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Best Quasi-convex Uniform Approximation

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1. INTRODUCTION

Let \( B = B[0, 1] \) be the linear space of all bounded real functions \( f \) on \([0, 1]\), with the uniform norm
\[
\| f \|_\infty = \sup_{x \in [0, 1]} |f(x)|.
\]

Let \( C[0, 1] \) denote the space of all continuous functions on \([0, 1]\).

**Definition 1.** A function \( g \in B \) is said to be quasi-convex [2] if
\[
g(x) \leq \max \{ g(s), g(t) \} \quad \text{for all } x, s, \text{ and } t \text{ such that } 0 \leq s \leq x \leq t \leq 1.
\]

Let \( K \subset B \) denote the set of all quasi-convex functions on \([0, 1]\).

Ubhaya [8] has proved that \( g \) is quasi-convex if and only if there exists a point \( p \in [0, 1] \), such that either

(i) \( g \) is nonincreasing on \([0, p)\) and is nondecreasing on \([p, 1]\) or

(ii) \( g \) is nonincreasing on \([0, p]\) and is nondecreasing on \((p, 1]\).

We call the point \( p \) (in either (i) or (iii)) a knot of \( g \). Let \( K_p \) denote the functions in \( K \) which have a knot at \( p \). Then,
\[
K = \bigcup_{p \in [0, 1]} K_p.
\]

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In general, the set of all the knots of a quasi-convex function is a closed subinterval of \([0, 1]\).

The problem of the best quasi-convex approximation is to find a \(g^* \in K\), such that

\[
\|f - g^*\|_\infty = \inf_{g \in K} \{\|f - g\|_\infty\}. \tag{1.1}
\]

This problem is considered in [8], where a sufficient condition for a best quasi-convex approximation to a bounded function is obtained, and some structural properties of best approximations are established. Algorithms for the computation of a best discrete quasi-convex approximation are presented in [1, 7].

Throughout this paper we shall assume that \(f \in C[0, 1]\), unless stated otherwise.

**Definition 2.** Given \(f \in C[0, 1]\), let

\[
G = G(f) = \{g^* \in K : \|f - g^*\|_\infty = \inf_{g \in K} \{\|f - g\|\}\} \tag{1.2}
\]

the set of best quasi-convex approximations to \(f\), and let

\[
P^* = \{p \in [0, 1] : p \text{ is a knot for some } g^* \in G\}. \tag{1.3}
\]

We call \(P^*\) the set of optimal knots.

We characterize both the best quasi-convex approximations and the optimal knots. In addition we describe the construction of the set of best approximations and prove that a best quasi-convex approximation is unique if and only if \(f\) is quasi-convex.

2. Preliminaries

Similar to the development in [5] we define two functionals \(\delta_r\) and \(\delta_\gamma\), which we use to obtain the error of the best quasi-convex approximation.

**Definition 3.** For \(f \in C[0, 1]\) and \(p \in [0, 1]\), let

\[
\delta_r(p) = \sup_{0 < x < y < p} \frac{\left|f(y) - f(x)\right|}{2}, \tag{2.1}
\]

and

\[
\delta_\gamma(p) = \sup_{p < x < y < 1} \frac{\left|f(x) - f(y)\right|}{2}. \tag{2.2}
\]
Thus, $\delta_-$ is a measure of the "decreasingness" of $f$ on $[0, p]$, and $\delta_+$ is a measure of the "increasingness" of $f$ on $(p, 1]$.

For $f \in C[0, 1]$ and $p \in [0, 1]$ (as in [8]), define

$$
\delta(p) = \max\{\delta_-(p), \delta_+(p)\}.
$$

(2.3)

Denote the minimum value of $\delta(p)$ on $[0, 1]$ by

$$
\delta^* = \inf_{0 \leq p \leq 1} \delta(p).
$$

(2.4)

Let

$$
P = \{ p \in [0, 1] : \delta(p) = \delta^* \}
$$

(2.5)

be the set of minima for $\delta$, and let

$$
S = \{ s \in [0, 1] : f(s) = \inf_{0 \leq x \leq 1} f(x) \}
$$

(2.6)

be the set of minima for $f$.

