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Best Quasi-convex Uniform Approximation

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1. INTRODUCTION

Let $\mathbf{B} = \mathbf{B}[0, 1]$ be the linear space of all bounded real functions f on $[0, 1]$, with the uniform norm

$$\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|.$$

Let $C[0, 1]$ denote the space of all continuous functions on $[0, 1]$.

DEFINITION 1. A function $g \in \mathbf{B}$ is said to be *quasi-convex* [2] if

$$g(x) \leqslant \max\{g(s), g(t)\} \qquad \text{for all } x, s, \text{ and } t \text{ such that } 0 \leqslant s \leqslant x \leqslant t \leqslant 1.$$

Let $\mathbf{K} \subset \mathbf{B}$ denote the set of all quasi-convex functions on $[0, 1]$.

Ubhaya [8] has proved that g is quasi-convex if and only if there exists a point $p \in [0, 1]$, such that either

(i) g is nonincreasing on $[0, p)$ and is nondecreasing on $[p, 1]$ or

(ii) g is nonincreasing on $[0, p]$ and is nondecreasing on $(p, 1]$.

We call the point p (in either (i) or (ii)) a *knot* of g. Let $\mathbf{K}_p$ denote the functions in $\mathbf{K}$ which have a knot at p. Then,

$$\mathbf{K} = \bigcup_{p \in [0,1]} \mathbf{K}_p.$$

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In general, the set of all the knots of a quasi-convex function is a closed subinterval of [0, 1].

The problem of the best quasi-convex approximation is to find a $g^* \in \mathbf{K}$, such that

$$\|f-g^*\|_\infty = \inf_{g \in \mathbf{K}} \{\|f-g\|_\infty\}. \tag{1.1}$$

This problem is considered in [8], where a sufficient condition for a best quasi-convex approximation to a bounded function is obtained, and some structural properties of best approximations are established. Algorithms for the computation of a best discrete quasi-convex approximation are presented in [1, 7].

Throughout this paper we shall assume that $f \in C[0, 1]$, unless stated otherwise.

DEFINITION 2. Given $f \in C[0, 1]$, let

$$\mathbf{G} = \mathbf{G}(f) = \{g^* \in \mathbf{K} : \|f-g^*\|_\infty = \inf_{g \in \mathbf{K}} \{\|f-g\|\}\} \tag{1.2}$$

the set of best quasi-convex approximations to f, and let

$$\mathbf{P}^* = \{p \in [0, 1] : p \text{ is a knot for some } g^* \in \mathbf{G}\}. \tag{1.3}$$

We call $\mathbf{P}^*$ the *set of optimal knots.*

We characterize both the best quasi-convex approximations and the optimal knots. In addition we describe the construction of the set of best approximations and prove that a best quasi-convex approximation is unique if and only if f is quasi-convex.

2. PRELIMINARIES

Similar to the development in [5] we define two functionals $\delta_\imath$ and δ_ℓ, which we use to obtain the error of the best quasi-convex approximation.

DEFINITION 3. For $f \in C[0, 1]$ and $p \in [0, 1]$, let

$$\delta_\ell(p) = \sup_{0 \leqslant x \leqslant y \leqslant p} \frac{[f(y)-f(x)]}{2}, \tag{2.1}$$

and

$$\delta_\imath(p) = \sup_{p < x \leqslant y \leqslant 1} \frac{[f(x)-f(y)]}{2}. \tag{2.2}$$

Thus, δ_ℓ is a measure of the "decreasingness" of f on $[0, p]$, and δ_r is a measure of the "increasingness" of f on $(p, 1]$.

For $f \in C[0, 1]$ and $p \in [0, 1]$ (as in [8]), define

$$\delta(p) = \max\{\delta_\ell(p), \delta_r(p)\}. \tag{2.3}$$

Denote the minimum value of $\delta(p)$ on $[0, 1]$ by

$$\delta^* = \inf_{0 \leqslant p \leqslant 1} \delta(p). \tag{2.4}$$

Let

$$\mathbf{P} = \{p \in [0, 1] : \delta(p) = \delta^*\} \tag{2.5}$$

be the set of minima for δ, and let

$$\mathbf{S} = \{s \in [0, 1] : f(s) = \inf_{0 \leqslant x \leqslant 1} f(x)\} \tag{2.6}$$

be the set of minima for f.

