Summer 2024

On Weighted Sequence Spaces

Gilbert D. Acheampong
Old Dominion University, acheampongdansogilbert@gmail.com

ON WEIGHTED SEQUENCE SPACES

by

Gilbert D. Acheampong
B.S. July 2013, University for Development Studies, Ghana
M.S. July 2020, The University of Texas at El Paso

A Dissertation Submitted to the Faculty of
Old Dominion University in Partial Fulfillment of the
Requirements for the Degree of

DOCTOR OF PHILOSOPHY

COMPUTATIONAL AND APPLIED MATHEMATICS

OLD DOMINION UNIVERSITY
August 2024

Approved by:

Raymond Cheng (Director)

Gordon Melrose (Member)

Sandipan Dutta (Member)

Tim J. Anderson (Member)

ABSTRACT

ON WEIGHTED SEQUENCE SPACES

Gilbert D. Acheampong
Old Dominion University, 2024
Director: Dr. Raymond Cheng

The space $\ell^{p,\alpha}$ of complex sequences $\mathbf{a} = (a_0, a_1, a_2, \dots)$ for which

$$\|\mathbf{a}\|_{p,\alpha} = \Big(\sum_{k=0}^{\infty} |a_k|^p (k+1)^{\alpha}\Big)^{1/p} < \infty$$

is studied. Each such sequence can be identified with the analytic function with power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

In this setting, the point evaluation and the difference quotient mappings are shown to be bounded; the cases are identified in which $\ell^{p,\alpha}$ is boundedly contained in $\ell^{r,\beta}$. Conditions on the parameters are derived for the analytic functions of $\ell^{p,\alpha}$ to have radial limits almost everywhere on the boundary of the domain, and for $\ell^{p,\alpha}$ to be an algebra. Smoothness properties of the boundary function are investigated. Basic properties of multipliers on $\ell^{p,\alpha}$ are established, and conditions on the multiplier norm and coefficient growth are derived. Multipliers having a certain extremal property are described. A discrete version of the Schur Test is obtained, and used to produce a family of examples of multipliers.

Dedicated to my family, whose love and support have been my greatest strength.

ACKNOWLEDGMENTS

To Elohim be the glory. For in Him we live, and move, and have our being.

I extend my deepest gratitude to my advisor, Professor Raymond Cheng, whose profound expertise in functional spaces has been invaluable throughout this journey. There was never a day that we lost track of the direction of this research, and for that, I am deeply grateful.

I would like to express my sincere appreciation to the members of my defense committee: Professors Melrose, Dutta, and Anderson. Your dedication, time, and thoughtful feedback have significantly contributed to the refinement and completion of this work. I am truly appreciative of your commitment and the expertise you have brought to this process.

My heartfelt thanks go to Old Dominion University and the Department of Mathematics & Statistics for their generous financial support through scholarships and tuition assistance. This support has enabled me to focus on my research without the burden of financial stress.

The sense of community I found within the mathematics departments at both Old Dominion University and The University of Texas at El Paso has been incredibly uplifting. To my colleagues at both institutions, thank you for the countless stimulating discussions and the shared passion for mathematics that have enriched my academic and professional experience over the past six years.

Finally, to my family, both here in the United States and back home in Ghana, your unwavering love, encouragement, and belief in me have been a constant source of strength and inspiration. I dedicate this achievement to you.

NOMENCLATURE

$\mathbb{N}_0$	*Set of non-negative integers*
$\mathbf{a}$	*Collection of complex sequences* $a_0, a_1, a_2, \ldots$
$\ell^{p,\alpha}$	$\mathbf{a} : \left(\sum_{k=0}^{\infty} \lvert a_k\rvert^p (k+1)^\alpha\right)^{1/p} < \infty$
$\lVert \cdot \rVert_{p,\alpha}$	*The norm or quasi-norm on* $\ell^{p,\alpha}$
$:=$	*Equality by definition*
S	*Forward shift operator*
B	*Backward shift operator*
$\mathbb{C}$	*Set of complex numbers*
$\mathbb{D}$	*The open unit disk in* $\mathbb{C}$ *defined as* $\{z \in \mathbb{C} : \lvert z\rvert < 1\}$
$\mathbb{T}$	*The unit circle in* $\mathbb{C}$ *defined as* $\{z \in \mathbb{C} : \lvert z\rvert = 1\}$
Λ_w	*The point evaluation fuctional*
Q_w	*The difference quotient mapping*
q	*Hölder's conjugate of p for which* $1/p + 1/q = 1$
H^p	*Hardy space on* $\mathbb{D}$
D	*Differentiation operator*
Δ_t	*Difference operator with increment t*
$\mathscr{M}_{p,\alpha}$	*Set of multipliers on* $\ell^{p,\alpha}$

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

Functional analysis is a branch of mathematics concerned with the study of spaces of functions with certain properties. These spaces are typically vector spaces, endowed with a metric, norm, or inner product. Of course the functions belonging to such a space are studied: What smoothness properties do they enjoy? Do they have interesting representations? Can their zero sets be characterized? In addition, it is worthwhile to study the space itself: What are its natural subspaces? Can we speak of its dual space? Are there any interesting dense subsets? Finally, as a rule we are interested in investigating operators or mappings on the space, especially those that arise in an organic way.

Function spaces can be rewarding to study as abstract mathematical objects. Moreover, the more we know about a particular class of functions, the more useful it can potentially be, as a tool for mathematical modeling in science, engineering, finance, medicine and so on. For example, the theoretical foundations of the Hardy space H^2 were laid down over a century ago; today, it is still used to model stationary Gaussian stochastic processes, with applications in signal processing and digital filter design [26].

This research focuses on a particular class of weighted sequence spaces, which arise in the following way. Let $0 < p < \infty$ and $\alpha \in \mathbb{R}$. The space $\ell^{p,\alpha}$ is defined to be the collection of complex sequences $\mathbf{a} = (a_0, a_1, a_2, \ldots)$ for which

$$\|\mathbf{a}\|_{p,\alpha} = \Big(\sum_{k=0}^{\infty} |a_k|^p (k+1)^\alpha \Big)^{1/p} < \infty.$$

Each $\ell^{p,\alpha}$ is a vector space, endowed with a norm (if $1 \le p < \infty$), or quasinorm (if $0 < p < 1$). It can be viewed as a special case of the Lebesgue space L^p, where the measure space is the index set $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, and for each $k \in \mathbb{N}_0$ the atom $\{k\}$ has mass $(k+1)^\alpha$.

But $\ell^{p,\alpha}$ is more than a sequence space: each $\mathbf{a} \in \ell^{p,\alpha}$ can be identified with the analytic function with power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

where the complex variable z lies in the open unit disk $\mathbb{D}$. We shall see that there is interplay between the behavior of the sequence (as such) with the properties of the associated analytic function. Furthermore, there will arise some natural operators on the space, including point evaluation, the forward and backward shifts, difference quotients, and multipliers. There will also emerge a broad pattern to the way the behavior of the space $\ell^{p,\alpha}$ varies with the parameters p and α.

Some special cases of $\ell^{p,\alpha}$ have been well studied. The Hardy space H^2 can be identified with $\ell^{2,0} = \ell^2$. There is quite a substantial literature on H^2 and its generalizations and applications, with numerous deep results; it stands as one of the great triumphs of a century of mathematical analysis. Our reference on this subject is the classic tome [8]. Significant advances have also been made to understand the Dirichlet space $\mathscr{D} = \ell^{2,1}$ [1], [11], [29], and the Bergman space $A_2 = \ell^{2,-1}$ [9], [17], [20]. More recently, a body of work concerning the spaces $\ell^p = \ell^{p,0}$ has been brought within a single volume [6], though many fundamental issues remain to be solved.

Elsewhere, there is work identifying upper and lower bounds for certain matrix operators on weighted sequence spaces [15], [16]. In [27], boundedness of point evaluation and cyclicity of polynomials in general weighted sequence spaces are covered; and in [28], the analyticity of formal power series. Further matters relating to cyclicity in weighted sequence spaces are addressed in [2], [14], [22]. A version of the Hardy inequality on weighted sequence spaces is obtained in [18].

Multiplication and composition operators on $\ell^{p,\alpha}$ are the subject of [10]. Notably, the dynamics of composition operators on $\ell^{p,\alpha}$ spaces $(1 \leq p < \infty)$ are explored in [24], [25], proving that no composition operator is hypercyclic on these spaces, and defining conditions for supercyclicity and cyclicity. The paper [13] studies the boundary behavior of $\ell^{p,p-1}$, and convergence of Taylor sums, treating the space as an extension of the classical Dirichlet space $\mathscr{D}$.

In the present work, we endeavor to contribute to the theory of $\ell^{p,\alpha}$ spaces in a systematic manner, culminating in a treatment of the (function) multipliers on these spaces. We first bring together some basic results about $\ell^{p,\alpha}$, including coverage of the shift and backward shift operators, the point evaluation functional, and difference quotient operators. This occupies the next two chapters. In Chapter 4, conditions on the parameters are obtained for the inclusion relation $\ell^{p,\alpha} \subseteq \ell^{r,\beta}$ to hold. Similarly, how $\ell^{p,\alpha}$ relates to the Hardy spaces is covered in Chapter 5. Chapter 6 identifies those cases in which each function in $\ell^{p,\alpha}$ has radial limits almost everywhere on the unit circle. We characterize in Chapter 7 all those $\ell^{p,\alpha}$ which constitute an algebra (under functional multiplication); this identifies cases in which the multiplier space of $\ell^{p,\alpha}$ coincides with the entirety of $\ell^{p,\alpha}$ itself. Next, the spaces $\ell^{p,\alpha}$ are partially characterized by a family of smoothness conditions. This is covered in Chapter 8. A family of algebras is identified by use of these smoothness conditions, which will later shed light on the multiplier space. The concluding section is devoted to multipliers on $\ell^{p,\alpha}$, a natural class of operators to study. We present some basic properties of multipliers, offer some norm and coefficient growth estimates, identify multipliers with a certain extremal property, and produce a family of examples using a discrete version of the Schur Test. Our hope is that these results raise further interest in weighted sequence spaces, and add value to their potential applications.

CHAPTER 2

BASIC PROPERTIES

Let us begin by setting forth some of the basic properties of the $\ell^{p,\alpha}$ spaces. We shall see that they are vector spaces, endowed with a norm or quasinorm. In the former case, the dual space is identified, along with the norming functions and a natural basis. The shift and backward shift operators are defined and shown to be bounded. We acknowledge that some of these results overlap with the papers [14], [27], as well as some other works concerned with weighted shift operators (see, for instance, [23]).

Let $0 < p < \infty$, and $\alpha \in \mathbb{R}$. By $\ell^{p,\alpha}$ we mean the collection of complex sequences $\mathbf{a} = (a_0, a_1, a_2, \ldots)$ such that

$$\|\mathbf{a}\|_{p,\alpha} := \left(\sum_{k=0}^{\infty} |a_k|^p (k+1)^\alpha \right)^{1/p} < \infty.$$

(This notation for the space $\ell^{p,\alpha}$ is not standard, but we shall see that it makes sense to treat the parameter pair (p,α) as a point in the right half-plane). Irrespective of α, we take $\ell^{\infty,\alpha}$ to be the collection of bounded sequences, and define

$$\|\mathbf{a}\|_{\infty,\alpha} := \sup_{k \geq 0} |a_k|.$$

Thus $\ell^{\infty,\alpha}$ coincides with the familiar unweighted space ℓ^∞; for that reason, our focus will be on the finite cases of p.

Remark 2.1. Notice that as p decreases toward zero, or as α increases, the convergence of the expression

$$\sum_{k=0}^{\infty} |a_k|^p (k+1)^\alpha$$

would require the sequence $\mathbf{a} = (a_k)_{k=0}^{\infty}$ to converge toward zero more rapidly. Thus, informally speaking, the members of $\ell^{p,\alpha}$ tend to be better behaved when the parameter pair (p,α) lies in the upper left extreme of the half plane $(0,\infty)\times(-\infty,\infty)$. This is a phenomenon that will play out repeatedly within this investigation.

From the general theory of L^p spaces, we know that $\ell^{p,\alpha}$ is a vector space over the complex scalars. Furthermore, we know that $\|\cdot\|_{p,\alpha}$ is a norm when $1 \le p < \infty$, under which $\ell^{p,\alpha}$ is a Banach space. If $0 < p < 1$, then $\|\cdot\|_{p,\alpha}$ is a quasinorm; in this situation,

$$d(\mathbf{a},\mathbf{b}) := \|\mathbf{a}-\mathbf{b}\|_{p,\alpha}^{p}$$

determines a metric under which $\ell^{p,\alpha}$ is complete. (See, for instance, [6, Chapter 1].) In what follows, convergence and continuity will be in the sense of this metric, when $0 < p < 1$.

If $1 \le p < \infty$, $p \neq 2$, and $1/p + 1/q = 1$, then the dual of $\ell^{p,\alpha}$ can be identified with $\ell^{q,\alpha}$ under the pairing

$$\langle \mathbf{a},\mathbf{b}\rangle := \sum_{k=0}^{\infty} a_k b_k (k+1)^{\alpha}, \tag{1}$$

where $\mathbf{a} = (a_0,a_1,a_2,\ldots) \in \ell^{p,\alpha}$ and $\mathbf{b} = (b_0,b_1,b_2,\ldots) \in \ell^{q,\alpha}$. If $p = 2$, the Hilbert space case, then b_k is replaced by its complex conjugate in the formula (1).

For any $k \ge 0$, let

$$\mathbf{e}_k := (\delta_{0,k},\delta_{1,k},\delta_{2,k},\ldots),$$

where $\delta_{j,k}$ is the Kronecker delta. Then $\{\mathbf{e}_k\}_{k=0}^{\infty}$ is a basis for $\ell^{p,\alpha}$.

Let $1 < p < \infty$, $p \neq 2$. Every nonzero vector $\mathbf{a} \in \ell^{p,\alpha}$ has a unique norming functional given by $T_{\mathbf{a}} : \mathbf{x} \mapsto \langle \mathbf{x}, \mathbf{b} \rangle$, where

$$b_k = \begin{cases} \bar{a}_k |a_k|^{p-2} / \|\mathbf{a}\|_{p,\alpha}^{p-1}, & a_k \neq 0 \\ 0, & a_k = 0 \end{cases}, \; k \geq 0.$$

Indeed, it is straightforward to check that $\|T_{\mathbf{a}}\| = 1$, and $T_{\mathbf{a}}(\mathbf{a}) = \|\mathbf{a}\|_{p,\alpha}$.

In the study of any function space it is of interest to explore the operators that arise in a natural way. Perhaps none is more natural than the shift S and the backward shift B operators. They are defined on any sequence $\mathbf{a} = (a_0, a_1, a_2, \ldots)$ by

$$S(a_0, a_1, a_2, \ldots) := (0, a_0, a_1, a_2, \ldots)$$

$$B(a_0, a_1, a_2, \ldots) := (a_1, a_2, a_3, \ldots).$$

Let us retain the symbols S and B for their restrictions to $\ell^{p,\alpha}$.

Proposition 2.1. *If $0 < p < \infty$, and $\alpha \in \mathbb{R}$, then S and B are bounded linear mappings on $\ell^{p,\alpha}$, with*

$$\sup_{\mathbf{a} \neq \mathbf{0}} \frac{\|S^n \mathbf{a}\|_{p,\alpha}}{\|\mathbf{a}\|_{p,\alpha}} = \begin{cases} (n+1)^{\alpha/p}, & \alpha \geq 0; \\ 1, & \alpha < 0 \end{cases} \tag{2}$$

$$\sup_{\mathbf{a} \neq \mathbf{0}} \frac{\|B^n \mathbf{a}\|_{p,\alpha}}{\|\mathbf{a}\|_{p,\alpha}} = \begin{cases} 1, & \alpha \geq 0; \\ (n+1)^{|\alpha|/p}, & \alpha < 0, \end{cases} \tag{3}$$

for $n \geq 1$.

Proof. The mappings S and B are obviously linear.

