p_A(c^p)} \leqslant \frac{1}{c - |w|}\|f\|_{\ell^p_A(c^p)}.$$

$\square$

Here is a criterion for extra zeros in a weighted space.

Lemma 5.8. *Suppose that $p \in (1, \infty)$, and $c > 0$. Let f be a polynomial with nonzero roots in the disk $|z| < c^p$, let $J = f - \widehat{f}$ in the geometry of $\ell^p_A(c^p)$, and let w be an extra zero of J. If $p \in (1, 2]$, then*

$$|w| \geqslant \frac{c^{p-1}(p-1) + c}{2};$$

if $p \in [2, \infty)$, *then*

$$1 - \frac{c^p}{(2^{p-1}-1)|w|^p} \geqslant \Big(\frac{c-|w|}{|w|}\Big)^p.$$

Proof. First, we have

$$\frac{J(z)}{1-z/w} = \frac{J(z)}{1-z/w}\Big(1 - z/w + z/w\Big) = J(z) + \frac{zJ(z)}{w(1-z/w)}.$$

Notice that

$$\begin{aligned}
\|zf(z)\|^p_{\ell^p_A(c^p)} &= \sum_{k=1}^{\infty} c^{kp}|f_{k-1}|^p \\
&= \sum_{k=0}^{\infty} c^{(k+1)p}|f_k|^p \\
&= c^p \sum_{k=0}^{\infty} c^{kp}|f_k|^p \\
&= c^p\|f\|^p_{\ell^p_A(c^p)}.
\end{aligned}$$

Lemma 5.4 carries over to the weighted space $\ell^p_A(c^p)$ in a straightforward way. Thus the assumption that w is an extra zero of J means that $J(z)/(z-w)$ lies in the span of f and its shifts. In particular,

$$J \perp_{\ell^p_A(c^p)} J(z)/(z-w).$$

Then by the Pythagorean inequality with parameters K and r (which applies to any L^p space),

$$\Big\|\frac{J(z)}{1-z/w}\Big\|^r_{\ell^p_A(c^p)} \geqslant \|J\|^r_{\ell^p_A(c^p)} + \frac{c^pK}{|w|^r}\Big\|\frac{J(z)}{1-z/w}\Big\|^r_{\ell^p_A(c^p)}$$

Now transpose the rightmost term, and apply the bound for difference quotients to get

$$\Big(1 - \frac{c^pK}{|w|^r}\Big)\Big\|\frac{J(z)}{1-z/w}\Big\|^r_{\ell^p_A(c^p)} \geqslant \|J(z)\|^r_{\ell^p_A(c^p)} \geqslant \Big(\frac{c-|w|}{|w|}\Big)^r\Big\|\frac{J(z)}{1-z/w}\Big\|^r_{\ell^p_A(c^p)}.$$

The conclusion is that

$$1 - \frac{c^pK}{|w|^r} \geqslant \Big(\frac{c-|w|}{|w|}\Big)^r.$$

When $p \in (1,2]$, we can use $K = p-1$ and $r = 2$. Then this gives

$$1 - \frac{c^p(p-1)}{|w|^2} \geqslant \Big(\frac{c-|w|}{|w|}\Big)^2$$

$$
\begin{aligned}
1 - \frac{c^p(p-1)}{|w|^2} &\geqslant \frac{c^2 - 2c|w| + |w|^2}{|w|^2} \\
|w|^2 - c^p(p-1) &\geqslant c^2 - 2c|w| + |w|^2 \\
-c^p(p-1) &\geqslant c^2 - 2c|w| \\
2c|w| &\geqslant c^2 + c^p(p-1)
\end{aligned}
$$

Hence we have the bound

$$
|w| \geqslant \frac{c^{p-1}(p-1)+c}{2} \tag{31}
$$

on the extra zero w of J.

In case $p \in [2, \infty)$, the Pythagorean parameter values are $K = 1/(2^{p-1}-1)$, and $r = p$. The condition on the extra zero w of a co-projection function J in $\ell^p_A(c^p)$ is

$$
1 - \frac{c^p}{(2^{p-1}-1)|w|^p} \geqslant \Big(\frac{c-|w|}{|w|}\Big)^p.
$$

□

Proof of Theorem 5.5 Now let us revert to the notation of the theorem, where f is a polynomial in ℓ^p_A, and $J = f - \widehat{f}$ has an extra zero w. Suppose that the roots of f lie inside the open disk of radius r, where $0 < r < 1$. Then we know from the formula for J that the kth Taylor coefficient of J is dominated by a factor of the form $R^{k(q-1)}$, where R is the largest modulus of a root of f. It follows that J belongs to the weighted space $\ell^p_A(1/r^{q-1})$. To be sure, the series

$$
\sum_{k=0}^{\infty} |J_k|^p (1/r^{k(q-1)p})
$$

converges geometrically. Furthermore, this J has the extra zero w. Finally, the bound (31) applies, with $c = 1/r^{q-1}$. That is,

$$
|w| \geqslant \frac{p-1}{2r} + \frac{1}{2r^{q-1}}.
$$

The expression on the right is continuous in r, so that this bound remains valid as r decreases to R. □

Note that when $R = 1$ in the bounding quantity we simply get $|w| \geqslant p/2$, which we already know from Theorem 5.3. Moreover, when $p = 2$, we find that J never has extra zeros, which is the expected result.

In case $p \in [2, \infty)$, the condition on the extra zero is

$$|w|^p - \Big(\frac{1}{R^{q-1}} - |w|\Big)^p \geqslant \frac{1}{R^q(2^{p-1}-1)}.$$

Once again, we see that any extra zero w cannot lie too close to the origin. As expected, when $p = 2$, extra zeros are ruled out altogether.

Theorem 5.5 gives rise to a simple sufficient condition for the p-inner function of a polynomial to have no extra zeros. A bit more can be said. We call a function $g \in \ell^p_A$ *cyclic* if $[g] = \ell^p_A$.

Corollary 5.9. *Let f be a polynomial, and let $J = f - \widehat{f}$ in ℓ^p_A. Suppose that all of the roots of f lie inside the disk $\{z : |z| < R\}$ for some $R \leqslant 1$. If $p \in (1, 2]$, and*

$$1 < \frac{p-1}{2R} + \frac{1}{2R^{q-1}},$$

then J cannot have any extra zeros in the closure of $\mathbb{D}$. If $p \in [2, \infty)$, and

$$1 - \Big(\frac{1}{R^{q-1}} - 1\Big)^p < \frac{1}{R^q(2^{p-1}-1)},$$

then J cannot have any extra zeros in the closure of $\mathbb{D}$. In either case, we have $[f] = [J]$, and $f = Jg$ for some cyclic vector $g \in \ell^p_A$.

To see this, note J is analytic in a neighborhood of the closed disk $\bar{\mathbb{D}}$, and by hypothesis has no zeros in $\bar{\mathbb{D}}$ apart from the roots of f. This implies $[f] = [J]$. Furthermore, $1/g = J/f$ is analytic in a neighborhood of the closed unit disk. Thus, the Taylor series $1/g(z) = \sum_{k=0}^{\infty} c_k z^k$ decays geometrically. Now $[g]$ contains $1 = \lim_{N\to\infty} g(z) \sum_{k=0}^{N} c_k z^k$, where the limit is in norm. Hence g is cyclic.

5.4 MULTIPLE EXTRA ROOTS

We end this chapter by claiming that we may modify our proof of Theorem 5.1 to produce polynomials whose p-inner functions have multiple extra roots. Instead of removing just the positive real root from our zero set, if we remove several roots, evenly distributed around the disc, we can tease in that same number of roots. More precisely, choose integers $m \geqslant 1$, and $n \geqslant 2$, and consider the polynomial

$$f(z) := \frac{1 - z^{mn}/r^{mn}}{1 - z^m/r^m}.$$

The roots of f are the mnth roots of unity scaled by r with scaled nth roots of unity removed.

For example, if $m = 3$ and $n = 4$, then we are simply removing 1, i, -1, and $-i$ from the 12th roots of unity and scaling the remaining roots by r.

A similar argument to the single extra root case yields n extra roots bound by

$$|w|^m \leqslant \frac{1}{r^{m(nq-1)}\left(1 + (\|J\|_p^p - 1)^{q-1}\right)}$$

Indeed, the argument is nearly identical with just the modifications that we replace n with mn, and $(n-1)$ with $(mn-m)$. We should also take some care in simplifying the various sums that appear, but the same symmetries that we leveraged to simplify them in the single extra root case apply here as well.

We have written a Mathematica code[13] that implements this scheme to search for the optimal choice of r and calculate the corresponding bound on $|w|$, given m, n, and p.

[13]See Appendix A.2 for the code, and Appendix D for data generated by this code. See also Appendix C.1 for plots of the bounds on these extra roots against the scaling factor r.

CHAPTER 6

FURTHER AREAS FOR STUDY

We finish with some thoughts on potential areas of further study.

6.1 EXTRA ZEROS FOR $p \in (1, 2)$

Our construction of a polynomial whose associated p-inner function has an extra root leverages the symmetries of scaled roots of unity to make our calculations tractable. We see no reason, however, that this regularity should be necessary to admit an extra root. Indeed, by the continuity of p-inner functions with respect to their zeros (Theorem 3.6), a small perturbation of the roots of our original function should not lead to a significant change in the overall structure of the problem. The primary issue with this would be that the calculations would become too do difficult to do analytically. We could attempt to perturb them in a way that maintains some symmetry (altering conjugate pairs in tandem, for example), but this still relies on structure that seems too stringent for a truly general concept of when these extra roots appear.

We should be able to take a p-inner function with one or more extra roots and continuously perturb its zero set until any extra roots are pushed out of the disc, since the extra root won't outright disappear. The question remains of how far we might be able to perturb the zero set, how much symmetry we can toss aside, before the extra roots are no longer in $\mathbb{D}$. From Theorem 5.5 we do know some about the bounds on any extra roots, but this only gives us an answer when we perturb elements of the zero set toward the boundary $\mathbb{T}$, and tells us nothing of what happens should we perturb the roots along the circle $|z| = r$.

6.2 EXTRA ZEROS FOR $p \in (2, \infty)$

The extra root problem remains open for the case $p \in (2, \infty)$. The examples we have constructed fail to produce any extra roots for such values of p. We suspect that the key to finding any examples will require the inclusion of roots of higher multiplicity, but our attempts thus far have failed. For example, one such failed attempt used the polynomial $f(z) = (1 - z/r)^m$ for some $r \in (0, 1)$. (More on this in a moment.) We suspect that making this idea work may require a scheme similar to what we have constructed in Chapter

5, utilizing the symmetries of the roots of unity, but raising the polynomial used in that scheme to some integer power $m \geqslant 2$.

It may be the case that no extra roots can be found for such values of p. In the theories of a number of different function spaces, including those of Hardy, Bergman, and Dirichlet, the value $p = 2$ acts as a sort of hard boundary between seemingly opposite behaviors. This may be the case here. Indeed, ℓ^2_A is precisely H^2, so there is at least one case where there are no extra zeros and the inner-outer factorization succeeds.

6.2.1 A CURIOUS ARTIFACT

While attempting to gather numerical data on the extra roots problem for $p \in (2, \infty)$, we attempted to estimate the p-inner part of a function with a given zero set as a polynomial.[14] This was ultimately fruitless, but a curious pattern emerged in our tests.

We tried starting with a polynomial with a single real root with multiplicity greater than 1. Our code attempted to utilize the same mechanism as we did for the case $p \in (1, 2)$, modifying the dual function G by including the appropriate reproducing kernel functions for the derivatives,[15] but we were still left with some coefficients that we could not determine (the number of these undetermined coefficients is precisely the multiplicity of our root minus one). So we tried to approximate J as a polynomial of large degree. As an artifact of this process we saw extra roots numbering precisely the degree of the approximating polynomial. This is to be expected since it is widely known that a polynomial of degree N will have precisely N roots counting multiplicities. What is odd about our results is that these fictitious extra roots appeared always to occur roughly evenly spaced on a circle centered at $z = 0$. Moreover, as we increased the degree of our estimating polynomial, this circle appears to converge to the unit circle, $\mathbb{T}$, the boundary of the domain of definition for ℓ^p_A.

The apparent convergence of these roots to $\mathbb{T}$ raises a tantalizing possibility. The canonical factorization theorem for Hardy spaces allows for the presence of a singular inner factor. The precise definition is not terribly important here;[16] what is relevant is that a singular inner function has no roots inside $\mathbb{D}$ and is unimodular almost everywhere on $\mathbb{T}$. In particular, such a function is allowed to vanish on a subset of $\mathbb{T}$ with measure zero. Our numerical experiments may hint at the possibility of a singular p-inner part in some cases.

The other option that seems likely is that the convergence of these fictitious roots to $\mathbb{T}$ is the result of some idea in the theory of polynomials with which we are unfamiliar. This may

[14] See Appendix A.3 for the Mathematica code.
[15] See equation (6).
[16] See Definition 2.7 for the precise definition of a singular inner function in the context of H^p.

be a known result, or one that is yet to be discovered. Perhaps something about minimizing the ℓ_A^p norm of a polynomial given a pre-zero set[17] naturally leads to this pattern.

Appendix C.2 contains some plots of these attempts at approximating J. The plots show the approximated roots and norm for each case. In the end, we abandoned this approach to approximating J not only because the computational expense of the minimizations involved was simply unreasonable,[18] but we were also unable to separate any potentially genuine extra roots from these fictitious roots.

[17] A pre-zero set is similar to a zero set, with the difference that we allow the function to have more zeros than are given. In essence, a pre-zero set is a subset of a zero set.

[18] For example, the 100 term case with $p = 4$ and multiplicity 10 uses an iterative algorithm to minimize the sum of the fourth power of the moduli of of 110 expressions each involving 100 unknowns. By default Mathematica will cease the minimization algorithm after 100 iterations if the required precision has not been reached, but this can be changed fairly easily. Unfortunately, even three digits of precision can require thousands of iterations for certain cases, which can translate to hours of computational time for each attempt. See Appendix C.2 for a brief description of our scheme for approximating J.

