1992

# On Sources in Comparability Graphs, With Applications

Stephan Olariu
*Old Dominion University*



## Original Publication Citation

Olariu, S. (1992). On sources in comparability graphs, with applications. *Discrete Mathematics, 110*(1-3), 289-292. doi:10.1016/0012-365x(92)90721-q

Discrete Mathematics 110 (1992) 289–292
North-Holland



# Note

# On sources in comparability graphs, with applications

S. Olariu

*Department of Computer Science, Old Dominion University, Norfolk, VA 23529-0162, USA*

Received 31 October 1989

*Abstract*

Olariu, S., On sources in comparability graphs, with applications, Discrete Mathematics 110 (1992) 289–292.

We characterize sources in comparability graphs and show that our result provides a unifying look at two recent results about interval graphs.

An orientation $\Theta$ of a graph $G$ is obtained by assigning unique directions to its edges. To simplify notation, we write $xy \in \Theta$, whenever the edge $xy$ receives the direction from $x$ to $y$. A vertex $w$ is called a *source* whenever $vw \in \Theta$ for no vertex $v$ in $G$. An orientation $\Theta$ is termed *transitive* if for every vertices $x$, $y$, $z$, $xy \in \Theta$ and $yz \in \Theta$ implies $xz \in \Theta$. It is well known that a graph $G$ that admits a transitive orientation is a *comparability* graph.

In this context, it makes sense to ask the following natural question:

for what vertices of a comparability graph can we find a transitive orientation that makes them into a source? (1)

The purpose of this note is to provide an answer to (1). As it turns out, our result provides a unifying look at two recent results concerning interval graphs [4, 6].

All the graphs in this work are simple with no self-loops nor multiple edges.

Familiarity with standard graph theoretical terminology compatible with Golumbic [5] is assumed. To specify our results, however, we need to define some new terms. For an arbitrary vertex $w$ of $G$, the graph $G^w$ is obtained from $G$ by adding a new vertex $w'$ and by making $w'$ adjacent to $w$ only. We claim that

*$w$ is a source in some transitive orientation of $G$ if, and only if, the graph $G^w$ is a comparability graph.* (2)

To justify (2), we note first that if $G^w$ is a comparability graph then, by reversing the orientation on all the edges if necessary, we guarantee that $w$ is a source in some transitive orientation $\Theta$ of $G^w$. In particular, $w$ is a source in the restriction of $\Theta$ to $G$.

Conversely, given a transitive orientation $\Theta$ of $G$ that makes $w$ into a source, the orientation $\Theta^w = \Theta \cup \{ww'\}$ of $G^w$ is transitive, and so $G^w$ is a comparability graph, as claimed.

A vertex $w$ of a graph $G$ is called *special* if $w$ coincides with one of the highlighted verteces in some graph $F_i$, $(1 \leq i \leq 4)$ or in the complement $\bar{F}_5$ of the graph $F_5$ featured in Fig. 1. A vertex that is not special is referred to as *regular.* As it turns out, regular vertices play an important role in the answer to (1). More precisely, we state the following result.

**Theorem 1.** *A vertex $w$ of a comparability graph $G$ is a source in some transitive orientation $\Theta$ of $G$ if, and only if, $w$ is regular.*

**Proof.** First, let $w$ be a regular vertex of $G$. If $w$ fails to be a source in any transitive orientation of $G$ then, by (2), the graph $G^w$ is not a comparability graph. Hence $G^w$ must contain an induced subgraph $H$ isomorphic to one of Gallai's forbidden graphs (for a list see Gallai [2] or Duchet [1]). Since, by

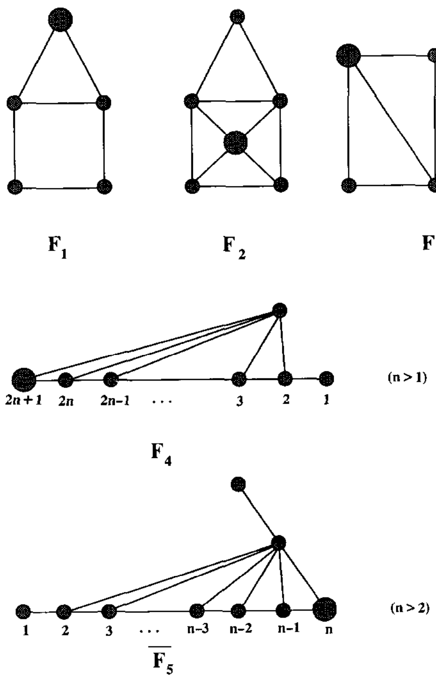


Fig. 1.

assumption, $G$ is a comparability graph, $w'$ must belong to $H$. But now, it is a straightforward observation that $w$ must be a special vertex, a contradiction.