If $\mathbf{a} \in \ell^{p,\alpha}$, then

$$\begin{aligned}
\|S^n \mathbf{a}\|_{p,\alpha}^p &= \|S^n(a_0, a_1, a_2, \ldots)\|_{p,\alpha}^p \\
&= \|(\underbrace{0, 0, \ldots, 0}_{n}, a_0, a_1, a_2, \ldots)\|_{p,\alpha}^p \\
&= \sum_{k=0}^{\infty} |a_k|^p (k+1+n)^\alpha \\
&= \sum_{k=0}^{\infty} |a_k|^p (k+1)^\alpha \cdot \left(\frac{k+1+n}{k+1}\right)^\alpha \\
&\leq \sup_{j\geq 0} \left(\frac{j+1+n}{j+1}\right)^\alpha \sum_{k=0}^{\infty} |a_k|^p (k+1)^\alpha \\
&= \sup_{j\geq 0} \left(\frac{j+1+n}{j+1}\right)^\alpha \|\mathbf{a}\|_{p,\alpha}^p.
\end{aligned}$$

This shows that

$$\sup_{\mathbf{a}\neq \mathbf{0}} \frac{\|S^n \mathbf{a}\|_{p,\alpha}}{\|\mathbf{a}\|_{p,\alpha}} \leq \sup_{j\geq 0} \left(\frac{j+1+n}{j+1}\right)^{\alpha/p}.$$

In fact, equality holds since

$$\|S^n \mathbf{e}_j\|_{p,\alpha} = \left(\frac{j+1+n}{j+1}\right)^{\alpha/p} \|\mathbf{e}_j\|_{p,\alpha}$$

for all $j \geq 0$, where $\mathbf{e}_j$ is a basis vector. Now we obtain (2) by observing that

$$\sup_{j\geq 0} \left(\frac{j+1+n}{j+1}\right)^{\alpha/p} = \begin{cases} (n+1)^{\alpha/p}, & \alpha \geq 0; \\ 1, & \alpha < 0. \end{cases}$$

Similarly, to establish (3) we start with

$$\begin{aligned}
\|B^n\mathbf{a}\|_{p,\alpha}^p &= \sum_{k=0}^{\infty} |a_{k+n}|^p (k+1)^\alpha \\
&= \sum_{k=0}^{\infty} |a_{k+n}|^p (k+n+1)^\alpha \cdot \left(\frac{k+1}{k+n+1}\right)^\alpha \\
&\leq \sup_{j\geq 0}\left(\frac{j+1}{j+n+1}\right)^\alpha \sum_{k=0}^{\infty} |a_{k+n}|^p (k+n+1)^\alpha \\
&\leq \sup_{j\geq 0}\left(\frac{j+1}{j+n+1}\right)^\alpha \sum_{k=0}^{\infty} |a_k|^p (k+1)^\alpha \\
&\leq \sup_{j\geq 0}\left(\frac{j+1}{j+n+1}\right)^\alpha \|\mathbf{a}\|_{p,\alpha}^p,
\end{aligned}$$

noting that for $j \geq 0$ we also have

$$\|B^n\mathbf{e}_{j+n}\|_{p,\alpha} = (j+1)^{\alpha/p} = \left(\frac{j+1}{j+n+1}\right)^{\alpha/p} \|\mathbf{e}_{j+n}\|_{p,\alpha}.$$

Therefore

$$\sup_{\mathbf{a}\neq\mathbf{0}} \frac{\|B^n\mathbf{a}\|_{p,\alpha}}{\|\mathbf{a}\|_{p,\alpha}} = \sup_{j\geq 0}\left(\frac{j+1}{j+n+1}\right)^{\alpha/p} = \begin{cases} 1, & \alpha \geq 0; \\ (n+1)^{|\alpha|/p}, & \alpha < 0. \end{cases}$$

as claimed. □

We note that Proposition 2.1 overlaps results contained in [14], [27], as well as some other works concerned with weighted shift operators (see, for instance, [23]).

When $1 \leq p < \infty$, the case $\ell^{p,\alpha}$ is a Banach space, then (2) and (3) give the usual operator norms of S^n and B^n, respectively. Let us abuse notation slightly and use the operator norm notation in the case $0 < p < 1$ as well. That is, for all cases $0 < p < \infty$ and $\alpha \in \mathbb{R}$, if T is a bounded linear mapping on $\ell^{p,\alpha}$, we write

$$\|T\| := \sup_{\mathbf{a}\neq\mathbf{0}} \frac{\|T\mathbf{a}\|_{p,\alpha}}{\|\mathbf{a}\|_{p,\alpha}}.$$

The following proposition shows that $\|\cdot\|$ is well behaved when $0 < p < 1$, even though it is not a norm in the strict sense.

Proposition 2.2. *Let $0 < p < 1$ and $\alpha \in \mathbb{R}$. If T_1 and T_2 are bounded linear mappings on $\ell^{p,\alpha}$, then so are $T_1 + T_2$ and $T_1 T_2$, with*

$$\|T_1 + T_2\| \leq \left(\|T_1\|^p + \|T_2\|^p\right)^{1/p} \text{ and } \|T_1 T_2\| \leq \|T_1\|\|T_2\|.$$

Proof. Linearity of the sum and composite is obvious; the bounds come from

$$\|(T_1 + T_2)\mathbf{a}\|_{p,\alpha}^p = \|T_1\mathbf{a} + T_2\mathbf{a}\|_{p,\alpha}^p \leq \|T_1\mathbf{a}\|_{p,\alpha}^p + \|T_2\mathbf{a}\|_{p,\alpha}^p \leq \|T_1\|^p\|\mathbf{a}\|_{p,\alpha}^p + \|T_2\|^p\|\mathbf{a}\|_{p,\alpha}^p$$

$$\|(T_1T_2)\mathbf{a}\|_{p,\alpha} = \|T_1(T_2\mathbf{a})\|_{p,\alpha} \leq \|T_1\|\|T_2\mathbf{a}\|_{p,\alpha}, \leq \|T_1\|\|T_2\|\|\mathbf{a}\|_{p,\alpha}$$

for any $\mathbf{a} \in \ell^{p,\alpha}$. □

Proposition 2.3. *Let $0 < p < 1$ and $\alpha \in \mathbb{R}$. A linear mapping on $\ell^{p,\alpha}$ is continuous if and only if it is bounded.*

The proof follows that for the corresponding result on a Banach space, *mutatis mutandis*.

In what follows we make repeated use of a basic inequality for the (unweighted) ℓ^p spaces: if $0 < p < r < \infty$, then for any complex sequence $(a_0, a_1, a_2, \ldots)$

$$\left(|a_0|^r + |a_1|^r + |a_2|^r + \cdots\right)^{1/r} \leq \left(|a_0|^p + |a_1|^p + |a_2|^p + \cdots\right)^{1/p}. \tag{4}$$

A recent source is [6, Proposition 1.5.2].

CHAPTER 3

ANALYTIC FUNCTIONS

In this section we see that each element of $\ell^{p,\alpha}$ can be associated with an analytic function, endowed with the (quasi-)norm that it inherits from its coefficient sequence. From this point forward, let us view the members of $\ell^{p,\alpha}$ primarily as analytic functions, rather than sequences. We adhere to the convention that if f is the name of a function, then its coefficient sequence will be $(f_0, f_1, f_2, \ldots)$. We shall see that point evaluation and difference quotients are bounded linear mappings. As usual, $\mathbb{D}$ stands for the open unit disk in the complex plane, and $\mathbb{T}$ is its boundary, the unit circle.

Proposition 3.1. *Let $0 < p < \infty$ and $\alpha \in \mathbb{R}$. If $\mathbf{f} = (f_0, f_1, f_2, \ldots) \in \ell^{p,\alpha}$, then the power series $f(z) := \sum_{k=0}^{\infty} f_k z^k$ converges in $\mathbb{D}$.*

Proof. By hypothesis, the series $\sum_{k=0}^{\infty} |f_k|^p (k+1)^\alpha$ converges. Consequently there is a constant $C > 0$ such that $|f_k| \leq C(k+1)^{-\alpha/p}$ for all $k \geq 0$. Thus the radius of convergence ρ of $f(z)$ is at least unity, as can be seen from

$$1/\rho = \limsup_{k} |f_k|^{1/k} \leq \lim_{k\to\infty} C^{1/k}(k+1)^{-\alpha/(pk)} = 1.$$

□

This result overlaps with [28], which considers more general domains and classes of weights.

As we have seen before with sequences, we will find that as α increases, or p decreases, the functions associated with $\ell^{p,\alpha}$ are “nicer".

The next proposition shows that the act of plugging a point into a function in $\ell^{p,\alpha}$ constitutes a bounded linear operator.

For $w \in \mathbb{D}$ we define the point evaluation functional Λ_w on $\ell^{p,\alpha}$ by

$$\Lambda_w(f) = f(w).$$

Proposition 3.2. *Let* $0 < p < \infty$ *and* $\alpha \in \mathbb{R}$*. For any* $w \in \mathbb{D}$*, the point evaluation functional* Λ_w *is a bounded linear mapping on* $\ell^{p,\alpha}$*.*

Proof. The linearity of Λ_w is obvious. Let $f \in \ell^{p,\alpha}$.

If $1 < p < \infty$, and $1/p + 1/q = 1$, then Hölder's inequality gives

$$\begin{aligned}
|\Lambda_w(f)| &= |f(w)| \\
&= \Big| \sum_{k=0}^{\infty} f_k w^k \Big| \\
&= \Big| \sum_{k=0}^{\infty} f_k (k+1)^{\alpha/p} (k+1)^{-\alpha/p} w^k \Big| \\
&\leq \Big(\sum_{k=0}^{\infty} |f_k|^p (k+1)^{\alpha} \Big)^{1/p} \Big(\sum_{k=0}^{\infty} (k+1)^{-\alpha q/p} w^{kq} \Big)^{1/q} \\
&= \|f\|_{p,\alpha} \Big(\sum_{k=0}^{\infty} (k+1)^{-\alpha q/p} w^{kq} \Big)^{1/q}.
\end{aligned}$$

The exponentially decaying factor dominates in the final series expression.

If $0 < p \leq 1$, then by (4) we have

$$\begin{aligned} |\Lambda_w(f)| &= |f(w)| \\ &= \Big| \sum_{k=0}^{\infty} f_k w^k \Big| \\ &\leq \sum_{k=0}^{\infty} |f_k| |w|^k \\ &= \sum_{k=0}^{\infty} |f_k| (k+1)^{\alpha/p} (k+1)^{-\alpha/p} |w|^k \\ &\leq \Big(\sum_{k=0}^{\infty} |f_k|^p (k+1)^{\alpha} (k+1)^{-\alpha} |w|^{kp} \Big)^{1/p} \\ &\leq \|f\|_{p,\alpha} \sup_{k \geq 0} (k+1)^{-\alpha/p} |w|^k. \end{aligned}$$

The factor $|w|^k$ ensures that the supremum is finite. □

The case $1 \leq p < \infty$ was previously covered in [27].

For an analytic function f on $\mathbb{D}$ and a point $w \in \mathbb{D}$, the difference quotient mapping Q_w is given by

$$Q_w(f)(z) := \frac{f(z) - f(w)}{z - w}.$$

The resulting function is again analytic in $\mathbb{D}$, a zero of the numerator having been removed.

More can be said in the context of $\ell^{p,\alpha}$.

Proposition 3.3. *Let $0 < p < \infty$ and $\alpha \in \mathbb{R}$. If $w \in \mathbb{D}$, then Q_w is a bounded linear mapping on $\ell^{p,\alpha}$.*

Proof. Once again, linearity is trivial to check.

We claim that

$$Q_w = \sum_{k=0}^{\infty} w^k B^{k+1},$$

where B is the backward shift. The right side represents a bounded mapping, since

$$\Big\|\sum_{k=0}^{\infty} w^k B^{k+1}\Big\| \le \begin{cases} \sum_{k=0}^{\infty} |w|^k \|B^{k+1}\|, & 1 \le p < \infty; \\ \left(\sum_{k=0}^{\infty} |w|^{kp} \|B^{k+1}\|^p\right)^{1/p}, & 0 < p < 1. \end{cases}$$

The exponentially decaying factors dominate both of the sums.

Moreover, for any $n \ge 1$

$$\begin{aligned} \sum_{k=0}^{\infty} w^k B^{k+1} \mathbf{e}_n &= \sum_{k=0}^{n-1} w^k B^{k+1} z^n \\ &= z^{n-1} + wz^{n-2} + \cdots + w^{n-2}z + w^{n-1} \\ &= \frac{z^n - w^n}{z - w} \\ &= Q_w \mathbf{e}_n. \end{aligned}$$

Now extend this equation by linearity and continuity to all of $\ell^{p,\alpha}$. □

This extends [6, Proposition 7.2.1].

From the point of view of analytic functions, the shift operator S could be viewed as multiplication by z, and the backward shift can be identified with Q_0.

For functions with non-negative coefficients, there is a reverse norm inequality.

Proposition 3.4. *Let $1 \le p < \infty$ and $\alpha < 0$. If the coefficients of f and g in $\ell^{p,\alpha}$ are non-negative, then*

$$\|fg\|_{p,\alpha} \ge \|f\|_{p,\alpha} \|g\|_{p,\alpha}.$$

Proof. For $n \ge 1$, consider the function $\Psi(x) = x(n+1-x)$, $1 \le x \le n$. It attains a minimum value at $x = n$. Thus for $1 \leq k \leq n$, we have

$$k(n+1-k) \geq n$$

$$(k+1)(n-k+1) \geq n+1$$

$$(k+1)^{\alpha}(n-k+1)^{\alpha} \leq (n+1)^{\alpha}. \tag{5}$$

Since the coefficients of f and g are non-negative, and $p \geq 1$,

$$\begin{aligned}
\|fg\|_{p,\alpha}^{p} &= \sum_{n=0}^{\infty} \Big| \sum_{k=0}^{n} f_k g_{n-k} \Big|^{p} (n+1)^{\alpha} \\
&\geq \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} |f_k|^{p} |g_{n-k}|^{p} \Big) (n+1)^{\alpha} \\
&\geq \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} |f_k|^{p} |g_{n-k}|^{p} (k+1)^{\alpha} (n-k+1)^{\alpha} \Big) \qquad (6) \\
&= \|f\|_{p,\alpha}^{p} \|g\|_{p,\alpha}^{p}.
\end{aligned}$$

The inequality (5) was used to obtain (6). $\square$

CHAPTER 4

INCLUSIONS

The main theorem of this chapter gives exact conditions on the parameters for the inclusion relation $\ell^{p,\alpha} \subseteq \ell^{r,\beta}$ to hold. Its proof will be built from a set of lemmas, handling the different cases as r,β varies through the right half-plane. The inclusion mappings will turn out to be bounded.

Previously we have argued that the members of $\ell^{p,\alpha}$ are nicer for small p and larger α. The following assertion bears this out.

Theorem 4.1. *Let $0 < p < \infty$ and $\alpha \in \mathbb{R}$. If $0 < r < \infty$, and $\beta \in \mathbb{R}$, then $\ell^{p,\alpha} \subseteq \ell^{r,\beta}$ if and only if the point (r,β) satisfies the condition*

$$\beta < -1 + (\alpha+1)r/p, \text{ if } 0 < r < p, \text{ or} \tag{7}$$

$$\beta \leq \alpha r/p, \text{ if } p \leq r < \infty. \tag{8}$$

In either case the inclusion mapping is bounded.

The shaded region in FIGURE 1 identifies the pairs (r,β) such that $\ell^{p,\alpha} \subseteq \ell^{r,\beta}$. We could interpret FIGURE 1 has telling us that the elements of $\ell^{p,\alpha}$ are nicer as the pair (p,α) lies higher and further to the left in the parameter set, in a manner consistent with Remark 2.1.

The horizontal axis represents the parameter r in $\ell^{r,\beta}$ which is a positive number; the vertical axis represents β, which can be any real number. Let the pair (p,α) in the half-plane be specified. Put a solid line from (p,α) to the origin; put a dashed line from (p,α) to the point $(0,-1)$. These two lines form the boundary of the shaded region. Then for all pairs (r,β) in the shaded region, we have $\ell^{p,\alpha} \subseteq \ell^{r,\beta}$. This makes sense because the more we go up and left in the diagram, the nicer the functions are.

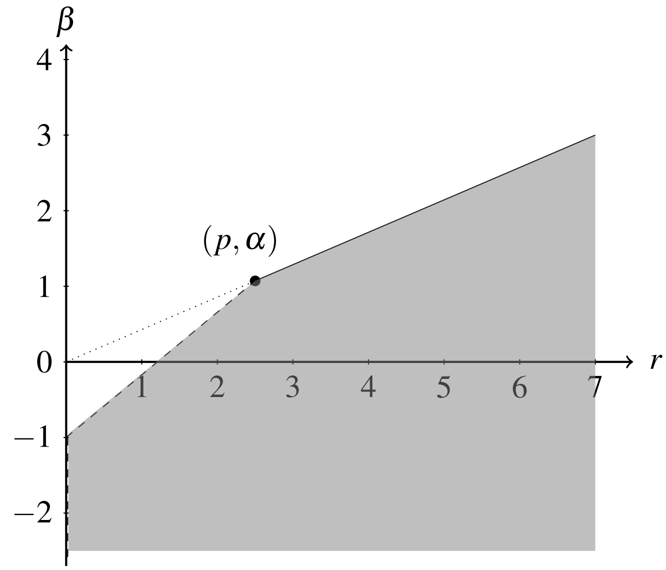

Figure 1. $\ell^{p,\alpha} \subseteq \ell^{r,\beta}$ exactly when (r,β) lies in the shaded region.