Let $[s_\ell, s_r]$ be the convex hull of $\mathbf{S}$. Then,

$$s_\ell = \inf \mathbf{S} \qquad \text{and} \qquad s_r = \sup \mathbf{S}. \tag{2.7}$$

Also, let $m = \inf\{f(x) : 0 \leqslant x \leqslant 1\}$, and then define

$$\eta_\ell = \inf\{x \in [0, s_\ell] : f(t) \leqslant m + 2\delta^*, \text{ for all } t \in [x, s_\ell]\}, \tag{2.8}$$

and

$$\eta_r = \sup\{x \in [s_r, 1] : f(t) \leqslant m + 2\delta^*, \text{ for all } t \in [s_r, x]\}. \tag{2.9}$$

Thus,

$$[s_\ell, s_r] \subseteq [\eta_\ell, \eta_r].$$

We shall prove that $\mathbf{P} = [\eta_\ell, \eta_r]$, and that $\mathbf{P} = \mathbf{P}^*$, the set of optimal knots.

Next, let $f \in \mathbf{B}$. For each $p \in [0, 1]$, similar to the delfinitions of U_p^- and V_p^- in [8] with θ_p^- replaced by δ^* we define the two functions

$$\underline{g}_p(x) = \begin{cases} \sup_{t \in [x, p]} f(t) - \delta^*, & x \in [0, p] \\ \sup_{t \in (p, x]} f(t) - \delta^*, & x \in (p, 1] \end{cases} \tag{2.10}$$

and

$$\tilde{g}_p(x) = \begin{cases} \inf\limits_{t \in [0,x]} f(t) + \delta^*, & x \in [0, p] \\ \inf\limits_{t \in [x,1]} f(t) + \delta^*, & x \in (p, 1]. \end{cases} \tag{2.11}$$

LEMMA 1. *Let $f \in C[0, 1]$. Then,*

(i) $|\Delta\delta_\ell(p)| \leqslant \frac{1}{2}\omega_f(|\Delta p|)$, *and* $|\Delta\delta_r(p)| \leqslant \frac{1}{2}\omega_f(|\Delta p|)$ (*where* $\omega_f(*)$ *denotes the modulus of continuity of* f). *Thus,* δ_ℓ *and* δ_r *are continuous.*

(ii) $\delta^* = 0$ *if and only if* $f \in \mathbf{K}$,

(iii) $\mathbf{S} \subset \mathbf{P}$.

Proof. (i) If $\Delta p > 0$ then

$$\delta_\ell(p + \Delta p) \leqslant \delta_\ell(p) + \sup_{p \leqslant x \leqslant y \leqslant p + |\Delta p|} \frac{[f(y) - f(x)]}{2},$$

and if $\Delta p < 0$ then,

$$\delta_\ell(p) \leqslant \delta_\ell(p - |\Delta p|) + \sup_{p - |\Delta p| \leqslant x \leqslant y \leqslant p} \frac{[f(y) - f(x)]}{2}.$$

It follows that

$$|\Delta\delta_\ell(p)| \leqslant \sup_{0 \leqslant y - x \leqslant |\Delta p|} \frac{[f(y) - f(x)]}{2} = \frac{1}{2}\,\omega_f(|\Delta p|).$$

Similarly, we may show the second inequality of (i).

(ii) First let $\delta^* = 0$. By (i) δ_ℓ and δ_r are continuous and thus so is δ, where $\delta(p) = \max\{\delta_\ell(p), \delta_r(p)\}$ for $p \in [0, 1]$. Hence, there exists a $p_0 \in [0, 1]$, such that $\delta(p_0) = \delta^* = 0$. Thus, $\delta_\ell(p_0) = \delta_r(p_0) = 0$, since δ_ℓ and δ_r are both nonnegative functions. Consequently, by the definitions of δ_ℓ and δ_r, f is nonincreasing on $[0, p_0]$, and nondecreasing on $(p_0, 1]$. Thus, $f \in \mathbf{K}$.