Let $[s_\ell, s_u]$ be the convex hull of $S$. Then,

$$
s_\ell = \inf S \quad \text{and} \quad s_u = \sup S.
$$

(2.7)

Also, let

$$
m = \inf\{f(x) : 0 \leq x \leq 1\},
$$

and then define

$$
\eta_\ell = \inf\{ x \in [0, s_\ell] : f(t) \leq m + 2\delta^*, \text{ for all } t \in [x, s_\ell] \},
$$

(2.8)

and

$$
\eta_u = \sup\{ x \in [s_u, 1] : f(t) \leq m + 2\delta^*, \text{ for all } t \in [s_u, x] \}.
$$

(2.9)

Thus,

$$
[s_\ell, s_u] \subseteq [\eta_\ell, \eta_u].
$$

We shall prove that $P = [\eta_\ell, \eta_u]$, and that $P = P^*$, the set of optimal knots.

Next, let $f \in B$. For each $p \in [0, 1]$, similar to the definitions of $U_p^-$ and $V_p^-$ in [8] with $\theta_p^-$ replaced by $\delta^*$ we define the two functions

$$
g_p(x) = \begin{cases} 
\sup_{t \in [x, p]} f(t) - \delta^*, & x \in [0, p] \\
\sup_{t \in (p, x]} f(t) - \delta^*, & x \in (p, 1] 
\end{cases}
$$

(2.10)
and

\[ g_p(x) = \begin{cases} \inf_{t \in [0,x]} f(t) + \delta^*, & x \in [0, p] \\ \inf_{t \in [x,1]} f(t) + \delta^*, & x \in (p, 1]. \end{cases} \quad (2.11) \]

**Lemma 1.** Let \( f \in C[0, 1] \). Then,

(i) \( |\Delta \delta^*(p)| = \frac{1}{2} \omega_f(|Ap|) \), and \( |\Delta \delta^*(p)| \leq \frac{1}{2} \omega_f(|Ap|) \) (where \( \omega_f(*) \) denotes the modulus of continuity of \( f \)). Thus, \( \delta^* \) and \( \delta \), are continuous.

(ii) \( \delta^* = 0 \) if and only if \( f \in K \).

(iii) \( S \subset P \).

**Proof:** (i) If \( Ap > 0 \) then

\[ \delta^*(p + Ap) \leq \delta^*(p) + \sup_{0 < x < y < p + |Ap|} \frac{|f(y) - f(x)|}{2}, \]

and if \( Ap < 0 \) then,

\[ \delta^*(p) \leq \delta^*(p - |Ap|) + \sup_{p - |Ap| < x < y < p} \frac{|f(y) - f(x)|}{2}. \]

It follows that

\[ |\Delta \delta^*(p)| \leq \sup_{0 < y - x \leq |Ap|} \frac{|f(y) - f(x)|}{2} = \frac{1}{2} \omega_f(|Ap|). \]

Similarly, we may show the second inequality of (i).

(ii) First let \( \delta^* = 0 \). By (i) \( \delta^* \) and \( \delta \), are continuous and thus so is \( \delta \), where \( \delta(p) = \max \{\delta^*(p), \delta^*(p)\} \) for \( p \in [0, 1] \). Hence, there exists a \( p_0 \in [0, 1] \), such that \( \delta(p_0) = \delta^* = 0 \). Thus, \( \delta^*(p_0) = \delta^*(p_0) = 0 \), since \( \delta^* \) and \( \delta \), are both nonnegative functions. Consequently, by the definitions of \( \delta^* \) and \( \delta \), \( f \) is nonincreasing on \( [0, p_0] \), and nondecreasing on \( (p_0, 1] \). Thus, \( f \in K \).