This theorem will be established via a set of lemmas.

Lemma 4.1. *Let* $0 < p < \infty$ *and* $\alpha \in \mathbb{R}$. *If* $r > p$, *then* $\ell^{p,\alpha} \subseteq \ell^{r,\alpha r/p}$.

Proof. By (4), we find that for any $f \in \ell^{p,\alpha}$ we have

$$\begin{aligned}
\|f\|_{p,\alpha} &= \Big(\sum_{k=0}^{\infty} |f_k|^p (k+1)^{\alpha} \Big)^{1/p} \\
&= \Big(\sum_{k=0}^{\infty} |f_k (k+1)^{\alpha/p}|^p \Big)^{1/p} \\
&\geq \Big(\sum_{k=0}^{\infty} |f_k (k+1)^{\alpha/p}|^{p \cdot r/p} \Big)^{(1/p)\cdot(p/r)} \\
&= \Big(\sum_{k=0}^{\infty} |f_k|^r (k+1)^{\alpha r/p} \Big)^{1/r} \\
&= \|f\|_{r,\alpha r/p}
\end{aligned}$$

□

Lemma 4.2. *Let* $0 < p < \infty$ *and* $\alpha \in \mathbb{R}$*. If* $\beta < \alpha$*, then* $\ell^{p,\alpha} \subseteq \ell^{p,\beta}$.

Proof. For any $f \in \ell^{p,\alpha}$, the comparison test gives

$$\|f\|_{p,\alpha} = \Big(\sum_{k=0}^{\infty} |f_k|^p (k+1)^{\alpha} \Big)^{1/p} \geq \Big(\sum_{k=0}^{\infty} |f_k|^p (k+1)^{\beta} \Big)^{1/p} = \|f\|_{p,\beta}.$$

□

Together, Lemmas 4.1 and 4.2 prove the sufficiency of (8). We check by inspection that the inclusion mappings have unit norm.

The next Lemma verifies the sufficiency of (7).

Lemma 4.3. *Let* $0 < p < \infty$ *and* $\alpha \in \mathbb{R}$*. If* $0 < r < p$*, and* $\beta \in \mathbb{R}$*, and* (r, β) *satisfies the condition*

$$\beta < -1 + (\alpha+1)r/p,$$

then $\ell^{p,\alpha} \subseteq \ell^{r,\beta}$*, and the inclusion mapping is bounded.*

Proof. Let

$$\gamma := -\beta + \alpha r/p.$$

The inequality (4.3) implies that

$$\frac{\gamma p}{p-r} > 1.$$

Next, from

$$\frac{1}{p/r} + \frac{1}{p/(p-r)} = 1$$

we see that p/r and $p/(p-r)$ are conjugate exponents. Thus, with $f \in \ell^{p,\alpha}$, Hölder's inequality yields

$$\begin{aligned}
\sum_{k=0}^{\infty} |f_k|^r (k+1)^{\beta} &= \sum_{k=0}^{\infty} |f_k|^r (k+1)^{\beta+\gamma} \frac{1}{(k+1)^{\gamma}} \\
&\le \Big(\sum_{k=0}^{\infty} |f_k|^{r \cdot p/r} (k+1)^{(\beta+\gamma)p/r} \Big)^{r/p} \Big(\sum_{k=0}^{\infty} \frac{1}{(k+1)^{\gamma p/(p-r)}} \Big)^{(p-r)/p} \\
&= \Big(\sum_{k=0}^{\infty} |f_k|^{p} (k+1)^{\alpha} \Big)^{r/p} \Big(\sum_{k=0}^{\infty} \frac{1}{(k+1)^{\gamma p/(p-r)}} \Big)^{(p-r)/p} \qquad (9) \\
&< \infty,
\end{aligned}$$

showing that $f \in \ell^{r,\beta}$.

Evidently from (9), the inclusion mapping is bounded by no more than the constant

$$\Big(\sum_{k=0}^{\infty} \frac{1}{(k+1)^{\gamma p/(p-r)}} \Big)^{(p-r)/pr}.$$

□

The remaining lemmas furnish counterexamples that establish the necessity of the conditions in Theorem 4.1.

Lemma 4.4. *Let* $0 < p < \infty$ *and* $\alpha \in \mathbb{R}$*. If* $p \le r < \infty$*, and*

$$\beta > \alpha r/p, \tag{10}$$

then $\ell^{p,\alpha} \not\subseteq \ell^{r,\beta}$*.*

Proof. Our strategy will be to exhibit a function $f \in \ell^{p,\alpha}$ for which $f \notin \ell^{r,\beta}$. Let

$$f_k = \begin{cases} (k+1)^{-\beta/r}, & \text{if } k+1 = 2^j \text{ for some } j \ge 0; \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\|f\|_{p,\alpha}^p = \sum_{k=0}^{\infty} |f_k|^p (k+1)^\alpha = \sum_{j=0}^{\infty} \frac{1}{2^{j\beta p/r}} 2^{j\alpha} < \infty,$$

since $\beta p/r > \alpha$.

However,

$$\|f\|_{r,\beta}^r = \sum_{k=0}^{\infty} |f_k|^r (k+1)^\beta = \sum_{j=0}^{\infty} \frac{1}{2^{j\beta r/r}} 2^{j\beta} = \infty.$$

Thus, f belongs to $\ell^{p,\alpha}$, but not $\ell^{r,\beta}$. □

Lemma 4.5. *Let* $0 < p < \infty$ *and* $\alpha \in \mathbb{R}$*. If* $0 < r < p$*, and*

$$\beta > -1 + (\alpha+1)r/p, \tag{11}$$

then $\ell^{p,\alpha} \not\subseteq \ell^{r,\beta}$*.*

Proof. Let $f_k = (k+1)^{-\gamma}$, where $\gamma = (\beta+1)/r$. Then (11) implies

$$\beta + 1 > (\alpha+1)r/p$$

$$\gamma > (\alpha+1)/p$$

$$\gamma p - \alpha > 1.$$

Consequently,

$$\|f\|_{p,\alpha}^p = \sum_{k=0}^{\infty} |f_k|^p (k+1)^{\alpha} = \sum_{k=0}^{\infty} \frac{1}{(k+1)^{\gamma p-\alpha}} < \infty.$$

On the other hand,

$$\|f\|_{r,\beta}^r = \sum_{k=0}^{\infty} |f_k|^r (k+1)^{\beta} = \sum_{k=0}^{\infty} \frac{1}{(k+1)^{\gamma r-\beta}} = \sum_{k=0}^{\infty} \frac{1}{(k+1)} = \infty.$$

□

It remains to handle the critical segment, the dashed oblique boundary line in FIGURE 1.

Lemma 4.6. *Let* $0 < p < \infty$ *and* $\alpha \in \mathbb{R}$. *If* $0 < r < p$, *and*

$$\beta = -1 + (\alpha+1)r/p, \tag{12}$$

then $\ell^{p,\alpha} \not\subseteq \ell^{r,\beta}$.

Proof. Let

$$f_k = \frac{1}{(k+1)^{(\beta+1)/r}[\log(k+2)]^{1/r}}.$$

Using $(\beta+1)/r = (\alpha+1)/p$, we find that

$$\begin{aligned}
\|f\|_{p,\alpha}^p &= \sum_{k=0}^{\infty} |f_k|^p (k+1)^{\alpha} \\
&= \sum_{k=0}^{\infty} \frac{1}{(k+1)^{p\cdot(\alpha+1)/p}[\log(k+2)]^{p/r}} (k+1)^{\alpha} \\
&= \sum_{k=0}^{\infty} \frac{1}{(k+1)[\log(k+2)]^{p/r}} \\
&< \infty.
\end{aligned}$$

By contrast,

$$\begin{aligned}
\|f\|_{r,\beta}^r &= \sum_{k=0}^{\infty} |f_k|^r (k+1)^{\beta} \\
&= \sum_{k=0}^{\infty} \frac{1}{(k+1)^{r\cdot(\beta+1)/r}[\log(k+2)]^{r/r}} (k+1)^{\beta} \\
&= \sum_{k=0}^{\infty} \frac{1}{(k+1)\log(k+2)} \\
&= \infty.
\end{aligned}$$

$\square$

CHAPTER 5

RELATION TO HARDY SPACES

For $0 < p < \infty$, the Hardy space H^p consists of those analytic functions f on the open unit disk $\mathbb{D}$ for which

$$\|f\|_{H^p} := \Big(\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \, \frac{d\theta}{2\pi} \Big)^{1/p} < \infty.$$

These spaces are well studied and have numerous applications [8].

Some classical inequalities identify the relationship between the Hardy space H^p and the unweighted sequence spaces ℓ^p. Here, we use the results of the previous section to describe which Hardy spaces are contained inside $\ell^{p,\alpha}$, and vice-versa.

The Hausdorff-Young inequality [8, p. 94] says that if $1 \leq p \leq 2$, and $1/p + 1/q = 1$, then

$$H^p \subseteq \ell^q \tag{13}$$

$$\ell^p \subseteq H^q, \tag{14}$$

with bounded inclusions.

A theorem of Hardy and Littlewood, and its dual [8, pages 95 and 97], state that

$$H^p \subseteq \ell^{p,p-2}, \text{ if } 0 < p \leq 2; \tag{15}$$

$$\ell^{p,p-2} \subseteq H^p, \text{ if } 2 \leq p < \infty. \tag{16}$$

We can extend these results as follows.

Theorem 5.1. *Let $1 \le p \le 2$ and $1/p + 1/q = 1$. If*

$$\alpha < r/q - 1, \ 0 < r < p;$$

$$\alpha \le r/q - 1, \ p \le r \le q; \ or$$

$$\alpha \le 0, \ q < r < \infty,$$

then $H^p \subseteq \ell^{r,\alpha}$.

Proof. The claim follows by applying Theorem 4.1 in conjunction with (13) and (15), except for the critical segment $\alpha = r/q - 1, \ p < r < q$. In that situation, write $\gamma = (2-p)/(q-p)$, and consider the mapping

$$T : \sum_{k=0}^{\infty} a_k z^k \longmapsto \sum_{k=0}^{\infty} (k+1)^{\gamma} a_k z^k,$$

which is densely defined on H^p and any $\ell^{p,\alpha}$. Boundedness of the inclusion $H^p \subseteq \ell^q$ is equivalent to the boundedness of T as a mapping from H^p to $\ell^{q,-\gamma q}$. Similarly, the boundedness of the inclusion $H^p \subseteq \ell^{p,p-2}$ is equivalent to the boundedness of T from H^p to $\ell^{p,-\gamma q}$.

Now apply the Marcinkiewicz Interpolation Theorem [19] to conclude that T is bounded from H^p to $\ell^{r,-\gamma q}$, whenever $p < r < q$. Thus if $f(z) = \sum_{k=0}^{\infty} f_k z^k \in H^p$, this implies that

$$\sum_{k=0}^{\infty} (k+1)^{-(2-p)q/(q-p)} \left| (k+1)^{(2-p)/(q-p)} f_k \right|^r = \sum_{k=0}^{\infty} (k+1)^{-(2-p)(q-r)/(q-p)} |f_k|^r < \infty;$$

that is, $H^p \subseteq \ell^{r,\alpha}$ for any (r,α) lying on the critical segment. $\square$

Theorem 5.2. *Let $2 \le p < \infty$ and $1/p + 1/q = 1$. If*

$$\alpha > 0, \ 0 < r < q;$$

$$\alpha \ge (p-2)r/(p-q) - 1, \ q \le r \le p; \ or$$

$$\alpha > (p-2)r/(p-q) - 1, \ p < r < \infty,$$

then $\ell^{r,\alpha} \subseteq H^p$.

Proof. The assertion arises from Theorem 4.1 together with (14) and (16), apart from the case $\alpha \geq (p-2)r/(p-q)-1$, $q<r<p$. On this critical segment, apply the Marcinkiewicz Interpolation Theorem, with

$$T: \sum_{k=0}^{\infty} b_k z^k \longmapsto \sum_{k=0}^{\infty} b_k (k+1)^{-\zeta} z^k,$$

where $\zeta = (p-2)/(p-q)$. The result is that T is a bounded mapping from $\ell^{r,-\zeta q}$ to H^p for all $q<r<p$. Thus, $\ell^{r,\alpha} \subseteq H^p$ when (r,α) lies on the critical segment. $\square$

These inclusions enable us to draw further inferences about the elements of $\ell^{p,\alpha}$. For example, under the conditions of Theorem 5.2, the zero sets of functions belonging to $\ell^{r,\alpha}$ must be Blaschke sequences [8, Theorem 2.4]. Furthermore, such functions have radial limits almost everywhere on the circle $\mathbb{T}$. We shall pursue the matter of radial limits further in the next chapter.

A similar course could be pursued with the Bergman spaces, A_p. See [9, Pages 81–83].

CHAPTER 6

RADIAL LIMITS

A function f associated with $\ell^{p,\alpha}$ is analytic in the open unit disk of the complex plane. Thus, it is defined on the interior of the circular region, but not on the boundary. But if f is nice enough, it might be possible to take limits as we tend toward the boundary along the radial lines. We will see which of the spaces $\ell^{p,\alpha}$ are sufficiently nice that all of the members have radial limits.

Let f be analytic on $\mathbb{D}$. We say that f has radial limits a.e. on $\mathbb{T}$ if

$$f(e^{i\theta}) := \lim_{r\to 1-} f(re^{i\theta})$$

exists for almost every $\theta \in [0,2\pi)$ in the sense of Lebesgue measure. If this occurs, we may then speak of the boundary function $f(e^{i\theta})$ that is associated with the analytic function $f(z)$. It will be our convention to use the same letter to denote the boundary function. The main theorem of this section identifies the parameter values (p,α) for which all members of $\ell^{p,\alpha}$ have radial limits a.e. on $\mathbb{T}$. Again, we see that this occurs when p is small and α is large, in some combination.

Later, in Chapter 8, we examine the radial limit functions for their smoothness properties.

Theorem 6.1. *Let $0 < p < \infty$ and $\alpha \in \mathbb{R}$. Every element of $\ell^{p,\alpha}$ has radial limits a.e. on $\mathbb{T}$ if and only if*

$$0 < p < 2 \;\; \textit{and} \;\; \alpha \geq 0; \;\; \textit{or} \tag{17}$$

$$2 \leq p < \infty \;\; \textit{and} \;\; \alpha > (p-2)/2. \tag{18}$$

Proof. Sufficiency follows from Theorem 4.1, as either condition (17) or (18) implies that $\ell^{p,\alpha} \subseteq \ell^{2,0} = H^2$.

Necessity will be obtained through a set of lemmas below. □

Every element of $\ell^{p,\alpha}$ has radial limits a.e. on $\mathbb{T}$ if and only if the point (p,α) lies in the shaded region of FIGURE 2. Yet again, the observation from Remark 2.1 is born out: The functions of $\ell^{p\alpha}$ have radial limits precisely when(p,α) lies sufficiently high and to the left in the parameter space.

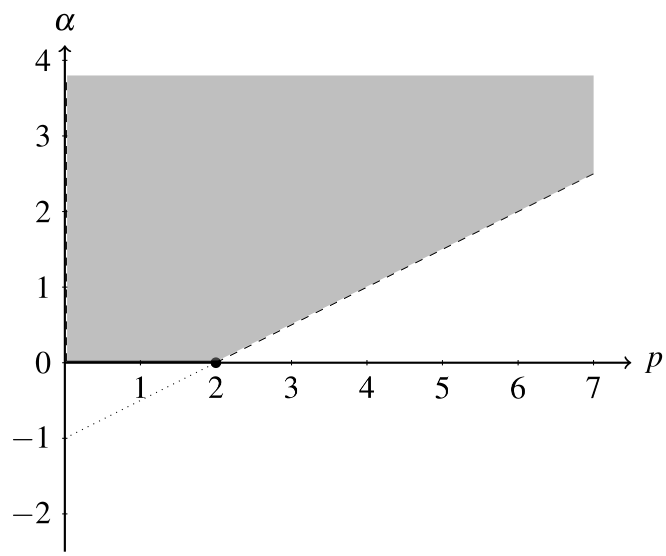

Figure 2. All members of $\ell^{p,\alpha}$ have radial limits a.e. exactly when (p,α) lies in the shaded region.

