1993

Quasi-Brittle Graphs, a New Class of Perfectly Orderable Graphs

Stephan Olariu
Old Dominion University

Original Publication Citation

Olariu, S. (1993). Quasi-brittle graphs, a new class of perfectly orderable graphs. *Discrete Mathematics, 113*(1-3), 143-153.
doi:10.1016/0012-365x(93)90513-s

Discrete Mathematics 113 (1992) 143–153
North-Holland

Quasi-brittle graphs, a new class of perfectly orderable graphs

Stephan Olariu*

Department of Computer Science, Old Dominion University, Norfolk, VA 23529-0162, USA

Received 7 January 1989
Revised 23 May 1991

Abstract

Olariu, S., Quasi-brittle graphs, a new class of perfectly orderable graphs, Discrete Mathematics 113 (1992) 143–153.

A graph G is quasi-brittle if every induced subgraph H of G contains a vertex which is incident to no edge extending symmetrically to a chordless path with three edges in either H or its complement $\bar{H}$. The quasi-brittle graphs turn out to be a natural generalization of the well-known class of brittle graphs. We propose to show that the quasi-brittle graphs are perfectly orderable in the sense of Chvátal: there exists a linear order $<$ on their set of vertices such that no induced path with vertices a, b, c, d and edges ab, bc, cd has $a < b$ and $d < c$.

1. Introduction

A linear order $<$ on the set of vertices of a graph G is *perfect* in the sense of Chvátal [4] if no induced path with vertices a, b, c, d and edges ab, bc, cd has $a < b$ and $d < c$.

Graphs which admit a perfect order are termed *perfectly orderable.* Chvátal [4] proved that if a graph G admits a perfect order, then an optimal coloring of G is obtained by using the greedy heuristic 'always use the smallest possible color'.

To this day, the structure of perfectly orderable graphs is not well understood. In particular, it is now known [10] that the recognition of perfectly orderable graphs is an NP-complete problem. Quite naturally, this motivated the study of particular classes of perfectly orderable graphs.

As a first step in this direction, Chvátal [3] suggested the study of *brittle* graphs which we are about to define. For this purpose, however, we need to define a few new terms.

Correspondence to: Stephan Olariu, Department of Computer Science, Old Dominion University, Norfolk, VA 23529-0162, USA.

* This work was supported by the National Science Foundation under grant CCR-8909996.

0012-365X/93/$06.00

It is customary to let P_k stand for the chordless path with k vertices. To simplify our notation, a P_4 with vertices a, b, c, d and edges ab, bc, cd will be denoted by $abcd$. In this context, we shall refer to a, d as *endpoints* and to b, c as *midpoints* of the P_4; the edges ab and cd are termed *wings* of the P_4 $abcd$. An edge ab of a graph G is a *symmetric wing* if there exist vertices c, d, p, q such that both $abcd$ and $bapq$ are P_4s in G. In the presence of a linear order $<$ on G, a P_4 $abcd$ is called an *obstruction* if $a<b$ and $d<c$. (In this notation, a graph is perfectly orderable if there exists an obstruction-free linear order on the vertices of G.)

Call a graph G *brittle* if every induced subgraph H of G contains a vertex which is either endpoint of midpoint of no P_4 in H.

It is an easy observation that brittle graphs are closed under complementation, and that they are perfectly orderable. Furthermore, they generalize triangulated graphs (i.e. graphs such that every cycle of length greater than three has a chord), and are recognizable in polynomial time Khouzam [9].

Several classes of brittle graphs were studied by Preissmann, de Werra, and Mahadev [12], Hoàng [6], Hoàng and Khouzam [7], Hertz and de Werra [5], Jamison and Olariu [8], and Olariu [11], among others.

The purpose of this work is to present a natural generalization of the class of brittle graphs, and to show that this new class of graphs is perfectly orderable. More precisely, a vertex w of a graph G is said to be *special* if w is incident with no symmetric wing in G and $\bar{G}$.