Conversely, assume that \( f \in K \). Then there exists a \( p_0 \in [0, 1] \) such that \( f \in K_{p_0} \). Therefore, \( \delta^*(p_0) = \delta^*(p_0) = 0 \), which implies that \( \delta(p_0) = 0 \). Hence, \( \delta^* = 0 \).

(iii) It is sufficient to show that if \( s \in S \), then,

\[ \delta^*(s) \leq \max \{\delta^*(p), \delta^*(p)\} \quad \text{for all } p \in [0, 1] \]

and

\[ \delta^*(s) \leq \max \{\delta^*(p), \delta^*(p)\} \quad \text{for all } p \in [0, 1]. \]

(2.12)
The proofs of (2.12) and (2.13) are similar; thus we only present the proof of (2.12).

If \( s = 0 \) then, since \( \delta_r(0) = 0 \), and since \( \delta_r \) and \( \delta \), are both nonnegative functions, (2.12) holds.

If \( s \in (0, 1] \), we consider two cases. First assume that \( p \geq s \). Then \( \delta_r(s) \leq \delta_r(p) \) and thus (2.12) holds.

Next, assume that \( p < s \), and \( \delta_r(p) < \delta_r(s) \). \( f \in C[0, 1] \) implies that

\[
2\delta_r(s) = f(y_1) - f(x_1) \quad \text{for some } x_1 \leq y_1 \text{ in } [0, s].
\]

It follows that \( 2\delta_r(p) < f(y_1) - f(x_1) \) and \( p < y_1 \). Hence,

\[
2\delta_r(s) \leq f(y_1) - f(s) \leq \sup_{p \leq x \leq y \leq s} [f(x) - f(y)] \leq 2\delta_r(p).
\]

Therefore, (2.12) holds.

**Lemma 2.** \( g_p \) and \( \tilde{g}_p \) as defined by (2.10) and (2.11) have the following properties:

(i) \( g_p, \tilde{g}_p \in K_p \) for all \( p \in [0, 1] \),

(ii) if \( f \in C[0, 1] \) then

(a) \( g_p \in C[0, 1] \) for all \( p \in [0, 1] \),

(b) \( \tilde{g}_p \in C[0, 1] \) if and only if \( p \in [s_r, s_s] \),

(c) if \( p \in [s_r, s_s] \), then \( g_p(x) = \tilde{g}_p(x) \) for all \( x \in [0, 1] \).

(d) if \( p \in [0, 1] \), then \( g_p(x) \leq \tilde{g}_p(x) \) for all \( x \in [0, 1] \).

**Proof.** (i) follows from the definitions (2.10) and (2.11).

(ii) (a) For all \( p \in [0, 1] \), (2.10) implies that \( g_p \) is continuous at any \( x \neq p \).

Next, to prove the continuity of \( g_p \) at \( x = p \), we observe that

\[
g_p(p-) = \lim_{\epsilon \to 0^+} \sup_{t \in [p-\epsilon, p]} f(t) - \delta^* = f(p) - \delta^*
\]

and

\[
g_p(p+) = \lim_{\epsilon \to 0^-} \sup_{t \in (p, p+\epsilon]} f(t) - \delta^* = f(p) - \delta^*,
\]

since \( f \in C[0, 1] \). Thus, \( g_p(p-) = g_p(p+) = g_p(p) \), and (a) is proved.