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APPENDIX A

MATHEMATICA CODES

A.1 EXTRA ROOTS WHEN $p \in (1,2)$

This program implements the scheme described in Section 5.2 to demonstrate the impossibility of an inner outer factorization for functions in ℓ^p_A. Define $f \in \ell^p_A$ with constant term and simple roots $re^{k \cdot 2\pi \mathrm{i}/n}$, $k = 1, 2, \ldots, n-1$. This program calculates the extra root of the coprojection J of f onto the space of all of its forward shifts, $[f]$ as well as the function J itself.

(*

Inputs :

p = index that defines the space ℓ^p_A

r = scaling factor for the roots of f (must satisfy 0 < r < 1)

n = number of zeros of (r−z)f;

Variables :

q = conjugate exponent to p

R = r^q used in calculating G

RR = r^(q−1) used in calculating J

Z = array of the nth roots of unity not including 1

G = the dual function for J

B = array of coefficients for G

b = the entries of B

GNorm = array of the minimum value of G and coefficients that attain this minimum

JNorm = norm of J

Y = 1+(JNorm^p−1)^(q−1) used in calculating w

w = the bound for the extra zero

JRoots = array storing all roots of J

```
J = the coprojection of f onto the space of its forward shifts
Ds = array of the coefficients of J
d = the entries of D
Js = system of equations used to calculate Ds
DD = intermediate array used to calculate the values in Ds
X = intermediate array used to calculate the values in Ds
*)

(* clears any previously used variables *)
ClearAll[p, q, r, n, R, RR, k, Z, G, B, b, GNorm, JNorm, Y, w, J, Ds, Js, d, DD, X,
    JRoots, z];

(* inputs *)
p = 4/3;
n = 4;
r = 0.9;

q = p/(p-1);
R = r^q;

(* populates Z *)
Z = ConstantArray[0, n - 1];
For[k = 1, k < n, k++,
    Z[[k]] = Exp[2*Pi*I*k/n];
  ];

(* defines G and B *)
G[B_] := Abs[1 + Sum[B[[i]], {i, 1, n-1}]]^q +
    Sum[(Abs[Sum[B[[i]] * (Z[[i]])^j, {i, 1, n-1}]]^q) * (R^j/(1-R^n)), {j, 1, n}];
B = Array[b, n-1];

(* simplifies the calculation by applying the symmetry in the b[i] *)
For[k = 1, k < n/2, k++,
    b[n-k] = b[k];
```

```
   ];

(* minimizes G with respect to the coefficients and calculates ||J|| and w *)
GNorm = Chop[Minimize[G[B], B]];
JNorm = GNorm[[1]]^(−1/q);
Y = 1+(JNorm^p−1)^(q − 1);
w = r/((R^n) Y);

(* collects all roots of J in one array, the nth element is w *)
JRoots = Flatten[{r*Z, w}];

(* collects the coefficients of the minimized G in the array B *)
ClearAll[k];
For[k = 1, k ⩽ n/2, k++,
      b[k] = GNorm[[2, k, 2]];
   ];

(* defines J, Ds, d *)
RR = r^(q − 1);
J[D_, z_] := 1 + (Sum[D[[i]] * (RR*z)^i, {i, 1, n}])/(1 − (RR*z)^n);
Ds = Array[d, n];
d[n] = 1−Y;

(* collects first n−1 entries of Ds in DD *)
DD = ConstantArray[0, n − 1];
ClearAll[k];
For[k = 1, k < n, k++,
      DD[[k]] = Ds[[k]];
   ];

(* defines the system Js *)
ClearAll[k];
Js = J@@{Ds, r*Z[[1]]} == 0;
For[k = 2, k < n, k++,
```

```
    Js = And[Js, J@@{Ds, r*Z[[k]]} == 0];
  ];
```

(* solves the system Js and stores resulting coefficients in Ds *)

```
X = Chop[NSolve[Js, DD]];
X = Flatten[X[[All, All, 2]]];
ClearAll[k];
For[k = 1, k < n, k++,
    Ds[[k]] = X[[k]];
  ];
```

(* prints results *)

```
Print["p = ", p, " q = ", q];
Print["n = ", n];
Print["r = ", r];
Print["||G|| ≈ ", NumberForm[GNorm[[1]]^(1/q), 16]];
Print["Coefficients of G ≈ ", B]; Print[" "];
Print["||J|| ≈ ", NumberForm[JNorm, 16]]; Print["Coefficients of J ≈ ", Ds]; Print[" "];
Print["J ≈ ", J[Ds, z]]; Print[" "];
Print["w ≈ ", NumberForm[w, 16], " J[w] ≈ ", J[Ds, w]];
```

A.2 MULTIPLE EXTRA ROOTS WHEN $p \in (1, 2)$

Section 5.4 describes a modification to the approach taken in Section 5.2. This code implements those modifications to calculate upper bounds for the shared modulus of multiple extra roots in the case where we remove every nth root out of $m \times n$ scaled roots of unity. If upper bounds less than 1 are found, the results are plotted against the scaling factor r. The value of r which minimizes the upper bound, w, is also determined and printed.
In the case $n = 3$, results are exact.

(* clears any previously used variables *)

```
ClearAll[p, q, n, m, rStep, MN, MQ, MNQ, rArray, wArray, bArray, gArray, jArray, w,
    R1, R2, M1, M2, G, B, B1, B2, b, gNorm, jNorm, Dmn, r, L, pic1, pic2, pic2a, pic3,
```

```
    loc, min, title, a, K1, K2, K3];

(* parameters; change these to evaluate different cases *)
p = 1.1;
n = 3;
m = 2;
rStep  = 2^(−10);

(* Hölder conjugate of p *)
q = p/(p−1);

(* intermediate calculations *)
MN = m * n;
K1 = MN−m;
MQ = N[m * q];
MNQ = N[m * n * q];
M1 = m^q;
M2 = K1^q − M1;

(* sets initial value of scaling factor r *)
r = 1 − rStep;

(* initialize arrays that will hold values of r, w, b, ‖G‖, and ‖J‖ *)
rArray = {};
wArray = {};
bArray = {};
gArray = {};
jArray = {};

(* loop calculates w for each r until w ⩾ 1, reducing r by rStep on each
      iteration *)
w = 0;
While[w < 1,
    rArray = Flatten[Catenate[{{r}, rArray}]]; (* adds r to rArray *)
```

```
    ClearAll[R1, R2, G, B, gNorm, jNorm, Dmn, w, K2, K3, B1, B2];

    (* intermediate calculations *)
    R1 = N[r^MQ];
    R2 = N[r^MNQ];
    K2 = M1 * R1/(1 − R1) + M2 * R2/(1 − R2);

    (* calculates estimate for ‖J‖ by minimizing norm of the dual function G *)
    G[b_] := Abs[1 − b * K1]^q + K2 * Abs[b]^q;
    K3 = (K1 + K2/K1 * Abs[K2/K1]^(p − 2))^(−1);
    B1 = {0, K3, 1/K1}; (* critical points of ‖G‖ *)[19]
    B2 = G[B1];
    B = Min[B2];
    b = B1 * [[Flatten[Position[B2, B]]]];

    gNorm = B^(1/q);
    jNorm = 1/gNorm;

    (* stores calculated values of ‖G‖, b, and ‖J‖ *)
    gArray = Flatten[Catenate[{{gNorm}, gArray}]];
    bArray = Flatten[Catenate[{b, bArray}]];
    jArray = Flatten[Catenate[{{jNorm}, jArray}]];

    (* calculates and stores w *)
    Dmn = − (Round[jNorm, .0001]^p − 1)^(q − 1);
    w = (R2 * r^ − m * (1 − Dmn))^(−1/m);
    wArray = Flatten[Catenate[{{w}, wArray}]];

    (* updates r *)
    r = r − rStep;
  ];

L = Length[rArray];
```

[19]See B.3 for a proof.

```
If[L == 1,
    (* prints message if all w ⩾ 1 for all r *)
    Print["p = ", p];
    Print["n = ", n];
    Print["m = ", m];
    Print["no extra roots found"],

    (* builds plot of w as a function of r *)
    title = StringForm["p = `, n = `, m = `", p, n, m];
    a = Min[rArray[[1]], Min[wArray]];
    pic1 = Plot[{x, 1}, {x, a , 1},
        PlotStyle → Directive[{Red, Dashed}],
        PlotRange → {{a, 1 }, {a, 1}},
        Frame → True]; (* plots lines w = 1 and w = r in red *)
    pic2 = ListPlot[Table[{rArray[[n]], wArray[[n]]}, {n, L}],
        PlotRange → {{a, 1}, {a, 1}},
        FrameLabel → {{w, None}, {r, None}},
        Frame → True]; (* plots each calculated point (r,w) *)
    pic2a = ListLinePlot[Table[{rArray[[n]], wArray[[n]]}, {n,L}],
        PlotRange → {{a, 1}, {a, 1}},
        PlotStyle → Thickness[0.003],
        Frame → True]; (* connects the points in pic2 with a continuous curve *)

    (* determines and prints certain minimal values *)
    loc = Position[wArray, Min[wArray]][[1, 1]];
    min = {rArray[[loc]], wArray[[loc]], bArray[[loc]], jArray[[loc]]};
    Print["p = ", p, "        q = ", q];
    Print["n = ", n];
    Print["m = ", m];
    Print["step size = ", rStep];
    Print["————————————————————"];
    Print["w is minimized at:"];
    Print["r = ", N[min[[1]]]];
    Print["w = ", min[[2]]];
```

```
    Print["b = ", min[[3]]];
    Print["||J|| ⩽ ", min[[4]]];
    Print["———————————————"];
    Print["minimum value of r with w < 1: "];
    Print[N[rArray[[1]]], " ⩽ r ⩽ ", N[rArray[[2]]]];
    Print["———————————————"];

    (*plots the minimal point in red*)
    pic3 = ListPlot[{{min[[1]], min[[2]]}},
        PlotStyle → Directive[Red, PointSize[Medium]]];

    (* Prints graph of w as a function of r *)
    Show[pic2, pic2a, pic1, pic3,
        PlotLabel → title,
        AspectRatio → 1];
];
```

A.3 EXTRA ROOTS FOR $p \in (2, \infty)$ (ATTEMPT)

The purpose of this code is to search for extra roots when $p > 2$, by starting with the polynomial $f(z) = (1 - z/r)^n$ and approximating the associated p-inner function J. The calculation of J is attempted by leveraging the extremal property of p-inner functions (that J is the unique function that has minimal norm, given a zero set) and approximation by polynomials. Specifically we let J be the product of f with an arbitrary polynomial of degree M and minimize the norm of the resulting polynomial. The code then numerically calculates the roots of the resulting $n+M$ degree polynomial and plots them along with the unit circle $\mathbb{T}$ for reference. The roots are stored in the array XX.

Ultimately this approach failed to be useful in our search for extra roots, but we include the code here for the unexpected artifact that appears in the calculations, referenced in Chapter 6.

```
(* clears any previously used variables *)
ClearAll[r, n, f, p, q, M, Cs, J, X, XX, CC, JJ, JNorm, pic, c, ar, ai, br, bi, AB, j, k];
```

```
(* parameters *)
r = .9; (* original root *)
p = 4; (* the ubiquitous p *)
n = 2; (* multiplicity of original root *)
M = 100; (* the number of terms used to approximate J *)

(* ar, ai, br, bi are the limits of the rectangle over which the results are
      plotted *)
AB = 1.5;
ar = −AB;
br = AB;
ai = −AB;
bi = AB;

f[z_] := (1-z/r)^n; (* the function f(z) *)
q = 1/(1-1/p); (* the Hölder conjugate of p *)

(* define J as f times a power series with M terms *)
Cs = Array[c, M];
JJ[z_, Cs_] := f[z] ∗ (1 + Sum[Cs[[k]]∗z^k,{k,1,M}]);

(* isolates the coefficients of J as defined above *)
X = CoefficientList[JJ[z, Cs], z];

(* minimizes the norm of the above J and determines the coefficients
the norm is stored as JNorm, the coefficients are stored in the array CC *)
CC = NMinimize[Sum[Abs[X[[j]]]^p, {j, 1, M+1}], Cs, WorkingPrecision → 10];
JNorm = CC[[1]]^(1/p);
CC = CC[[2, All, 2]];

(* uses calculated coefficients to define J(z) *)
J[z_] := JJ[z, CC];
```

```
(* numerically solves J(z)=0 and stores results in the array XX *)
XX = NSolve[J[z] == 0, z][[All,1,2]];

(* plots the resulting roots in the rectangle [ar,br]×[ai,bi] in the complex
      plane *)
title = StringTemplate[" p =`1`, r =`2`, multiplicity =`3`, terms used: `4`"][p,r,n,M];
pic = ListPlot[{Re[#], Im[#]}&/@XX,
      PlotRange → {{ar, br}, {ai, bi}},
      AspectRatio → 1,
      Frame → True,
      FrameLabel → {{Im, None}, {Re, title}},
      PlotStyle → Directive[Red, PointSize[.006]]
   ];

(* plots the unit circle on graph for reference *)
Show[pic, Graphics@Circle[{0, 0}, 1], ImageSize → Full]

(* prints JNorm *)
Print[StringTemplate["‖J‖ =`1`"][JNorm]]
```

APPENDIX B

ADDITIONAL PROOFS

B.1 PROPERTIES OF $a^{\langle s\rangle}$

We present here proofs of properties 1-3 of lemma 2.13.

Let $p \in (1,\infty)$, $r, s \in \mathbb{R}$, and $z, w \in \mathbb{C}$.

