1991

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Original Publication Citation

Jamison, B., & Olariu, S. (1991). On a unique tree-representation for P_4-extendible graphs. *Discrete Applied Mathematics, 34*(1-3), 151-164. doi:10.1016/0166-218x(91)90085-b

Discrete Applied Mathematics 34 (1991) 151–164
North-Holland

On a unique tree representation for P_4-extendible graphs

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Received 10 June 1989
Revised 29 April 1990

Abstract

Jamison, B. and S. Olariu, On a unique tree representation for P_4-extendible graphs, Discrete Applied Mathematics 34 (1991) 151–164.

Several practical applications in computer science and computational linguistics suggest the study of graphs that are unlikely to have more than a few induced paths of length three. These applications have motivated the notion of a cograph, defined by the very strong restriction that no vertex may belong to an induced path of length three. The class of P_4-extendible graphs that we introduce in this paper relaxes this restriction, and in fact properly contains the class of cographs, while still featuring the remarkable property of admitting a unique tree representation. Just as in the case of cographs, the class of P_4-extendible graphs finds applications to clustering, scheduling, and memory management in a computer system. We give several characterizations for P_4-extendible graphs and show that they can be constructed from single-vertex graphs by a finite sequence of operations. Our characterization implies that the P_4-extendible graphs admit a tree representation unique up to isomorphism. Furthermore, this tree representation can be obtained in polynomial time.

1. Introduction

Finding a wide array of applications in communications, transportation, VLSI design, program optimization, database design, and other areas of computer science and engineering, graph problems often require fast solutions. A powerful tool for obtaining efficient solutions to graph problems is the *divide-and-conquer* paradigm, one of whose manifestations is graph decomposition.

An increasingly popular approach to graph decomposition involves associating with a given graph G a rooted tree $T(G)$ whose leaves are subgraphs of G (e.g. ver-

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** This author's work was supported by the National Science Foundation under grant CCR-8909996.

tices, edges, cliques, stable sets, cutsets) and whose internal nodes correspond to certain prescribed graph operations. Of a particular interest are classes of graphs G for which the following conditions hold:

- $T(G)$ can be obtained *efficiently*, that is, in time polynomial in the size of G;
- $T(G)$ is unique up to labelled tree isomorphism.

One of the earliest and best known examples of graphs satisfying both conditions mentioned above is the class of *cographs* discovered and investigated independently by various researchers. (The interested reader is referred to [3, 12] for a wealth of information about cographs.) Lerchs [9] showed that the cographs are precisely the graphs containing no chordless path on four vertices (termed a P_4). In addition, he showed that with every cograph G one can associate a unique rooted tree $T(G)$, called the *cotree* of G, whose leaves are precisely the vertices of G; the internal nodes are labeled by 0 or 1 in such a way that two vertices x, y are adjacent in G if and only if their lowest common ancestor in $T(G)$ is labeled 1. Later, Stewart [12] proved that the tree representations of a cograph can be obtained in polynomial time.

Tree representation satisfying the conditions mentioned above have been obtained for several other classes of graphs including the interval graphs [2], chordal graphs [11], maximal outerplanar graphs [1], TSP digraphs [8], P_4-reducible graphs [5], and P_4-sparse graphs [7], among others.

Several practical applications in computer science and computational linguistics suggest the study of graphs that are unlikely to have more than a few induced P_4's. Examples include examination scheduling and semantic clustering of index terms (see [3]). In examination scheduling, a *conflict graph* is readily constructed: the vertices represent different courses offered, while courses x and y are linked by an edge if and only if some student takes both of them. (In the weighted version, the weight of edge xy stands for the number of students taking both x and y.) Clearly, in any coloring of the conflict graph, vertices that are assigned the same color correspond to courses whose examinations can be held concurrently. It is usually anticipated that very few paths of length three will occur in the conflict graph. In the second application, we construct a graph whose vertices are the index terms; an edge occurs between two index terms to denote self-referencing or semantic proximity. Again, very few P_4 are expected to occur.

These applications have motivated Jamison and Olariu [5] to introduce the notion of a P_4-reducible graph: this is a graph none of whose vertices belongs to more than one P_4. Clearly, P_4-reducible graphs strictly contain the class of cographs. As it turns out, a remarkable property of the P_4-reducible graphs is their unique tree representation up to (labelled) tree isomorphism.