In the diagram, if the pair (p,α) belongs to the shaded region, then all of the members have radial limits almost everywhere on the boundary. Notice that this happens for pairs that are sufficiently far up and to the left, as previously expected.

The forthcoming counterexamples rely on a theorem of Littlewood. Here is a version of it that suits our purpose (see, for instance, [6, Proposition 6.5.2]).

Theorem 6.2. *Let $(a_0, a_1, a_2, \ldots)$ be a sequence of complex numbers such that*

$$\limsup_{k\to\infty} |a_k|^{1/k} = 1 \quad \text{and} \quad \sum_{k=0}^{\infty} |a_k|^2 = \infty.$$

Then there are infinitely many choices of sign $\varepsilon_k = \pm 1$, $k \geq 0$, such that the function

$$f(z) = \sum_{k=0}^{\infty} \varepsilon_k a_k z^k$$

fails to have radial limits almost everywhere on $\mathbb{T}$.

Lemma 6.1. *Let $2 \leq p < \infty$, and $\alpha < (p-2)/2$. There exist functions $f \in \ell^{p,\alpha}$ such that f fails to have radial limits a.e. on $\mathbb{T}$.*

Proof. Let $f_k = (k+1)^{-1/2}$. Then the hypotheses of Theorem 6.2 are met, and so f fails to have radial limits a.e. In addition,

$$\|f\|_{p,\alpha}^p = \sum_{k=0}^{\infty} \left| \frac{1}{(k+1)^{1/2}} \right|^p (k+1)^{\alpha} = \sum_{k=0}^{\infty} \frac{1}{(k+1)^{(p/2)-\alpha}}.$$

The series converges, since the hypothesis $\alpha < (p-2)/2$ implies that $1 < (p/2) - \alpha$, which places f in $\ell^{p,\alpha}$. □

Lemma 6.2. *Let $2 < p < \infty$, and $\alpha = (p-2)/2$. There exist functions $f \in \ell^{p,\alpha}$ such that f fails to have radial limits a.e. on $\mathbb{T}$.*

Proof. Let $f_k = (k+1)^{-1/2}[\log(k+2)]^{-1/2}$. Again, the hypotheses of Theorem 6.2 are met, and so f fails to have radial limits a.e. Furthermore,

$$
\begin{aligned}
\|f\|_{p,\alpha}^{p} &= \sum_{k=0}^{\infty} \left| \frac{1}{(k+1)^{1/2}[\log(k+2)]^{1/2}} \right|^{p} (k+1)^{\alpha} \\
&= \sum_{k=0}^{\infty} \frac{1}{(k+1)^{(p/2)-\alpha}[\log(k+2)]^{p/2}} \\
&= \sum_{k=0}^{\infty} \frac{1}{(k+1)[\log(k+2)]^{p/2}} \\
&< \infty.
\end{aligned}
$$

□

Lemma 6.3. *Let $0 < p < 2$ and $\alpha < 0$. There exist functions $f \in \ell^{p,\alpha}$ such that f fails to have radial limits a.e. on $\mathbb{T}$.*

Proof. Let N be a positive integer such that $N\alpha < -1$. Define

$$
a_k = \begin{cases} 1, \ \text{if} \ k+1 = j^N & \text{for some} \ \ j \geq 1; \\ (k+1)^{-1/pa}, & \text{otherwise.} \end{cases}
$$

Then once again Theorem 6.2 applies. It remains to check that

$$
\begin{aligned}
\sum_{k=0}^{\infty} |a_k|^p (k+1)^{\alpha} &\leq \sum_{j=0}^{\infty} |1|^p j^{\alpha N} + \sum_{k=0}^{\infty} \frac{1}{(k+1)} (k+1)^{\alpha} \\
&= \sum_{j=0}^{\infty} \frac{1}{j^{|\alpha| N}} + \sum_{k=0}^{\infty} \frac{1}{(k+1)^{1+|\alpha|}} \\
&< \infty.
\end{aligned}
$$

□

CHAPTER 7

ALGEBRAS

We say that $\ell^{p,\alpha}$, viewed as a space of analytic functions, is an algebra if it is closed under multiplication (in addition to being a vector space). It has long been established that $\ell^p = \ell^{p,0}$ is an algebra when $0 < p \leq 1$ (for an exposition, see [6, Section 6.6]). Here our main theorem characterizes those pairs (p,α) for which $\ell^{p,\alpha}$ is an algebra. In such cases, multiplication satisfies a norm inequality

$$\|fg\|_{p,\alpha} \leq c\|f\|_{p,\alpha}\|g\|_{p,\alpha}$$

for some constant c depending only on p and α. Its proof relies on a set of lemmas that complete the chapter.

Knowing which of the spaces $\ell^{p,\alpha}$ are algebras will inform our work in Section 9 on multipliers.

Theorem 7.1. *Let $0 < p < \infty$ and $\alpha \in \mathbb{R}$. The space $\ell^{p,\alpha}$ is an algebra if and only if*

$$0 < p \leq 1, \text{ and } \alpha \geq 0; \text{ or} \tag{19}$$

$$1 < p < \infty, \text{ and } \alpha > p-1. \tag{20}$$

If $\alpha \geq 0$, then $\ell^{1,\alpha}$ is a Banach algebra; if (20) holds, then $\ell^{p,\alpha}$ is a Banach algebra.

The pairs (p,α) for which $\ell^{p,\alpha}$ is an algebra lie in the shaded region of FIGURE 3. Yet again, we see that the property of being an algebra favors the parameter pairs (p,α) that are higher and toward the left of the half plane.

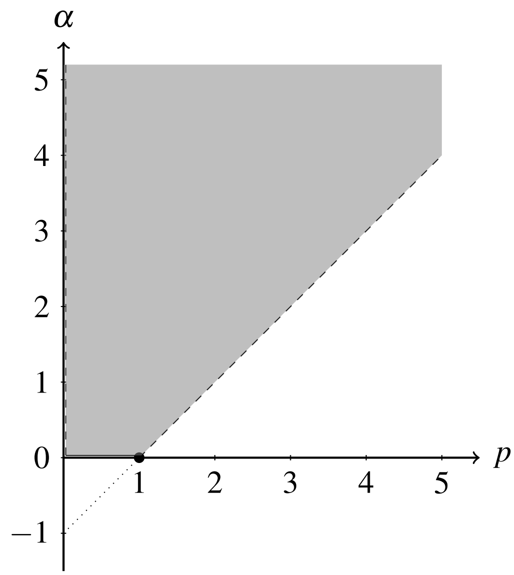

Figure 3. $\ell^{p,\alpha}$ is an algebra exactly when (p,α) lies in the shaded region.

Lemma 7.1. *Let $0 < p \le 1$. If $\alpha \ge 0$, then $\ell^{p,\alpha}$ is an algebra. Moreover, $\ell^{1,\alpha}$ is a Banach algebra.*

Proof. Let f and g be members of $\ell^{p,\alpha}$. Then by (4), we have

$$\begin{aligned}
\|fg\|_{p,\alpha}^p &= \sum_{n=0}^{\infty}\Big|\sum_{k=0}^{n} f_k g_{n-k}\Big|^p (n+1)^\alpha \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} |f_k|^p |g_{n-k}|^p (n+1)^\alpha \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} |f_k|^p (k+1)^\alpha |g_{n-k}|^p (n-k+1)^\alpha \frac{(n+1)^\alpha}{(k+1)^\alpha (n-k+1)^\alpha} \\
&\le \|f\|_{p,\alpha}^p \|g\|_{p,\alpha}^p \cdot \sup_{0\le k\le n} \frac{(n+1)^\alpha}{(k+1)^\alpha (n-k+1)^\alpha}. \qquad (21)
\end{aligned}$$

By elementary calculus, the function $\Phi(x) := (x+1)(n-x+1)$ on the interval $[0,n]$ is critical at $x=0$, $x=n/2$ and $x=n$, with its minimum occurring at the endpoints 0 and n. Thus, with $\alpha \ge 0$, the supremum in (21) is uniformly bounded by 1 for all k and n. In particular, we see that $\ell^{1,\alpha}$ is a Banach algebra. □

This proves sufficiency of condition (19). Note that when $0 < p < 1$ and $\alpha \geq 0$, $\ell^{p,\alpha}$ is not a Banach algebra in the strict sense (it is not a Banach space), but multiplication is bounded in norm by a constant: $\|fg\|_{p,\alpha} \leq c\|f\|_{p,\alpha}\|g\|_{p,\alpha}$.

The next lemma handles sufficiency of the condition (20). The method used here was previously described in [27]. Here and elsewhere we adopt a typical convention: within a chain of estimates, the letter C will denote an evolving constant, the precise value of which is not important for the conclusion.

Lemma 7.2. *Let $1 < p < \infty$. If $\alpha > p-1$, then $\ell^{p,\alpha}$ is a Banach algebra.*

Proof. Let f and g belong to $\ell^{p,\alpha}$. Then with $1/p + 1/q = 1$, Hölder's inequality says

$$\begin{aligned}
\|fg\|_{p,\alpha}^p &= \sum_{n=0}^{\infty} \Big| \sum_{k=0}^{n} f_k g_{n-k} \Big|^p (n+1)^\alpha \\
&= \sum_{n=0}^{\infty} \Big| \sum_{k=0}^{n} f_k (k+1)^{\alpha/p} g_{n-k} (n-k+1)^{\alpha/p} \Big[\frac{(n+1)}{(k+1)(n-k+1)} \Big]^{\alpha/p} \Big|^p \\
&\leq \sum_{n=0}^{\infty} \Big\{ \sum_{k=0}^{n} |f_k|^p (k+1)^\alpha |g_{n-k}|^p (n-k+1)^\alpha \Big\}^{p/p} \Big\{ \sum_{j=0}^{n} \Big[\frac{(n+1)}{(k+1)(n-k+1)} \Big]^{\alpha q/p} \Big\}^{p/q} \\
&\leq \sum_{n=0}^{\infty} \Big\{ \sum_{k=0}^{\infty} |f_k|^p (k+1)^\alpha |g_n|^p (n+1)^\alpha \Big\} \cdot \sup_{N \geq 0} \Big\{ \sum_{j=0}^{N} \Big[\frac{(N+1)}{(k+1)(N-k+1)} \Big]^{\alpha q/p} \Big\}^{p/q} \\
&= \|f\|_{p,\alpha}^p \|g\|_{p,\alpha}^p \cdot \sup_{N \geq 0} \Big\{ \sum_{j=0}^{N} \Big[\frac{(N+1)}{(k+1)(N-k+1)} \Big]^{\alpha q/p} \Big\}^{p/q}. \qquad (22)
\end{aligned}$$

Thus we are done if the supremum in (22) is finite.

Write $s := \alpha q/p$. The sum in (22) is comparable to the integral

$$\begin{aligned}
\int_{x=1}^{x=N} \Big[\frac{N+1}{x(N+1-x)} \Big]^s dx &= \int_{x=1}^{x=N} \Big[\frac{1}{x} + \frac{1}{N+1-x} \Big]^s dx \\
&\leq C \int_{x=1}^{x=N} \Big[\frac{1}{x^s} + \frac{1}{(N+1-x)^s} \Big] dx \\
&\leq \frac{2C}{s-1} \Big[1 - \frac{1}{N^{s-1}} \Big].
\end{aligned}$$

This is bounded if $s > 1$. This shows that $\ell^{p,\alpha}$ is a Banach algebra if $\alpha q/p > 1$, or equivalently, $\alpha > p-1$. $\square$

In fact, under the conditions of Lemma 7.2, the elements of $\ell^{p,\alpha}$ have absolutely summable coefficients.

Proposition 7.1. *If $1 < p < \infty$, and $\alpha > p-1$, then $\ell^{p,\alpha}$ is boundedly contained in ℓ^1.*

Proof. Let $f \in \ell^{p,\alpha}$ and $1/p + 1/q = 1$. Then

$$\begin{aligned}\sum_{k=0}^{\infty} |f_k| &= \sum_{k=0}^{\infty} |f_k|(k+1)^{\alpha/p}(k+1)^{-\alpha/p} \\ &\le \Big(\sum_{k=0}^{\infty} |f_k|^p (k+1)^{\alpha}\Big)^{1/p} \Big(\sum_{k=0}^{\infty} \frac{1}{(k+1)^{\alpha q/p}}\Big)^{1/q} \\ &= \|f\|_{p,\alpha}^{p} \Big(\sum_{k=0}^{\infty} \frac{1}{(k+1)^{\alpha/(p-1)}}\Big)^{1/q}.\end{aligned}$$

The second factor converges since $\alpha/(p-1) > 1$. It serves as a bound for the inclusion mapping of $\ell^{p,\alpha}$ into ℓ^1. $\square$

It remains to prove the necessity implication of Theorem 7.1. This will be accomplished by different methods, depending on the region of the parameter space.

Lemma 7.3. *Let $1 < p < \infty$. If $0 \le \alpha < p-1$, then $\ell^{p,\alpha}$ is not an algebra.*

Proof. Since $\alpha < p-1$, we have

$$\frac{\alpha+1}{p} = \frac{2\alpha+2}{2p} < \frac{p-1+\alpha+2}{2p} = \frac{p+\alpha+1}{2p}.$$

Choose

$$\beta := \frac{p+\alpha+1}{2p},$$

and define f via the coefficients

$$f_k = \frac{1}{(k+1)^\beta}, \ k \geq 0.$$

Then

$$\|f\|_{p,\alpha}^p = \sum_{k=0}^\infty |f_k|^p (k+1)^\alpha = \sum_{k=0}^\infty \frac{1}{(k+1)^{\beta p - \alpha}} < \infty,$$

since $\beta p - \alpha > 1$. We may conclude that $f \in \ell^{p,\alpha}$.

On the other hand, crude estimates yield

$$\begin{aligned}
\|f^2\|_{p,\alpha}^p &= \sum_{n=0}^\infty \Big| \sum_{k=0}^n f_k f_{n-k} \Big|^p (n+1)^\alpha \\
&= \sum_{n=0}^\infty \Big| \sum_{k=0}^n \frac{1}{(k+1)^\beta (n-k+1)^\beta} \Big|^p (n+1)^\alpha \\
&\geq \sum_{n=0}^\infty \Big[(n+1) \cdot \frac{1}{(n/2)^{2\beta}} \Big]^p (n+1)^\alpha \\
&\geq C \sum_{n=0}^\infty \frac{1}{(n+1)^{2\beta p - p - \alpha}} \\
&= C \sum_{n=0}^\infty \frac{1}{(n+1)} \\
&= \infty.
\end{aligned}$$

That is, f^2 fails to belong to $\ell^{p,\alpha}$, which therefore fails to be an algebra. □

To extend the necessity result to the case $\alpha < 0$, we need the following general fact about Banach algebras.

Proposition 7.2. *If $\mathscr{X}$ is a Banach algebra of analytic functions on $\mathbb{D}$ such that point evaluation at any point of $\mathbb{D}$ is bounded, then $\mathscr{X}$ consists of bounded functions.*

Proof. Let $f \in \mathscr{X}$. For any positive integer n, and any $w \in \mathbb{D}$,

$$|f(w)|^n = |\Lambda_w(f^n)| \leq \|\Lambda_w\| \|f^n\|_{\mathscr{X}} \leq \|\Lambda_w\| \|f\|_{\mathscr{X}}^n.$$

Hence $|f(w)| \le \|\Lambda_w\|^{1/n}\|f\|_{\mathscr{X}}$. Take $n \to \infty$ to complete the verification. $\square$

Proposition 7.3. *If $0 < p < \infty$, and $\alpha < 0$, then $\ell^{p,\alpha}$ contains unbounded functions.*

Proof. For all indices k, define

$$f_k = \begin{cases} 2^{j|\alpha|/p}, & \text{if } k = 3^j \text{ for some } j \ge 1; \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\|f\|_{p,\alpha}^p = \sum_{k=0}^{\infty} |f_k|^p (k+1)^\alpha = \sum_{j=1}^{\infty} (2^{j|\alpha|/p})^p (3^j+1)^\alpha \le \sum_{j=1}^{\infty} \left(\frac{2}{3}\right)^{j|\alpha|} < \infty.$$

Next, for $0 < r < 1$,

$$f(r) = \sum_{k=0}^{\infty} f_k r^k = \sum_{j=1}^{\infty} 2^{j|\alpha|/p} r^{3^j}.$$

This is unbounded as r increases to 1. $\square$

Here is an immediate consequence.

Lemma 7.4. *If $1 \le p < \infty$, and $\alpha < 0$, then $\ell^{p,\alpha}$ fails to be an algebra.*

Next, we handle the critical line on which $\alpha = p - 1$.