(b) Similarly, for all \( p \in [0, 1] \), \( \tilde{g}_p \) is continuous where \( x \neq p \). Next, if \( x = p \) and \( p \in [s_r, s_s] \), then

\[
\tilde{g}_p(p-) = \lim_{\epsilon \to 0^+} \inf_{t \in [0, p-\epsilon]} f(t) + \delta^* = f(s_r) + \delta^*
\]
and
\[ \tilde{g}_p(p+) = \lim_{\epsilon \to 0} \inf_{t \in [p + \epsilon, 1]} f(t) + \delta^* = f(s_r) + \delta^*. \]

Hence, \( \tilde{g}_p(p-) = \tilde{g}_p(p) = \tilde{g}_p(p+) \).
Conversely, suppose that \( p \notin [s_r, s_s] \). If \( p < s_r \), then
\[ \tilde{g}_p(p-) = \lim_{\epsilon \to 0} \inf_{t \in [0, p - \epsilon]} f(t) + \delta^* > f(s_r) + \delta^* \]
\[ = \lim_{\epsilon \to 0} \inf_{t \in [p - \epsilon, 1]} f(t) + \delta^* = \tilde{g}_p(p +). \]

While if \( p > s \), then
\[ \tilde{g}_p(p-) = \lim_{\epsilon \to 0} \inf_{t \in [0, p - \epsilon]} f(t) + \delta^* = f(s_r) + \delta^* < \lim_{\epsilon \to 0} \inf_{t \in [p + \epsilon, 1]} f(t) + \delta^* = \tilde{g}_p(p +). \]

(c) Let \( p \in [s_r, s_s] \). For \( x \in [s_r, p] \),
\[ \tilde{g}_p(x) = \inf_{t \in [0, x]} f(t) + \delta^* = f(s_r) + \delta^* = m + \delta^*, \]
for \( x \in (p, s_s] \),
\[ \tilde{g}_p(x) = \inf_{t \in [x, 1]} f(t) + \delta^* = f(s_s) + \delta^* = m + \delta^*, \]
and for \( x \notin [s_r, s_s] \),
\[ \tilde{g}_p(x) = \tilde{g}_{s_r}(x). \]

Thus, \( \tilde{g}_p = \tilde{g}_{s_r} \).

(d) Assume that \( p \notin [s_r, s_s] \). If \( p < s_r \), then
\[ \tilde{g}_p(x) = \tilde{g}_{s_r}(x) \quad \text{for all } x \in [0, p], \]
\[ \tilde{g}_p(x) = \inf_{t \in [x, 1]} f(t) + \delta^* = f(s_r) + \delta^* < \inf_{t \in [0, x]} f(t) + \delta^* = \tilde{g}_{s_r}(x) \quad \text{for all } x \in (p, s_s) \]
and \( \tilde{g}_p(x) = \tilde{g}_{s_r}(x) \) for all \( x \in [s_s, 1] \). If \( p > s_s \), then \( \tilde{g}_p(x) = \tilde{g}_{s_s}(x) \) for all \( x \in [0, s_s] \),
\[ \tilde{g}_p(x) = \inf_{t \in [0, x]} f(t) + \delta^* = f(s_s) + \delta^* < \inf_{t \in [x, 1]} f(t) + \delta^* = \tilde{g}_{s_s}(x) \quad \text{for all } x \in (s_s, p) \]
and $g_p(x) = \tilde{g}_s(x)$ for all $x \in [p, 1]$. Thus, by (c) if $p \in [0, 1]$, then $g_p(x) \leq \tilde{g}_s(x)$ for all $x \in [0, 1]$.

**Theorem 1.** Let $f \in C[0, 1]$, and let $P$ be the set of minimum points for $\delta$. Then,

$$P = [\eta_\epsilon, \eta_*],$$

where $\eta_\epsilon$ and $\eta_*$ are defined by (2.8) and (2.9), respectively.