The purpose of this paper is to generalize the notion of P_4-reducibility, by relaxing in a natural way the constraints prescribing how the P_4's interact: we allow a P_4 to "extend" in a sense that will be made precise later. Just as in the case of P_4-reducible graphs, the class of P_4-extendible graphs that we introduce and investigate finds applications to clustering, scheduling, and memory management in a computer system.

Our main result gives a constructive characterization of the P_4-extendible graphs. To anticipate, all the P_4-extendible graphs turn out to be constructible from single vertices by a finite sequence involving three graph operations. Our characterization implies that P_4-extendible graphs are uniquely tree representable. Our result implies, trivially, that the isomorphism problem can be decided in polynomial time for P_4-extendible graphs, since it reduces to labelled tree isomorphism. An interesting feature of the class of P_4-extendible graphs is that they can be "reduced" in a canonical way to P_4-reducible graphs, by using a very simple greedy algorithm, that performs local changes only. The details are spelled out by Theorem 3.2.

The paper is organized as follows: Section 2 presents the main result, including two characterizations of P_4-extendible graphs. Section 3 deals with the details of the tree representation, as suggsted by Theorems 2.12 and 2.14. Finally, Section 4 summarizes the results and presents open problems.

2. Basics

All the graphs in this work are finite, with no loops or multiple edges. We assume familiarity with standard graph-theoretical terminology compatible with Golumbic [4]. At the same time, to specify our results we use some new terms that we are about to define.

Let $G=(V,E)$ be an arbitrary graph. For a vertex x of G, we let $N_G(x)$ denote the set of all the vertices of G which are adjacent to x: we assume adjacency to be nonreflexive, and so $x \notin N_G(x)$; $|N_G(w)|$ is termed the *degree* of w. If S is a subset of the vertex set of G, we let G_S stand for the subgraph of G induced by S. Occasionally, to simplify the notation, we shall blur the distinction between a set H of vertices and the graph G_H it induces, using the same symbol for both.

A vertex z is said to *distinguish* vertices u and v, whenever z is adjacent to precisely one of u, v. We let P_k (C_k) stand for the chordless path (cycle) on k vertices. In a P_4 with vertices, a, b, c, d and edges ab, bc, cd the vertices a and d are referred to as *endpoints* while b and c are termed *midpoints*.

Let W be a proper subset of V. For a vertex x outside W, write $x \in S(W)$ whenever x belongs to a P_4 sharing vertices with W. In case $S(W)$ contains at most one vertex, we shall say that W has a *proper extension*. A set D is said to be an *extension set* if $D = W \cup S(W)$ for a set W with a proper extension, inducing a P_4 in G.

For later reference we take note of the following simple results whose justification is immediate.

Observation 2.1. *Let W be a set with a proper extension, inducing a P_4 in G. If a vertex x outside W belongs to $S(W)$ then x together with three vertices in W induces a P_4.*

(Follows directly from the fact that W has a proper extension.)

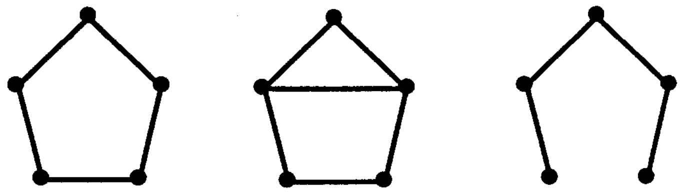

Fig. 1.

Observation 2.2. *Let W be a set with a proper extension, inducing a P_4 in G. A vertex x outside W belongs to $S(W)$ if and only if at least one of the following is satisfied:*

(1.1) *x distinguishes the midpoints of W;*

(1.2) *x distinguishes the endpoints of W;*

(1.3) *x is adjacent to both endpoints and nonadjacent to both midpoints.*

(First, if x belongs to $S(W)$ then, by Observation 2.1, it must induce a P_4 with three vertices in W. It is easy to see that whenever this happens, one of the conditions (1.1), (1.2) or (1.3) must be satisfied. Conversely, if one of the conditions (1.1)–(1.3) is satisfied, then x together with three vertices in W induce a P_4, implying that $x \in S(W)$.)