Conversely, assume that $f \in \mathbf{K}$. Then there exists a $p_0 \in [0, 1]$ such that $f \in \mathbf{K}_{p_0}$. Therefore, $\delta_l(p_0) = \delta_r(p_0) = 0$, which implies that $\delta(p_0) = 0$. Hence, $\delta^* = 0$.

(iii) It is sufficient to show that if $s \in \mathbf{S}$, then,

$$\delta_\ell(s) \leqslant \max\{\delta_\ell(p), \delta_r(p)\} \qquad \text{for all } p \in [0, 1] \tag{2.12}$$

and

$$\delta_r(s) \leqslant \max\{\delta_\ell(p), \delta_r(p)\} \qquad \text{for all } \quad p \in [0, 1]. \tag{2.13}$$

The proofs of (2.12) and (2.13) are similar; thus we only present the proof of (2.12).

If $s=0$ then, since $\delta_\ell(0)=0$, and since δ_ℓ and δ_r are both nonnegative functions, (2.12) holds.

If $s\in(0,1]$, we consider two cases. First assume that $p\geqslant s$. Then $\delta_\ell(s)\leqslant\delta_\ell(p)$ and thus (2.12) holds.

Next, assume that $p<s$, and $\delta_\ell(p)<\delta_r(s)$. $f\in C[0,1]$ implies that

$$2\delta_\ell(s)=f(y_1)-f(x_1) \qquad \text{for some} \quad x_1\leqslant y_1 \quad \text{in } [0,s].$$

It follows that $2\delta_\ell(p)<f(y_1)-f(x_1)$ and $p<y_1$. Hence,

$$2\delta_\ell(s)\leqslant f(y_1)-f(s)\leqslant \sup_{p\leqslant x\leqslant y\leqslant s}[f(x)-f(y)]\leqslant 2\delta_r(p).$$

Therefore, (2.12) holds.

LEMMA 2. *$\underline{g}_p$ and $\bar{g}_p$ as defined by* (2.10) *and* (2.11) *have the following properties*:

(i) $\underline{g}_p, \bar{g}_p\in\mathbf{K}_p$ *for all* $p\in[0,1]$,

(ii) *if* $f\in C[0,1]$ *then*

(a) $\underline{g}_p\in C[0,1]$ *for all* $p\in[0,1]$,

(b) $\bar{g}_p\in C[0,1]$ *if and only if* $p\in[s_\ell,s_r]$,

(c) *if* $p\in[s_\ell,s_r]$, *then* $\bar{g}_p(x)=\bar{g}_{s_\ell}(x)$ *for all* $x\in[0,1]$.

(d) *if* $p\in[0,1]$, *then* $\bar{g}_p(x)\leqslant\bar{g}_{s_\ell}(x)$ *for all* $x\in[0,1]$.

Proof. (i) follows from the definitions (2.10) and (2.11).

(ii) (a) For all $p\in[0,1]$, (2.10) implies that $\underline{g}_p$ is continuous at any $x\neq p$.

Next, to prove the continuity of $\underline{g}_p$ at $x=p$, we observe that

$$\underline{g}_p(p-)=\lim_{\varepsilon\to 0}\sup_{t\in[p-\varepsilon,p]} f(t)-\delta^*=f(p)-\delta^*$$

and

$$\underline{g}_p(p+)\lim_{\varepsilon\to 0}\sup_{t\in(p,p+\varepsilon]} f(t)-\delta^*=f(p)-\delta^*,$$

since $f\in C[0,1]$. Thus, $\underline{g}_p(p-)=\underline{g}_p(p+)=\underline{g}_p(p)$, and (a) is proved.

(b) Similarly, for all $p\in[0,1]$, $\bar{g}_p$ is continuous where $x\neq p$. Next, if $x=p$ and $p\in[s_\ell,s_r]$, then

$$\bar{g}_p(p-)=\lim_{\varepsilon\to 0}\inf_{t\in[0,p-\varepsilon]} f(t)+\delta^*=f(s_\ell)+\delta^*$$

and

$$\bar{g}_p(p+) = \lim_{\varepsilon \to 0} \inf_{t \in [p+\varepsilon, 1]} f(t) + \delta^* = f(s_r) + \delta^*.$$

Hence, $\bar{g}_p(p-) = \bar{g}_p(p) = \bar{g}_p(p+)$.