Conversely, let $w$ be a source in some transitive orientation $\Theta$ of $G$. Again, it is routine to check that if $w$ is special, then $G^w$ contains one of the forbidden graphs in Gallai's catalog, contradicting (2). This completes the proof of Theorem 1. □

A graph $G = (V, E)$ is termed an *interval graph* if there exists a family $\{I_v\}_{v \in V}$ of intervals such that for distinct vertices $u$, $v$ in $G$

$$uv \in E \text{ if, and only if, } I_u \cap I_v \neq \emptyset.$$

Such a family $\{I_v\}_{v \in V}$ of intervals is commonly referred to as an *interval representation* of $G$. An interval $I_x = [a_x, b_x]$, is called an *end interval* if $a_x \leqslant a_y$ for every $y \in V$; the vertex $x$ itself is termed an *end vertex*.

An early characterization of interval graphs was proposed by Gilmore and Hoffman [3]: they showed that a graph $G$ is an interval graph if, and only if, $G$ itself is triangulated and its complement $\bar{G}$ is a comparability graph.

Let $\Theta$ be a transitive orientation of a graph $G$. A linear order $<$ on the vertex-set of $G$ is said to be *consistent* with $\Theta$ if

$$u < v \text{ whenever } uv \in \Theta.$$

(Note that such a linear order is readily available: we only need apply a topological sort to $G$. Furthermore, every source in $\Theta$ can be placed first in $<$.)

We are now in a position to show that Theorem 1 implies the following result.

**Corollary 1.1** (Gimbel [4]). *A vertex $w$ in an interval graph $G$ is an end vertex if, and only if, $G$ contains an induced subgraphs none of the graphs $\bar{F}_1$, $\bar{F}_2$, or $F_5$ featured in Fig. 1, with $w$ one of the highlighted vertices.*

**Proof.** The 'only if' implication is easy: we only need observe that if $G$ contains an induced subgraph isomorphic to one of the graphs, $\bar{F}_1$, $\bar{F}_2$, or $F_5$ featured in Fig. 1, then the highlighted vertices cannot correspond to an end interval in any interval representation of $G$.

To prove the 'if' implication, assume that $G$ contains no induced subgraph isomorphic to one of the graphs $\bar{F}_1$, $\bar{F}_2$, or $F_5$ featured in Fig. 1. Since $G$ must be a triangulated graph, $G$ cannot contain an induced subgraph isomorphic to the complement of the graph $F_3$ or $F_4$ of Fig. 1. Consequently, $w$ is a regular vertex in $\bar{G}$. By Theorem 1, $w$ is a source in some transitive orientation $\Theta$ of $\bar{G}$. Now a result of Gilmore and Hoffman [3] guarantees that

> *for every transitive orientation $\Theta$ of the edges of $\bar{G}$, there exists a linear order $<$ on the set of the maximal cliques of $G$ such that $<$ is consistent with $\Theta$ and such that for every vertex $x$ of $G$ the maximal cliques containing $x$ occur consecutively in $<$.*

(For a proof the interested reader is referred to Golumbic [5, pp. 172–173].)

Note, furthermore, that by virtue of (3), the set $I_x$ of all the maximal clique containing $x$ is an *interval.* Since $w$ is a source in $\Theta$, $I_w$ becomes an end interval in $\{I_v\}_{v\in V}$, as claimed. □

Recently, Skrien and Gimbel [6] proposed to call an interval graph $G$ *homogeneously representable* if for every vertex $v$ of $G$, there exists an interval representation of $G$ in which $I_v$ is an end interval.

Theorem 1 implies the following characterization of the homogeneously representable interval graphs.

**Corollary 1.2** (Skrien and Gimbel [6]). *An interval graph $G$ is homogeneously representable if, and only if, $G$ contains no induced subgraph isomorphic to one of the graphs $\bar{F}_1$ and $F_5$ with $n = 3$.*

**Proof.** The 'only if' implication is immediate; to settle the 'if' implication, we only need show that every vertex of $\bar{G}$ can be a source in some transitive orientation of $\bar{G}$, for then the conclusion follows by (3). Since $G$ is triangulated, $\bar{G}$ cannot have an induced subgraph iomorphic to one of the graphs $F_3$ and $F_4$; further, $F_1$ is an induced subgraph of $F_2$. Consequently, every vertex of $\bar{G}$ must be regular, and Theorem 1 implies that every vertex of $\bar{G}$ is a source in some transitive orientation of $\bar{G}$. □

## References


[1] P. Duchet, Classical Perfect Graphs, Ann. Discrete Math. 21 (North-Holland, Amsterdam, 1984).
[2] T. Gallai, Transitiv orientierbare Graphen, Acta Math. Acad. Sci. Hungar. 18 (1967) 25–66.
[3] P.C. Gilmore and A.J. Hoffman, A characterization of comparability graphs and interval graphs, Canad. J. of Math. 16 (1964) 539–548.
[4] J. Gimbel, End vertices in interval graphs, Discrete Appl. Math. 21 (1988) 257–259.
[5] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs (Academic Press, New York, 1980).
[6] D. Skrien and J. Gimbel, Homogeneously representable interval graphs, Discrete Math. 55 (1985) 213–216.