Observation 2.3. *If D is an extension set and some set $B \subseteq D$ induces a P_4 in G, then $D = B \cup S(B)$.*

(Since D is an extension set, we find a set W with a proper extension, inducing a P_4 in G such that $D = W \cup S(W)$. If B and W coincide, then there is nothing to prove. Otherwise, by Observation 2.1, B and W have three vertices in common. The conclusion follows.)

Observation 2.3 asserts that for an extension set D the role of W in the definition of D can be played by any subset B of D inducing a P_4 in G. This property of extension sets will be frequently used in this work with no further explanation.

A graph G is termed *P_4-extendible* if every set W inducing a P_4 in G has a proper extension. Trivially, every P_4-reducible graph is P_4-extendible: to see this, note that for every set W inducing a P_4, $S(W)$ must be empty. It is easy to see, however, that all the graphs featured in Fig. 1 are P_4-extendible but not P_4-reducible. Hence the class of P_4-extendible graphs strictly contains the class of P_4-reducible graphs. (It should also be noted that none of the graphs in Fig. 1 is P_4-sparse; conversely, the graph with vertices a, a', b, b', c, c' and edges aa', bb', cc', ab, bc, ca is P_4-sparse but not P_4-extendible.)

In the remainder of this paper we shall often rely directly or indirectly on the following simple observations. A P_4-extendible graph $G=(V,E)$ along with a set $W=\{w_0, w_1, w_2, w_3\}$ inducing a P_4 in G with edges w_0w_1, w_1w_2, w_2w_3 is assumed. We let D stand for the extension set $W \cup S(W)$.

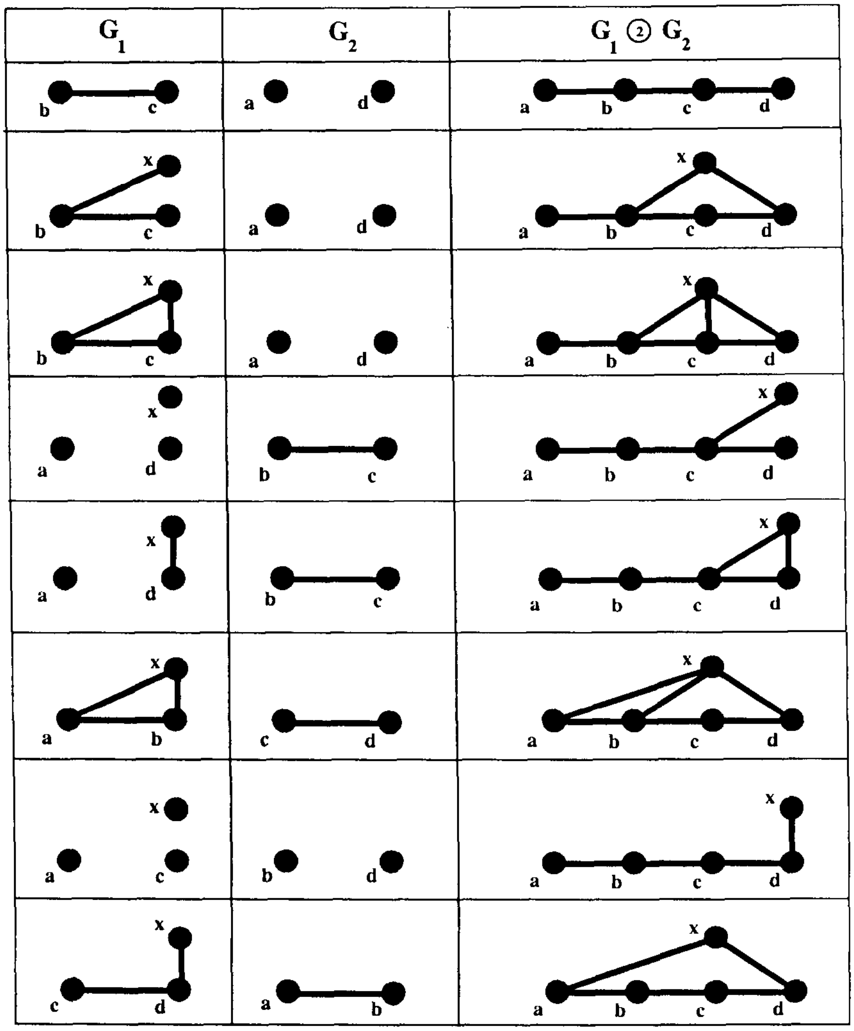

Fig. 2.