**Proof.** Assume that $x_0 \in [\eta_\epsilon, \eta_*]$. We consider three cases.

**Case 1.** $x_0 \in [\eta_\epsilon, s_\epsilon]$. Then, $\delta_\epsilon(x_0) \leq \delta_\epsilon(s_\epsilon)$. However, since $s_\epsilon \in S \subseteq P$,

$$\delta_\epsilon(x_0) = \max \left\{ \sup_{x_0 < x < y < s_\epsilon} \frac{|f(x) - f(y)|}{2}, \sup_{s_\epsilon < x < y < 1} \frac{|f(x) - f(y)|}{2} \right\} \leq \delta_*.$$

**Case 2.** $x_0 \in (s_\epsilon, s_*]$. Then, $\sup_{s_\epsilon < x, y < s_*} \frac{|f(y) - f(x)|}{2} \leq \delta_*$. Since $s_\epsilon \in P$,

$$\delta_\epsilon(x_0) = \max \left\{ \delta_\epsilon(s_\epsilon), \sup_{s_\epsilon < x < y < x_0} \frac{|f(y) - f(x)|}{2} \right\} \leq \delta_*,$$

and

$$\delta_\epsilon(x_0) = \max \left\{ \delta_\epsilon(s_\epsilon), \sup_{x_0 < x < y < s_*} \frac{|f(x) - f(y)|}{2} \right\} \leq \delta_*.$$

**Case 3.** $x_0 \in [s_\epsilon, \eta_*]$. Then $\delta_\epsilon(x_0) \leq \delta_\epsilon(s_\epsilon) \leq \delta_*$. Also, since $s_\epsilon \in P$,

$$\delta_\epsilon(x_0) = \max \left\{ \delta_\epsilon(s_\epsilon), \sup_{s_\epsilon < x < y < x_0} \frac{|f(y) - f(x)|}{2} \right\} \leq \delta_*.$$
Combining all three cases,
\[ \delta(x_0) = \max \{ \delta_\epsilon(x_0), \delta_1(x_0) \} \leq \delta^*, \quad \text{for } x_0 \in [\eta_\epsilon, \eta_1]. \]

Hence, \( x_0 \in \mathbb{P} \), and thus, \([\eta_\epsilon, \eta_1] \subseteq \mathbb{P}\).

Next, assume that \( x_0 \not\in [\eta_\epsilon, \eta_1] \). If \( x_0 < \eta_\epsilon \), then by the definition of \( \eta_\epsilon \), there exists a \( t_0 \in [x_0, s_\epsilon] \) such that \( \frac{1}{2} f(t_0) > \frac{1}{2} m + \delta^* \). Hence,
\[ \delta_1(x_0) \geq \sup_{x_0 \leq x \leq y \leq s_\epsilon} \frac{|f(x) - f(y)|}{2} \geq \frac{1}{2} f(t_0) - \frac{1}{2} f(s_\epsilon) > \delta^*. \]

This implies that \( x_0 \not\in \mathbb{P} \). If \( x_0 > \eta_1 \), then by the definition of \( \eta_1 \), there exists a \( t_0 \in [s_1, x_0] \) such that \( \frac{1}{2} f(t_0) > \frac{1}{2} m + \delta^* \). Hence,
\[ \delta_1(x_0) \geq \sup_{s_1 \leq x \leq y \leq x_0} \frac{|f(y) - f(x)|}{2} \geq \frac{1}{2} f(t_0) - \frac{1}{2} f(s_1) > \delta^*, \]

which implies that \( x_0 \not\in \mathbb{P} \). Thus, \( \mathbb{P} \subseteq [\eta_\epsilon, \eta_1] \).

3. Duality

In this section we prove that for \( p \in [\eta_\epsilon, \eta_1] \), \( g_p \) and \( \tilde{g}_p \) are both best quasi-convex approximations to \( f \in C[0, 1] \), and that \( \delta^* \) is the error of best approximation.