Lemma 7.5. *If $1 < p < \infty$, then $\ell^{p,p-1}$ fails to be an algebra.*

Proof. Let $\gamma = (p+1)/p$ and consider f determined by the coefficients

$$f_k = \frac{1}{(k+1)[\log(k+2)]^\gamma}.$$

Then

$$\begin{aligned}\|f\|_{p,p-1}^p &= \sum_{k=0}^{\infty} |f_k|^p (k+1)^{p-1} \\ &= \sum_{k=0}^{\infty} \frac{(k+1)^{p-1}}{[\log(k+2)]^{\gamma p}(k+1)^p} \\ &= \sum_{k=0}^{\infty} \frac{1}{[\log(k+2)]^{\gamma p}(k+1)}.\end{aligned}$$

Since $\gamma p = (p+1)/2 > 1$, the series converges, and we have $f \in \ell^{p,p-1}$.

On the other hand, routine estimates yield

$$\begin{aligned}\|f^2\|_{p,p-1}^p &= \sum_{n=0}^{\infty} \Big| \sum_{k=0}^{n} f_k f_{n-k} \Big|^p (n+1)^{p-1} \\ &\geq \sum_{n\in 2\mathbb{N}} \Big| 2 \sum_{k=0}^{n/2} f_k f_{n-k} \Big|^p (n+1)^{p-1} \\ &= \sum_{n\in 2\mathbb{N}} \Big| 2 \sum_{k=0}^{n/2} \frac{1}{(k+1)[\log(k+2)]^{\gamma}(n-k+1)[\log(n-k+2)]^{\gamma}} \Big|^p (n+1)^{p-1} \\ &\geq \sum_{n\in 2\mathbb{N}} \left[2 \frac{1}{(1+n/2)^p [\log(2+n/2)]^{\gamma p}} \Big(\sum_{k=0}^{n/2} \frac{1}{(k+1)[\log(k+2)]^{\gamma}} \Big)^p \right] (n+1)^{p-1} \\ &\geq C \sum_{n\in 2\mathbb{N}} \frac{1}{n^p [\log(n+2)]^{\gamma p}} \Big(\int_{x=1}^{n/2} \frac{dx}{x[\log(x+1)]^{\gamma}} \Big)^p (n+1)^{p-1} \\ &= C \sum_{n\in 2\mathbb{N}} \frac{1}{n^p [\log(n+2)]^{\gamma p}} \Big(\frac{1}{1-\gamma} [\log n]^{1-\gamma} \Big)^p (n+1)^{p-1} \qquad (23) \\ &= C \sum_{n\in 2\mathbb{N}} \frac{1}{n \log n} \\ &= \infty,\end{aligned}$$

where in the step (23) we use $\gamma p - (p - \gamma p) = 2\gamma p - p = 1$. □

It remains to treat the case that (p, α) lies in the strip $0 < p < 1$, $\alpha < 0$.

Lemma 7.6. *Let $0 < p < 1$. If $\alpha < 0$, then $\ell^{p,\alpha}$ fails to be an algebra.*

Proof. As before we exhibit a function that lies in $\ell^{p,\alpha}$ whose square does not. Let $N \geq 2$ be an integer such that $N|\alpha| > 1$. Suppose that $0 \leq t < 1$, and define

$$\beta = t\frac{N|\alpha|-1}{p}.$$

With that, take f to have the coefficients

$$f_k = \begin{cases} j^{\beta}, & \text{if } k = j^N \text{ for some } j \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\|f\|_{p,\alpha}^p = \sum_{k=0}^{\infty} |f_k|^p (k+1)^{\alpha} = \sum_{j=1}^{\infty} j^{\beta p} (j^N+1)^{\alpha} < \infty,$$

since $\alpha N + \beta p < -1$. This confirms that $f \in \ell^{p,\alpha}$.

Next, since $0 < p < 1$, $N \geq 2$ and $\alpha < 0$, we have

$$\begin{aligned}
\|f^2\|_{p,\alpha}^p &= \sum_{n=0}^{\infty} \Big| \sum_{m=0}^{n} f_m f_{n-m} \Big|^p (n+1)^{\alpha} \\
&= \sum_{n=0}^{\infty} \Big| \sum_{m=0}^{n} f_m f_{n-m} (n+1)^{\alpha/p} \Big|^p \\
&\geq \Big(\sum_{n=0}^{\infty} \Big[\sum_{m=0}^{n} f_m f_{n-m} (n+1)^{\alpha/p} \Big] \Big)^p \\
&\geq \Big(\sum_{j=1}^{\infty} \sum_{k=0}^{\infty} j^{\beta} k^{\beta} (j^N + k^N)^{\alpha/p} \Big)^p \\
&\geq \Big(\sum_{j=1}^{\infty} \sum_{k=0}^{\infty} j^{\beta} k^{\beta} (j^2 + k^2)^{N\alpha/2p} \Big)^p \\
&\geq C \Big(\int_1^{\infty} \int_1^{\infty} \frac{x^{\beta} y^{\beta}}{(x^2+y^2)^{N|\alpha/2p}} \, dx \, dy \Big)^p \\
&\geq C \Big(\int_1^{\infty} \int_0^{\pi/2} \frac{r^{2\beta} [\cos\theta]^{\beta} [\sin\theta]^{\beta}}{r^{N|\alpha|/p}} \, r \, dr \, d\theta \Big)^p \\
&\geq C \Big(\int_1^{\infty} \frac{dr}{r^{(N|\alpha|/p) - 2\beta - 1}} \Big)^p.
\end{aligned} \tag{24}$$

There are now two cases. First, if $p = 1/2$, then we choose $t = 1/2$, so that

$$\beta = \frac{N|\alpha|-1}{2p}.$$

We then have

$$\begin{aligned}\frac{N|\alpha|}{p} - 2\beta - 1 &= \frac{N|\alpha|}{p} - 2\frac{N|\alpha|-1}{2p} - 1 \\ &= 1.\end{aligned}$$

Hence the integral diverges in (24), and it follows $f^2 \notin \ell^{p,\alpha}$.

In the second case, $p \neq 1/2$. Consider the family of lines

$$y = \frac{2}{N(2t-1)}x - \frac{2t}{N(2t-1)} \tag{25}$$

in the x,y-plane, with t varying from 0 to $1/2$, and from $1/2$ to 1. Their slopes range from $-\infty$ to $-2/N$, and then from $2/N$ to ∞. Each of these lines passes through the point $(1/2, -1/N)$. Thus, if $0 < p < 1$, $p \neq 1/2$, and $\alpha < 0$, we can find an integer $N \geq 2$ with $N|\alpha| > 1$ (as previously required), and a value of t so that the point (p, α) lies on the line (25).

Consequently, for this particular t,

$$\begin{aligned}\alpha &= \frac{2}{N(2t-1)}p - \frac{2t}{N(2t-1)} \\ N|\alpha|(2t-1) &= -2p + 2t \\ \frac{N|\alpha|}{p} &= 2 + 2\left(t\frac{N|\alpha|-1}{p}\right) \\ \frac{N|\alpha|}{p} &= 2 + 2\beta \\ \frac{N|\alpha|}{p} - 2\beta - 1 &= 1.\end{aligned}$$

Once again, the integral diverges in (24). $\square$

CHAPTER 8

SMOOTHNESS

In the special case that f has radial limits a.e., it may happen that the boundary limit function $f(e^{i\theta})$ enjoys certain smoothness properties. Here we show that for some parameter values, the members of $\ell^{p,\alpha}$ are nearly characterized by a sort of mean-square Lipschitz condition on their boundary limit function. This is used to identify another family of Banach algebras at the end of the section.

To prove the main smoothness results, we borrow some methods and ideas from [4], which trace further back to [3]. Accordingly, the smoothness classes of functions that arise here have some independent interest.

If $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is a power series, define

$$\|f\|'_{p,\alpha} := \Big(|a_0|^p + \sum_{n=0}^{\infty} 2^{n\alpha} \sum_{k=2^n}^{2^{n+1}-1} |a_k|^p\Big)^{1/p}.$$

This breaks the infinite series into blocks of length 2^n, and adds a weight $2^{n\alpha}$ to each block. It turns out that the resulting norm is (topologically) equivalent to the norm on $\ell^{p,\alpha}$.

Proposition 8.1. *If $0 < p < \infty$ and $\alpha \in \mathbb{R}$, then $\|\cdot\|'_{p,\alpha}$ is equivalent to $\|\cdot\|_{p,\alpha}$.*

Proof. Let $(a_0, a_1, a_2, \ldots)$ be a complex sequence. Then for $0 < p < \infty$ and $\alpha \geq 0$,

$$\begin{aligned}\sum_{k=0}^{\infty} |a_k|^p (k+1)^\alpha &= |a_0|^p + \sum_{n=0}^{\infty} \sum_{k=2^n}^{2^{n+1}-1} |a_k|^p (k+1)^\alpha \\ &\geq |a_0|^p + \sum_{n=0}^{\infty} 2^{n\alpha} \sum_{k=2^n}^{2^{n+1}-1} |a_k|^p.\end{aligned}$$

Similarly,

$$\sum_{k=0}^{\infty}|a_k|^p(k+1)^{\alpha} \le |a_0|^p + \sum_{n=0}^{\infty} 2^{[n+1]\alpha} \sum_{k=2^n}^{2^{n+1}-1} |a_k|^p$$
$$\le 2^{\alpha}\Big(|a_0|^p + \sum_{n=0}^{\infty} 2^{n\alpha} \sum_{k=2^n}^{2^{n+1}-1} |a_k|^p\Big).$$

The inequalities reverse when $\alpha < 0$, and again we obtain the claimed equivalence. □

We use the alternate norm to derive smoothness conditions associated with $\ell^{p,\alpha}$. Let us express the differentiation operator D and the difference operator Δ_t on the analytic polynomials by

$$De^{ik\theta} = ike^{ik\theta}$$
$$\Delta_t e^{ik\theta} = e^{ik(\theta+t)} - e^{ik\theta}$$

and extending linearly. Thus D is the differentiation operator on square-integrable functions on the unit circle; Δ_t is the first-order difference operator with increment t.

The following theorem asserts that if an analytic function in the disk is sufficiently smooth on the boundary circle, then it belongs to $\ell^{p,\alpha}$.

Theorem 8.1. *Let $1 < p \le 2$ and $\alpha > 0$. Choose $b \in (0,1]$, and integers M and N such that*

$$0 \le M < M+b < M+N, \quad \textit{and}$$
$$\alpha < (M+b+\tfrac{1}{2})p - 1.$$

If for some $c > 0$, the function $f(e^{i\theta}) = \sum_{k=0}^{\infty} a_k e^{ik\theta} \in H^2$ satisfies the smoothness condition

$$\int_0^{2\pi} \left|\Delta_t^N(D^M f)(e^{i\theta})\right|^2 \frac{d\theta}{2\pi} < c|t|^{2b}, \quad -\pi \le t \le \pi, \tag{26}$$

then $f \in \ell^{p,\alpha}$.

To make sense of the condition (26), consider the special case $M = 0$ and $N = 1$. Then the condition is

$$\int_0^{2\pi} \left| f(e^{i(\theta+t)}) - f(e^{i\theta}) \right|^2 \frac{d\theta}{2\pi} < c|t|^{2b};$$

This says that f satisfies a mean-square Lipschitz condition with parameter b. The general case merely says that f is even nicer.

Proof. Parseval's identity says that condition (26) can be expressed as

$$\sum_{k=0}^{\infty} |e^{ikt} - 1|^{2N} k^{2M} |a_k|^2 \leq c|t|^{2b}. \tag{27}$$

We will also need the elementary inequality

$$\frac{2}{\pi} \leq \left| \frac{e^{ix} - 1}{x} \right|, \quad 0 < x \leq \pi. \tag{28}$$

Fix a non-negative integer n, and put $t = 2^{-(n+1)}$. For k satisfying $2^n < k \leq 2^{n+1}$, we have $0 < kt < 1$, and hence (28) holds with $x = kt$. Utilizing this with (27), we find that for any $u > 0$ we have

$$\begin{aligned}
\sum_{k=2^n+1}^{2^{n+1}-1} k^u |a_k|^2 &\leq \sum_{k=2^n+1}^{2^{n+1}-1} k^{u-2M} k^{2M} \left(\frac{\pi}{2}\right)^{2N} \left| \frac{e^{ikt} - 1}{kt} \right|^{2N} |a_k|^2 \\
&\leq \left(\frac{\pi}{2}\right)^{2N} \frac{2^{|u-2M|} 2^{n(u-2M)}}{(2^n+1)^{2N} |t|^{2N}} \sum_{k=2^n+1}^{2^{n+1}-1} k^{2M} |e^{ikt} - 1|^{2N} |a_k|^2 \\
&\leq c \left(\frac{\pi}{2}\right)^{2N} \frac{2^{|u-2M|} 2^{n(u-2M)} |t|^{2b}}{(2^n+1)^{2N} |t|^{2N}} \\
&= c \left(\frac{\pi}{2}\right)^{2N} \frac{2^{|u-2M|} 2^{n(u-2M)} 2^{2(n+1)(N-b)}}{(2^n+1)^{2N}}.
\end{aligned}$$

(Note: the factor $2^{|u-2M|}$ appears in order to manage the possibility that $u - 2M$ is negative.) This expression is summable over n, provided that

$$u - 2M + 2(N-b) - 2N < 0, \quad \text{or} \quad u < 2(M+b).$$

That is, we have shown that the condition $0 < u < 2(M+b)$ implies that

$$\sum_{k=2}^{\infty} k^u |a_k|^2 = \sum_{n=0}^{\infty} \Big(\sum_{k=2^n+1}^{2^{n+1}-1} k^u |a_k|^2 \Big) < \infty. \tag{29}$$

Next, we observe that if $1 < p < 2$, the parameters $2/p$ and $2/(2-p)$ constitute a pair of conjugate exponents. Hölder's inequality then enables us to estimate as follows for any $v > 0$.

$$\begin{aligned} \sum_{k=0}^{\infty} |a_k|^p (k+1)^{\alpha} &= \sum_{k=0}^{\infty} |a_k|^p (k+1)^{\alpha} (k+1)^{vp/2} \cdot \frac{1}{(k+1)^{vp/2}} \\ &\leq \left(\sum_{k=0}^{\infty} |a_k|^2 (k+1)^{v+(2\alpha/p)} \right)^{p/2} \left(\sum_{k=0}^{\infty} (k+1)^{-vp/(2-p)} \right)^{(2-p)/2}. \end{aligned} \tag{30}$$

The second factor is finite if $vp/(2-p) > 1$.

According to the calculation leading up to (29), the first factor in (30) is bounded provided that

$$u = \frac{2\alpha}{p} + v < 2(M+b).$$

By hypothesis, the condition $\alpha < (M+b+\frac{1}{2})p - 1$ holds, which is equivalent to

$$\frac{2\alpha}{p} + \frac{2}{p} - 1 < 2(M+b).$$

Thus we can select

$$v = \frac{2}{p} - 1 + \varepsilon$$

for some $\varepsilon > 0$ sufficiently small, ensuring that the first factor in (30) is bounded.

With this selection, we also have

$$\frac{vp}{2-p} = \frac{2-p+\varepsilon p}{2-p} > 1,$$

and hence the second factor in (30) is also bounded.

If $p = 2$, then we can take $u = \alpha$, and the condition $\alpha < 2(M+b) = 2(M+b+\frac{1}{2}) - 1$ suffices, as claimed. $\square$

A reverse containment is possible when $2 \le p < \infty$.

Theorem 8.2. *Let $2 \le p < \infty$, and let $b \in (0,1]$. Choose integers M and N such that*

$$0 \le M < M+b < M+N.$$

If $f \in \ell^{p,\alpha}$ with $\alpha > p(M+b+\frac{1}{2})-1$, or if $f \in \ell^{2,\alpha}$ with $\alpha \ge 2(M+b)$, then f has boundary values almost everywhere satisfying the smoothness condition

$$\int_0^{2\pi} \left|\Delta_t^N (D^M f)(e^{i\theta})\right|^2 \frac{d\theta}{2\pi} \le c|t|^{2b}$$

for some $c > 0$.

Proof. By hypothesis,

$$\alpha = p(M+b) + \frac{p-2}{2} + \frac{\varepsilon p}{2}$$

for some $\varepsilon > 0$.