Observation 2.4. *D induces a connected subgraph of both G and $\bar{G}$.*

(Follows from Observation 2.1 together with the fact that every P_4 is self-complementary.)

For a vertex z outside D write

- $z \in T_G(D)$ whenever z is adjacent to all the vertices in D;
- $z \in P_G(D)$ whenever z is adjacent to some, but not all the vertices of G;
- $z \in I_G(D)$ whenever z is adjacent to no vertices of D.

Whenever possible, we shall drop the reference to the graph G writing, simply, $T(D)$, $P(D)$, and $I(D)$.

An extension set D in an arbitrary graph G is said to be *separable* if no vertex of D is both endpoint of some P_4 in G_D and midpoint of some P_4 in G_D.

Our next result shows that an extension set D is separable as soon as $P_G(D)$ is nonempty. More precisely, we have the following.

Lemma 2.5. *Let D be an extension set in a P_4-extendible graph G. If $P_G(D)$ is nonempty, then D is separable. Moreover, every vertex in $P_G(D)$ is adjacent to all the midpoints in D and nonadjacent to all the endpoints in D.*

Proof. We only need show that for every set $A \subseteq D$ inducing a P_4 in G, every vertex in $P_G(D)$ is adjacent to both its midpoints and nonadjacent to both its endpoints.

By virtue of Observation 2.3, we can write $D = A \cup S(A)$. If A and D coincide, then there is nothing to prove. We shall, therefore, assume that $S(A)$ is nonempty. Let $p \in P(D)$ be a counterexample to our claim. Since $p \notin S(A)$, the characterization provided by Observation 2.2 implies that p is adjacent (nonadjacent) to all the vertices in A. Since p is adjacent to some, but not all the vertices in D, p must be nonadjacent (adjacent) to the unique vertex in x in $S(A)$.

By Observation 2.1, x belongs to a P_4 A' in D sharing three vertices with A. Hence, p is adjacent to an odd number of vertices in A'. Now Observation 2.2 implies $p \in S(A') \subset D$, a contradiction. This completes the proof of the lemma. □

Observation 2.6. *Let D be a separable extension set in a P_4-extendible graph $G=(V,E)$. If $V = D \cup P_G(D)$, then D contains at least one vertex of degree* 1.

(Let a subset A of D induce a P_4 with vertices a, b, c, d and edges ab, bc, cd. We propose to show that at least one of the vertices a or d has degree 1 in G. First, by Lemma 2.5, no vertex in $P_G(D)$ is adjacent to either a or d. Hence, we only need show that no vertex in $S(A)$ is adjacent to both a and d. Since this is, trivially, true if $S(A)$ is empty, we shall assume that $\{x\} = S(A)$. Suppose that x is adjacent to both a and d. Since D is separable, neither a nor d can be midpoints of a P_4. In particular neither of $\{b,a,x,d\}$, $\{c,d,x,a\}$ can induce a P_4. But now, xb, $xc \in E$, contradicting Observation 2.2.)

Observation 2.7. *No vertex in $I(D)$ is adjacent to a vertex in $P(D)$.*

(Consider adjacent vertices i in $I(D)$ and p in $P(D)$; by Lemma 2.5, $\{w_0, w_1, p, i\}$ induces a P_4 in G, implying that p, $i \in S(W)$, a contradiction.)

Observation 2.8. *Every vertex in $T(D)$ is nonadjacent to all the vertices in $P(D)$.*

(Let vertices t in $T(D)$ and p in $P(D)$ be adjacent. By Lemma 2.5, $pw_2 \in E$ and $pw_0 \notin E$. By the definition of $T(D)$, tw_0, $tw_2 \in E$. But now, $\{w_0, t, w_2, p\}$ induces a P_4 in G, implying that t, $p \in S(W)$, a contradiction.)

Observation 2.9. *No vertex in $I(D)$ distinguishes nonadjacent vertices in $T(D)$.*

(Else, if a vertex i in $I(D)$ distinguishes nonadjacent vertices t, t' in $T(D)$, then $\{w_0, t, t', i\}$ induces a P_4 in G, implying that $t, t', i \in S(W)$, a contradiction.)