Conversely, suppose that $p \notin [s_\ell, s_r]$. If $p < s_\ell$, then

$$\bar{g}_p(p-) = \lim_{\varepsilon \to 0} \inf_{t \in [0, p-\varepsilon]} f(t) + \delta^* > f(s_\ell) + \delta^*$$

$$= \lim_{\varepsilon \to 0} \inf_{t \in [p+\varepsilon, 1]} f(t) + \delta^* = \bar{g}_p(p+).$$

While if $p > s_r$ then

$$\bar{g}_p(p-) = \lim_{\varepsilon \to 0} \inf_{t \in [0, p-\varepsilon]} f(t) + \delta^* = f(s_l) + \delta^*$$

$$< \lim_{\varepsilon \to 0} \inf_{t \in [p+\varepsilon, 1]} f(t) + \delta^* = \bar{g}_p(p+).$$

(c) Let $p \in [s_\ell, s_r]$. For $x \in [s_\ell, p]$,

$$\bar{g}_p(x) = \inf_{t \in [0, x]} f(t) + \delta^* = f(s_\ell) + \delta^* = m + \delta^*,$$

for $x \in (p, s_r]$,

$$\bar{g}_p(x) = \inf_{t \in [x, 1]} f(t) + \delta^* = f(s_r) + \delta^* = m + \delta^*,$$

and for $x \notin [s_\ell, s_r]$,

$$\bar{g}_p(x) = \bar{g}_{s_\ell}(x).$$

Thus, $\bar{g}_p \equiv \bar{g}_{s_\ell}$.

(d) Assume that $p \notin [s_\ell, s_r]$. If $p < s_\ell$, then

$$\bar{g}_p(x) = \bar{g}_{s_\ell}(x) \qquad \text{for all} \quad x \in [0, p],$$

$$\bar{g}_p(x) = \inf_{t \in [x, 1]} f(t) + \delta^* = f(s_r) + \delta^*$$

$$< \inf_{t \in [0, x]} f(t) + \delta^* = \bar{g}_{s_\ell}(x) \qquad \text{for all} \quad x \in (p, s_\ell)$$

and $\bar{g}_p(x) = \bar{g}_{s_\ell}(x)$ for all $x \in [s_\ell, 1]$. If $p > s_r$, then $\bar{g}_p(x) = \bar{g}_{s_\ell}(x)$ for all $x \in [0, s_r]$,

$$\bar{g}_p(x) = \inf_{t \in [0, x]} f(t) + \delta^* = f(s_r) + \delta^*$$

$$< \inf_{t \in [x, 1]} f(t) + \delta^* = \bar{g}_{s_\ell}(x) \qquad \text{for all} \quad x \in (s_r, p)$$

and $\bar{g}_p(x)=\bar{g}_{s_\ell}(x)$ for all $x\in[p,1]$. Thus, by (c) if $p\in[0,1]$, then $\bar{g}_p(x)\leqslant\bar{g}_{s_\ell}(x)$ for all $x\in[0,1]$.

THEOREM 1. *Let $f\in C[0,1]$, and let $\mathbf{P}$ be the set of minimum points for δ. Then,*

$$\mathbf{P}=[\eta_\ell,\eta_r],$$

where η_ℓ and η_r are defined by (2.8) *and* (2.9), *respectively.*

Proof. Assume that $x_0\in[\eta_\ell,\eta_r]$. We consider three cases.

Case 1. $x_0\in[\eta_\ell,s_\ell]$. Then, $\delta_\ell(x_0)\leqslant\delta_\ell(s_\ell)$. However, since $s_\ell\in\mathbf{S}\subset\mathbf{P}$,

$$\delta_r(x_0)=\max\left\{\sup_{x_0<x\leqslant y\leqslant s_\ell}\frac{[f(x)-f(y)]}{2},\ \sup_{s_\ell\leqslant x\leqslant y\leqslant 1}\frac{[f(x)-f(y)]}{2},\right.$$
$$\left.\times\sup_{x_0<x\leqslant s_\ell<y\leqslant 1}\frac{[f(x)-f(y)]}{2}\right\}$$
$$=\max\left\{\sup_{x_0\leqslant x\leqslant s_\ell}\frac{f(x)}{2}-\frac{f(s_\ell)}{2},\ \delta_r(s_\ell)\right\}\leqslant\delta^*.$$