In case $2 < p < \infty$, the parameters $2/p$ and $-2/(p-2)$ constitute a pair of conjugate exponents, the second being negative. The Reverse Hölder's Inequality [6, Proposition 1.4.11] then applies as follows for $v = \varepsilon + (p-2)/p > 0$.

$$\begin{aligned}\sum_{k=0}^{\infty} |a_k|^p (k+1)^{\alpha} &= \sum_{k=0}^{\infty} |a_k|^p (k+1)^{\alpha} (k+1)^{-vp/2} \cdot \frac{1}{(k+1)^{-vp/2}} \\ &\ge \left(\sum_{k=0}^{\infty} |a_k|^2 (k+1)^{(2\alpha/p)-v}\right)^{p/2} \left(\sum_{k=0}^{\infty} (k+1)^{-vp/(p-2)}\right)^{-(p-2)/2}.\end{aligned}$$

The second factor is nonzero (i.e., the series converges) since $vp/(p-2) > 1$. Consequently the first factor is finite. Since the quantity $(2\alpha/p) - v$ is positive, the coefficients a_k must be square-summable by comparison; this confirms that f has boundary values almost everywhere.

Suppose that s and n are non-negative integers, $u = 2(M+b) > 0$, and $2^{-(s+1)} < |t| \leq 2^{-s}$. Notice that

$$u = 2(M+b) = \frac{2\alpha}{p} - v.$$

For any $n \leq s$ we have

$$\begin{aligned}\sum_{k=2^n}^{2^{n+1}-1} k^u |a_k|^2 &= \sum_{k=2^n}^{2^{n+1}-1} k^u (k|t|)^{-2N} (k|t|)^{2N} k^{-2M} k^{2M} |a_k|^2 \\ &= \sum_{k=2^n}^{2^{n+1}-1} k^{u-2M-2N} |t|^{-2N} (k|t|)^{2N} k^{2M} |a_k|^2 \\ &\geq \sum_{k=2^n}^{2^{n+1}-1} k^{u-2M-2N} |t|^{-2N} |e^{ikt}-1|^{2N} k^{2M} |a_k|^2.\end{aligned}$$

In the last step we used the fact that $|e^{ix}-1| \leq |x|$ for any real x, in particular $x = k|t|$. Taking into account that $u-2M-2N = 2(b-N) < 0$, we may continue the estimates with

$$\begin{aligned}&\geq 2^{(n+1)(u-2M-2N)} |t|^{-2N} \sum_{k=2^n}^{2^{n+1}-1} |e^{ikt}-1|^{2N} k^{2M} |a_k|^2 \\ &\geq 2^{(s+1)(u-2M-2N)} |t|^{-2N} \sum_{k=2^n}^{2^{n+1}-1} |e^{ikt}-1|^{2N} k^{2M} |a_k|^2 \\ &\geq 2^{(u-2M-2N)} |t|^{-2N-(u-2M-2N)} \sum_{k=2^n}^{2^{n+1}-1} |e^{ikt}-1|^{2N} k^{2M} |a_k|^2,\end{aligned}$$

which gives

$$\sum_{k=2^n}^{2^{n+1}-1} |e^{ikt}-1|^{2N} k^{2M} |a_k|^2 \leq 2^{2M+2N-u} |t|^{u-2M} \sum_{k=2^n}^{2^{n+1}-1} k^u |a_k|^2. \tag{31}$$

For $n > s$ the estimate is similar, except that we use the crude bound $|e^{ikt}-1| \leq 2$. The result is

$$\begin{aligned}\sum_{k=2^n}^{2^{n+1}-1} k^u |a_k|^2 &= \sum_{k=2^n}^{2^{n+1}-1} k^u 2^{-2N} 2^{2N} k^{-2M} k^{2M} |a_k|^2 \\ &\geq \sum_{k=2^n}^{2^{n+1}-1} k^{u-2M} 2^{-2N} |e^{ikt}-1|^{2N} k^{2M} |a_k|^2 \\ &\geq (2^n)^{u-2M} 2^{-2N} \sum_{k=2^n}^{2^{n+1}-1} |e^{ikt}-1|^{2N} k^{2M} |a_k|^2.\end{aligned}$$

Considering that $u-2M>0$, this tells us that

$$\sum_{k=2^n}^{2^{n+1}-1} |e^{ikt}-1|^{2N} k^{2M} |a_k|^2 \le 2^{2N}\left(\frac{1}{2^n}\right)^{u-2M} \sum_{k=2^n}^{2^{n+1}-1} k^u |a_k|^2$$

$$\le 2^{2N} 2^{u-2M} |t|^{u-2M} \sum_{k=2^n}^{2^{n+1}-1} k^u |a_k|^2. \tag{32}$$

It follows from (31) and (32) that, by summing over $n \ge 0$, we can find a constant $c>0$ such that

$$\sum_{k=1}^{\infty} |e^{ikt}-1|^{2N} k^{2M} |a_k|^2 \le c|t|^{u-2M} = c|t|^{2b}. \tag{33}$$

That is, f satisfies the claimed smoothness condition.

If $p=2$, then there is no need to introduce v, and the calculations go through using $u=\alpha \ge 2(M+b)$. □

Near converses of the above two results also hold.

Theorem 8.3. *Let $1<p\le 2$, and let $b\in(0,1]$. Choose integers M and N such that*

$$0 \le M < M+b < M+N.$$

If $f \in \ell^{p,\alpha}$, with $\alpha \ge p(M+b)$, then f satisfies the smoothness condition

$$\int_0^{2\pi} \left|\Delta_t^N (D^M f)(e^{i\theta})\right|^2 \frac{d\theta}{2\pi} \le c|t|^{2b}$$

for some $c>0$.

Proof. We begin with the norm inequality

$$\left(\sum_{k=0}^{\infty} |a_k|^p (k+1)^{\alpha}\right)^{1/p} \ge \left(\sum_{k=0}^{\infty} |a_k|^2 (k+1)^{2\alpha/p}\right)^{1/2}.$$

Next, apply the calculation leading to (33), except that $u = 2\alpha/p$. It remains to check that with this choice of u, the smoothness condition requires that

$$2b \le u - 2M = \frac{2\alpha}{p} - 2M, \quad \text{or}$$

$$\alpha \ge (M+b)p,$$

as claimed. $\square$

Theorem 8.4. *Let $2 \le p < \infty$ and $\alpha > 0$. Choose $b \in (0,1]$, and integers M and N such that*

$$0 \le M < M+b < M+N, \quad \textit{and}$$

$$\alpha \le p(M+b).$$

If for some $c > 0$, the function $f(e^{i\theta}) = \sum_{k=0}^{\infty} a_k e^{ik\theta} \in H^2$ satisfies the smoothness condition

$$\int_0^{2\pi} \left|\Delta_t^N (D^M f)(e^{i\theta})\right|^2 \frac{d\theta}{2\pi} < c|t|^{2b}, \quad -\pi \le t \le \pi, \tag{34}$$

then $f \in \ell^{p,\alpha}$.

Proof. Again, by the basic norm inequality (4) we have

$$\left(\sum_{k=0}^{\infty} |a_k|^p (k+1)^{\alpha}\right)^{1/p} \le \left(\sum_{k=0}^{\infty} |a_k|^2 (k+1)^{2\alpha/p}\right)^{1/2}.$$

The calculation preceding (29) remains applicable, with the identification $u = 2\alpha/p$. Thus the criterion for the (29) to hold in the present case is

$$\frac{2\alpha}{p} \le 2(M+b), \quad \text{or}$$

$$\alpha \le p(M+b).$$

$\square$

Remark 8.1. We can see from Theorems 8.2 and 8.3 that if the pair (p,α) satisfies

$$\alpha > 0,\ 1 < p \leq 2;\ \text{or}$$

$$\alpha > \frac{p-2}{2},\ 2 \leq p < \infty,$$

then every $f \in \ell^{p,\alpha}$ satisfies some smoothness condition of the form (26). This collection of pairs very nearly coincides with the parameter set for which members of $\ell^{p,\alpha}$ have radial limits, the latter differing by the inclusion of the line segment $\{(p,\alpha):\ \alpha = 0,\ 0 < p \leq 2\}$.

When $p = 2$ there is an exact smoothness criterion for membership in $\ell^{p,\alpha}$.

Corollary 8.1. *Let $b \in (0,1]$, let M and N be integers such that*

$$0 \leq M < M+b < M+N,$$

and let $\alpha = 2(M+b)$. There exists $c > 0$ such that the function $f(e^{i\theta}) \in H^2$ satisfies the smoothness condition

$$\int_0^{2\pi} \left|\Delta_t^N (D^M f)(e^{i\theta})\right|^2 \frac{d\theta}{2\pi} < c|t|^{2b}, \quad -\pi \leq t \leq \pi, \tag{35}$$

if and only if $f \in \ell^{p,\alpha}$.

In addition, we get a basket of equivalent norms for $\ell^{2,\alpha}$. Here $\|\cdot\|_{H^2}$ is the norm on the Hardy space H^2.

Corollary 8.2. *Let $b \in (0,1]$, let M and N be integers such that*

$$0 \leq M < M+b < M+N,$$

and let $\alpha = 2(M+b)$. Then the norm given by

$$\|f\|_{[b,M,N]} = \|f\|_{H^2} + \sup_{0<|t|\leq\pi} \frac{\|\Delta_t^N D^M f\|_{H^2}}{|t|^b},$$

is equivalent to $\|\cdot\|_{2,\alpha}$.

Having a set of equivalent norms on a space is extremely useful. For example, the equivalent norm enables us to identify another family of Banach algebras.

Corollary 8.3. *If $\alpha > 0$, then $\ell^{2,\alpha} \cap H^\infty$ is a Banach algebra under the norm*

$$\|f\|_{\ell^{2,\alpha}\cap H^\infty} := \|f\|_{2,\alpha} + \|f\|_{H^\infty}.$$

Proof. When $\alpha > 1$, Theorem 7.1 and Proposition 7.1 tell us that $\ell^{p,\alpha} \cap H^\infty = \ell^{p,\alpha}$ is a Banach algebra. If $0 < \alpha \leq 1$, choose $M = 0$, $N = 1$ and $b = \alpha/2$. Then for any $f, g \in \ell^{2,\alpha} \cap H^\infty$, we have

$$\begin{aligned}
\|\Delta_t(fg)\|_{H^2}^2 &\leq 2\big(\|f\Delta_t g\|_{H^2}^2 + \|g\Delta_t f\|_{H^2}\big) \\
&\leq 2\big(\|f\|_{H^\infty}^2 \|\Delta_t g\|_{H^2} 2^2 + \|g\|_{H^\infty}^2 \|\Delta_t f\|_{H^2}^2\big) \\
&\leq c|t|^{2b}
\end{aligned}$$

for some constant c. In addition, $\|fg\|_{H^2} \leq \|f\|_{H^2}\|g\|_{H^\infty} < \infty$. We may thus conclude that $\|fg\|_{[\alpha/2,0,1]} < \infty$, and hence $fg \in \ell^{2,\alpha}$. $\square$

This extends [11, Theorem 1.3.2], in which the $\alpha = 1$ case is handled.

CHAPTER 9

MULTIPLIERS

The investigation of any function space must include an examination of the operators on that space, particularly those that emerge in an organic way. Multipliers constitute such a class of operators, and we conclude this project by studying the multipliers on $\ell^{p,\alpha}$.

Let $0 < p < \infty$ and $\alpha \in \mathbb{R}$. An analytic function h on $\mathbb{D}$ is a multiplier on $\ell^{p,\alpha}$ if $hf \in \ell^{p,\alpha}$ for all $f \in \ell^{p,\alpha}$. By the Closed Graph Theorem (i.e., a generalized version applicable to functions in a complete metric space), the mapping

$$M_h : f \longmapsto hf$$

is bounded and continuous (it is obviously linear). The set $\mathscr{M}_{p,\alpha}$ of multipliers on $\ell^{p,\alpha}$ is a vector space. To each $h \in \mathscr{M}_{p,\alpha}$ we associate the norm

$$\|h\|_{\mathscr{M}_{p,\alpha}} = \|M_h\| := \sup_{f \neq 0} \frac{\|hf\|_{p,\alpha}}{\|f\|_{p,\alpha}},$$

recognizing that when $0 < p < 1$, $\|\cdot\|_{\mathscr{M}_{p,\alpha}}$ is not an operator norm in the strict sense.

We shall lay out some basic properties of multipliers, then relate the operator norm of a multiplier to the behavior of its coefficients. Next, we describe the multipliers that exhibit a certain extremal property. The chapter concludes with a version of the Schur Test, which is used to extract a family of examples of multipliers. These results extend some of the material in [5], which surveys multipliers on $\ell^p = \ell^{p,0}$.

This paper treats only function multipliers. There are multipliers of other senses throughout the literature. See, for example, [6, page 92] and [8, Section 6.4].

We begin by establishing some basic properties of multipliers.

Proposition 9.1. *If $0 < p < \infty$, and $\alpha \in \mathbb{R}$, then $\mathscr{M}_{p,\alpha} \subseteq \ell^{p,\alpha}$.*

Proof. Since $1 \in \ell^{p,\alpha}$, we have $h = h1 \in \ell^{p,\alpha}$. □

Proposition 9.2. *If $0 < p < \infty$, and $\alpha \in \mathbb{R}$, then the elements of $\mathscr{M}_{p,\alpha}$ are bounded functions.*

Proof. Let $h \in \mathscr{M}_{p,\alpha}$ and $w \in \mathbb{D}$. Then for every $n \geq 1$

$$\begin{aligned} |h(w)|^n &= |\Lambda_w(M_h^n 1)| \\ &\leq \|\Lambda_w\| \cdot \|M_h\|^n \cdot \|1\|_{p,\alpha} \\ |h(w)| &\leq \|\Lambda_w\|^{1/n} \cdot \|M_h\| \cdot \|1\|_{p,\alpha}^{1/n}. \end{aligned}$$

Take $n \longrightarrow \infty$ to complete the proof. □

The difference quotients turn out to be bounded linear mappings on the multiplier space.

Proposition 9.3. *Let $0 < p < \infty$, and $\alpha \in \mathbb{R}$. If $w \in \mathbb{D}$, and $h \in \mathscr{M}_{p,\alpha}$, then $Q_w h \in \mathscr{M}_{p,\alpha}$, and*

$$\|Q_w h\|_{\mathscr{M}_{p,\alpha}} \leq \left(\|Q_w\|^p \|M_h\|^p + \|\Lambda_w\|^p \|h\|_{p,\alpha}^p \|Q_w\|^p\right)^{1/p},\ 0 < p < 1;$$

$$\|Q_w h\|_{\mathscr{M}_{p,\alpha}} \leq \|Q_w\| \cdot \|M_h\| + \|\Lambda_w\| \cdot \|h\|_{p,\alpha} \|Q_w\|,\ 1 \leq p < \infty.$$

Proof. If $f \in \ell^{p,\alpha}$, then

$$\begin{aligned} (Q_w h) f &= \frac{h(z) - h(w)}{z - w} f(z) \\ &= \frac{h(z) f(z) - h(w) f(z)}{z - w} \\ &= \frac{h(z) f(z) - h(w) f(w)}{z - w} + \frac{h(w) f(w) - h(w) f(z)}{z - w} \\ &= Q_w(hf) - h(w) Q_w f \\ &\in \ell^{p,\alpha}. \end{aligned}$$

The norm bounds can be derived from this by inspection. Linearity is evident. □

Proposition 9.4. *If* $1 \leq p < \infty$, *and* $\alpha \in \mathbb{R}$, *then* $\mathscr{M}_{p,\alpha}$ *is closed in the operator norm.*

Proof. Let $h^{(1)}, h^{(2)}, h^{(3)}, \ldots$ be a Cauchy sequence in $\mathscr{M}_{p,\alpha}$. Since the space of operators on $\ell^{p,\alpha}$ is closed, we know that $h^{(k)}$ converges in operator norm to some operator T. It follows that $h^{(k)}1$ converges to $T1$ in $\ell^{p,\alpha}$. Let $h := T1$. Then $h^{(k)}f \longrightarrow Tf$ in $\ell^{p,\alpha}$, while at the same time $h^{(k)}f$ converges uniformly to hf on compact subsets of $\mathbb{D}$. This forces $Tf = hf$, and we conclude that the Cauchy sequence converges to an element of $\mathscr{M}_{p,\alpha}$. □

A similar result could be fashioned when $0 < p < 1$, with care taken to identify a metric on the multipliers on $\ell^{p,\alpha}$.

In some cases it is possible to describe the multiplier space completely. For example, it is well known that $\mathscr{M}_{2,0} = H^\infty$ (see [6, Proposition 12.2.6] for an exposition). For some pairs (p, α), the multiplier space $\mathscr{M}_{p,\alpha}$ coincides with $\ell^{p,\alpha}$.

Proposition 9.5. *If*

$$0 < p \leq 1, \text{ and } \alpha \geq 0; \text{ or}$$

$$1 < p < \infty, \text{ and } \alpha > p - 1,$$

then $\mathscr{M}_{p,\alpha} = \ell^{p,\alpha}$.