Observation 2.10. *No vertex in $T(D)$ distinguishes adjacent vertices in $I(D)$.*

(Otherwise, if a vertex t in $T(D)$ distinguishes adjacent vertices i, i' in $I(D)$, then $\{w_0, t, i, i'\}$ induces a P_4 in G, a contradiction.)

Observation 2.11. *Let G be a graph whose vertex set V partitions into nonempty, disjoint sets V' and V'' such that no P_4 in G contains vertices from both V' and V''. Then G is P_4-extendible as soon as the subgraphs of G induced by V' and V'' are.*

(Let G', G'' be the subgraphs of G induced by V', V'', respectively. Assume that both G' and G'' are P_4-extendible graphs and let A be an arbitrary set of vertices of G inducing a P_4. By assumption, $A \subseteq V'$ or $A \subseteq V''$. The conclusion follows.)

We are now in a position to state the first characterization of P_4-extendible graphs which is at the heart of all subsequent results presented in this paper. In particular, Theorem 2.12 suggests a constructive characterization of P_4-extendible graphs which will be specified in Theorem 2.14.

Theorem 2.12. *A graph G is P_4-extendible if and only if for every induced subgraph $H = (V_H, E_H)$ of G, precisely one of the following conditions is satisfied:*

(i) *H is disconnected;*

(ii) *$\bar{H}$ is disconnected;*

(iii) *H is an extension set;*

(iv) *there is a unique separable extension set $D \subset V_H$ such that every vertex outside D is adjacent to all midpoints and nonadjacent to all endpoints of D.*

Proof. Write $G = (V, E)$. The proof of the "if" part is by induction: assuming the statement true for all graphs with fewer vertices than G, we only need show that G is a P_4-extendible graph as soon as one of conditions (i)-(iv) is satisfied.

To begin, if (iii) is satisfied, then there is nothing to prove. Next, if one of the conditions (i) or (ii) is satisfied, then V can be partitioned into two nonempty sets with no P_4 in G containing vertices from both, and we are done by the induction hypothesis together with Observation 2.11.

We may, therefore, assume that (iv) holds. Let D be the extension set featured in (iv). Again, consider the partition of V into D and $V - D$. Since D is an extension set, no P_4 in G contains vertices from both D and $V - D$. Now the conclusion follows from Observation 2.11 together with the induction hypothesis.

To prove the "only if" part, suppose that G is a P_4-extendible graph and let $H = (V_H, E_H)$ be an arbitrary induced subgraph of G. Since P_4-extendible graphs are hereditary, it follows that H is P_4-extendible. By Observation 2.4 it follows easily that conditions (i)–(iv) are pairwise incompatible. Thus, we only need show that (iv) must hold true whenever conditions (i)–(iii) fail.

For this purpose, we shall assume that both H and $\bar{H}$ are connected and that H itself is not an extension set. Since both H and $\bar{H}$ are connected, a result of Seinsche [10] guarantees that H contains a P_4. This, in turn, implies that G must contain at least one extension set.

Our proof of Theorem 2.12 relies, in part, on the following intermediate result.

Lemma 2.13. *Let D be an extension set in G with both $T_H(D)$ and $I_H(D)$ nonempty. If no vertex in $T_H(D)$ is adjacent to all the vertices in $I_H(D)$, then $T_H(D) \cup I_H(D)$ contains an extension set D' with $P_H(D) \subset P_H(D')$.*

Proof. Choose a vertex t in $T_H(D)$ such that

$$|N_H(t) \cap I_H(D)| \text{ is as large as possible.} \tag{1}$$

We claim that:

$$\text{If a vertex } x \text{ in } T_H(D) \text{ is nonadjacent to a vertex in some component } Z \text{ of } I_H(D), \text{ then } x \text{ is adjacent to no vertices in } Z. \tag{2}$$

(Follows by the connectedness of Z and Observation 2.10 combined.)

Since, by assumption, no vertex in $T_H(D)$ is adjacent to all the vertices of $I_H(D)$, (2) guarantees the existence of a component Z' of $I_H(D)$ such that t is adjacent to no vertices in Z'. The connectedness of H, together with Observation 2.7 guarantees that some vertex z' in Z' is adjacent to some vertex t' in $T_H(D)$.

Our choice of t, expressed in (1), implies the existence of a vertex z in some component Z distinct from Z' such that $tz \in E_H$ and $t'z \notin E_H$. We note that since z distinguishes t and t', Observation 2.9 guarantees that t and t' are adjacent, and so the set $B = \{t, t', z, z'\}$ induces a P_4 in H. Let D' stand for $B \cup S(B)$: since H is a P_4-extendible graph, D' is an extension set in H.