1. $\quad (zw)^{\langle s\rangle} = z^{\langle s\rangle} w^{\langle s\rangle}$

Proof. Let $z := re^{i\alpha}$, $w := te^{i\beta}$, where $r, t, \alpha, \beta \in \mathbb{R}$; $r, t > 0$

$$\begin{aligned}
(zw)^{\langle s\rangle} &= \left(re^{i\alpha}te^{i\beta}\right)^{\langle s\rangle} \\
&= \left(rte^{i(\alpha+\beta)}\right)^{\langle s\rangle} \\
&= (rt)^s e^{-i(\alpha+\beta)} \\
&= r^s t^s e^{-i\alpha} e^{-i\beta} \\
&= r^s e^{-i\alpha} t^s e^{-i\beta} \\
&= z^{\langle s\rangle} w^{\langle s\rangle}
\end{aligned}$$

□

2. $\quad |z|^s = z^{\langle s-1\rangle} z$

Proof.

$$\begin{aligned}
z^{\langle s-1\rangle} z &= |z|^{s-2}\bar{z}z \\
&= |z|^{s-2}|z|^2 \\
&= |z|^s
\end{aligned}$$

□

3. $\left(z^{\langle s\rangle}\right)^r = (z^r)^{\langle s\rangle}$

Proof. Let $z := te^{i\alpha}$, where $t, \alpha \in \mathbb{R}$, $t > 0$

$$\begin{aligned}\left(z^{\langle s\rangle}\right)^r &= \left(t^s e^{-i\alpha}\right)^r \\ &= t^{sr} e^{-ir\alpha} \\ &= (t^r)^s e^{-ir\alpha} \\ &= \left(t^r e^{ir\alpha}\right)^{\langle s\rangle} \\ &= (z^r)^{\langle s\rangle}\end{aligned}$$

□

B.2 POINT EVALUATION FUNCTIONALS, $k_w^{(n)}$, FOR $f^{(n)}(z)$ IN ℓ_A^p

First we show that the point valuation functionals are indeed the indicated sum.

Theorem B.1. *Let $w, z \in \mathbb{D}$, $f \in \ell_A^p$, and $k_w^{(n)} := \sum_{i=n}^{\infty} \frac{i!}{(i-n)!} w^{i-n} z^i$. Then*

$$\left\langle f, k_w^{(n)} \right\rangle = f^{(n)}(w)$$

Proof. Let $f(z) = \sum_{i=0}^{\infty} a_i z^i$. We begin by calculating the derivative of this sum:

$$\begin{aligned}f^{(n)}(z) &= \left(\frac{d}{dz}\right)^n \sum_{i=0}^{\infty} a_i z^i \\ &= \left(\frac{d}{dz}\right)^{n-1} \sum_{i=1}^{\infty} a_i i z^{i-1} \\ &= \left(\frac{d}{dz}\right)^{n-2} \sum_{i=2}^{\infty} a_i i(i-1) z^{i-2} \\ &\vdots \\ &= \sum_{i=n}^{\infty} a_i i(i-1)\cdots(i-n+1) z^{i-n} \\ &= \sum_{i=n}^{\infty} a_i \frac{i!}{(i-n)!} z^{i-n}\end{aligned}$$

Next we calculate

$$\begin{aligned}
\langle f, k_w^n \rangle &= \left\langle \sum_{i=0}^{\infty} a_i z^i,\ \sum_{i=n}^{\infty} \frac{i!}{(i-n)!} w^{i-n} z^i \right\rangle \\
&= \sum_{i=n}^{\infty} \frac{i!}{(i-n)!} w^{i-n} w^i \\
&= f^{(n)}(w)
\end{aligned}$$

$\square$

We now show that the closed form is valid.

Theorem B.2. *Let $|w|, |z| < 1$, then*

$$\sum_{j=n}^{\infty} \frac{j!}{(j-n)!} w^{j-n} z^j = \frac{n! z^n}{(1-wz)^{n+1}}$$

Proof. First we show that $\sum_{j=0}^{m} \binom{n-1+j}{j} = \binom{n+m}{m}$: this is trivial for $m = 0$. Suppose $\sum_{j=0}^{m-1} \binom{n-1+j}{j} = \binom{n+m-1}{m-1}$, then we have the following:

$$\begin{aligned}
\sum_{j=0}^{m-1} \binom{n-1+j}{j} &= \binom{n+m-1}{m-1} \\
\sum_{j=0}^{m} \binom{n-1+j}{j} &= \binom{n+m-1}{m-1} + \binom{n+m-1}{m} \\
&= \frac{(n+m-1)!}{n!(m-1)!} + \frac{(n+m-1)!}{(n-1)!m!} \\
&= \frac{(n+m-1)!m}{n!m!} + \frac{(n+m-1)!n}{n!m!} \\
&= \frac{(n+m-1)!(n+m)}{n!m!} \\
&= \binom{n+m}{m}
\end{aligned}$$

Next we show that $\sum_{j=0}^{\infty} \binom{n+j}{j} x^j = \frac{1}{(1-x)^{n+1}}$ whenever $|x| < 1$: For $n = 0$, this is just the usual geometric series. Suppose for $n > 0$ we have $\sum_{j=0}^{\infty} \binom{n-1+j}{j} x^j = \frac{1}{(1-x)^n}$. Then we have

$$\frac{1}{(1-x)^{n+1}} = \frac{1}{1-x} \sum_{j=0}^{\infty} \binom{n-1+j}{j} x^j$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} x^k \sum_{j=0}^{\infty} \binom{n-1+j}{j} x^j \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \binom{n-1+j}{j} x^{j+k} \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{j} \binom{n-1+k}{k} x^j \\
&= \sum_{j=0}^{\infty} \binom{n+j}{j} x^j
\end{aligned}
$$

Finally we show that the closed form is as claimed:

$$
\begin{aligned}
\sum_{j=n}^{\infty} \frac{j!}{(j-n)!} w^{j-n} z^j &= n!z^n + (n+1)!wz^{n+1} + \frac{(n+2)!}{2!} w^2 z^{n+2} + \cdots \\
&= n!z^n \sum_{j=0}^{\infty} \frac{n+j!}{n!j!} w^j z^j \\
&= n!z^n \sum_{j=0}^{\infty} \binom{n+j}{j} w^j z^j \\
&= \frac{n!z^n}{(1-wz)^{n+1}}
\end{aligned}
$$

$\square$

B.3 MINIMIZATION OF $\|G\|_q$

Let $H(b) := |1-bD|^q + C\,|b|^q$, where b, C, and D are all real-valued and $1 < q < \infty$. Then if $b \neq 0, \frac{1}{D}$, we have the following:

$$
\begin{aligned}
\frac{dH}{db} &= -qD(1-bD)\,|1-bD|^{q-2} + qCb\,|b|^{q-2} \\
&= q\left(Cb^{\langle q-1\rangle} - D\,[1-bD]^{\langle q-1\rangle}\right)
\end{aligned}
$$

Setting this equal to zero gives

$$
\begin{aligned}
0 &= Cb^{\langle q-1\rangle} - D\,[1-bD]^{\langle q-1\rangle} \\
D\,[1-bD]^{\langle q-1\rangle} &= Cb^{\langle q-1\rangle}
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{1-bD}{b}\right)^{\langle q-1\rangle} &= \frac{C}{D} \\
\frac{1-bD}{b} &= \left(\frac{C}{D}\right)^{\langle p-1\rangle} \\
\frac{1}{b} &= D + \left(\frac{C}{D}\right)^{\langle p-1\rangle} \\
b &= \left[D + \left(\frac{C}{D}\right)^{\langle p-1\rangle}\right]^{-1}
\end{aligned}
$$

Thus H is minimized when b is either 0, $\frac{1}{D}$, or $\left[D + \left(\frac{C}{D}\right)^{\langle p-1\rangle}\right]^{-1}$.

If we let $D = mn - m$ and $C = \frac{m^q r^{mq}}{1-r^{mq}} + \frac{[(mn-m)^q - m^q]r^{mnq}}{1-r^{mnq}}$, then H is precisely our estimate for $\|G\|_q^q$ in the multiple extra zero scheme. Thus we have analytically minimized this estimate. (This is useful for numerically estimating J(z) and any extra roots it may have.)

APPENDIX C

ADDITIONAL FIGURES

C.1 MULTIPLE EXTRA ROOTS OF J, FOR $p \in (1, 2)$

We present here a selection of plots[20] representing the multiple extra root scheme described at the end of Chapter 5. Each figure plots the bound for the extra roots w, against the scaling factor r. The point at which w is minimized is plotted in red,[21] while the rest of the points are blue. The figures also include the line $r = w$ plotted in red for reference. In most cases, the curves formed by the calculated values of w appear to tangentially approach this line as r approaches to 1. The code used to generate these plots can be found in Appendix A.2.

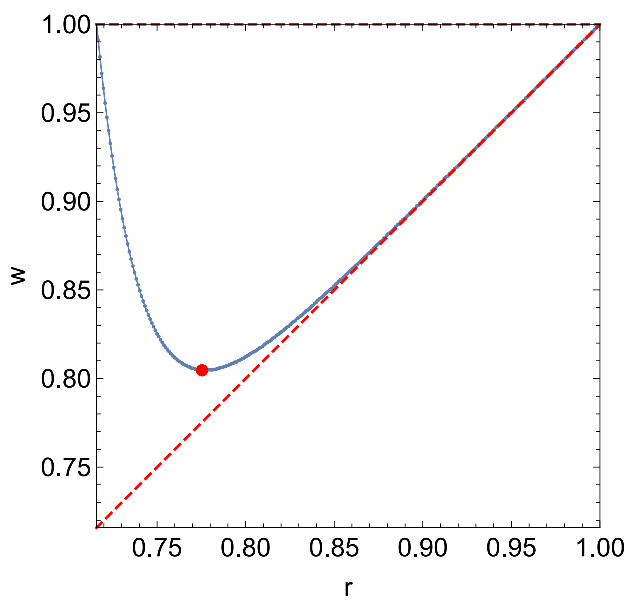

FIG. 1: Bounds on Multiple Extra Roots, $p = \frac{10}{9}$, $n = 4$, $m = 1$

[20]We have generated plots for each case represented in the table in Appendix D. We have not included all of them here because there are simply too many to include, and they are mostly quite similar.

[21]the coordinates of each minimizing point can be found in the corresponding table in Appendix D.

