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Some aspects of the semi-perfect elimination

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Abstract

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Several efficient algorithms have been proposed to construct a perfect elimination ordering of the vertices of a chordal graph. We study the behaviour of two of these algorithms in relation to a new concept, namely the semi-perfect elimination ordering, which provides a natural generalization of chordal graphs.

1. Introduction

A graph G is said to be *chordal* (also *triangulated* or *rigid circuit*) if every cycle in G of length at least four has a chord. Chordal graphs arise naturally in a wide spectrum of applications including the study of evolutionary trees [1], the facility location problem [2], scheduling [10], and solving sparse systems of linear equations [12, 13]. For a wealth of results concerning chordal graphs the interested reader is referred to Duchet [4] or Golumbic [6].

Dirac [3] proved that chordal graphs contain a vertex whose neighbours are pairwise adjacent: such a vertex is termed *simplicial*. An ordering $x_1, x_2, \ldots, x_n$ of the vertices of G is said to be a *perfect elimination ordering* (PEO, for short) if the corresponding linear order $<$ with $x_i < x_j$ iff $i < j$ satisfies

$$x_i \text{ is a simplicial vertex in } G_{\{x_i, x_{i+1}, \ldots, x_n\}} \text{ for every } i.$$

Fulkerson and Gross [5] proved that a graph G is triangulated if and only if it admits a perfect elimination ordering. Later, Rose, Tarjan and Leuker [13], Tarjan and Yannakakis [15], Shier [14], and Hoffman and Sakarovitch (see [14]) proposed efficient algorithms to find perfect elimination orderings in chordal graphs. They all prove particular instances of the following template theorem:

A graph G is chordal if and only if any ordering of the vertices of G produced by algorithm A is a PEO. (1)

Here, of course, A stands for one of lexicographic breadth-first search (or LBFS) [13], maximum cardinality search (or MCS) [15], maximum element in component (or MEC) [14], maximum neighbourhood in component (or MCC) [14], or the Hoffman–Sakarovitch algorithm mentioned in [14].

A natural extension of the class of chordal graphs is obtained by relaxing the condition related to the existence of the simplicial vertex, as we are about to explain.

For this purpose, however, we need to introduce some new terms. As usual, we let C_k (P_k) stand for the chordless cycle (path) on k vertices. If $\{a,b,c,d\}$ induces a P_4 in G with edges ab, bc, cd, then we shall refer to b and c as the *midpoints* of this P_4.

Call a vertex x in G *semi-simplicial* if x is midpoint of no P_4 in G. Clearly, every simplicial vertex is semi-simplicial, but not conversely.

An ordering $x_1, x_2, \dots, x_n$ of the vertices of G is said to be a *semi-perfect elimination ordering* (SPEO, for short) if the corresponding linear order $<$ with $x_i < x_j$ iff $i<j$ satisfies

$$x_i \text{ is a semi-simplicial vertex in } G_{\{x_i, x_{i+1}, \dots, x_n\}} \text{ for every } i. \tag{2}$$

The present work was motivated by a search for a result in the spirit of (1). More precisely, we want an answer to the following natural question:

What is the class C_A of graphs for which every ordering produced by algorithm A is a SPEO? (3)

Jamison and Olariu [7] and Olariu [8] have provided an answer to (3) with A standing for LBFS and MCS. They also show how to solve the four classical optimization problems on C_{LBFS} and C_{MCS} in linear time.

The purpose of this paper is to answer question (3) for MEC and MCC. Our main results states that

$$C_{\text{MCS}} = C_{\text{MEC}} = C_{\text{MCC}}.$$

This common class of graphs strictly contains all chordal graphs, all Welsh–Powell opposition graphs (see Olariu and Randall [9]) and all superfragile graphs (see Preissmann, de Werra and Mahadev [11]).