Case 2. $x_0\in(s_\ell,s_r)$. Then, $\sup_{s_\ell\leqslant x,y\leqslant s_r}([f(y)-f(x)]/2)\leqslant\delta^*$. Since $s_\ell\in\mathbf{P}$,

$$\delta_\ell(x_0)=\max\left\{\delta_\ell(s_\ell),\ \sup_{s_\ell\leqslant x\leqslant y\leqslant x_0}\frac{[f(y)-f(x)]}{2},\right.$$
$$\left.\times\sup_{0\leqslant x\leqslant s_\ell<y\leqslant x_0}\frac{[f(y)-f(x)]}{2}\right\}\leqslant\delta^*,$$

and

$$\delta_r(x_0)=\max\left\{\delta_r(s_r),\ \sup_{x_0\leqslant x\leqslant y\leqslant s_r}\frac{[f(x)-f(y)]}{2},\right.$$
$$\left.\times\sup_{x_0\leqslant x\leqslant s_r\leqslant y\leqslant 1}\frac{[f(x)-f(y)]}{2}\right\}\leqslant\delta^*.$$

Case 3. $x_0\in[s_r,\eta_r]$. Then $\delta_r(x_0)\leqslant\delta_r(s_r)\leqslant\delta^*$. Also, since $s_r\in\mathbf{P}$,

$$\delta_\ell(x_0)=\max\left\{\delta_\ell(s_r),\ \sup_{s_r\leqslant x\leqslant y\leqslant x_0}\frac{[f(y)-f(x)]}{2},\right.$$
$$\left.\times\sup_{0\leqslant x\leqslant s_r\leqslant y\leqslant x_0}\frac{[f(y)-f(x)]}{2}\right\}$$
$$=\max\left\{\delta_\ell(s_r),\ \sup_{s_r\leqslant x\leqslant x_0}\frac{f(x)}{2}-\frac{f(s_r)}{2}\right\}\leqslant\delta^*.$$

Combining all three cases,

$$\delta(x_0) = \max\{\delta_\ell(x_0), \delta_r(x_0)\} \leqslant \delta^*, \qquad \text{for} \quad x_0 \in [\eta_\ell, \eta_r].$$

Hence, $x_0 \in \mathbf{P}$, and thus, $[\eta_\ell, \eta_r] \subseteq \mathbf{P}$.

Next, assume that $x_0 \notin [\eta_\ell, \eta_r]$. If $x_0 < \eta_\ell$, then by the definition of η_ℓ, there exists a $t_0 \in [x_0, s_\ell]$ such that $\frac{1}{2}f(t_0) > \frac{1}{2}m + \delta^*$. Hence,

$$\delta_r(x_0) \geqslant \sup_{x_0 \leqslant x \leqslant y \leqslant s_\ell} \frac{[f(x) - f(y)]}{2} \geqslant \frac{1}{2} f(t_0) - \frac{1}{2} f(s_\ell) > \delta^*.$$

This implies that $x_0 \notin \mathbf{P}$. If $x_0 > \eta_r$, then by the definition of η_r, there exists a $t_0 \in [s_r, x_0]$ such that $\frac{1}{2}f(t_0) > \frac{1}{2}m + \delta^*$. Hence,

$$\delta_r(x_0) \geqslant \sup_{s_r \leqslant x \leqslant y \leqslant x_0} \frac{[f(y) - f(x)]}{2} \geqslant \frac{1}{2} f(t_0) - \frac{1}{2} f(s_r) > \delta^*,$$

which implies that $x_0 \notin \mathbf{P}$. Thus, $\mathbf{P} \subseteq [\eta_\ell, \eta_r]$.

3. Duality

In this section we prove that for $p \in [\eta_\ell, \eta_r]$, g_p and $\bar{g}_p$ are both best quasi-convex approximations to $f \in C[0, 1]$, and that δ^* is the error of best approximation.