A graph G is said to be *quasi-brittle* if every induced subgraph H of G contains a special vertex. It is easy to see that every brittle graph is quasi-brittle: if some vertex z is endpoint of no P_4 in G, or midpoint of no P_4 in G, then z must be special. Fig. 1 features a graph that is quasi-brittle but not brittle. Hence the class of quasi-brittle graphs strictly contains the class of brittle graphs.

In addition, it turns out that the quasi-brittle graphs are perfectly orderable and can be recognized in polynomial time.

2. The main result

All the graphs in this work are finite, with no loops or multiple edges. In addition to standard graph-theoretical terminology compatible with Berge [1], we use some new terms that we are about to define.

Let $G=(V, E)$ be an arbitrary graph. For a vertex x of G, we let $N_G(x)$ denote the set of all the vertices of G which are adjacent to x: we assume adjacency to be non-reflexive, and so $x \notin N_G(x)$; we let $N'_G(x)$ stand for the set of vertices adjacent to x in the complement $\bar{G}$ of G. (The notation will be shortened to $N(x)$ and $N'(x)$ when the underlying graph is understood and no confusion is possible.) A proper subset H ($|H| \geqslant 2$) of vertices of G will be referred to as *homogeneous* if every vertex outside H is either adjacent to all the vertices in H or to none of them.

We are now in a position to state our main result.

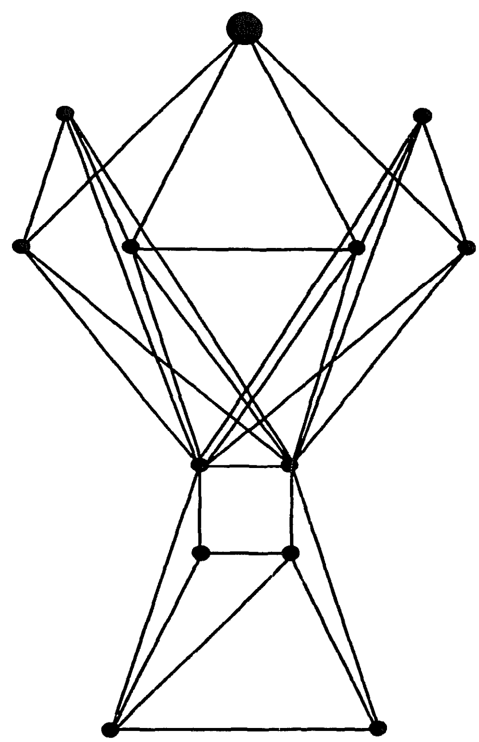

Fig. 1.

Theorem 1. *Every quasi-brittle graph is perfectly orderable.*

Proof of Theorem 1. Let $G=(V, E)$ be a quasi-brittle graph. Assuming the statement true for all quasi-brittle graphs with fewer vertices than G, we need only prove that G itself is perfectly orderable.

For this purpose, we shall find it convenient to rely on a number of intermediate results that we present as facts.

Fact 1. *If G contains a homogeneous set, then G is perfectly orderable.*

Proof of Fact 1. Let H be a homogeneous set in G, and let h stand for an arbitrary vertex in H. By the induction hypothesis, we find a perfect order

$$h_1 <_H h_2 <_H \cdots <_H h_{|H|}$$

on the vertices of H. Similarly, there exists a perfect order

$$x_1 < x_2 < \cdots < h = x_j < \cdots < x_{|V|-|H|+1}$$

on the vertices of $G-(H-h)$.

But now, it is easy to see that

$$x_1 < x_2 < \cdots < x_{j-1} < h_1 < \cdots < h_{|H|} < x_{j+1} < \cdots < x_{|V|-|H|+1}$$

is a perfect order on G, as claimed. □

Fact 1 allows us to assume that G contains no homogeneous set.