2. The result

Let G be a graph. We shall let V stand for the vertex set of G; E will denote the set of edges of G. For a vertex x in G let $N(x)$ stand for the set of all the vertices adjacent to x in G. (We assume adjacency to be nonreflexive, and so $x \notin N(x)$.) We let $N'(x)$ stand for the set of all the vertices adjacent to x in the complement $\bar{G}$ of G.

To make our exposition self-contained, we shall reproduce here the details of MEC and MCC.

```
Procedure MEC(G);
/ Input: the adjacency list of G;
  Output: an ordering σ of the vertices of G /

begin
  S ← ∅;
  for i ← n downto 1 do
    begin
      let C be an arbitrary component of G − S;
      pick x in C such that N(x) ∩ S ⊂ N(y) ∩ S for no vertex y in C;
      σ(x) ← i; / assign to x number i /
      S ← S ∪ {x}
    end
end;

Procedure MCC(G);
/ Input: the adjacency list of G;
  Output: an ordering σ of the vertices of G /

begin
  S ← ∅;
  for i ← n downto 1 do
    begin
      let C be an arbitrary component of G − S;
      pick x in C such that |N(x) ∩ S| < |N(y) ∩ S| for no vertex y
        in C;
      σ(x) ← i; / assign to x number i /
      S ← S ∪ {x}
    end
end;
```

Note that we can think of the output of both MEC and MCC as a linear order $<$ on V by setting

$$u<v \text{ whenever } \sigma(u)<\sigma(v).$$

Our arguments rely on the following results that we present as lemmas.

Lemma 1. *Let σ be produced by MEC or MCC and let $<$ be the corresponding linear order. Let vertices a, b, c satisfy $a<b$, $b<c$, $ac \in E$, $bc \notin E$, and let S stand for the set of ordered vertices in G just before b is about to be ordered. If a and b are in the same component of $G-S$, then there exists a vertex b' in G with $bb' \in E$, $ab' \notin E$ and $b<b'$.*

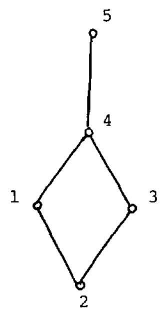

Fig. 1.

Proof. Let σ be produced by MEC (MCC) and let $N_S(a)$, $N_S(b)$ stand for $N(a) \cap S$, $N(b) \cap S$, respectively. Since b is chosen before a, it must be the case that

$$N_S(b) \not\subset N_S(a) \qquad (|N_S(b)| \not< |N_S(a)|).$$

Thus, since $ac \in E$ and $bc \notin E$, there must exist a vertex, say, b' in S with $ab' \notin E$, $bb' \in E$, as claimed. □

Lemma 2. *Let G be a graph with no induced $\bar{P}_5$, C_k $(k \geq 5)$ or the graph F in Fig.* 1, *let σ be an ordering of the vertices of G produced by MEC or MCC, and let $<$ stand for the corresponding linear order. Then, for every choice of vertices a, b, c, d satisfying*

$$a<b, \quad b<c, \quad a<d, \quad ab, ac, bd \in E, \quad bc, ad \notin E, \tag{4}$$

we have $cd \in E$.

Proof. Write $G=(V,E)$. If $<$ is a semi-perfect elimination, then the conclusion follows trivially.

We may, therefore, assume that $<$ is not a semi-perfect elimination. If the statement is false, then we shall let a stand for the last vertex in the linear order $<$ for which there are vertices b, c, d with $cd \notin E$ satisfying (4). Next, we let c stand for the *largest* vertex in $N(a)$ for which there exist vertices b and d with $cd \notin E$ satisfying (4). Further, with a and c chosen as before, let b stand for the largest vertex in $<$ for which there is a vertex d, $cd \notin E$, such that (4) is satisfied. Finally, with a, b, c chosen as above, we let d be the largest vertex in the linear order $<$ which is adjacent to b and nonadjacent to both a and c.