**Lemma 3.** Let \( f \in C[0, 1] \) and \( p \in [\eta_\epsilon, \eta_1] \). Then,
\[ \|f - g_p\|_\infty \leq \delta^* \quad \text{and} \quad \|f - \tilde{g}_p\|_\infty \leq \delta^*. \]

**Proof.** The proofs of these two inequalities are similar. Thus, we present only the proof of the second.

If \( x \in [0, p] \) then \( \tilde{g}_p(x) \leq f(x) + \delta^* \). Also, for each \( \epsilon > 0 \), there exists a \( t \in [0, x] \) such that \( \tilde{g}_p(x) > f(t) + \delta^* - \epsilon \). Since \( p \in \mathbb{P} \), \( \delta(p) = \max \{ \delta_\epsilon(p), \delta_1(p) \} = \delta^* \), and thus \( \delta^* \geq |f(x) - f(t)|/2 \). Hence, \( \tilde{g}_p(x) > f(t) + \delta^* - \epsilon \geq f(x) - \delta^* - \epsilon \).

Consequently, if \( x \in [0, p] \), then \( |f(x) - \tilde{g}_p(x)| \leq \delta^* \). Similarly, we can show that if \( x \in (p, 1] \), then \( |f(x) - \tilde{g}_p(x)| \leq \delta^* \). Thus, \( \|f - \tilde{g}_p\|_\infty \leq \delta^* \).

The following theorem shows that \( \delta^* \) is the measure of the best quasi-convex approximation to \( f \in C[0, 1] \).

**Theorem 2 (Duality).** Let \( f \in C[0, 1] \). Then,
\[ \inf_{g \in \mathcal{K}} \|f - g\|_\infty = \delta^*, \]

with \( \delta^* \) as defined by (2.4).
Proof: For each $g \in K$, there exists a $p \in [0, 1]$ such that $g \in K_p$. Hence, for $0 \leq x \leq y \leq p$ (or $0 \leq x \leq y < p$),

$$f(y) - f(x) \leq f(y) - f(x) + g(x) - g(y)$$
$$\leq |f(y) - g(y)| + |f(x) - g(x)| \leq 2 \|f - g\|_\infty,$$

and for $p < x \leq y \leq 1$ (or $p \leq x \leq y \leq 1$),

$$f(x) - f(y) \leq f(x) - f(y) + g(y) - g(x)$$
$$\leq |f(x) - g(x)| + |f(y) - g(y)| \leq 2 \|f - g\|_\infty.$$

It follows that $\delta_r(p) \leq \|f - g\|_\infty$ and $\delta_s(p) \leq \|f - g\|_\infty$. Therefore, for each $g \in K$,

$$\|f - g\|_\infty \geq \max\{\delta_r(p), \delta_s(p)\} = \delta(p) \geq \delta^*,$$

and thus $\inf_{g \in K} \|f - g\|_\infty \geq \delta^*$.

By Lemma 3 we also have $\|f - \tilde{g}_p\|_\infty \leq \delta^*$, and by Lemma 2 $\tilde{g}_p \in K_p \subseteq K$. Consequently, $\inf_{g \in K} \|f - g\|_\infty = \delta^*$.

Theorem 2 can be extended to bounded $f$ by using Theorem 4.2 of [8] and (A) of Theorem 1 of [5].

Corollary 1. If $f \in C[0, 1]$ and $p \in P = [\eta_r, \eta_s]$, then

$$\|f - g_p\|_\infty = \|f - \tilde{g}_p\|_\infty = \delta^*.$$

Therefore, $g_p$ and $\tilde{g}_p$ are both best approximations to $f$, and

$$P \subseteq P^*.$$

4. Optimal Knots

We now characterize $P^*$, the set of optimal knots.

Lemma 4. If $g$ is a best quasi-convex approximation to $f \in C[0, 1]$, and $p$ is a knot for $g$, then $p \in P = [\eta_r, \eta_s]$. Thus, $P^* \subseteq P$.

Proof. Assume that $p \notin P$; then by the definition of $P$ either $\delta_r(p) > \delta^*$ or $\delta_s(p) > \delta^*$.