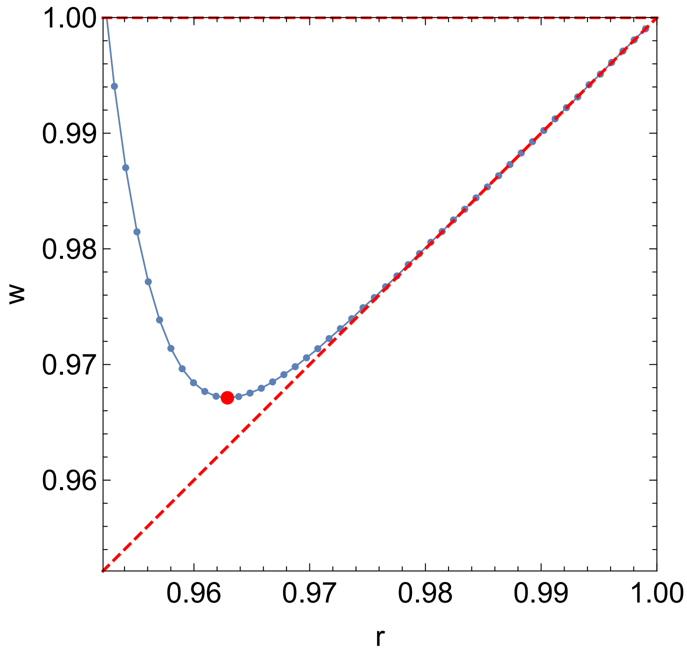

FIG. 2: Bounds on Multiple Extra Roots, $p = \frac{8}{7}$, $n = 6$, $m = 6$

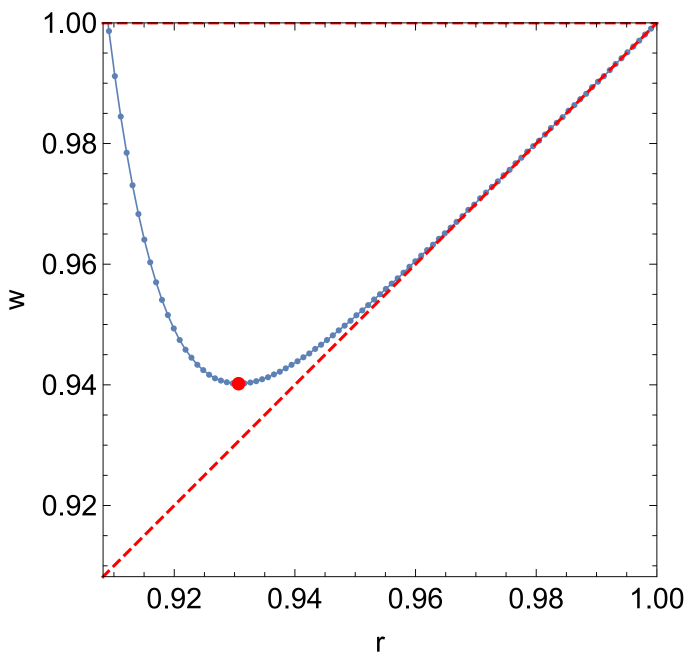

FIG. 3: Bounds on Multiple Extra Roots, $p = \frac{7}{6}$, $n = 6$, $m = 3$

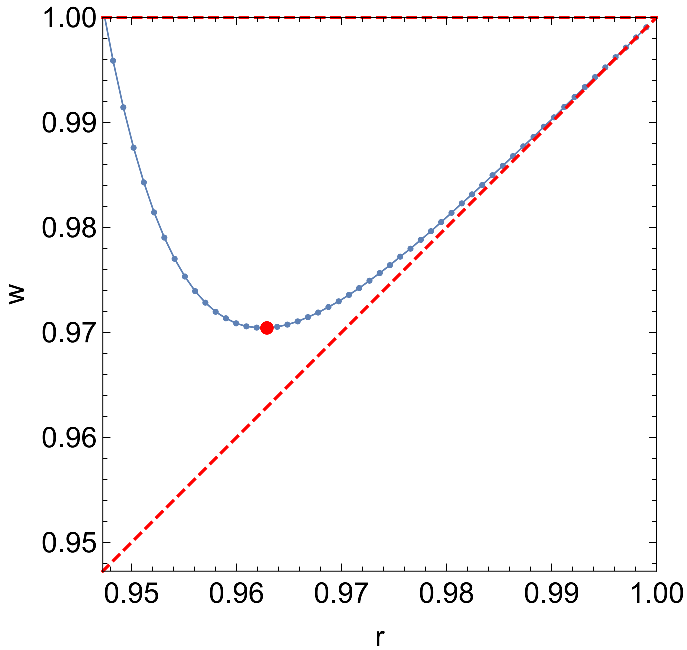

FIG. 4: Bounds on Multiple Extra Roots, $p = \frac{11}{9}$, $n = 6$, $m = 5$

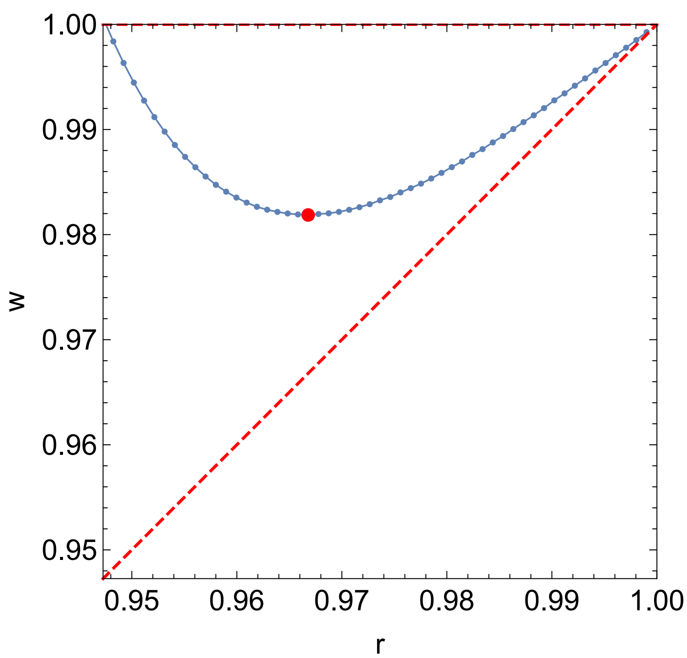

FIG. 5: Bounds on Multiple Extra Roots, $p = \frac{5}{4}$, $n = 4$, $m = 5$

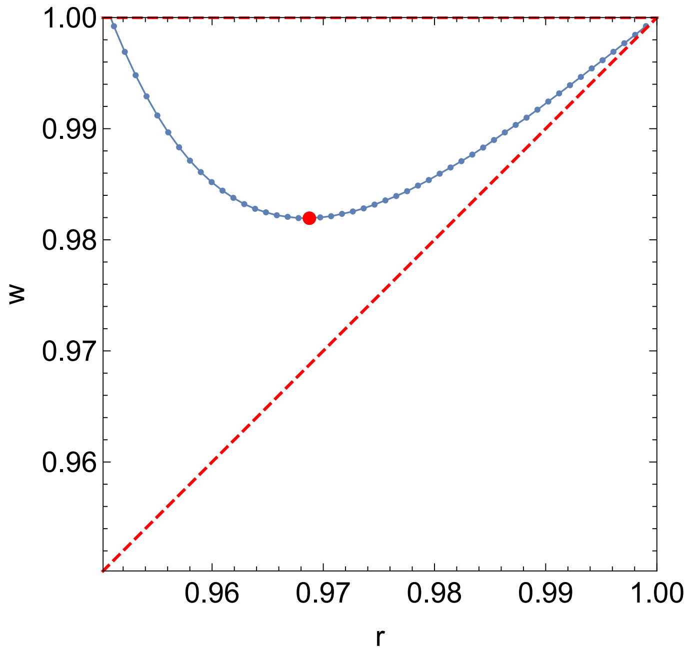

FIG. 6: Bounds on Multiple Extra Roots, $p = \frac{13}{10}$, $n = 5$, $m = 5$

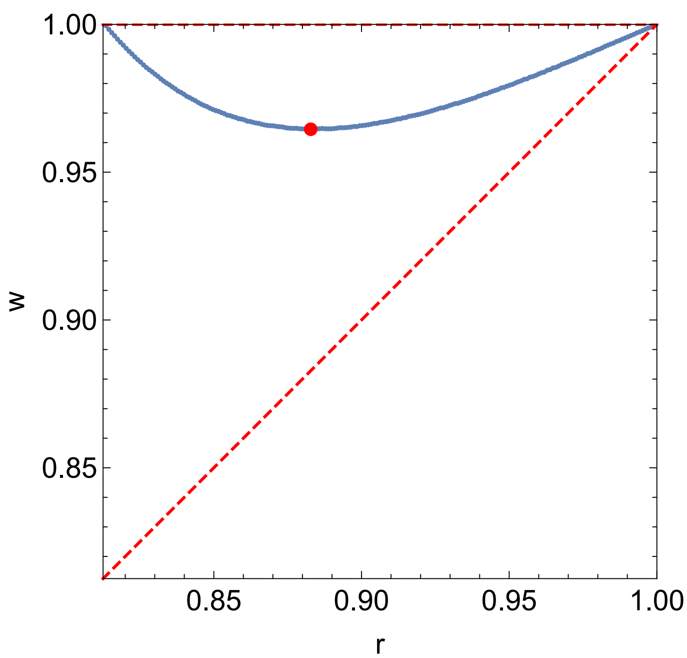

FIG. 7: Bounds on Multiple Extra Roots, $p = \frac{4}{3}$, $n = 4$, $m = 1$

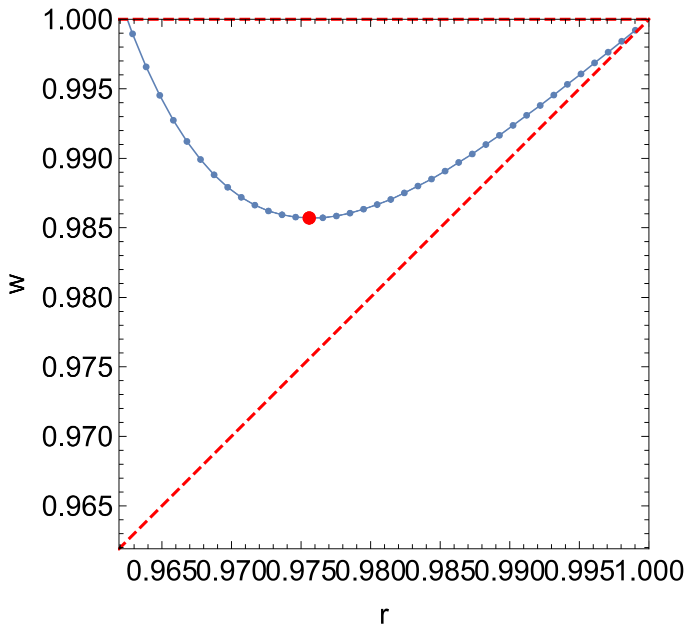

Fig. 8: Bounds on Multiple Extra Roots, $p = \frac{4}{3}$, $n = 6$, $m = 6$

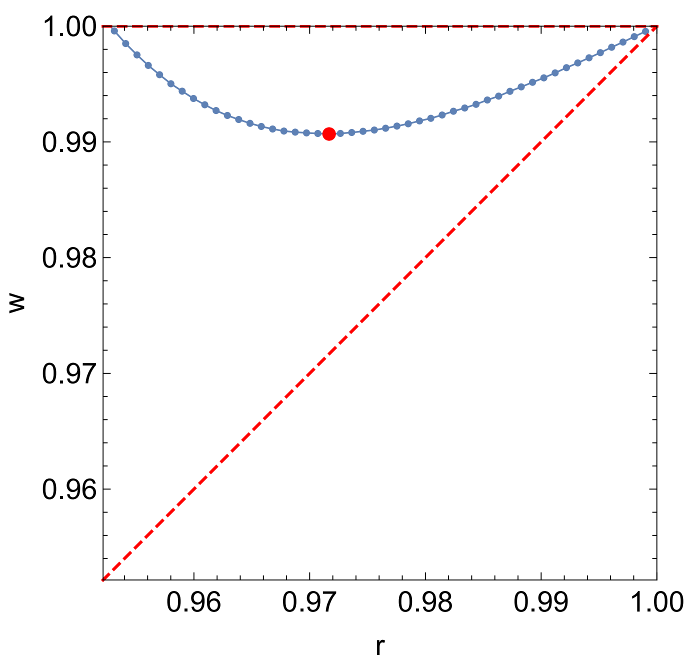

Fig. 9: Bounds on Multiple Extra Roots, $p = \frac{7}{5}$, $n = 5$, $m = 4$

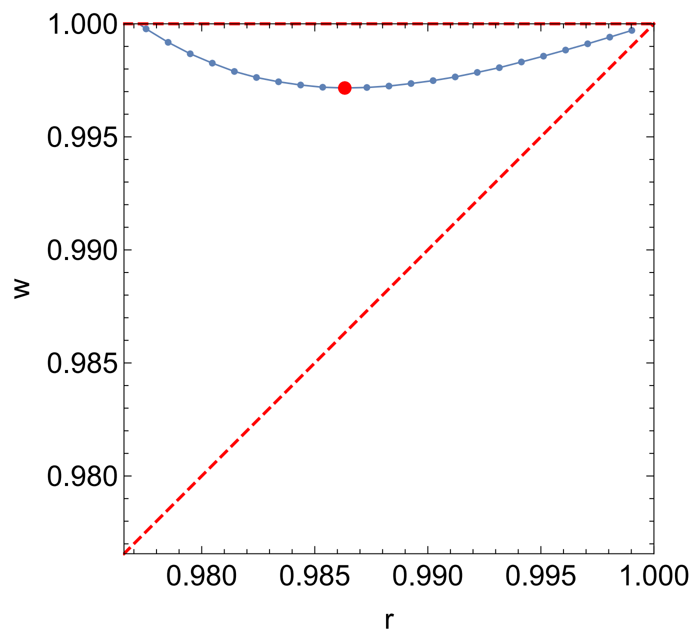

FIG. 10: Bounds on Multiple Extra Roots, $p = \frac{3}{2}$, $n = 6$, $m = 6$

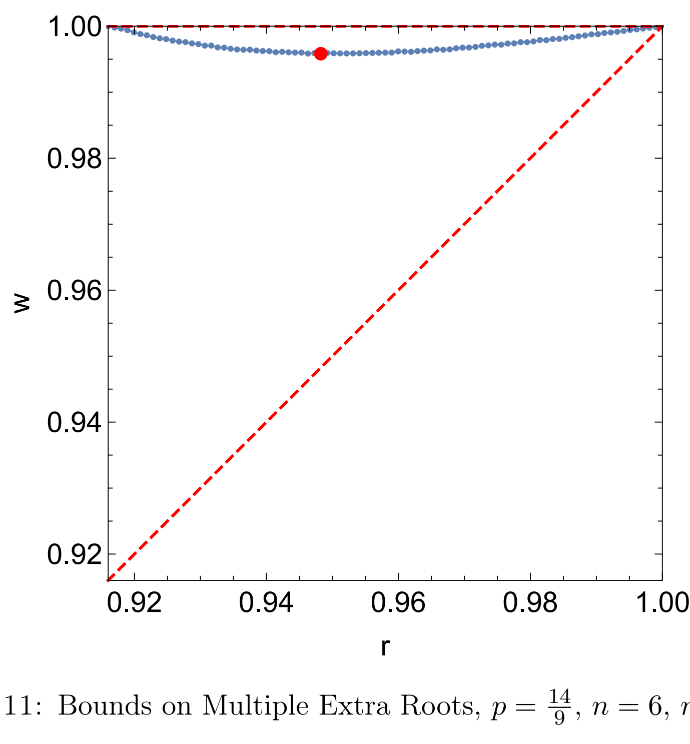

FIG. 11: Bounds on Multiple Extra Roots, $p = \frac{14}{9}$, $n = 6$, $m = 1$

C.2 APPROXIMATIONS OF THE ROOTS OF J FOR $p \in (2, \infty)$

As discussed in Chapter 6, we attempted to numerically identify polynomials f whose associated p-inner part $J = f - \widehat{f}$ have an extra root by representing J itself as a polynomial. Specifically, we let $J(z) = f(z)g(z)$, where $f = \left(1 - \frac{z}{r}\right)^n$ and g is an arbitrary polynomial of degree M. We then found the choice of coefficients of g that minimize $\|J\|_p$. The Mathematica code we used to do this can be found in A.3. We present here some plots of the roots of these attempts. In each of these cases we chose the root of f to be $r = 0.9$ and chose $M = 10,\ 50$, and 100.