Proof. This follows immediately from Theorem 7.1, which states that $\ell^{p,\alpha}$ is an algebra under the identified conditions, with multiplication being norm bounded in the sense $\|fg\|_{p,\alpha} \leq c\|f\|_{p,\alpha}\|g\|_{p,\alpha}$. □

Here is another case in which the multiplier space for $\ell^{p,\alpha}$ is completely known.

Proposition 9.6. *If $0 < p \le 1$, and $\alpha < 0$, then $\mathscr{M}_{p,\alpha} = \ell^p = \ell^{p,0}$, with equal norms.*

Proof. Let $f \in \ell^{p,\alpha}$ and $h \in \mathscr{M}_{p,\alpha}$. Then

$$\begin{aligned}
\|hf\|_{p,\alpha}^p &= \sum_{n=0}^{\infty} \Big| \sum_{k=0}^{n} h_{n-k} f_k \Big|^p (n+1)^\alpha \\
&\le \sum_{n=0}^{\infty} \sum_{k=0}^{n} |h_{n-k}|^p |f_k|^p (n+1)^\alpha \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} |h_{n-k}|^p \Big(\frac{k+1}{n+1}\Big)^{|\alpha|} |f_k|^p (k+1)^\alpha \\
&\le \sum_{n=0}^{\infty} \sum_{k=0}^{n} |h_{n-k}|^p \cdot 1 \cdot |f_k|^p (k+1)^\alpha \\
&= \|h\|_{p,0}^p \|f\|_{p,\alpha}^p.
\end{aligned}$$

Conversely, for every $n \ge 1$ we have

$$\|hz^N\|_{p,\alpha}^p = \Big\| \sum_{k=0}^{\infty} h_k z^{k+N} \Big\|_{p,\alpha}^p = \sum_{k=0}^{\infty} |h_k|^k (k+N+1)^\alpha \le \|M_h\|^p \cdot \|z^N\|_{p,\alpha}^p = \|M_h\|^p (N+1)^\alpha.$$

From this we we find that

$$\sum_{k=0}^{\infty} |h_k|^p \Big(\frac{N+1}{k+N+1}\Big)^{|\alpha|} \le \|M_h\|^p$$

for all N. Take $N \to \infty$ to conclude that

$$\Big(\sum_{k=0}^{\infty} |h_k|^p\Big)^{1/p} \le \|M_h\|.$$

Thus equality is forced, and we conclude $\|h\|_{p,0} = \|M_h\|$. □

In Corollary 8.3 we saw that $\ell^{2,\alpha} \cap H^\infty$ is a Banach algebra when $0 < \alpha < 1$. It is natural to wonder whether $\ell^{2,\alpha} \cap H^\infty$ coincides with the multiplier space for $\ell^{2,\alpha}$. The following example shows that it does not, however.

Proposition 9.7. *Let $0 < \alpha \le 1$. There exists $h \in \ell^{2,\alpha} \cap H^\infty$ such that $h \notin \mathscr{M}_{2,\alpha}$.*

Proof. For $\alpha = 1$, this was proved as [11, Theorem 5.1.6].

Assume that $0 < \alpha < 1$. Let $n_k = 2^{2^k}$, $k \geq 0$, and define

$$h_m = \frac{1}{k^2 n_k^\alpha},\ n_{k-1}^\alpha \leq m < n_k^\alpha$$

for all $k \geq 1$.

Thus

$$\sum_{k=2}^{\infty} h_m = \sum_{k=1}^{\infty} \Big(\sum_{n_{k-1}^\alpha \leq m < n_k^\alpha} h_m \Big) \leq \sum_{k=1}^{\infty} (n_k^\alpha - n_{k-1}^\alpha) \frac{1}{k^2 n_k^\alpha} < \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

This shows that $h \in H^\infty$.

Next,

$$\sum_{m=2}^{\infty} h_m^2 m^\alpha = \sum_{k=1}^{\infty} \Big(\sum_{n_{k-1}^\alpha \leq m < n_k^\alpha} h_m^2 m^\alpha \Big) \leq \sum_{k=1}^{\infty} (n_k^\alpha - n_{k-1}^\alpha) \frac{n_k^\alpha}{k^4 n_k^{2\alpha}} \leq \sum_{k=1}^{\infty} \frac{1}{k^4} < \infty,$$

showing that $h \in \ell^{2,\alpha}$ as well.

Continuing, suppose that $f_k = 1/(k[\log k]^{3/4})$ for all $k \geq 2$. Then $f \in \ell^{2,\alpha}$, since

$$\sum_{k=2}^{\infty} f_k^2 k^\alpha = \sum_{k=2}^{\infty} \frac{k^\alpha}{k^2 (\log k)^{3/2}} < \infty.$$

Finally, for some constant $C > 0$, and $n \geq 5$,

$$\sum_{j=2}^{n-2} f_j \geq \int_1^{n-2} \frac{1}{x(\log x)^{3/4}}\, dx \geq C(\log n)^{1/4}.$$

Consequently

$$\begin{aligned}
\|hf\|_{2,\alpha}^2 &\geq \sum_{n=5}^{\infty} \Big|\sum_{j=2}^{n-2} h_{n-j} f_j\Big|^2 (n+1)^{\alpha} \\
&\geq \sum_{n=5}^{\infty} h_n^2 (n+1)^{\alpha} \Big(\sum_{j=2}^{n-2} f_j\Big)^2 \\
&\geq \sum_{n=5}^{\infty} h_n^2 (n+1)^{\alpha} C^2 (\log n)^{1/2} \\
&\geq C^2 \sum_{k=2}^{\infty} \Big(\sum_{n_k/2 \leq n < n_k} (n+1)^{\alpha} (\log n)^{1/2} h_n^2\Big) \\
&\geq C^2 \sum_{k=2}^{\infty} \frac{n_k^{\alpha} (n_k^{\alpha}/2)}{2 n_k^{2\alpha} k^4} [\log(2^{2^{k-1}})]^{1/2} \\
&= \infty,
\end{aligned}$$

from which we see that h fails to be a multiplier on $\ell^{2,\alpha}$. □

We have pointed out some instances for which the multipliers on $\ell^{p,\alpha}$ are explicitly characterized. In other cases, however, we can only offer partial descriptions in terms of their coefficient growth or decay, and provide bounds for the multiplier norm. The next several results are of this nature.

Proposition 9.8. *Let* $0 < p < \infty$ *and* $\alpha \in \mathbb{R}$. *If* $h \in \mathscr{M}_{p,\alpha}$, *then*

$$\|M_h\| \leq \begin{cases} \big(|h_0|^p + |h_1|^p \|S\|^p + |h_2|^p \|S^2\|^p + \cdots\big)^{1/p}, & 0 < p < 1; \\ |h_0| + |h_1| \cdot \|S\| + |h_2| \cdot \|S^2\| + \cdots, & 1 \leq p < \infty. \end{cases}$$

Proof. This follows immediately from

$$h(z)f(z) = h_0 f(z) + h_1 S f(z) + h_2 S^2 f(z) + \cdots$$

for any $f \in \ell^{p,\alpha}$. □

Thus, for example, if $1 \le p < \infty$ and $\alpha \ge 0$, then

$$\|M_h\| \le 1^{\alpha/p}|h_0| + 2^{\alpha/p}|h_1| + 3^{\alpha/p}|h_2| + \cdots,$$

and we may conclude that $\ell^{1,\alpha/p} \subseteq \mathscr{M}_{p,\alpha}$. Similar expressions arise in the other cases.

Proposition 9.9. *Let $1 < p < \infty$, $1/p + 1/q = 1$, and $\alpha \in \mathbb{R}$. If $h \in \mathscr{M}_{p,\alpha}$, then for every $n \ge 0$,*

$$|h_0| \cdot 1^\alpha + |h_1| \cdot 2^\alpha + \cdots + |h_n| \cdot (n+1)^\alpha \le \|M_h\| \big(1^\alpha + 2^\alpha + \cdots + (n+1)^\alpha\big)^{1/q} \tag{36}$$

and

$$|h_0| + |h_1| + \cdots + |h_n| \le \|M_h\| \Big(\frac{1}{1^{\alpha(q-1)}} + \frac{1}{2^{\alpha(q-1)}} + \cdots + \frac{1}{(n+1)^{\alpha(q-1)}} \Big)^{1/q}. \tag{37}$$

Proof. Let

$$c_k = \begin{cases} \overline{h_k}/|h_k|, & \text{if } h_k \ne 0; \\ 0, & \text{otherwise.} \end{cases}$$

For any $g \in \ell^{q,\alpha}$, we have

$$|\langle h, g\rangle| = |\langle M_h 1, g\rangle| \le \|M_h\| \cdot \|1\|_{p,\alpha} \cdot \|g\|_{q,\alpha}.$$

Then (36) follows by taking

$$g(z) = c_0 + c_1 z + \cdots + c_n z^n,$$

and (37) derives from the choice

$$g(z) = \frac{c_0}{1^\alpha} + \frac{c_1}{2^\alpha} z + \cdots + \frac{c_n}{(n+1)^\alpha} z^n.$$

$\square$

Proposition 9.10. *Let $1 < p < \infty$, $1/p + 1/q = 1$, and $\alpha \in \mathbb{R}$. If $h \in \mathscr{M}_{p,\alpha}$, then for every $n \geq 0$, we have*

$$\begin{aligned}\Bigg| h_0 + \frac{2^\alpha + \cdots + [n+1]^\alpha}{1^\alpha \cdots + [n+1]^\alpha} h_1 + \frac{3^\alpha + \cdots + [n+1]^\alpha}{1^\alpha + \cdots + [n+1]^\alpha} h_2 \\ + \cdots + \frac{[n+1]^\alpha}{1^\alpha + \cdots + [n+1]^\alpha} h_n \Bigg| \leq \|M_h\| \end{aligned} \tag{38}$$

and

$$\begin{aligned}\big|(n+1)h_0 + nh_1 + \cdots + 1h_n\big| \leq \|M_h\| \big(1^\alpha + 2^\alpha + \cdots + [n+1]^\alpha\big)^{1/p} \\ \times \Big(\frac{1}{1^{\alpha(q-1)}} + \frac{1}{2^{\alpha(q-1)}} \cdots + \frac{1}{[n+1]^{\alpha(q-1)}}\Big)^{1/q}. \end{aligned} \tag{39}$$

Proof. Take

$$f(z) = 1 + z + z^2 + \cdots + z^n$$
$$g(z) = \frac{1}{1^\alpha} + \frac{z}{2^\alpha} + \cdots + \frac{z^n}{[n+1]^\alpha}.$$

Then (38) results from $|\langle hf, f\rangle| \leq \|M_h\| \cdot \|f\|_{p,\alpha} \|f\|_{q,\alpha}$, and (39) derives from $|\langle hf, g\rangle| \leq \|M_h\| \cdot \|f\|_{p,\alpha} \|g\|_{q,\alpha}$. □

Corollary 9.1. *Let $1 < p < \infty$ and $\alpha \geq 0$. If $h \in \mathscr{M}_{p,\alpha}$ has non-negative coefficients, then $h \in \ell^1$.*

Proof. Take $n \longrightarrow \infty$ in (38), and invoke the monotone convergence theorem. □

The next result essentially makes use of the adjoint of a multiplier.

Proposition 9.11. *Let* $1 < p < \infty$*,* $1/p + 1/q = 1$*, and* $\alpha \in \mathbb{R}$*. If* $h \in \mathscr{M}_{p,\alpha}$*, then*

$$
\begin{aligned}
&\|M_h\| \cdot \|h\|_{p,\alpha}^{p-1} \\
&\geq \Bigg\{ \Bigg| |h_0|^p \left(\frac{0+1}{1}\right)^\alpha + |h_1|^p \left(\frac{1+1}{1}\right)^\alpha + |h_2|^p \left(\frac{2+1}{1}\right)^\alpha + \cdots \Bigg|^q (0+1)^\alpha \\
&\quad + \Bigg| |h_1|^{p-2}\overline{h_1} h_0 \left(\frac{1+1}{2}\right)^\alpha + |h_2|^{p-2}\overline{h_2} h_1 \left(\frac{2+1}{2}\right)^\alpha + |h_3|^{p-2}\overline{h_3} h_2 \left(\frac{3+1}{2}\right)^\alpha + \cdots \Bigg|^q (1+1)^\alpha \\
&\quad + \Bigg| |h_2|^{p-2}\overline{h_2} h_0 \left(\frac{2+1}{3}\right)^\alpha + |h_3|^{p-2}\overline{h_3} h_1 \left(\frac{3+1}{3}\right)^\alpha + |h_4|^{p-2}\overline{h_4} h_2 \left(\frac{4+1}{3}\right)^\alpha + \cdots \Bigg|^q (2+1)^\alpha \\
&\quad + \cdots \Bigg\}^{1/q},
\end{aligned}
\tag{40}
$$

where $|h_k|^{p-2}\overline{h_k}$ *is understood to be zero when* $h_k = 0$*.*

Proof. Let $f \in \ell^{p,\alpha}$. For any $n \geq 0$,

$$\langle hf, z^n \rangle = \sum_{k=0}^{n} h_{n-k} f_k (n+1)^\alpha = \sum_{k=0}^{n} h_{n-k} f_k \left(\frac{n+1}{k+1}\right)^\alpha (k+1)^\alpha = \Big\langle f, u_n(z) \Big\rangle,$$

where

$$u_n(z) = h_n \left(\frac{n+1}{1}\right)^\alpha + h_{n-1} \left(\frac{n+1}{2}\right)^\alpha z + \cdots + h_0 \left(\frac{n+1}{n+1}\right)^\alpha z^n.$$

Then for any $g \in \ell^{q,\alpha}$, we may write

$$\langle hf, g \rangle = \Big\langle f, \sum_{k=0}^{\infty} g_k u_k(z) \Big\rangle.$$

The Riesz Representation Theorem [7, Theorem 5.5] compels the expression $\sum_{k=0}^{\infty} g_k u_k(z)$ to belong to $\ell^{q,\alpha}$, since the mapping $f \mapsto \langle hf, g \rangle$ is a bounded linear functional on $\ell^{p,\alpha}$ with norm not exceeding $\|M_h\| \cdot \|g\|_{q,\alpha}$.

Consequently

$$\begin{aligned}
\infty &> \Big\| \sum_{k=0}^{\infty} g_k u_k(z) \Big\|_{q,\alpha}^q \\
&= \Big\| g_0 h_0 \Big(\frac{0+1}{1}\Big)^{\alpha} \\
&\qquad + g_1 h_1 \Big(\frac{1+1}{1}\Big)^{\alpha} + g_1 h_0 \Big(\frac{1+1}{2}\Big)^{\alpha} \\
&\qquad + g_2 h_2 \Big(\frac{2+1}{1}\Big)^{\alpha} + g_2 h_1 \Big(\frac{2+1}{2}\Big)^{\alpha} + g_2 h_0 \Big(\frac{2+1}{3}\Big)^{\alpha} \\
&\qquad + \cdots \Big\|_{q,\alpha}^q \\
&= \Big| g_0 h_0 \Big(\frac{0+1}{1}\Big)^{\alpha} + g_1 h_1 \Big(\frac{1+1}{1}\Big)^{\alpha} + g_2 h_2 \Big(\frac{2+1}{1}\Big)^{\alpha} + \cdots \Big|^q (0+1)^{\alpha} \\
&\qquad + \Big| g_1 h_0 \Big(\frac{1+1}{2}\Big)^{\alpha} + g_2 h_1 \Big(\frac{2+1}{2}\Big)^{\alpha} + g_3 h_2 \Big(\frac{3+1}{2}\Big)^{\alpha} + \cdots \Big|^q (1+1)^{\alpha} \\
&\qquad + \Big| g_2 h_0 \Big(\frac{2+1}{3}\Big)^{\alpha} + g_3 h_1 \Big(\frac{3+1}{3}\Big)^{\alpha} + g_4 h_2 \Big(\frac{4+1}{3}\Big)^{\alpha} + \cdots \Big|^q (2+1)^{\alpha} \\
&\qquad + \cdots .
\end{aligned}$$

Now choose

$$g_k = \begin{cases} |h_k|^{p-2}\overline{h_k}, & \text{if } h_k \neq 0; \\ 0, & \text{if } h_k = 0. \end{cases}$$

Then $\|g\|_{q,\alpha}^q = \|h\|_{p,\alpha}^p$, and the claim follows. □

Again, the above results describe how the coefficients of a multiplier grow or decay, and how they relate to the multiplier norm.