If $\delta_r(p) > \delta^*$, then there exist $x_1 < y_1$ in $[0, p]$ such that $\frac{1}{2}|f(y_1) - f(x_1)| > \delta^*$. Since $g$ is a best approximation, it follows from Theorem 2 (duality) that

$$-\delta^* \leq g(x_1) - f(x_1) \leq \delta^*.$$
Hence,
\[ g(y_1) - f(y_1) \leq g(x_1) - f(y_1) \]
\[ = g(x_1) - f(x_1) + f(x_1) - f(y_1) \]
\[ < \delta^* - 2\delta^* = -\delta^*. \]

Similarly, if \( \delta_1(p) > \delta^* \) then there exist \( x_2 < y_2 \) in \( (p, 1] \) such that \( \frac{1}{2}[f(x_2) - f(y_2)] > \delta^* \) and, as above, \( g(y_2) - f(y_2) > \delta^* \). Consequently, \( g \) is not a best quasi-convex approximation to \( f \). This contradiction implies that \( p \in P \).

Combining Corollary 1 and Lemma 4 we have the following:

**Theorem 3.** If \( f \in C[0, 1] \) then,
\[ P^* = P, \]
where \( P^* \) is the set of optimal knots and, \( P = [\eta, \eta] \) is the set of minimum points for \( \delta \).

5. The Characterization of the Best Approximations

In this section we present a characterization of best quasi-convex approximations to \( f \in C[0, 1] \).

**Lemma 5.** Let \( f \in C[0, 1] \) and let \( g \) be a best quasi-convex approximation to \( f \). Then, there exists a \( p \in [\eta, \eta] \) such that
\[ g_p(x) \leq g(x), \quad \text{for all } x \in [0, 1]. \]

**Proof.** By Lemma 4, if we let \( p_0 \) be a knot of \( g \), then \( p_0 \in [\eta, \eta] \). Next, we show \( g_{p_0}(x) \leq g(x) \) for all \( x \in [0, 1] \). Assume, to the contrary, that there exists an \( x_0 \in [0, 1] \) such that
\[ g(x_0) < g_{p_0}(x_0). \]

If \( x_0 \in [0, p_0] \), then \( g(x_0) < g_{p_0}(x_0) = \sup_{t \in [x_0, p_0]} f(t) - \delta^* \). Hence, there exists a \( t_0 \in [x_0, p_0] \) such that \( g(x_0) < f(t_0) - \delta^* \). Thus,
\[ g(t_0) \leq g(x_0) < f(t_0) - \delta^*. \]

If \( x_0 \in [p_0, 1] \), then \( g(x_0) < g_{p_0}(x_0) = \sup_{t \in [p_0, x_0]} f(t) - \delta^* \). Hence, there exists a \( t_0 \in [p_0, x_0] \) such that
\[ g(x_0) < f(t_0) - \delta^*. \]
Thus, there exists a \( t_0 \in [0, 1] \) such that
\[
g(t_0) < f(t_0) - \delta^*. \]

Hence, \( g \) cannot be a best approximation to \( f \) (contra).

**Theorem 4 (Characterization of Best Approximation).** Let \( f \in C[0, 1] \). Then, \( g \) is a best uniform quasi-convex approximation to \( f \) on \([0, 1]\) if and only if there exists a \( p \in [\eta_*, \eta_+] \) such that
\[
g_p(x) \leq g(x) \leq \tilde{g}_s(x), \quad \text{for all } x \in [0, 1]. \tag{5.1} \]

**Proof. Necessity.** Let \( g \) be a best approximation to \( f \) from \( K \). The first inequality follows from Lemma 5. It remains to show that \( g(x) \leq \tilde{g}_s(x) \), for all \( x \in [0, 1] \).