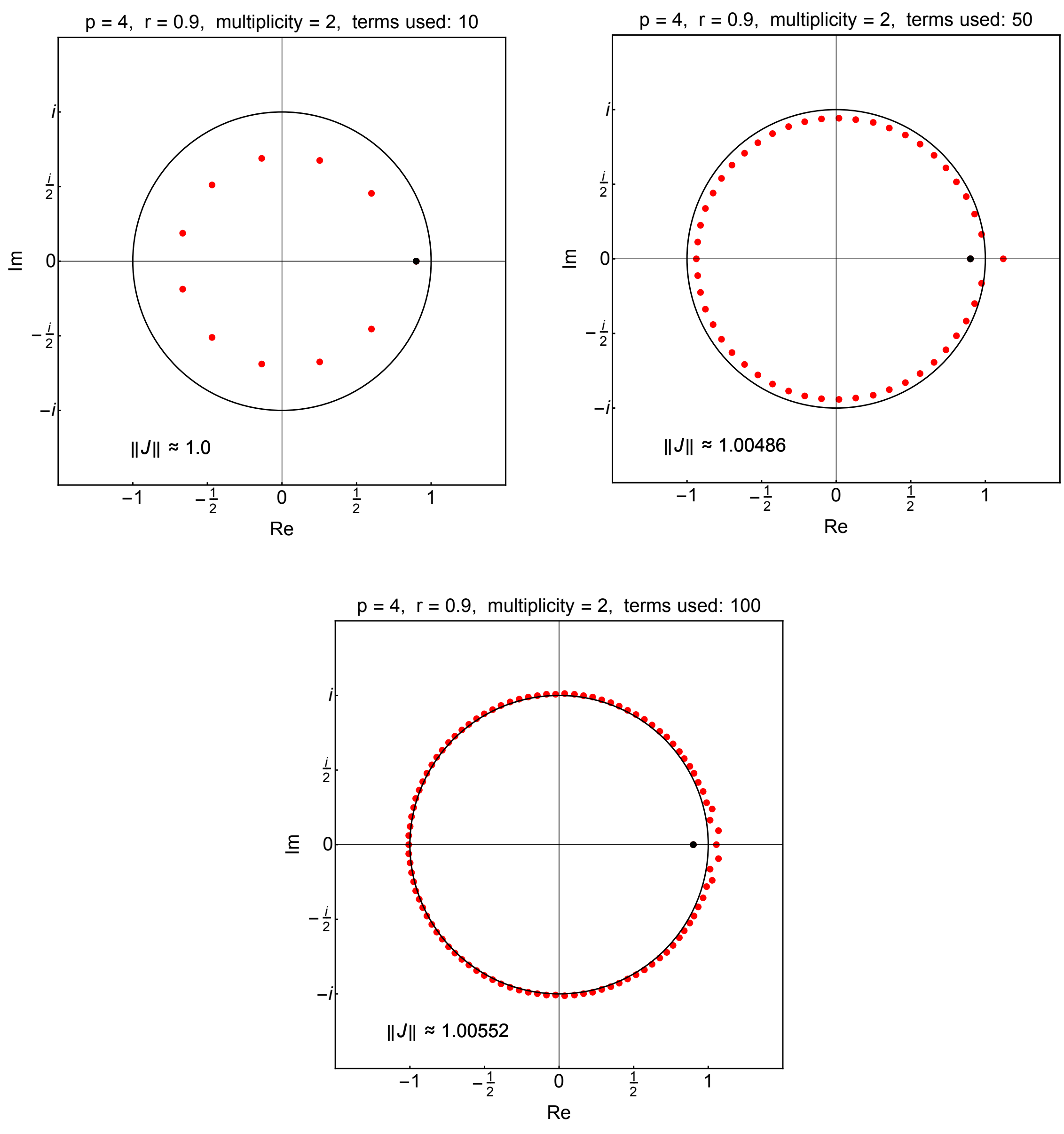

FIG. 12: Approximation of the roots of J, $p = 4$, $n = 2$

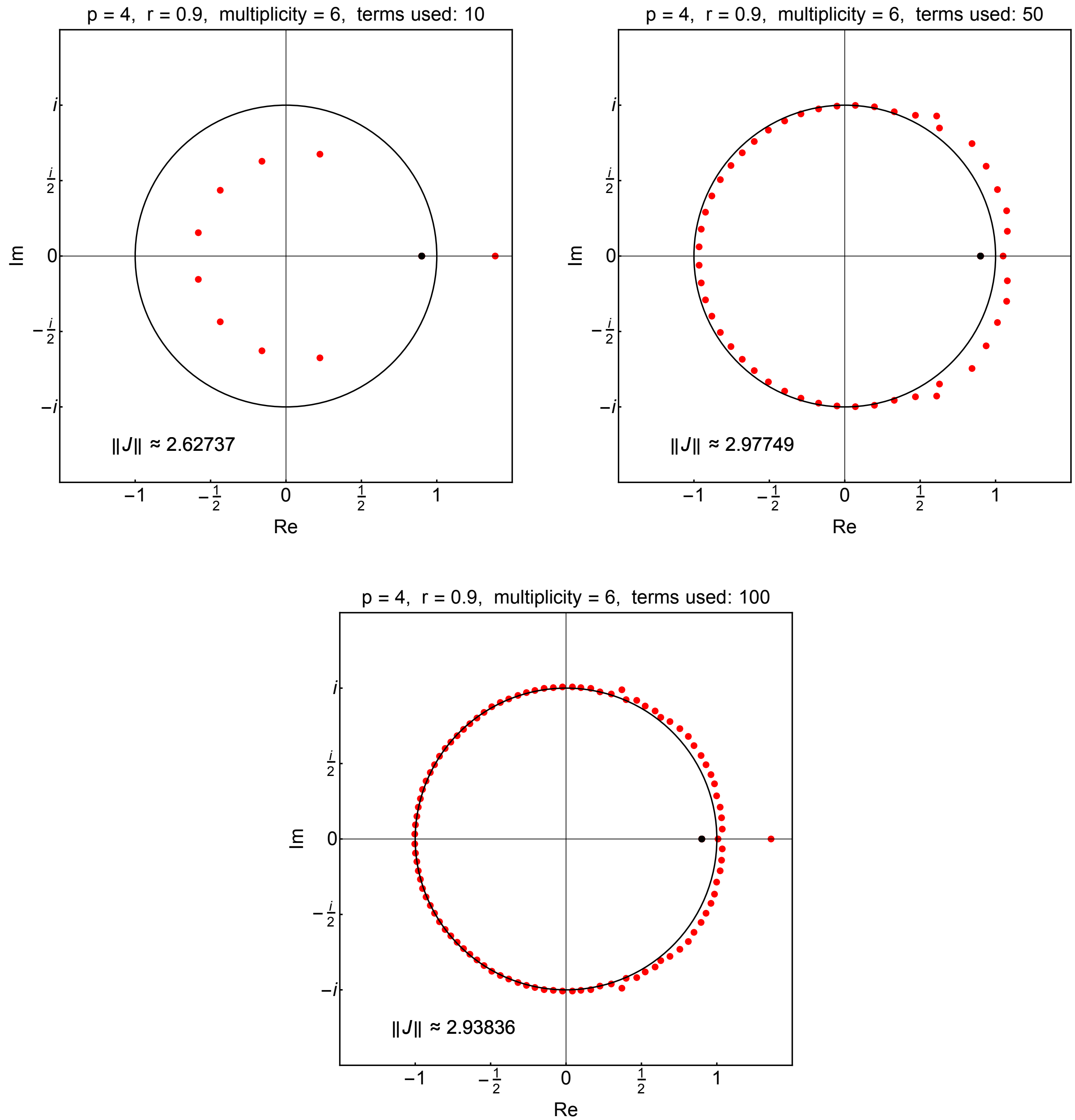

FIG. 13: Approximation of the roots of J, $p = 4$, $n = 6$

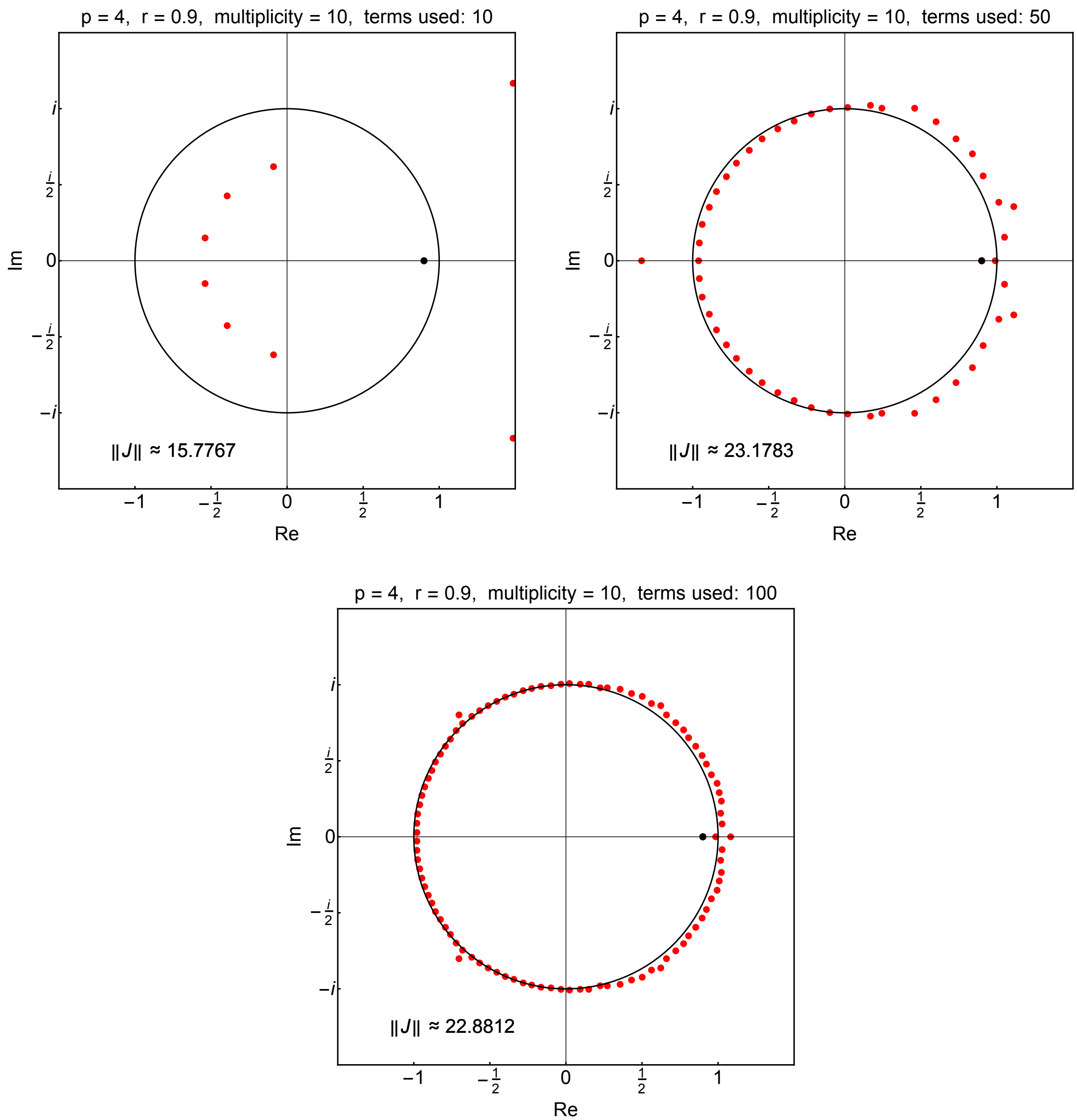

FIG. 14: Approximation of the roots of J, $p = 4$, $n = 10$

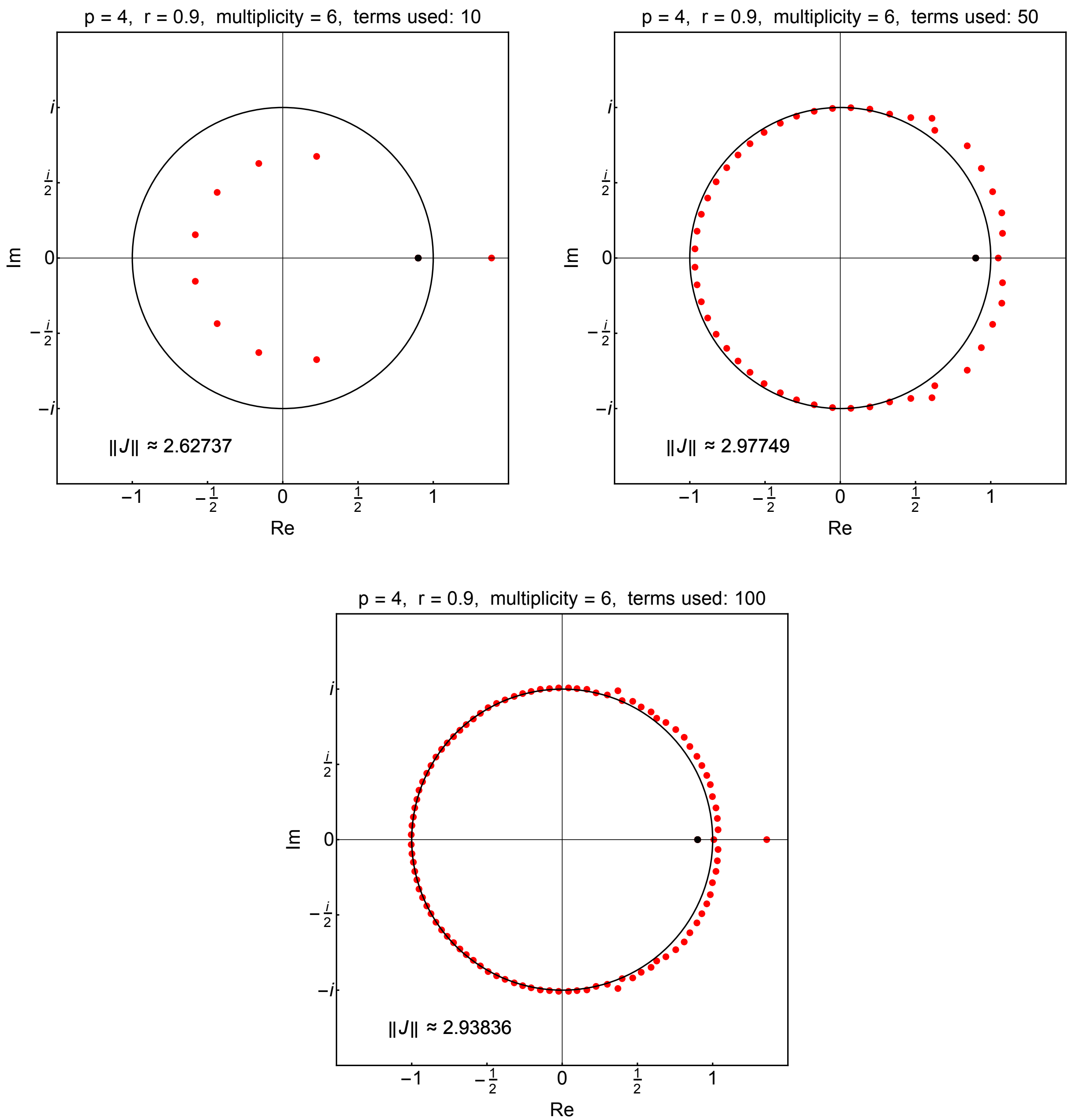

FIG. 15: Approximation of the roots of J, $p = 10$, $n = 2$

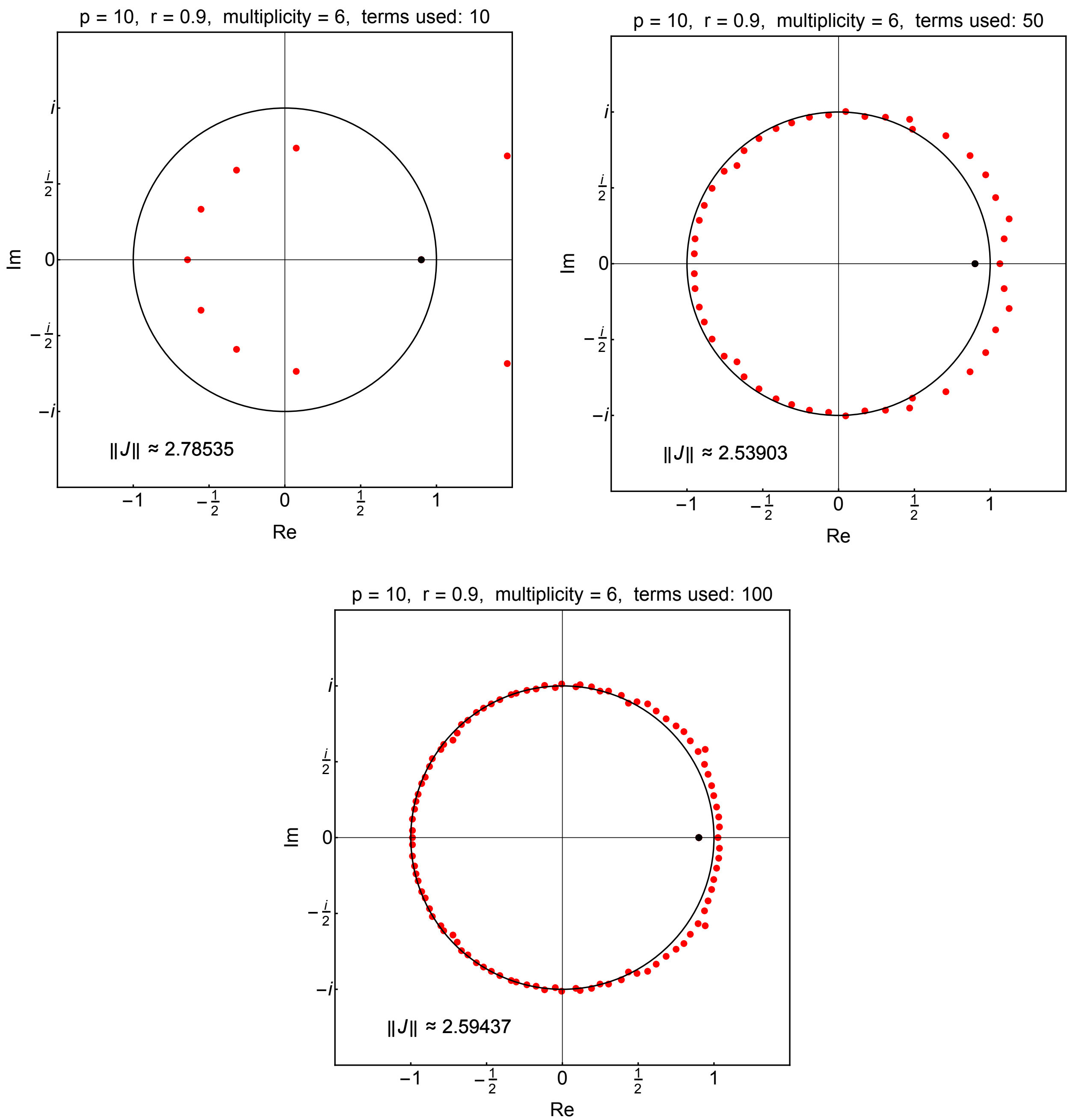

FIG. 16: Approximation of the roots of J, $p = 10$, $n = 6$

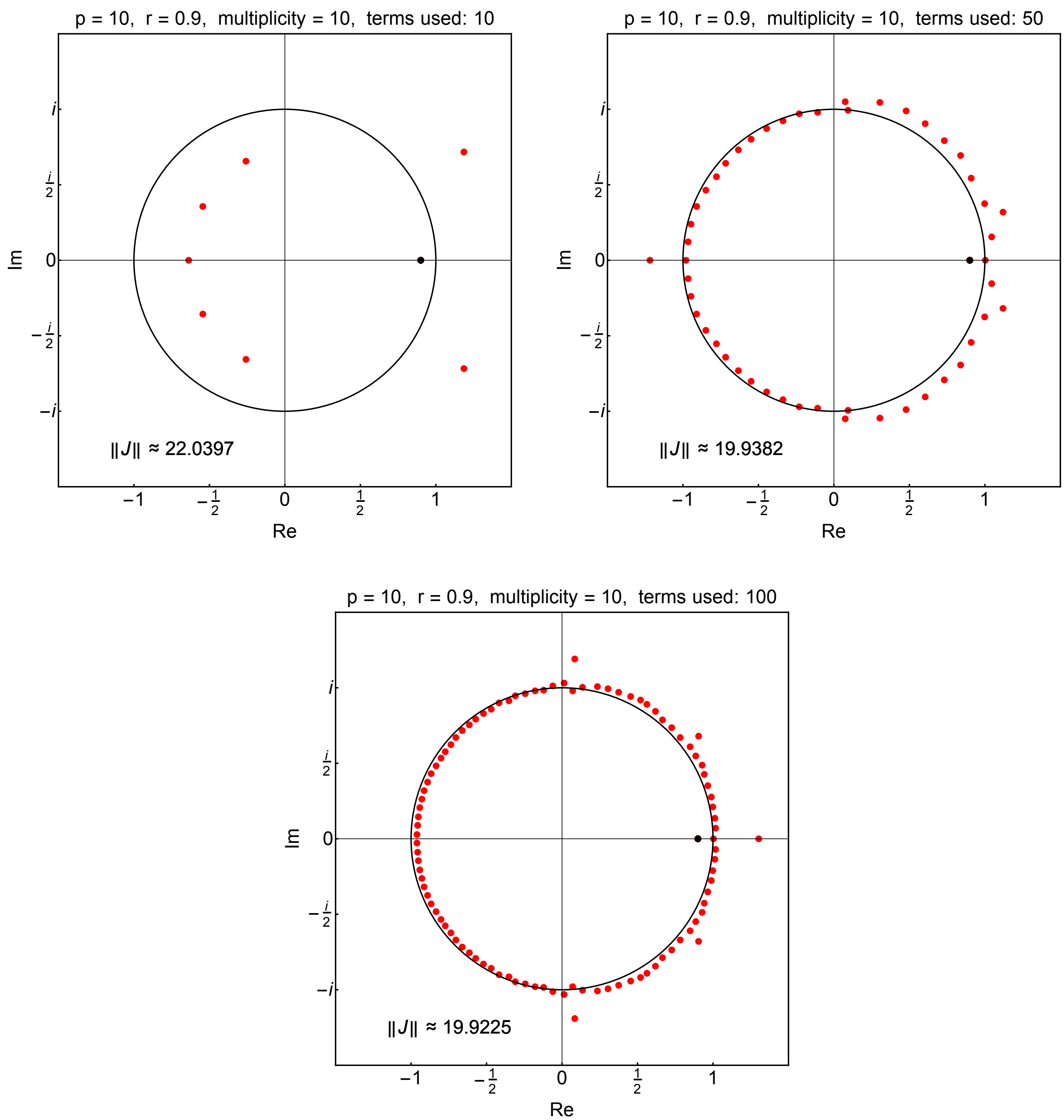

FIG. 17: Approximation of the roots of J, $p = 10$, $n = 10$

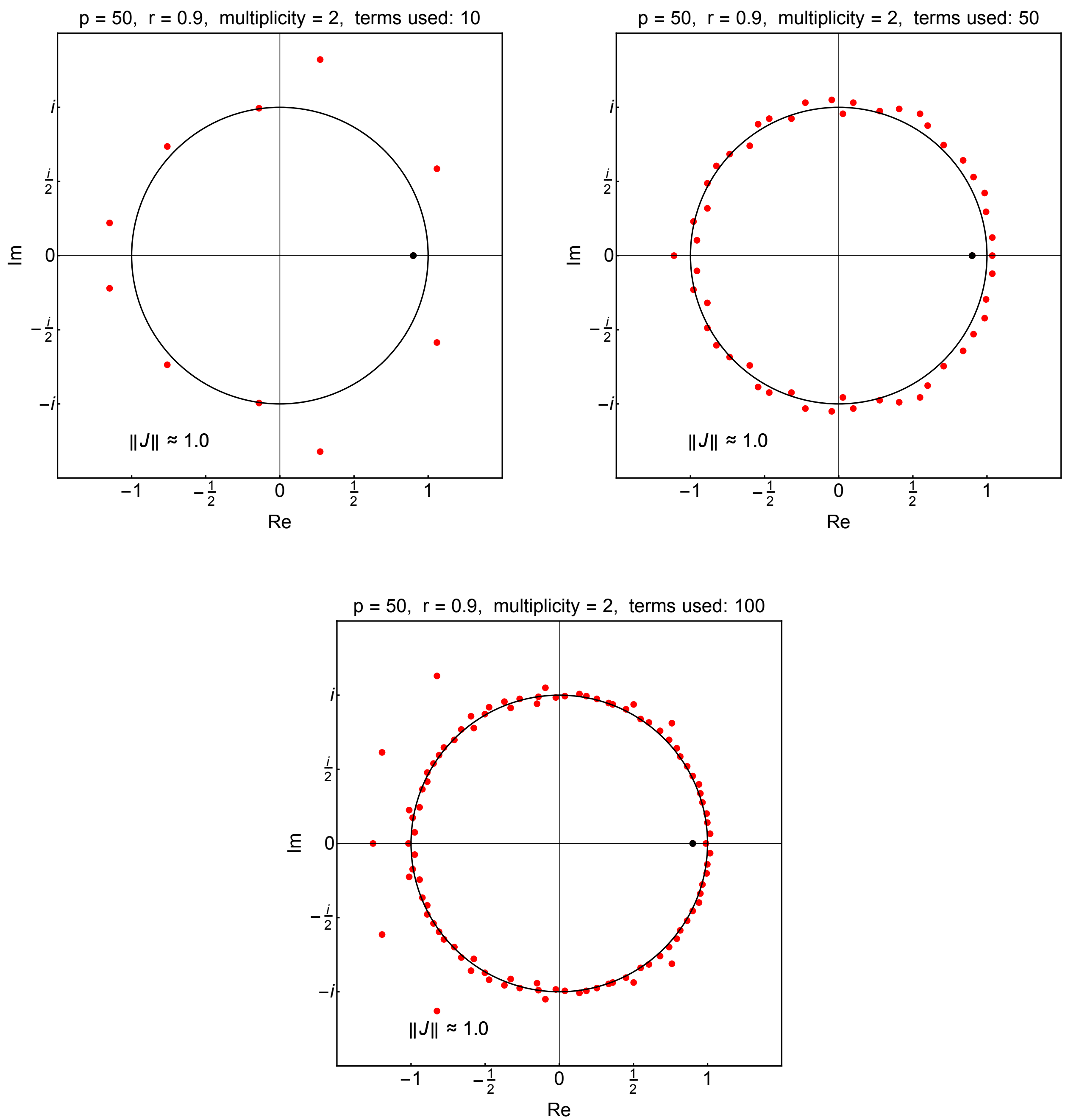

FIG. 18: Approximation of the roots of J, $p = 50$, $n = 2$

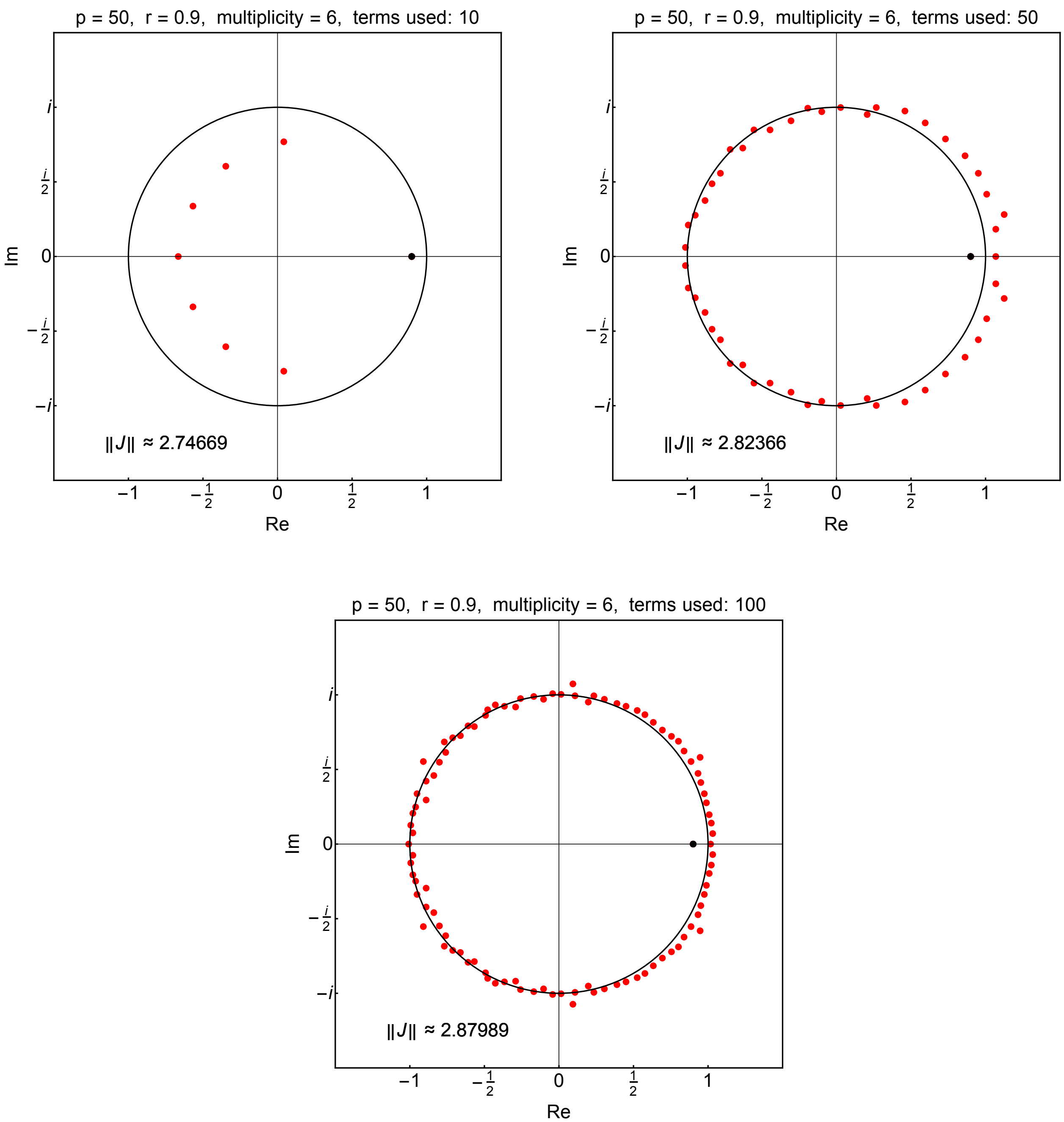

FIG. 19: Approximation of the roots of J, $p = 50$, $n = 6$

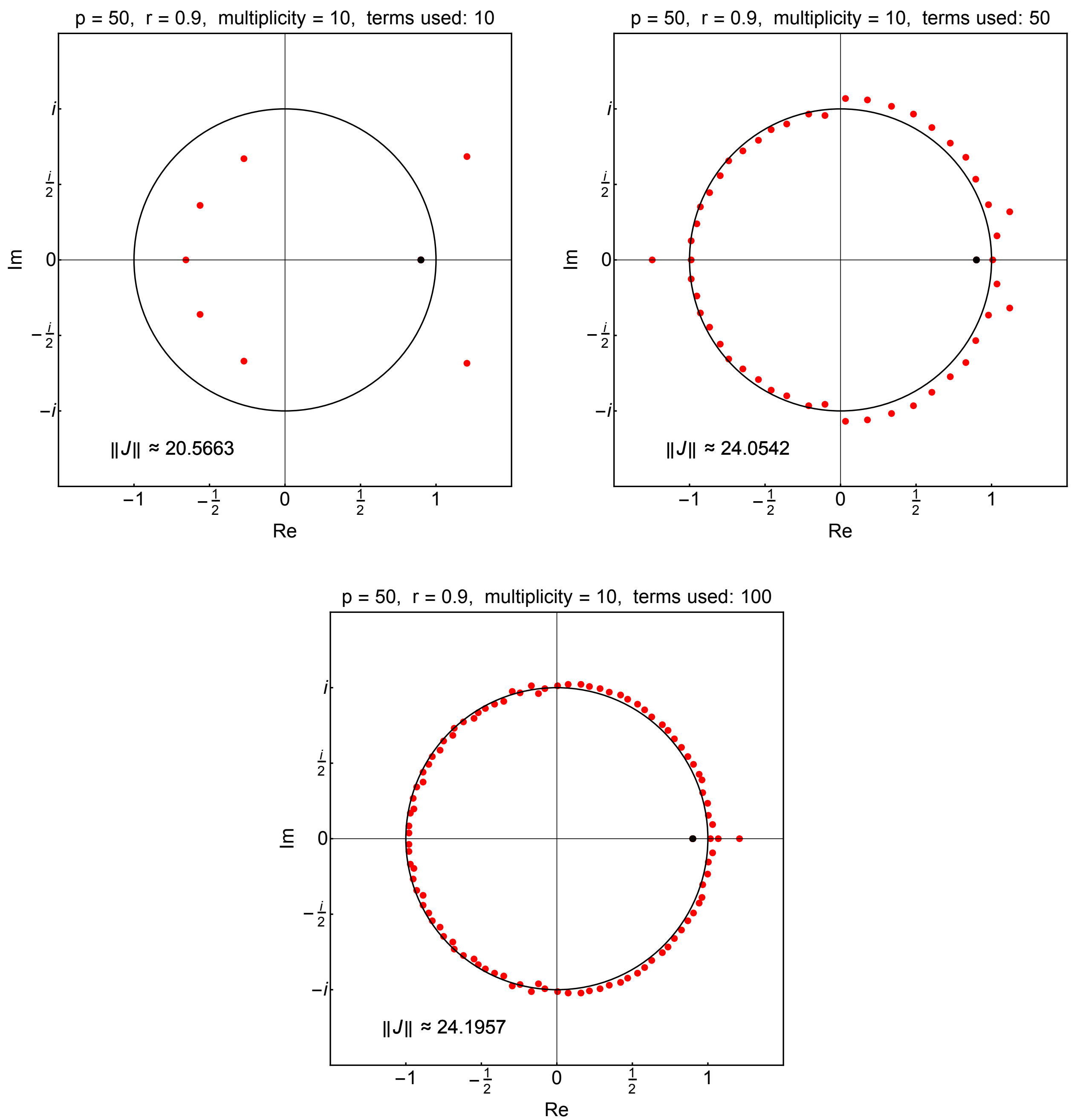

FIG. 20: Approximation of the roots of J, $p = 50$, $n = 10$

APPENDIX D

ADDITIONAL DATA

We present here data on the multiple extra root scheme presented at the end of Chapter 5. Specifically, we have calculated the value of the scaling factor r which minimizes the bound on the extra roots w for several choices of m and n as described in the text. We have chosen m and n to range over the values 1 to 6 and 4 to 6 respectively. We have done this for every rational choice of $p \in (1,2)$ such that the denominator in reduced form is at most 10. Such cases where the corresponding table is missing indicates that no extra roots were identified, and any instance of "NA" in a table indicates that no extra roots were found for that case. Each table also shows a bound on the norm of the corresponding p-inner function as well as a lower bound on choices of r that will yield extra roots.[22]

TABLE 1: Bounds on the multiple extra roots of J; $p = \frac{11}{10}$, $q = 11$

n	m	r	$\lvert w \rvert$	bound for $\lVert J \rVert$	lower bound on r
4	1	0.770508	0.795529	3.63757	0.712891
4	2	0.878906	0.891969	3.61018	0.84375
4	3	0.917969	0.926651	3.59548	0.892578
4	4	0.9375	0.944441	3.61018	0.918945
4	5	0.949219	0.955317	3.63621	0.93457
4	6	0.958008	0.962623	3.60209	0.944336
5	1	0.771484	0.78914	4.44208	0.722656
5	2	0.87793	0.888388	4.45862	0.849609
5	3	0.916992	0.924127	4.45141	0.897461