Observations 2.7 and 2.8, combined, imply that

$$P_H(D) \subseteq P_H(D').$$

To see that the inclusion is strict, note that by the definition of $T_H(D)$ and $I_H(D)$, every vertex in D belongs to $P_H(D')$. With this, the proof of the lemma is complete. □

Proof of Theorem 2.12 (continued). Choose an extension set D in H such that

$$|P_H(D)| \text{ is as large as possible.} \tag{3}$$

We claim that

$$\text{both } T_H(D) \text{ and } I_H(D) \text{ are empty.} \tag{4}$$

If precisely one of the sets $T_H(D)$ and $I_H(D)$ is nonempty, then by Observations 2.7 and 2.8 combined, either H or $\bar{H}$ is disconnected, contrary to our assumption. Hence, if (4) is false, then both $T_H(D)$ and $I_H(D)$ are nonempty.

Further, if no vertex of $T_H(D)$ is adjacent to all the vertices of $I_H(D)$, then Lemma 2.13 guarantees the existence of an extension set D' with $P_H(D) \subset P_H(D')$, contradicting our choice of D in (1).

It must be the case, therefore, that some vertex t in $T_H(D)$ is adjacent to all the vertices in $I_H(D)$. Let F stand for the connected component of the subgraph of $\bar{H}$ induced by $T_H(D)$, containing t. Note that by Observation 2.9, every vertex in F is adjacent to all the vertices in $I_H(D)$. But now, by the definition of $T_H(D)$, together with Observation 2.8, it follows that $\bar{H}$ is disconnected (since every vertex in F is adjacent to all the vertices in $V_H - F$), a contradiction. Thus, (4) must hold true.

By virtue of (4) we can write

$$V_H = D \cup P_H(D).$$

Since, by assumption, H itself is not an extension set, (4) guarantees that

$$P_H(D) \neq \emptyset.$$

Note that by virtue of Lemma 2.5, D is separable, and every vertex in $P_H(D)$ is adjacent to all the midpoints and nonadjacent to all the endpoints in D. Finally, the uniqueness of D follows directly from Observation 2.6: D contains the only vertices of degree 1 in H.

With this, the proof of Theorem 2.12 is complete. □

Our constructive characterization of the P_4-extendible graphs relies, in part, on two graph operations devised by Lerchs [8] for the purpose of characterizing the class of cographs. More precisely, let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be disjoint graphs. Define

- $G_1 ⓪ G_2 = (V_1 \cup V_2, E_1 \cup E_2)$;
- $G_1 ① G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup \{xy \mid x \in V_1, y \in V_2\})$.

It is easy to see that the operations ⓪ and ① reflect the conditions (i) and (ii), respectively, in Theorem 2.12. For the purpose of constructing the P_4-extendible graphs, we need to introduce two new operations to reflect conditions (iii) and (iv).

The ② operation is defined in Fig. 2: taking graphs G_1 and G_2 as input, it constructs a new graph $G_1 ② G_2$ which is an extension set. It is easy to verify that the ② operation is well defined and admits a unique inverse: given an arbitrary graph G that is an extension set, the graphs G_1 and G_2 of Fig. 2 are uniquely determined.

The ③ operation will reflect condition (iv) in Theorem 2.12. More precisely, let $G_1 = (V_1, E_1)$ be a graph such that V_1 is a separable extension set and let $G_2 = (V_2, E_2)$ be an arbitrary graph disjoint from G_1. Define

- $G_1 ③ G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup \{xy \mid x \text{ a midpoint of } V_1, y \in V_2\}$.

As it turns out, all P_4-extendible graphs are constructible by means of the operations ⓪, ①, ②, and ③. More precisely, we have the following result.

Theorem 2.14. *For a graph G the following statements are equivalent*:

(i) *G is a P_4-extendible graph*;

(ii) *G is obtained from single-vertex graphs by a finite sequence of operations* ⓪, ①, ②, ③.

