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SCATTERING PARAMETERS FOR AN EPSTEIN PROFILE IN A HALF-SPACE

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Abstract—The reflection of waves for a stratified medium in $(-\infty, \infty)$ can be studied by transforming the hypergeometric differential equation into a wave equation. In the context of an astrophysical problem [1], the corresponding analysis is carried out for an Epstein profile in $(0, \infty)$. This profile represents a scattering potential in quantum mechanical terminology: its properties are briefly discussed.

The most general second-order ordinary differential equation with regular singular points at 0, 1, and ∞, is the hypergeometric differential equation:

$$\xi(1-\xi)\frac{d^2F}{d\xi^2} + \{\gamma - (\alpha+\beta+1)\}\frac{dF}{d\xi} - \alpha\beta F = 0, \tag{1}$$

where α, β, and γ are parameters, in general complex. The following substitutions [2,3]

$$F = r(z)Z(z), \quad \xi = P(z), \tag{2}$$

where $P(z)$ is an arbitrary function, and

$$r(z) = \xi^{-\gamma/2}(1-\xi)^{(\gamma-\alpha-\beta-1)/2}\left(\frac{d\xi}{dz}\right)^{1/2} \tag{3}$$

reduce (1) to wave equation form

$$\frac{d^2Z}{dz^2} + g(z)Z = 0, \tag{4}$$

where

$$g(z) = \frac{1}{2}\frac{d^2}{dz^2}\left(\ln\frac{dP}{dz}\right) - \frac{1}{4}\left[\frac{d}{dz}\left(\ln\frac{dP}{dz}\right)\right]^2 - \left(\frac{d}{dz}\ln P\right)^2\left\{K_1 + \frac{K_2P}{1-P} + \frac{K_3P}{(1-P)^2}\right\} \tag{5}$$

and

$$4K_1 = \gamma(\gamma-2) \tag{6}$$
$$4K_2 = 1 - (\alpha-\beta)^2 + \gamma(\gamma-2) \tag{7}$$
$$4K_3 = (\alpha+\beta-\gamma)^2 - 1, \tag{8}$$

which can of course be solved explicitly for α, β and γ.

In equation (4), $g(z)$ may be viewed as the square of the refractive index for a medium (in the context of wave propagation or scattering problems). By choosing

$$P(z) = -e^{mz}, \tag{9}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX the general asymmetric Epstein is obtained, wherein

$$g(z) = -m^2\left(K_1 + \tfrac{1}{4}\right) + \frac{m^2 K_2 e^{mz}}{1+e^{mz}} + \frac{m^2 K_3 e^{mz}}{(1+e^{mz})^2}. \tag{10}$$

(Other choices of P are possible: see [3]. A generalization of (9) was provided by Burman and Gould [4]). Writing $K_1 = -g_0/(m^2 - 1/4)$, $K_2 = -g_0 N/m^2$, $K_3 = -4g_0 M/m^2$, a simpler form of (10) emerges:

$$\frac{g(z)}{g_0} = 1 - Ne^{mz}(1+e^{mz})^{-1} - 4Me^{mz}(1+e^{mz})^{-2}. \tag{11}$$

In what follows, we derive analytic expressions for the reflection and transmission coefficients for waves encountering the potential (11) in $(0,\infty)$. This is to be contrasted with those coefficients corresponding to $z \in (-\infty,\infty)$; different representations of hypergeometric functions are required. The physics of the situation is discussed briefly below.

Corresponding to the transformations (2) and (9), the interval $z \in (0,\infty)$ is mapped into $\xi \in (-1,-\infty)$, so we seek the analytic continuation of the appropriate "transmitted" solution ($z \to \infty$, $\xi \to -\infty$) valid in the domain of the "incident" and "reflected" solutions ($z \to 0$, $\xi \to -1$). The appropriate connection formula can be obtained by a judicious use of formula (15.3.8) in [5], yielding

$$\begin{aligned}\xi^{-\alpha}F(\alpha,\alpha-\gamma+1,\alpha-\beta+1;\xi^{-1}) &= N(\xi-1)^{-\alpha}F\left(\alpha,\gamma-\beta,\gamma;\frac{\xi}{\xi-1}\right)\\ &\quad + M(\xi-1)^{\gamma-\alpha-1}\xi^{1-\gamma}F\left(\alpha-\gamma+1,1-\beta,2-\gamma;\frac{\xi}{\xi-1}\right),\end{aligned} \tag{12}$$