5	4	0.936523	0.94256	4.49311	0.921875
5	5	0.949219	0.953751	4.4622	0.936523
5	6	0.958008	0.961338	4.40631	0.947266
6	1	0.780273	0.794682	5.20079	0.738281
6	2	0.882813	0.891454	5.23109	0.859375
6	3	0.920898	0.926303	5.17783	0.90332
6	4	0.939453	0.944185	5.2452	0.926758
6	5	0.951172	0.955099	5.25749	0.94043
6	6	0.958984	0.962451	5.28307	0.950195

[22] The lower bounds on r are accurate within 2^{-10}.

TABLE 2: Bounds on the multiple extra roots of J; $p = \frac{10}{9}$, $q = 10$

n	m	r	$\|w\|$	bound for $\|J\|$	lower bound on r
4	1	0.775391	0.804627	3.54855	0.71582
4	2	0.880859	0.897123	3.54153	0.845703
4	3	0.918945	0.930136	3.54019	0.894531
4	4	0.938477	0.947137	3.5444	0.918945
4	5	0.951172	0.957487	3.5062	0.93457
4	6	0.958984	0.964446	3.51631	0.945313
5	1	0.776367	0.796355	4.30966	0.724609
5	2	0.880859	0.892425	4.31959	0.851563
5	3	0.918945	0.926945	4.3174	0.898438
5	4	0.938477	0.944668	4.32427	0.921875
5	5	0.950195	0.955516	4.349	0.9375
5	6	0.958984	0.962778	4.27847	0.947266
6	1	0.785156	0.800924	5.0189	0.740234
6	2	0.885742	0.894889	5.03835	0.860352
6	3	0.922852	0.928677	4.99397	0.904297
6	4	0.941406	0.946048	5.01025	0.927734
6	5	0.952148	0.956589	5.10107	0.941406
6	6	0.959961	0.963695	5.10118	0.951172

TABLE 3: Bounds on the multiple extra roots of J; $p = \frac{9}{8}$, $q = 9$

n	m	r	$\|w\|$	bound for $\|J\|$	lower bound on r
4	1	0.783203	0.815895	3.41957	0.71875
4	2	0.885742	0.903225	3.40265	0.847656
4	3	0.922852	0.93447	3.38491	0.895508
4	4	0.941406	0.950445	3.39142	0.920898
4	5	0.953125	0.960157	3.3769	0.935547
4	6	0.959961	0.966665	3.42767	0.946289
5	1	0.78418	0.8054	4.11616	0.727539
5	2	0.884766	0.897487	4.14433	0.852539
5	3	0.921875	0.930382	4.13096	0.899414
5	4	0.94043	0.947358	4.15743	0.922852
5	5	0.952148	0.95764	4.14929	0.9375
5	6	0.959961	0.964561	4.14937	0.948242
6	1	0.790039	0.808532	4.83917	0.743164
6	2	0.888672	0.899222	4.84821	0.861328
6	3	0.923828	0.931651	4.88709	0.905273
6	4	0.943359	0.948287	4.78178	0.927734
6	5	0.954102	0.95841	4.82153	0.942383
6	6	0.961914	0.96524	4.77593	0.951172

TABLE 4: Bounds on the multiple extra roots of J; $p = \frac{8}{7}$, $q = 8$

n	m	r	$\|w\|$	bound for $\|J\|$	lower bound on r
4	1	0.793945	0.830239	3.25508	0.723633
4	2	0.889648	0.911196	3.28437	0.850586