Proof. Let $G=(V,E)$ be obtained from single-vertex graphs by a finite sequence σ of zero or more operations ⓪, ①, ②, ③. We prove the implication (ii) → (i) by induction on the length of σ. Assume the statement true for graphs obtained by sequences involving fewer operations than σ. If G arises from the nonempty graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ by some operation $s\in\sigma$, then, by the induction hypothesis, both G_1 and G_2 are P_4-extendible graphs. If s is a ② operation, then G is trivially P_4-extendible.

Furthermore, if s is one of the operations ⓪, ①, or ③, then no P_4 in G can have vertices from both G_1 and G_2; since by the induction hypothesis both G_1 and G_2 are P_4-extendible graphs, Observation 2.11 guarantees that G is P_4-extendible.

To prove the implication (i) → (ii), we proceed by induction on the size of G. Assuming the implication true for all graphs with fewer vertices than G, we propose to prove the implication for G itself.

For this purpose, note that if G or $\bar{G}$ is disconnected, then G arises from two of its proper induced subgraphs by a ⓪ or a ① opertion, and the conclusion is guaranteed by the induction hypothesis. If G is an extension set, then G arises from two of its proper induced subgraphs by a ② operation as in Fig. 2. Finally, by Theorem 2.12, if both G and $\bar{G}$ are connected and if G itself is not an extension set, then there exists a unique separable extension set D in G such that every vertex in $V-D$ is adjacent to every midpoint in D and nonadjacent to every endpoint in D.

But now, it is obvious that G arises from the graphs G_D and G_{V-D} by a ③ operation, and the proof of the theorem is complete. □

3. The tree representation

Theorems 2.12 and 2.14 suggest a natural way of associating with every P_4-extendible graph G a tree $T(G)$ (called the *px-tree* of G). To anticipate, the leaves of $T(G)$ are precisely the vertices of G; an internal node λ of $T(G)$ is labelled i $(0\le i\le 3)$ whenever the subgraph G' of G corresponding to the subtree T' of $T(G)$ rooted at λ arises from two of its proper induced subgraphs by an ⓘ operation.

As a preliminary step, however, given a P_4-extendible graph G distinct from an extension set with G and $\bar{G}$ connected, we present an algorithm to compute the unique separable extension set featured in condition (iv) of Theorem 2.12. The details are spelled out by the following procedure.

Procedure Find_Separable_Extension_Set(G,D);
{*Input*: a P_4-extendible graph $G=(V,E)$ distinct from an extension set with both G and $\bar{G}$ connected;

```
       Output: the unique separable extension set D featured in condition
               (iv) of Theorem 2.12}
1      begin
2        a ← an arbitrary vertex of degree 1 in G;
3        b ← N_G(a);
4        T ← V − N_G(b);
5        N_G(T) ← {v ∈ V − T | v is adjacent to some vertex in T};
6        D ← {a, b} ∪ T ∪ N_G(T)
7      end; {Find_Separable_Extension_Set}
```

Theorem 3.1. *Given a P_4-extendible graph G distinct from an extension set and such that both G and $\bar{G}$ are connected, Procedure Find_Separable_Extension_Set correctly computes the unique separable extension set D featured in condition* (iv) *of Theorem* 2.12.

Proof. Write $G=(V,E)$. By assumption, $P_G(D)$ is nonempty. By Lemma 2.5, every vertex in $P_G(D)$ is adjacent to all the midpoints and nonadjacent to all the endpoints in D. Consequently, all the vertices of degree one (which exist by virtue of Observation 2.6) must belong to D, each of them being an endpoint in D.

Let a be an arbitrary vertex of degree one in G, and let b stand for the unique neighbor of a. Consider the set $T=V-N_G(b)$ computed in line 4 of the procedure. We claim that

$$\text{every vertex in } T \text{ belongs to } D. \tag{5}$$

(To justify (5), note that b must be a midpoint in D, and so by Lemma 2.5 we have $P_G(D)=V-D\subset N_G(b)$. Hence, $T=V-N_G(b)\subsetneq D$, as claimed.)

Next, we claim that

$$\text{no vertex in } T \text{ is adjacent to a vertex in } P_G(D). \tag{6}$$

(Otherwise, if a vertex t in T were adjacent to a vertex p in $P_G(D)$, then $\{a,b,p,t\}$ would induce a P_4 in G, implying that $p\in D$, a contradiction.)