To begin, we claim that

$$\textit{b and c have no common neighbour e with } a<e \textit{ and } ae \notin E. \tag{5}$$

(Let e be a common neighbour of b and c with $a<e$ and $ae \notin E$. But now, $\{a, b, c, d, e\}$ induces a $\bar{P}_5$ or an F, depending on whether or not $de \in E$.)

Next, we claim that

$$b<d. \tag{6}$$

(To justify (6), we note that Lemma 1 applied to the vertices a, b, c implies the existence of a vertex b' with $bb' \in E$, $ab' \notin E$ and $b<b'$. If b' coincides with d, then we are done. Otherwise, by virtue of (5), we have $cb' \notin E$. But now, b' contradicts our choice of d.)

Write $x \in B$ whenever there exists a path

$$b = w_0, w_1, \dots, w_s = x,$$

joining b and x, with

$$w_{i-1} < w_i \quad \text{and} \quad aw_i \notin E \quad (1 \le i \le s). \tag{7}$$

Trivially, $b \in B$. We note that (6) implies that $d \in B$.

Similarly, write $y \in C$ whenever there exists a path

$$c = v_0, v_1, \dots, v_t = y,$$

joining c and y, with

$$v_{i-1} < v_i \quad \text{and} \quad av_i \notin E \quad (1 \le i \le t). \tag{8}$$

Let b', c' stand for the largest vertex in $<$ which belongs to B, C, respectively. By the definition of B, we find a chordless path

$$b = b_0, b_1, \dots, b_p = b',$$

in B, joining b and b', with the b_i's satisfying (7) in place of the w_i's.

Similarly, the definition of C guarantees the existence of a chordless path

$$c = c_0, c_1, \dots, c_q = c'$$

in C, joining c and c', with the c_i's satisfying (8) in the place of the v_i's.

For further reference, we note that

$$cb_i \notin E \quad (0 \le i \le p). \tag{9}$$

(To justify (9), let i stand for the smallest subscript for which $cb_i \in E$. Since $bc \notin E$, we have $i \ge 1$; by (5), we have $i \ge 2$. But now, $\{a, c, b_0, b_1, \dots, b_i\}$ induces a C_k with $k \ge 5$.)

It is easy to see that

$$c<b'. \tag{10}$$

(Otherwise, Lemma 1 applied to the vertices a, b', c implies the existence of a vertex b$''$ with $b'b'' \in E$, $ab'' \notin E$ and $b'<b''$, contradicting the maximality of b'.)

Further, we claim that

$$C \neq \{c\}. \tag{11}$$

To justify (11), let i stand for the smallest subscript such that $c<b_i$. Such a subscript must exist by the assumption that $b<c$ together with (10).

Note that (9) together with the fact that the b_i's satisfy (7) guarantees that we can apply Lemma 1 to the vertices b_{i-1}, c, b_i. We find a vertex x with $xb_{i-1} \notin E$, $xc \in E$ and $c<x$. We may assume that $ax \in E$, otherwise we are done.

Observe that $xb_0 \in E$, for otherwise either $\{a, b_0, b_1, c, x\}$ induces a $\bar{P}_5$ or x contradicts our choice of c, depending on whether or not $xb_1 \in E$. Let j $(0 \leq j < i-1)$ be the largest subscript such that $xb_j \in E$. But now, (7), (9) together with our choice of the subscript i guarantee that b_j contradicts our choice of a. Thus (11) must hold.

Next, we claim that

$$B \cap C \neq \emptyset. \tag{12}$$

To prove (12), we may assume that

no edge in G has one endpoint in B and the other in C

for otherwise we are done.

Symmetry in the following argument allows us to assume that

$$b' < c'. \tag{13}$$

Let i be the smallest subscript for which $b'<c_i$ (such a subscript must exist by virtue of (10) and (13) combined).

Lemma 1 applied to the vertices c_{i-1}, b', c_i guarantees the existence of a vertex b'' with $b'b'' \in E$, $c_{i-1}b'' \notin E$ and $b'<b''$. We must have $ab'' \in E$, else we contradict the maximality of b'.