For each \( t \in [0, 1] \), \(-\delta^* \leq f(t) - g(t) \leq \delta^* \). By the definition of \( \tilde{g}_s(x) \), for \( x \in [0, s_*] \) and for all \( \varepsilon > 0 \) there exists a \( t \in [0, x] \) satisfying \( \tilde{g}_s(x) > f(t) + \delta^* - \varepsilon \). Also, for \( x \in (s_*, 1] \) and for all \( \varepsilon > 0 \) there exists a \( t \in [x, 1] \) satisfying \( \tilde{g}_s(x) > f(t) + \delta^* - \varepsilon \). Let \( p_0 \) be a knot for \( g \). If \( p_0 \leq s_* \), then \( g(x) < g(t) \) for \( 0 \leq t \leq x < p_0 \) (or \( 0 \leq t \leq x < p_0 \)), and moreover \( g(x) \leq g(t) \leq f(t) + \delta^* < \tilde{g}_s(x) + \varepsilon \), for \( x \in [0, p_0] \) (or \( x \in [0, p_0] \)). It follows that \( g(x) \leq \tilde{g}_s(x) \) for \( x \in [0, p_0] \) (or \( x \in [0, p_0] \)). Also, \( g(x) \leq g(t) \) for \( s_* < x \leq t < 1 \) (or \( s_* < x \leq t < 1 \)), and \( g(x) \leq g(t) \leq f(t) + \delta^* < \tilde{g}_s(x) + \varepsilon \), for \( x \in (s_*, 1] \) (or \( x \in (s_*, 1] \)). Thus, \( g(x) \leq \tilde{g}_s(x) \), for \( x \in (s_*, 1] \) (or \( x \in (s_*, 1] \)). In either case \( g(s_* + ) \leq \tilde{g}_s(s_* + ) \). Hence for \( x \in (p_0, s_*] \) (or \( [p_0, s_*] \)), by Lemma 2,
\[
g(x) \leq g(s_* + ) \leq \tilde{g}_s(s_* + ) = \tilde{g}_s(s_*) \leq \tilde{g}_s(x). \]

Therefore, if \( p_0 \leq s_* \), then \( g(x) \leq \tilde{g}_s(x) \), for all \( x \in [0, 1] \).

If \( p_0 > s_* \), then we can similarly prove that
\[
g(x) \leq \tilde{g}_s(x), \quad \text{for all } x \in [0, 1]. \]

**Sufficiency.** If \( g \in K \) and there exists a \( p \in [\eta_*, \eta_+] \) such that (5.1) holds, then by Corollary 1, \( \| f - \tilde{g}_p \|_\infty = \| f - \tilde{g}_s \|_\infty = \delta^* \). Thus, \( \| f - g \|_\infty = \delta^* \), and \( g \) is a best approximation to \( f \).

The following corollary gives the structure of \( G \), the set of best approximations:

**Corollary 2.** Let \( f \in C[0, 1] \), then
\[
G = \bigcup_{p \in [\eta_*, \eta_+]} \{ g^* \in K : g_p(x) \leq g^*(x) \leq \tilde{g}_s(x), \text{ for all } x \in [0, 1] \}. \]
THEOREM 5 (Nonuniqueness of the Best Quasi-convex approximation). Let $f \in C[0,1]$. Then $f$ has a unique best uniform quasi-convex approximation if and only if $f$ is quasi-convex.

Proof: If $f \in K$ then $f$ is its own unique best approximation from $K$.

Next, assume that $G$ has a unique element. Then by Corollary 2 for all $p \in [\eta, \eta]$, $g_p(x) = \bar{g}_s(x)$, for all $x \in [0,1]$. In particular, we find that $g_{s_i}(s_i) = \bar{g}_{s_i}(s_i)$. Hence, by the definitions of $g_{s_i}$ and $\bar{g}_{s_i}$, $f(s_i) - \delta^* = f(s_i) + \delta^*$. Hence, $\delta^* = 0$, and by Lemma 1, $f \in K$.

Theorem 5 can also be derived from Theorem 5.1 of [8].

REFERENCES

3. J. J. Swetits, S. E. Weinstein, and Yuesheng Xu, On the characterization and computation of best monotone approximation in $L_p[0,1]$ for $1 \leq p < \infty$, J. Approx. Theory 60 (1990), 58–69.