where

$$F(a,b,c;x) = 1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)x^2}{1.2c(c+1)} + \cdots \tag{13}$$

is the hypergeometric series, converging for $|x| < 1$, and in terms of the gamma function Γ (with appropriate restrictions on arguments)

$$M = \frac{\Gamma(\alpha-\beta+1)\Gamma(\gamma-1)}{\Gamma(\alpha)\Gamma(\gamma-\beta)} \tag{14}$$

and

$$N = \frac{\Gamma(\alpha-\beta+1)\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)\Gamma(1-\beta)}. \tag{15}$$

Utilizing the transformations (2) and (9), we find that as $z \to \infty$, the left-hand side of (12) corresponds to

$$Z \to m^{(-1/2)}(-1)^{(1/2)(\gamma-1)-\alpha}e^{(1/2)m(\beta-\alpha)z}, \tag{16}$$

and as $z \to 0$, the right-hand side of (12) corresponds to

$$Z \to m^{-1/2}(-1)^{(1/2)(\gamma-1)-\alpha}e^{1/2m(\beta-\alpha)z}\left\{2^p N F_a e^{(1/2)m(\gamma-1)z} + 2^q M F_b e^{(-1/2)m(\gamma-1)z}\right\}, \tag{17}$$

where

$$F_a = F(\alpha,\gamma-\beta,\gamma;\tfrac{1}{2}) \quad \text{and} \quad F_b = F(\alpha-\gamma+1,1-\beta,2-\gamma;\tfrac{1}{2}) \tag{18}$$

and

$$2p = 1+\beta-\alpha-\gamma, \qquad 2q = \gamma-\alpha+\beta-1. \tag{19}$$

Thus, the expression (17), valid as $z \to 0$, is to be regarded as the analytic continuation of the expression (16), valid as $z \to \infty$. Further, if the parameters α, β and γ are such that $(\beta-\alpha)/i$ and $(\gamma-1)/i$ are real positive quantities, (16) represents a plane undamped transmitted wave propagating in the direction $z \to \infty$, and (17) represents a superposition of similar but incident and reflected waves as $z \to 0$. Comparing coefficients in front of the various exponential terms, we find for reflection coefficient R and transmission coefficient T, the following expressions:

$$R = \frac{2^{\gamma-1} F(\alpha-\gamma+1,\, 1-\beta,\, 2-\gamma;\, (1/2))\Gamma(\gamma-1)\Gamma(1-\beta)\Gamma(1+\alpha-\gamma)}{F(\alpha,\, \gamma-\beta,\, \gamma;\, (1/2))\Gamma(\alpha)\Gamma(\gamma-\beta)\Gamma(1-\gamma)} \tag{20}$$

and

$$T = \frac{2^{(1/2)(\alpha+\gamma-\beta-1)}\Gamma(1+\alpha-\gamma)\Gamma(1-\beta)}{F(\alpha,\, \gamma-\beta,\, \gamma;\, (1/2))\Gamma(\alpha-\beta+1)\Gamma(1-\gamma)}. \tag{21}$$

The analysis carried out here has been with a specific astrophysical context in mind: scattering of waves in a stellar interior (see [1]). Thus $z = 0$ corresponds to $r = 0$, the center of the spherical star, and the governing Schrödinger equation for the dependent variable is such that as $r \to 0$ ($z \to 0$), the waves become in fact spherical waves: the spherical dependence is removed via the transformation to the Cartesian equation (4). In this context, equation (11) may be regarded more as a scattering potential $V(z)$, with the following properties:

(i) $V(0) = 1 - \frac{N}{2} - M$

(ii) $\lim_{z \to \infty} V(z) = 1 - N$

(iii) $V'(z_m) = 0, \quad z_m = m^{-1} \ln \left| \frac{4M+N}{4M-N} \right|$

(iv) $V(z_m) = 1 - \frac{(N+4M)}{8} - \frac{(16M^2-N^2)}{16N}$.

To fit this qualitatively to a particular class of potentials, the limiting values $V(0)$ and $V(\infty)$ (or $V(z_0), z_0 < \infty$) specify M and N; locating the position of the extremum $V(z_m)$ at $z = z_m$ specifies m via M and N. Alternatively, given $V(z_0)$ and $V(z_m)$, we may find M, N, and m.

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