LEMMA 3. *Let $f \in C[0, 1]$ and $p \in [\eta_\ell, \eta_r]$. Then,*

$$\|f - g_p\|_\infty \leqslant \delta^* \qquad \textit{and} \qquad \|f - \bar{g}_p\|_\infty \leqslant \delta^*.$$

Proof. The proofs of these two inequalities are similar. Thus, we present only the proof of the second.

If $x \in [0, p]$ then $\bar{g}_p(x) \leqslant f(x) + \delta^*$. Also, for each $\varepsilon > 0$, there exists a $t \in [0, x]$ such that $\bar{g}_p(x) > f(t) + \delta^* - \varepsilon$. Since $p \in \mathbf{P}$, $\delta(p) = \max\{\delta_\ell(p), \delta_r(p)\} = \delta^*$, and thus $\delta^* \geqslant [f(x) - f(t)]/2$. Hence, $\bar{g}_p(x) > f(t) + \delta^* - \varepsilon \geqslant f(x) - \delta^* - \varepsilon$.

Consequently, if $x \in [0, p]$, then $|f(x) - \bar{g}_p(x)| \leqslant \delta^*$. Similarly, we can show that if $x \in (p, 1]$, then $|f(x) - \bar{g}_p(x)| \leqslant \delta^*$. Thus, $\|f - \bar{g}_p\| \leqslant \delta^*$.

The following theorem shows that δ^* is the measure of the best quasi-convex approximation to $f \in C[0, 1]$.

THEOREM 2 (Duality). *Let $f \in C[0, 1]$. Then,*

$$\inf_{g \in \mathbf{K}} \|f - g\|_\infty = \delta^*,$$

with δ^* as defined by (2.4).

Proof. For each $g \in \mathbf{K}$, there exists a $p \in [0, 1]$ such that $g \in \mathbf{K}_p$. Hence, for $0 \leqslant x \leqslant y \leqslant p$ (or $0 \leqslant x \leqslant y < p$),

$$f(y) - f(x) \leqslant f(y) - f(x) + g(x) - g(y)$$
$$\leqslant |f(y) - g(y)| + |f(x) - g(x)| \leqslant 2 \, \|f - g\|_\infty,$$

and for $p < x \leqslant y \leqslant 1$ (or $p \leqslant x \leqslant y \leqslant 1$),

$$f(x) - f(y) \leqslant f(x) - f(y) + g(y) - g(x)$$
$$\leqslant |f(x) - g(x)| + |f(y) - g(y)| \leqslant 2 \, \|f - g\|_\infty.$$

It follows that $\delta_\ell(p) \leqslant \|f - g\|_\infty$ and $\delta_r(p) \leqslant \|f - g\|_\infty$. Therefore, for each $g \in \mathbf{K}$,

$$\|f - g\|_\infty \geqslant \max\{\delta_\ell(p), \delta_r(p)\} = \delta(p) \geqslant \delta^*,$$

and thus $\inf_{g \in \mathbf{K}} \|f - g\|_\infty \geqslant \delta^*$.

By Lemma 3 we also have $\|f - \bar{g}_p\|_\infty \leqslant \delta^*$, and by Lemma 2 $\bar{g}_p \in \mathbf{K}_p \subset \mathbf{K}$.

Consequently, $\inf_{g \in \mathbf{K}} \|f - g\|_\infty = \delta^*$.

Theorem 2 can be extended to bounded f by using Theorem 4.2 of [8] and (A) of Theorem 1 of [5].

COROLLARY 1. *If $f \in C[0, 1]$ and $p \in \mathbf{P} = [\eta_\ell, \eta_r]$, then*

$$\|f - \underline{g}_p\|_\infty = \|f - \bar{g}_p\|_\infty = \delta^*.$$

Therefore, $\underline{g}_p$ and $\bar{g}_p$ are both best approximations to f, and

$$\mathbf{P} \subseteq \mathbf{P}^*.$$

4. Optimal Knots

We now characterize $\mathbf{P}^*$, the set of optimal knots.

LEMMA 4. *If g is a best quasi-convex approximation to $f \in C[0, 1]$, and p is a knot for g, then $p \in \mathbf{P} = [\eta_\ell, \eta_r]$. Thus, $\mathbf{P}^* \subseteq \mathbf{P}$.*

Proof. Assume that $p \notin \mathbf{P}$; then by the defintion of $\mathbf{P}$ either $\delta_\ell(p) > \delta^*$ or $\delta_r(p) > \delta^*$.