Note that $b''c_0 \in E$, for otherwise either $\{a, c_0, c_1, b', b''\}$ induces an F (in case $c_1b'' \in E$), or with the assignment $b \leftarrow c_0$, $d \leftarrow c_1$, $c \leftarrow b''$ we contradict our original choice of the vertices c, b and d (in case $c_1b'' \notin E$). Let j $(0 \leq j < i-1)$ stand for the largest subscript such that $b''c_j \in E$. But now, our choice of i guarantees that c_j contradicts our choice of a. Thus, (12) must hold true.

Let w be the first vertex in the linear order $<$ which belongs to $B \cap C$. By the definition of B, there exists a chordless path P_B in B joining w and b satisfying (7);

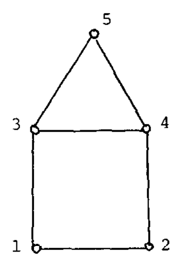

Fig. 2.

similarly the definition of C implies the existence of a chordless path P_C in C joining w and c, and satisfying (8).

By our choice of the vertex w, $P_B \cap P_C = \{w\}$. By (5), w is adjacent to at most one of the vertices b and c, and thus G must contain a chordless cycle of length at least five induced by $\{a, b, c\}$ together with $P_B \cup P_C$. With this the proof of Lemma 2 is complete. □

We are now in a position to state our main result.

Theorem 3. *For a graph G the following two statements are equivalent:*

(i) *G contains no induced subgraph isomorphic to a* $\bar{P}_5$, *a* C_k ($k \geq 5$) *or to the graph F in Fig.* 1,

(ii) *for every induced subgraph H of G, every ordering of the vertices of H produced by MEC or MCC is a semi-perfect elimination ordering.*

Proof. Write $G=(V,E)$. The implication (ii) ⇒ (i) is trivial: no ordering produced by MEC or MCC on a C_k with $k \geq 5$, is a semi-simplicial elimination; furthermore, it is a routine matter to check that the orderings implied by the labelings of the graph F in Fig. 1 and $\bar{P}_5$ suggested in Fig. 2 are produced by both MEC and MCC and yet not a semi-perfect elimination.

Assuming the implication (i) ⇒ (ii) true for all the graphs with fewer vertices than G, we only need to show that G itself satisfies the implication.

If this is not the case, then some linear order $<$ on V produced by MEC or MCC is not a semi-perfect elimination. We shall let a stand for the last vertex in the linear order $<$ which contradicts (2). Write $x \in A$ whenever $a<x$.

Let c be the *largest* vertex in $N(a) \cap A$ for which there exist a vertex b in $N(a) \cap A$ with $bc \notin E$, and a vertex in $N'(a) \cap A$ which is adjacent to precisely one of the vertices b and c. Our choice implies, trivially, that $b<c$.

Lemma 2 guarantees that every vertex w in $N(b) \cap N'(a) \cap A$ is adjacent to c. Therefore, by our choice of a, we find a vertex d in A, with $cd \in E$ and ad, $bd \notin E$.

Lemma 1 applied to the vertices a, b, c guarantees the existence of a vertex b' with $ab' \notin E$, $bb' \in E$ and $b<b'$. By Lemma 2, $b'c \in E$. However, now $\{a, b, b', c, d\}$ induces a $\bar{P}_5$ or an F, depending on whether or not $b'd \in E$. □

Jamison and Olariu [7] proved that the class of graphs containing no induced subgraph isomorphic to one of the graphs $\bar{P}_5$, C_k ($k \geq 5$) or F in Fig. 1 is *precisely* the class of graphs for which every ordering produced by the algorithm MCS of Tarjan and Yannakakis [15] is a semi-perfect elimination ordering.

Thus, in the terminology of (3) we can write

$$C_{\text{MCS}} = C_{\text{MEC}} = C_{\text{MCC}}.$$

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