4	3	0.925781	0.93989	3.2608	0.897461
4	4	0.943359	0.954553	3.27847	0.921875
4	5	0.955078	0.963491	3.2464	0.936523
4	6	0.961914	0.969485	3.2761	0.947266
5	1	0.790039	0.816706	3.96435	0.731445
5	2	0.888672	0.903778	3.9702	0.854492
5	3	0.923828	0.934759	3.99535	0.900391
5	4	0.942383	0.950695	3.99052	0.924805
5	5	0.954102	0.96035	3.95292	0.938477
5	6	0.960938	0.966852	4.01686	0.949219
6	1	0.797852	0.818307	4.57689	0.746094
6	2	0.892578	0.904575	4.60917	0.863281
6	3	0.926758	0.935341	4.62948	0.90625
6	4	0.945313	0.951099	4.55755	0.928711
6	5	0.956055	0.96069	4.55186	0.942383
6	6	0.962891	0.967121	4.60047	0.952148

TABLE 5: Bounds on the multiple extra roots of J; $p = \frac{7}{6}$, $q = 7$

n	m	r	$\|w\|$	bound for $\|J\|$	lower bound on r
4	1	0.806641	0.849236	3.07018	0.730469
4	2	0.897461	0.921521	3.08318	0.854492
4	3	0.930664	0.947003	3.07621	0.900391
4	4	0.947266	0.959953	3.08607	0.923828
4	5	0.958008	0.967849	3.06689	0.938477
4	6	0.963867	0.973144	3.12216	0.948242
5	1	0.799805	0.831818	3.73777	0.736328
5	2	0.894531	0.912049	3.73098	0.858398
5	3	0.928711	0.940452	3.71625	0.902344
5	4	0.945313	0.955006	3.76025	0.925781
5	5	0.957031	0.963858	3.68415	0.94043
5	6	0.963867	0.969808	3.70123	0.950195
6	1	0.806641	0.831081	4.29627	0.750977
6	2	0.897461	0.91162	4.32716	0.866211
6	3	0.930664	0.940188	4.31058	0.908203
6	4	0.947266	0.95483	4.33404	0.930664
6	5	0.958008	0.963665	4.28845	0.944336
6	6	0.964844	0.969656	4.29334	0.953125

TABLE 6: Bounds on the multiple extra roots of J; $p = \frac{6}{5}$, $q = 6$

n	m	r	$\|w\|$	bound for $\|J\|$	lower bound on r
4	1	0.822266	0.875364	2.85499	0.742188
4	2	0.904297	0.935631	2.89896	0.861328
4	3	0.936523	0.956625	2.86332	0.905273
4	4	0.951172	0.967287	2.89129	0.927734
4	5	0.960938	0.973721	2.88317	0.941406
4	6	0.967773	0.978083	2.86174	0.951172
5	1	0.814453	0.852691	3.43285	0.745117
5	2	0.902344	0.923426	3.43647	0.862305
5	3	0.933594	0.948287	3.44449	0.90625
5	4	0.950195	0.960958	3.42147	0.928711
5	5	0.958984	0.96865	3.48685	0.942383
5	6	0.96582	0.973822	3.47722	0.952148
6	1	0.817383	0.848838	3.97487	0.757813
6	2	0.904297	0.921337	3.96639	0.870117
6	3	0.935547	0.946861	3.94171	0.911133
6	4	0.951172	0.959865	3.94859	0.932617
6	5	0.960938	0.967765	3.92979	0.945313
6	6	0.966797	0.973071	3.99281	0.954102