We know from Proposition 9.1 that in general $\|M_h\| \geq \|h\|_{p,\alpha}$ for any $h \in \mathscr{M}_{p,\alpha}$. It is interesting to ask for which multipliers h does equality hold between its operator and vector norms, a sort of extremal property. It is known, for example, that in $\ell^{2,0} = H^2$, the extremal multipliers are exactly the constant multiples of inner functions [12, Section 7]. The following theorem describes the multipliers on $\ell^{p,\alpha}$ which enjoy the extremal property.

Theorem 9.1. *Let* $1 < p < \infty$ *and* $\alpha \in \mathbb{R}$. *Suppose that* $h \in \mathscr{M}_{p,\alpha}$, *h is not identically zero, and define G by*

$$G(z)\|h\|_{p,\alpha}^{p-1} = |h_0|^{p-2}\overline{h_0} + |h_1|^{p-2}\overline{h_1}z + |h_2|^{p-2}\overline{h_2}z^2 + \cdots \in \ell^{q,\alpha}.$$

If $\|M_h\| = \|h\|_{p,\alpha}$, *then for every* $n \geq 1$ *we have* $\langle S^n h, G\rangle = 0$.

Proof. If the condition $\|M_h\| = \|h\|_{p,\alpha}$ holds, then in the expression (40), each line except the first must vanish. The claim follows. □

In Theorem 9.1, the function G is the norming functional of h. Note that the condition $\langle S^n h, G\rangle = 0$, $n \geq 1$, is a way to say that h is "inner" in $\ell^{p,\alpha}$ in a certain sense (cf. [6, Definition 8.3.2]). The converse of Theorem 9.1 is known to fail, however, even for $\ell^p = \ell^{p,0}$; see [12, Example 7.1 and Theorem 7.3].

The Schur Test supplies a way to estimate the norm of an integral operator on L^2, based on its kernel [21]. Here is a discrete version of the Schur Test, which can be used to identify operators on $\ell^{p,\alpha}$. In particular, we use it to produce a class of multipliers. For this purpose, we temporarily view elements of $\ell^{p,\alpha}$ as column vectors of coefficients, and matrices $B = [b_{j,k}]_{j,k\geq 0}$ as operators by left multiplication.

Theorem 9.2. *Assume that* $1 < p < \infty$, $1/p + 1/q = 1$, *and* $\alpha \in \mathbb{R}$. *Let* $B = [b_{j,k}]_{j,k\geq 0}$ *be a matrix of non-negative entries. Suppose that there are constants* C_1 *and* C_2, *and positive sequences* $(s_j)_{j\geq 0}$ *and* $(t_k)_{k\geq 0}$ *such that*

$$\sum_{k=0}^{\infty} b_{j,k} t_k^{-1/p} \leq C_1 \frac{s_j^{-1/p}}{(j+1)^{\alpha q/p}} \text{ for all } j \geq 0; \tag{41}$$

$$\sum_{j=0}^{\infty} b_{j,k} s_j^{-1/q} \leq C_2 t_k^{-1/q} (k+1)^{\alpha} \text{ for all } k \geq 0. \tag{42}$$

Then B is a bounded linear operator on $\ell^{p,\alpha}$, *with* $\|B\| \leq C_1^{1/q} C_2^{1/p}$.

Proof. B is obviously linear. Suppose that $x \in \ell^{p,\alpha}$, and $y = Bx$. Then by Hölder's inequality and (41),

$$\begin{aligned} y_j &= \sum_{k=0}^{\infty} b_{j,k} x_k \\ &= \sum_{k=0}^{\infty} b_{j,k} t_k^{-1/pq} t_k^{1/pq} x_k \\ &= \sum_{k=0}^{\infty} b_{j,k}^{1/q} t_k^{-1/pq} b_{j,k}^{1/p} t_k^{1/pq} x_k \\ &\leq \Big(\sum_{k=0}^{\infty} b_{j,k} t^{-1/p} \Big)^{1/q} \Big(\sum_{k=0}^{\infty} b_{j,k} t_k^{1/q} |x_k|^p \Big)^{1/p} \\ &\leq \Big(C_1 \frac{s_j^{-1/p}}{(j+1)^{\alpha q/p}} \Big)^{1/q} \Big(\sum_{k=0}^{\infty} b_{j,k} t_k^{1/q} |x_k|^p \Big)^{1/p}. \end{aligned}$$

Therefore (42) allows for

$$\begin{aligned} \|y\|_{p,\alpha}^p &\leq \sum_{j=0}^{\infty} |y_j|^p (j+1)^{\alpha} \\ &\leq \sum_{j=0}^{\infty} \Big(C_1 \frac{s_j^{-1/p}}{(j+1)^{\alpha q/p}} \Big)^{p/q} (j+1)^{\alpha} \sum_{k=0}^{\infty} b_{j,k} t_k^{1/q} |x_k|^p \\ &= C_1^{p/q} \sum_{k=0}^{\infty} t_k^{1/q} |x_k|^p \sum_{j=0} s_j^{-1/q} b_{j,k} \\ &\leq C_1^{p/q} \sum_{k=0}^{\infty} t_k^{1/q} |x_k|^p C_2 t_k^{-1/q} (k+1)^{\alpha} \\ &= C_1^{p/q} C_2 \sum_{k=0}^{\infty} |x_k|^p (k+1)^{\alpha} \\ &= C_1^{p/q} C_2 \|x\|_{p,\alpha}^p. \end{aligned}$$

□

Here is an example of applying Theorem 9.2 to get a class of multipliers on $\ell^{p,\alpha}$.

Proposition 9.12. *Let $1 < p < \infty$ and $\beta < -1$. If $\alpha < |\beta| - 1$ and $|h_k| \leq (k+1)^{\beta}$ for all $k \geq 0$, then $h \in \mathscr{M}_{p,\alpha}$.*

Proof. First, assume that $h_k \geq 0$ for all k. We apply Theorem 9.2, with $t_k = 1$ for all k, and $s_j = (j+1)^{-\alpha q}$ for all j. Let the infinite matrix B be defined by

$$b_{j,k} = \begin{cases} h_{j-k}, & \text{if } j \geq k; \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\sum_{k=0}^{\infty} b_{j,k} t^{-1/p} = \sum_{k=0}^{\infty} h_{j-k} 1^{-1/p} \leq C \sum_{k=0}^{j} (j-k+1)^{\beta} = C[1^{\beta} + 2^{\beta} + \cdots + j^{\beta}].$$

Since $\beta < -1$, the last expression is uniformly bounded over j. Hence we can find a constant C_1 such that

$$\sum_{k=0}^{\infty} b_{j,k} t^{-1/p} \leq C_1 = C_1 (j+1)^{\alpha q/p} \frac{1}{(j+1)^{\alpha q/p}} = C_1 s_j^{-1/p} \frac{1}{(j+1)^{\alpha q/p}}.$$

This checks that condition (41) holds.

Next, suppose $0 < \alpha < |\beta| - 1$. Then

$$\begin{aligned} \sum_{j=0}^{\infty} b_{j,k} s_j^{-1/q} &\leq \sum_{j=k}^{\infty} (j-k+1)^{\beta} (j+1)^{\alpha} \\ &= \sum_{j=k} (j-k+1)^{\beta+\alpha} \left(\frac{j+1}{j-k+1} \right)^{\alpha} \\ &\leq \sum_{j=1}^{\infty} j^{\beta+\alpha} (k+1)^{\alpha}, \end{aligned}$$

where we used the elementary inequality $k + m \leq (k+1)m$, for all $m \geq 1$. Thus condition (42) holds, with $C_2 = \sum_{j=1}^{\infty} j^{\beta+\alpha} < \infty$.

Finally, if $\alpha \leq 0$, then

$$\sum_{j=k}^{\infty} (j-k+1)^{\beta} (j+1)^{\alpha} \leq \sum_{j=k}^{\infty} (j-k+1)^{\beta} (k+1)^{\alpha} = \sum_{j=1}^{\infty} j^{\beta} (k+1)^{\alpha},$$

and once again (42) holds.

By Theorem 9.2, B is a bounded operator on $\ell^{p,\alpha}$. But B is simply the matrix for multiplication by h.

Finally, for h having complex coefficients, the lemma below shows that if $|h_0| + |h_1|z + |h_2|z^2 + \cdots$ is a multiplier, then so is $h(z) = h_0 + h_1 z + h_2 z^2 + \cdots$. $\square$

If $f(z) = f_0 + f_1 z + f_2 z^2 + \cdots$, let us write $\breve{f}(z) := |f_0| + |f_1|z + |f_2|z^2 + \cdots$. Obviously $f \in \ell^{p,\alpha}$ if and only if $\breve{f} \in \ell^{p,\alpha}$, and their norms coincide.

Lemma 9.1. *Let $1 < p < \infty$ and $\alpha \in \mathbb{R}$. If $\breve{h} \in \mathscr{M}_{p,\alpha}$, then $h \in \mathscr{M}_{p,\alpha}$.*

Proof. For any $f \in \ell^{p,\alpha}$,

$$\begin{aligned}\|hf\|_{p,\alpha}^p &= \sum_{n=0}^{\infty} \Big| \sum_{k=0}^{n} h_k f_{n-k} \Big|^p (n+1)^\alpha \\ &\leq \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} |h_k||f_{n-k}| \Big)^p (n+1)^\alpha \\ &= \|\breve{h}\breve{f}\|_{p,\alpha}^p \\ &\leq \|M_{\breve{h}}\|^p \|\breve{f}\|_{p,\alpha}^p \\ &= \|M_{\breve{h}}\|^p \|f\|_{p,\alpha}^p.\end{aligned}$$

This shows that $\|M_h\| \leq \|M_{\breve{h}}\|$. $\square$

We have laid out some elementary properties of multipliers, presented some examples of multipliers, and furnished estimates for multiplier coefficients. Multipliers enjoying a certain extremal property were shown to exhibit a certain orthogonality property.

CHAPTER 10

CONCLUSION AND FUTURE WORK

This chapter concludes our goal to systematically study the $\ell^{p,\alpha}$ spaces. Numerous issues remain to be solved. They include the characterization of zero sets; problems of interpolation; problems about sampling and bases; canonical factorization and invariant subspaces. There are numerous other classes of operators that merit exploration.

The applications, largely yet to be identified, will drive many other questions. These matters will be the subject of future projects.

REFERENCES

[1] L. V. Ahlfors, “Complex analysis: An introduction to the theory of analytic functions of one complex variable,” in *International Series in Pure and Applied Mathematics*, New York-Auckland-London: McGraw-Hill Book Co., 1979, ch. 5.

[2] C. Bénéteau, A. A. Condori, C. Liaw, D. Seco, and A. A. Sola, “Cyclicity in dirichlet-type spaces and extremal polynomials,” *J. Anal. Math.*, vol. 126, pp. 259–286, 2015.

[3] J. Bergh and J. Löfström, *Interpolation Spaces*. New York: Springer-Verlag, 1976.

[4] R. Cheng and C. B. Harris, “Mixed-norm spaces and prediction of sαs moving averages,” *J. Time Series Anal.*, vol. 36, pp. 853–875, 2015.

[5] R. Cheng, J. Mashreghi, and W. T. Ross, “Multipliers of sequence spaces,” *Concr. Oper.*, vol. 4, pp. 76–108, 2017.

[6] R. Cheng, J. Mashreghi, and W. T. Ross, *Function Theory and ℓ^p Spaces* (University Lecture Series). Providence, RI: AMS, 2020, vol. 75.

[7] J. B. Conway, *A Course in Functional Analysis* (Graduate Texts in Mathematics). New York: Springer-Verlag, 1985, vol. 96.

[8] P. L. Duren, *Theory of H^p Spaces*. New York: Academic Press, 1970.

[9] P. L. Duren and A. Schuster, *Bergman Spaces* (Mathematical Surveys and Monographs). Providence, RI: AMS, 1962, vol. 100.

[10] Y. Estaremi and M. R. Jabbaradeh, “Multiplication and composition operators on the space of formal power series,” *Rend. Circ. Mat. Palermo (2)*, vol. 70, pp. 641–651, 2021.

[11] O. El-Fallah, K. Kellay, J. Mashreghi, and T. Ransford, *A Primer on the Dirichlet Space*. New York: Cambridge University Press, 2014.

[12] C. Felder and R. Cheng, "On the geometry of the multiplier space of ℓ^p_A," *Concr. Oper.*, vol. 9, pp. 41–52, 2022.

[13] P. J. Gerlach-Mena and J. Müller, "ℓ^p-type dirichlet spaces," *Mediterr. J. Math.*, vol. 19, article no. 105, 2022.

[14] K. Hedayatian, "On cyclicity in the space $H^p(\beta)$," *Taiwanese J. Math.*, vol. 8, pp. 429–442, 2004.

[15] G. J. O. Jameson and R. Lashkaripour, "Lower bounds of operators on weighted ℓ_p spaces and lorentz sequence spaces," *Glasg. Math. J.*, vol. 42, pp. 211–223, 2000.

[16] G. J. O. Jameson and R. Lashkaripour, "Norms of certain operators on weighted ℓ_p spaces and lorentz sequence spaces," *J. Inequal. Pure Appl. Math.*, vol. 3, pp. 137–144, 2002.

[17] W. A. Kozek, "Lectures on frames," in *Applied and Numerical Harmonic Analysis Series*, Boston, MA: Birkhäuser Boston, Inc., 2006, ch. 2.

[18] R. Lashkaripour and D. Foroutannia, "Extension of hardy inequality on weighted sequence spaces," *Sci. Islam. Repub. Iran*, vol. 20, pp. 159–166, 2009.

[19] J. Marcinkiewicz, "Sur l'interpolation d'operations," *C. R. Acad. Sci. Paris*, vol. 208, pp. 1272–1273, 1939.

[20] W. Rudin, "Functional analysis," in New York-London: McGraw-Hill Book Co., 1973, ch. 10.

[21] I. Schur, “Bemerkungen zur theorie der beschränkten bilinearformen mit unendlich vielen veränderlichen,” *J. reine angew. Math.*, vol. 140, pp. 1–28, 1911.

[22] D. Seco and R. Téllez, “A polynomial approach to cyclicity for weighted ℓ^p_A,” *Banach J. Math. Anal.*, vol. 15, p. 1, 2021.

[23] A. L. Shields, “Weighted shift operators and analytic function theory,” in *Topics in Operator Theory*, ser. Mathematical Surveys and Monographs, C. Pearcy, Ed., vol. 13, Providence, RI: AMS, 1974.

[24] V. K. Srivastava, “Hypercyclicity, supercyclicity, and cyclicity of a weighted composition operator on ℓ^p spaces,” *Georgian Mathematical Journal*, vol. 28, no. 6, pp. 971–979, 2021.

[25] V. K. Srivastava and H. Chandra, “Hypercyclicity, supercyclicity and cyclicity of composition operators on L^p spaces,” *The Journal of the Indian Mathematical Society*, vol. 86, no. 1-2, pp. 187–198, 2019.

[26] R. M. Young, *An introduction to signal processing*. Prentice Hall, 2002.

[27] B. Yousefi, “On the space $\ell^p(\beta)$,” *Rend. Circ. Mat. Palermo (2)*, vol. 49, pp. 115–120, 2000.

[28] B. Yousefi, “Bounded analytic structure of the banach space of formal power series,” *Rend. Circ. Mat. Palermo (2)*, vol. 51, pp. 403–410, 2002.

[29] K. Zhu, “Spaces of analytic functions in the unit disk,” in *Graduate Texts in Mathematics*, vol. 33, New York: Springer-Verlag, 2007, ch. 8.

VITA

Gilbert D. Acheampong

Department of Mathematics & Statistics

Old Dominion University

Norfolk, VA 23529

EDUCATION

2020-2024, Ph.D. Computational & Applied Mathematics, Old Dominion University (ODU), Norfolk VA, USA.

2018-2020, M.S. Mathematics, The University of Texas at El Paso (UTEP), El Paso TX, USA.

2016-2017, M.S. Mathematical Sciences, AIMS-Next Einstein Initiative.

2009-2013, B.S. Mathematics, University for Development Studies (UDS), Tamale, Ghana.

PROFESSIONAL EXPERIENCE

2023-Present, Site Reliability Engineer, WebstaurantStore, Lititz Pennsylvania, USA.

2020-2023, DevOps Cloud Engineer, CloudFift, Washington D.C., USA.

2018-2020, System Support Engineer, UTEP Math Sciences Department, El Paso TX, USA.

2014-2016, Linux System Administrator and Research Assistant, Ghana Mathematics Society, Accra Ghana.

PUBLICATION

- R. Cheng and G. Acheampong, *"Multipliers on Weighted Sequence Spaces,"* Taiwanese Journal of Mathematics (submitted).