If $\delta_\ell(p) > \delta^*$, then there exist $x_1 < y_1$ in $[0, p]$ such that $\frac{1}{2}[f(y_1) - f(x_1)] > \delta^*$. Since g is a best approximation, it follows from Theorem 2 (duality) that

$$-\delta^* \leqslant g(x_1) - f(x_1) \leqslant \delta^*.$$

Hence,

$$\begin{aligned} g(y_1)-f(y_1) &\leqslant g(x_1)-f(y_1) \\ &= g(x_1)-f(x_1)+f(x_1)-f(y_1) \\ &< \delta^* - 2\delta^* = -\delta^*. \end{aligned}$$

Similarly, if $\delta_r(p) > \delta^*$ then there exist $x_2 < y_2$ in $(p, 1]$ such that $\frac{1}{2}[f(x_2)-f(y_2)] > \delta^*$ and, as above, $g(y_2)-f(y_2) > \delta^*$. Consequently, g is not a best quasi-convex approximation to f. This contradiction implies that $p \in \mathbf{P}$

Combining Corollary 1 and Lemma 4 we have the following:

THEOREM 3. *If $f \in C[0, 1]$ then,*

$$\mathbf{P}^* = \mathbf{P},$$

where $\mathbf{P}^*$ is the set of optimal knots and, $\mathbf{P} = [\eta_\ell, \eta_r]$ is the set of minimum points for δ.

5. The Characterization of the Best Approximations

In this section we present a characterization of best quasi-convex approximations to $f \in C[0, 1]$.

LEMMA 5. *Let $f \in C[0, 1]$ and let g be a best quasi-convex approximation to f. Then, there exists a $p \in [\eta_\ell, \eta_r]$ such that*

$$\underline{g}_p(x) \leqslant g(x), \qquad \text{for all} \quad x \in [0, 1].$$

Proof. By Lemma 4, if we let p_0 be a knot of g, then $p_0 \in [\eta_\ell, \eta_r]$. Next, we show $\underline{g}_{p_0}(x) \leqslant g(x)$ for all $x \in [0, 1]$. Assume, to the contrary, that there exists an $x_0 \in [0, 1]$ such that

$$g(x_0) < \underline{g}_{p_0}(x_0).$$

If $x_0 \in [0, p_0]$, then $g(x_0) < \underline{g}_{p_0}(x_0) = \sup_{t \in [x_0, p_0]} f(t) - \delta^*$. Hence, there exists a $t_0 \in [x_0, p_0]$ such that $g(x_0) < f(t_0) - \delta^*$. Thus,

$$g(t_0) \leqslant g(x_0) < f(t_0) - \delta^*.$$

If $x_0 \in [p_0, 1]$, then $g(x_0) < \underline{g}_{p_0}(x_0) = \sup_{t \in [p_0, x_0]} f(t) - \delta^*$. Hence, there exists a $t_0 \in [p_0, x_0]$ such that

$$g(x_0) < f(t_0) - \delta^*.$$

Thus, there exists a $t_0 \in [0, 1]$ such that

$$g(t_0) < f(t_0) - \delta^*.$$

Hence, g cannot be a best approximation to f (contra).

THEOREM 4 (Characterization of Best Approximation). *Let $f \in C[0, 1]$. Then, g is a best uniform quasi-convex approximation to f on $[0, 1]$ if and only if there exists a $p \in [\eta_\ell, \eta_r]$ such that*

$$\underline{g}_p(x) \leqslant g(x) \leqslant \bar{g}_{s_r}(x), \qquad \textit{for all} \quad x \in [0, 1]. \tag{5.1}$$

Proof. Necessity. Let g be a best approximation to f from $\mathbf{K}$. The first inequality follows from Lemma 5. It remains to show that $g(x) \leqslant \bar{g}_{s_r}(x)$, for all $x \in [0, 1]$.