TABLE 7: Bounds on the multiple extra roots of J; $p = \frac{11}{9}$, $q = \frac{11}{2}$

n	m	r	$\|w\|$	bound for $\|J\|$	lower bound on r
4	1	0.832031	0.892255	2.72767	0.750977
4	2	0.912109	0.944581	2.72846	0.866211
4	3	0.939453	0.962734	2.75398	0.908203
4	4	0.955078	0.971918	2.72739	0.930664
4	5	0.963867	0.977472	2.72859	0.944336
4	6	0.969727	0.981186	2.73188	0.953125
5	1	0.824219	0.866523	3.24745	0.750977
5	2	0.90625	0.930884	3.29058	0.866211
5	3	0.936523	0.953385	3.2889	0.908203
5	4	0.952148	0.964846	3.28153	0.930664
5	5	0.961914	0.971753	3.25747	0.944336
5	6	0.967773	0.976407	3.28648	0.953125
6	1	0.826172	0.860577	3.74257	0.762695
6	2	0.90918	0.927674	3.73337	0.873047
6	3	0.938477	0.951222	3.73454	0.913086
6	4	0.953125	0.963161	3.76158	0.93457
6	5	0.962891	0.970425	3.70997	0.947266
6	6	0.96875	0.975306	3.73454	0.955078

TABLE 8: Bounds on the multiple extra roots of J; $p = \frac{5}{4}$, $q = 5$

n	m	r	$\|w\|$	bound for $\|J\|$	lower bound on r
4	1	0.84375	0.912582	2.58136	0.763672
4	2	0.917969	0.955292	2.59046	0.873047
4	3	0.944336	0.969986	2.59498	0.914063
4	4	0.958008	0.977415	2.59339	0.93457
4	5	0.966797	0.981866	2.57385	0.947266
4	6	0.972656	0.984895	2.55639	0.956055
5	1	0.833008	0.883478	3.07895	0.759766
5	2	0.912109	0.940001	3.09341	0.871094
5	3	0.941406	0.959579	3.06129	0.912109
5	4	0.956055	0.969526	3.04586	0.933594
5	5	0.963867	0.975556	3.09362	0.946289
5	6	0.969727	0.979581	3.09859	0.955078
6	1	0.835938	0.875204	3.49932	0.769531
6	2	0.914063	0.93549	3.50758	0.876953
6	3	0.941406	0.956531	3.53079	0.916016
6	4	0.956055	0.967198	3.50836	0.936523
6	5	0.964844	0.973684	3.49519	0.948242
6	6	0.970703	0.978035	3.48653	0.957031

TABLE 9: Bounds on the multiple extra roots of J; $p = \frac{9}{7}$, $q = \frac{9}{2}$

n	m	r	$\|w\|$	bound for $\|J\|$	lower bound on r
4	1	0.862305	0.936828	2.37796	0.782227
4	2	0.928711	0.967889	2.37645	0.883789
4	3	0.952148	0.978483	2.37115	0.920898
4	4	0.962891	0.983838	2.39841	0.94043
4	5	0.969727	0.987033	2.41455	0.951172
4	6	0.975586	0.98919	2.37898	0.958984
5	1	0.84668	0.904697	2.84735	0.771484
5	2	0.920898	0.951179	2.83066	0.87793
5	3	0.945313	0.967184	2.87107	0.916992
5	4	0.958984	0.975308	2.85862	0.936523
5	5	0.966797	0.98018	2.87227	0.949219
5	6	0.972656	0.983452	2.84647	0.957031
6	1	0.847656	0.893523	3.22866	0.779297
6	2	0.920898	0.945276	3.22178	0.882813
6	3	0.946289	0.963182	3.23369	0.919922
6	4	0.959961	0.97227	3.20189	0.939453
6	5	0.967773	0.977748	3.20678	0.951172
6	6	0.972656	0.981422	3.2444	0.958984

TABLE 10: Bounds on the multiple extra roots of J; $p = \frac{13}{10}$, $q = \frac{13}{3}$

n	m	r	$\lvert w \rvert$	bound for $\lVert J \rVert$	lower bound on r
4	1	0.864258	0.945739	2.34135	0.790039
4	2	0.930664	0.972495	2.32737	0.888672
4	3	0.953125	0.981594	2.32939	0.923828
4	4	0.96582	0.986152	2.29795	0.942383
4	5	0.972656	0.988912	2.29486	0.953125
4	6	0.976563	0.990757	2.3183	0.960938
5	1	0.853516	0.912928	2.74426	0.776367
5	2	0.921875	0.955499	2.78728	0.880859
5	3	0.949219	0.970124	2.72406	0.918945
5	4	0.959961	0.977491	2.79496	0.938477
5	5	0.96875	0.981942	2.74775	0.950195
5	6	0.973633	0.98495	2.76353	0.958008
6	1	0.853516	0.900759	3.10786	0.78418
6	2	0.922852	0.949078	3.13877	0.884766
6	3	0.948242	0.965782	3.12261	0.921875
6	4	0.960938	0.974208	3.12187	0.94043
6	5	0.96875	0.979333	3.11282	0.952148
6	6	0.973633	0.982751	3.13524	0.959961

TABLE 11: Bounds on the multiple extra roots of J; $p = \frac{4}{3}$, $q = 4$

n	m	r	$\lvert w \rvert$	bound for $\lVert J \rVert$	lower bound on r
4	1	0.882813	0.964522	2.15805	0.8125
4	2	0.94043	0.982115	2.14706	0.901367
4	3	0.959961	0.988033	2.14555	0.932617
4	4	0.970703	0.991035	2.12346	0.949219
4	5	0.975586	0.992831	2.15161	0.958984
4	6	0.979492	0.994027	2.15612	0.96582
5	1	0.863281	0.931262	2.58508	0.790039
5	2	0.929688	0.965016	2.57388	0.888672
5	3	0.951172	0.976554	2.61486	0.923828
5	4	0.963867	0.982353	2.58688	0.942383
5	5	0.97168	0.985867	2.55379	0.954102
5	6	0.976563	0.988218	2.54139	0.960938
6	1	0.862305	0.917174	2.91907	0.794922
6	2	0.929688	0.957689	2.88855	0.891602
6	3	0.952148	0.971595	2.90538	0.925781
6	4	0.962891	0.978645	2.96037	0.943359
6	5	0.970703	0.98287	2.926	0.955078
6	6	0.975586	0.985707	2.92112	0.961914

TABLE 12: Bounds on the multiple extra roots of J; $p = \frac{11}{8}$, $q = \frac{11}{3}$

n	m	r	$\lvert w \rvert$	bound for $\lVert J \rVert$	lower bound on r
4	1	0.90918	0.983508	1.92356	0.848633
4	2	0.952148	0.991733	1.94017	0.920898
4	3	0.96875	0.994468	1.92375	0.946289
4	4	0.975586	0.995892	1.94482	0.959961
4	5	0.981445	0.99668	1.91446	0.967773
4	6	0.984375	0.997234	1.91936	0.972656
5	1	0.879883	0.952061	2.35446	0.80957
5	2	0.936523	0.975757	2.38205	0.899414
5	3	0.958008	0.983754	2.36075	0.931641
5	4	0.96875	0.987827	2.34613	0.948242
5	5	0.975586	0.990252	2.31709	0.958008
5	6	0.978516	0.991858	2.37468	0.964844
6	1	0.87793	0.936596	2.63557	0.80957
6	2	0.935547	0.967777	2.67208	0.899414
6	3	0.958008	0.97842	2.61798	0.931641
6	4	0.967773	0.983786	2.64555	0.948242
6	5	0.973633	0.986984	2.67637	0.958008
6	6	0.978516	0.989156	2.63702	0.964844

TABLE 13: Bounds on the multiple extra roots of J; $p = \frac{7}{5}$, $q = \frac{7}{2}$

n	m	r	$\lvert w \rvert$	bound for $\lVert J \rVert$	lower bound on r
4	1	0.922852	0.992107	1.80295	0.878906
4	2	0.963867	0.996032	1.76426	0.9375
4	3	0.974609	0.997348	1.78486	0.957031
4	4	0.982422	0.998015	1.74875	0.967773
4	5	0.985352	0.998432	1.76531	0.973633
4	6	0.987305	0.998683	1.78198	0.978516
5	1	0.889648	0.96325	2.22696	0.823242
5	2	0.942383	0.981466	2.24146	0.907227
5	3	0.961914	0.987594	2.22335	0.9375
5	4	0.97168	0.990683	2.21045	0.952148
5	5	0.976563	0.992558	2.24059	0.961914
5	6	0.980469	0.993811	2.23865	0.967773
6	1	0.884766	0.947406	2.51135	0.819336
6	2	0.941406	0.973393	2.49249	0.905273
6	3	0.958984	0.982176	2.54796	0.935547
6	4	0.969727	0.986599	2.51739	0.951172
6	5	0.975586	0.989284	2.52444	0.960938
6	6	0.979492	0.99105	2.53289	0.966797

TABLE 14: Bounds on the multiple extra roots of J; $p = \frac{10}{7}$, $q = \frac{10}{3}$

n	m	r	$\|w\|$	bound for $\|J\|$	lower bound on r
4	1	0.961914	0.998634	1.51645	0.929688
4	2	0.977539	0.999356	1.55899	0.963867
4	3	0.986328	0.999563	1.53258	0.975586
4	4	0.988281	0.999668	1.56976	0.981445
4	5	0.992188	0.999742	1.51899	0.985352
4	6	0.993164	0.999775	1.53166	0.987305
5	1	0.90332	0.974661	2.06636	0.84082
5	2	0.950195	0.987241	2.07025	0.916992
5	3	0.96582	0.991502	2.08719	0.943359
5	4	0.975586	0.993614	2.04446	0.957031
5	5	0.979492	0.994883	2.08147	0.96582
5	6	0.982422	0.99574	2.10356	0.970703
6	1	0.895508	0.958992	2.34066	0.832031
6	2	0.945313	0.97929	2.36327	0.912109
6	3	0.962891	0.986152	2.37368	0.94043
6	4	0.97168	0.989606	2.38958	0.955078
6	5	0.977539	0.991668	2.37517	0.963867
6	6	0.981445	0.993068	2.36214	0.969727

TABLE 15: Bounds on the multiple extra roots of J; $p = \frac{13}{9}$, $q = \frac{13}{5}$

n	m	r	$\|w\|$	bound for $\|J\|$	lower bound on r
4	1	NA	NA	NA	NA
4	2	NA	NA	NA	NA
4	3	NA	NA	NA	NA
4	4	NA	NA	NA	NA
4	5	NA	NA	NA	NA
4	6	NA	NA	NA	NA
5	1	0.912109	0.980286	1.97105	0.852539
5	2	0.956055	0.990131	1.95497	0.922852
5	3	0.969727	0.993409	1.97279	0.948242
5	4	0.977539	0.995049	1.96247	0.960938
5	5	0.980469	0.996055	2.0219	0.967773
5	6	0.984375	0.996704	1.99011	0.973633
6	1	0.900391	0.965001	2.26288	0.838867
6	2	0.949219	0.982361	2.25569	0.916016
6	3	0.964844	0.988205	2.28828	0.943359
6	4	0.974609	0.991152	2.2417	0.957031
6	5	0.979492	0.99292	2.24922	0.964844
6	6	0.982422	0.994074	2.27825	0.970703

TABLE 16: Bounds on the multiple extra roots of J; $p = \frac{3}{2}$, $q = 3$

n	m	r	$\|w\|$	bound for $\|J\|$	lower bound on r
4	1	NA	NA	NA	NA
4	2	NA	NA	NA	NA
4	3	NA	NA	NA	NA
4	4	NA	NA	NA	NA
4	5	NA	NA	NA	NA
4	6	NA	NA	NA	NA
5	1	0.944336	0.99542	1.64688	0.905273
5	2	0.973633	0.997724	1.61877	0.951172
5	3	0.980469	0.998465	1.66076	0.966797
5	4	0.985352	0.998863	1.65969	0.974609
5	5	0.988281	0.999076	1.65906	0.979492
5	6	0.991211	0.999244	1.61528	0.983398
6	1	0.920898	0.983073	1.97176	0.870117
6	2	0.959961	0.991501	1.96536	0.932617
6	3	0.972656	0.994352	1.979	0.954102
6	4	0.979492	0.995746	1.97627	0.96582
6	5	0.983398	0.996614	1.98402	0.972656
6	6	0.986328	0.99716	1.97357	0.976563

TABLE 17: Bounds on the multiple extra roots of J; $p = \frac{14}{9}$, $q = \frac{14}{5}$

n	m	r	$\|w\|$	bound for $\|J\|$	lower bound on r
4	1	NA	NA	NA	NA
4	2	NA	NA	NA	NA
4	3	NA	NA	NA	NA
4	4	NA	NA	NA	NA
4	5	NA	NA	NA	NA
4	6	NA	NA	NA	NA
5	1	NA	NA	NA	NA
5	2	NA	NA	NA	NA
5	3	NA	NA	NA	NA
5	4	NA	NA	NA	NA
5	5	NA	NA	NA	NA
5	6	NA	NA	NA	NA
6	1	0.948242	0.995828	1.64517	0.916016
6	2	0.973633	0.997917	1.64777	0.957031
6	3	0.982422	0.998613	1.64567	0.970703
6	4	0.988281	0.998957	1.59256	0.977539
6	5	0.989258	0.999171	1.65268	0.982422
6	6	0.991211	0.999315	1.6436	0.985352

TABLE 18: Bounds on the multiple extra roots of J; $p = \frac{11}{7}$, $q = \frac{11}{4}$

n	m	r	$\|w\|$	bound for $\|J\|$	lower bound on r
4	1	NA	NA	NA	NA
4	2	NA	NA	NA	NA
4	3	NA	NA	NA	NA
4	4	NA	NA	NA	NA
4	5	NA	NA	NA	NA
4	6	NA	NA	NA	NA
5	1	NA	NA	NA	NA
5	2	NA	NA	NA	NA
5	3	NA	NA	NA	NA
5	4	NA	NA	NA	NA
5	5	NA	NA	NA	NA
5	6	NA	NA	NA	NA
6	1	0.964844	0.998177	1.47656	0.936523
6	2	0.980469	0.999087	1.50906	0.967773
6	3	0.987305	0.99939	1.49916	0.977539
6	4	0.989258	0.999557	1.54206	0.983398
6	5	0.993164	0.999649	1.46318	0.986328
6	6	0.993164	0.999713	1.52411	0.988281

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