Finally, note that by (5) and (6) combined, it follows that the set D computed in line 5 is the desired one, thus completing the proof of the theorem. □

Next, we describe the formal construction of the px-tree of a P_4-extendible graph G by the following recursive procedure.

```
Procedure Build_tree(G);
{Input: a P_4-extendible graph G = (V, E);
 Output: the px-tree T(G) corresponding to G}
  begin
    if |V| = 1 then
      return the tree T consisting of the unique vertex of G;
```

if G (respectively $\bar{G}$) is disconnected **then**
begin
let $G_1, G_2, \ldots, G_p$ $(p \geq 2)$ denote the components of G (respectively $\bar{G}$);
let $T_1, T_2, \ldots, T_p$ be the corresponding *px*-trees rooted at $r_1, r_2, \ldots, r_p$;
return the tree $T(G)$ obtained by adding $r_1, r_2, \ldots, r_p$ as children of a node labelled 0 (1);
end;
if G is an extension set **then**
begin
write $G = G_1 \textcircled{2} G_2$ as in Fig. 2;
let T_1, T_2 be the corresponding *px*-trees rooted at r_1 and r_2;
return the tree $T(G)$ obtained by adding r_1, r_2 as children of a node labelled 2
end;
Find_Separable_Extension_Set(G, D);
let T_1, T_2 be the *px*-trees corresponding to G_D and G_{V-D}, respectively, rooted at r_1 and r_2;
return the tree $T(G)$ obtained by adding r_1, r_2 as the left and right children, respectively, of a node labelled 3
end; {Build_tree}

By Theorems 2.12 and 2.14 it follows immediately that the *px*-tree of a P_4-extendible graph G is unique up to labelled tree isomorphism.

An important property of P_4-extendible graphs is that they can be "reduced" easily to P_4-reducible graphs in a natural way. More precisely, given an arbitrary P_4-extendible graph G, the *reduced graph* $R(G)$ associated with G is the induced subgraph of G obtained by the following procedure.

Procedure Collapse(G);
{*Input*: a P_4-extendible graph G;
Output: the reduced graph $R(G)$}
begin
$H \leftarrow G$;
while there exists an extension set $D = W \cup S(W)$ in H **do**
$H \leftarrow H - S(W)$;
return(H)
end;

The definition of a P_4-extendible graph guarantees that the graph $R(G)$ returned by Procedure Collapse is a P_4-reducible graph. The uniqueness implied by the definition of the reduced graph is justified by the following result.

Theorem 3.2. *The reduced graph of a P_4-extendible graph is unique up to isomorphism.*

Proof. Assume the statement true for all P_4-extendible graphs with fewer vertices than G.

If G or $\bar{G}$ is disconnected, then we are done by the induction hypothesis since no P_4 in G has vertices in distinct components of G or $\bar{G}$. We may assume, therefore, that both G and $\bar{G}$ are connected. Note that if G is an extension set, then the reduced graph is always isomorphic to the P_4.

Otherwise, Theorem 2.12 guarantees the existence of a unique separable extension set D such that every vertex outside of D is adjacent to every midpoint and nonadjacent to every endpoint of D. By the induction hypothesis, the reduced graph $R(G')$ of the graph G' induced by $V-D$ is unique up to isomorphism. Let A be some subset of D inducing a P_4. By Observation 2.3, we can write $D=A\cup S(A)$. Now the conclusion follows from the observation that $R(G)$ is obtained from $R(G')$ by replacing D by A. □

4. Conclusion and open questions

In this paper we have introduced and investigated the class of P_4-extendible graphs which is a natural generalization of the class of P_4-reducible graphs with applications to clustering and scheduling. As it turns out, the P_4-extendible graphs feature the remarkable property of being uniquely tree representable.

Furthermore, the conversion between a P_4-extendible graph and the corresponding tree representation can be carried out in polynomial time and, consequently, the graph isomorphism problem can be solved in polynomial time for P_4-extendible graphs. It would be of interest to further investigate this tree structure for the purpose of solving efficiently other computational problems important in applications such as: clustering, minimum fill-in, minimum weight dominating set, hamiltonicity and others. Linear-time recognition algorithms for cographs and P_4-reducible graphs are known to exist (see [3,6]). It would be very interesting to see whether the same techniques can be applied for the purpose of recognizing P_4-extendible graphs efficiently. We conjecture that a linear-time recognition algorithm for P_4-extendible graphs is achievable, and pose it as an open problem.

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