For each $t \in [0, 1]$, $-\delta^* \leqslant f(t) - g(t) \leqslant \delta^*$. By the definition of $\bar{g}_{s_r}(x)$, for $x \in [0, s_r]$ and for all $\varepsilon > 0$ there exists a $t \in [0, x]$ satisfying $\bar{g}_{s_r}(x) > f(t) + \delta^* - \varepsilon$. Also, for $x \in (s_r, 1]$ and for all $\varepsilon > 0$ there exists a $t \in [x, 1]$ satisfying $\bar{g}_{s_r}(x) > f(t) + \delta^* - \varepsilon$. Let p_0 be a knot for g. If $p_0 \leqslant s_r$, then $g(x) \leqslant g(t)$ for $0 \leqslant t \leqslant x \leqslant p_0$ (or $0 \leqslant t \leqslant x < p_0$), and moreover $g(x) \leqslant g(t) \leqslant f(t) + \delta^* < \bar{g}_{s_r}(x) + \varepsilon$, for $x \in [0, p_0]$ (or $x \in [0, p_0)$). It follows that $g(x) \leqslant \bar{g}_{s_r}(x)$ for $x \in [0, p_0]$ (or $x \in [0, p_0)$). Also, $g(x) \leqslant g(t)$ for $s_r < x \leqslant t \leqslant 1$ (or $s_r \leqslant x \leqslant t \leqslant 1$), and $g(x) \leqslant g(t) \leqslant f(t) + \delta^* < \bar{g}_{s_r}(x) + \varepsilon$, for $x \in (s_r, 1]$ (or $x \in [s_r, 1]$). Thus, $g(x) \leqslant \bar{g}_{s_r}(x)$, for $x \in (s_r, 1]$ (or $x \in [s_r, 1]$). In either case $g(s_r+) \leqslant \bar{g}_{s_r}(s_r+)$. Hence for $x \in (p_0, s_r]$ (or $[p_0, s_r)$), by Lemma 2,

$$g(x) \leqslant g(s_r+) \leqslant \bar{g}_{s_r}(s_r+) = \bar{g}_{s_r}(s_r) \leqslant \bar{g}_{s_r}(x).$$

Therefore, if $p_0 \leqslant s_r$ then $g(x) \leqslant \bar{g}_{s_r}(x)$, for all $x \in [0, 1]$.

If $p_0 > s_r$, then we can similarly prove that

$$g(x) \leqslant \bar{g}_{s_r}(x), \qquad \text{for all} \quad x \in [0, 1].$$

Sufficiency. If $g \in \mathbf{K}$ and there exists a $p \in [\eta_\ell, \eta_r]$ such that (5.1) holds, then by Corollary 1, $\|f - \underline{g}_p\|_\infty = \|f - \bar{g}_{s_r}\|_\infty = \delta^*$. Thus, $\|f - g\|_\infty = \delta^*$, and g is a best approximation to f.

The following corollary gives the structure of $\mathbf{G}$, the set of best approximations:

COROLLARY 2. *Let $f \in C[0, 1]$, then*

$$\mathbf{G} = \bigcup_{p \in [\eta_\ell, \eta_r]} \{g^* \in \mathbf{K} : \underline{g}_p(x) \leqslant g^*(x) \leqslant \bar{g}_{s_r}(x), \textit{for all } x \in [0, 1]\}.$$

THEOREM 5 (Nonuniqueness of the Best Quasi-convex approximation). *Let $f \in C[0, 1]$. Then f has a unique best uniform quasi-convex approximation if and only if f is quasi-convex.*

Proof. If $f \in \mathbf{K}$ then f is its own unique best approximation from $\mathbf{K}$.

Next, assume that $\mathbf{G}$ has a unique element. Then by Corollary 2 for all $p \in [\eta_\ell, \eta_r]$, $\underline{g}_p(x) = \bar{g}_{s_r}(x)$, for all $x \in [0, 1]$. In particular, we find that $\underline{g}_{s_r}(s_r) = \bar{g}_{s_r}(s_r)$. Hence, by the definitions of $\underline{g}_{s_r}$ and $\bar{g}_{s_r}$, $f(s_r) - \delta^* = f(s_r) + \delta^*$. Hence, $\delta^* = 0$, and by Lemma 1, $f \in \mathbf{K}$.

Theorem 5 can also be derived from Theorem 5.1 of [8].

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