Next, we note that

$$\text{every special vertex in } G \text{ is both midpoint of some } P_4 \text{ in } G \text{ and endpoint of some } P_4 \text{ in } G. \tag{1}$$

[To justify (1), consider a vertex x that is endpoint of no P_4 in G, and let $z_1 < z_2 < \cdots < z_{|V|-1}$ be a perfect order on $G - x$. It is easy to see that $x < z_1 < z_2 < \cdots < z_{|V|-1}$ is a perfect order on G. Similarly, if x is midpoint of no P_4 in G, then the linear order $z_1 < z_2 < \cdots < z_{|V|-1} < x$ is a perfect order on G.]

Let w be a special vertex in G and let $F_1, F_2, \ldots, F_k$ $(k \geq 1)$ stand for the connected components of the subgraph of $\bar{G}$ induced by $N(w)$. We may assume without loss of generality that

$$|F_1| \leq |F_2| \leq \cdots \leq |F_k|. \tag{2}$$

Fact 2. *Let x be an arbitrary vertex in $N(w)$. If wx extends to a P_4 $wxpq$ in G, then the component F_i containing x satisfies $F_i = \{x\}$.*

Proof of Fact 2. Clearly, $p, q \in N'(w)$. We claim that

$$wypq \text{ is a } P_4, \text{ for every choice of the vertex } y \text{ in } F_i. \tag{3}$$

[Let y be an arbitrary vertex in F_i. To begin, assume that $xy \notin E$; note that if $yp \notin E$ then, in $\bar{G}$, both $wpyx$ and $pwqx$ are P_4s contradicting that w is special. Thus $yp \in E$. Similarly, if $yq \in E$ then, in G, the edge wx is symmetric wing, a contradiction. Next, if $xy \in E$, then the conclusion follows by an easy inductive argument on the length of the shortest path in $\bar{F}_i$ joining x and y.]

To complete the proof of Fact 2, we need only show that

If $|F_i| \geq 2$, then F_i is a homogeneous set.

[Suppose not; now some vertex u in $V - F_i$ is adjacent to some, but not all the vertices in F_i. Clearly, u belongs to $N'(w)$. By the connectedness of F_i in $\bar{G}$, we find vertices z, z' in F_i with uz', $zz' \notin E$ and $uz \in E$. But now, the edge $z'w$ extends to a P_4, namely $z'wzu$. By (3), wz' also extends to a P_4, contradicting that w is special.]

With this, the proof of Fact 2 is complete. □

Fact 2 can be rephrased as follows.

Corollary 2a. *If $|F_i| \geq 2$ for all $i = 1, 2, \ldots, k$, then w is endpoint of no P_4 in G.*

Note that (1) and (2), together with Corollary 2a imply the existence of a subscript i_0 $(2 \leq i_0 \leq k)$ such that

$$|F_i| \geq 2 \text{ if, and only if, } i \geq i_0.$$

Next, we enumerate the connected components of the subgraph of G induced by $N'(w)$ as

$$H_1, H_2, \ldots, H_m \quad (m \geq 1)$$

such that

$$|H_1| \leq |H_2| \leq \cdots \leq |H_m|. \tag{4}$$

Fact 3. *Let x' be an arbitrary vertex in $N'(w)$. If wx' extends to a P_4 in $\bar{G}$, then the component H_j of $N'(w)$ containing x' satisfies $H_j = \{x'\}$.*

The proof of Fact 3 mirrors that of Fact 2 and is, therefore, omitted.

An equivalent way of stating Fact 3 goes as follows.

Corollary 3a. *If $|H_j| \geq 2$ for all $j = 1, 2, \ldots, m$, then w is midpoint of no P_4 in G.*

[By Fact 3, w is endpoint of no P_4 in $\bar{G}$. Since every P_4 is self-complementary, w is midpoint of no P_4 in G.]

Note that (1), (4), together with Corollary 3a, imply the existence of a subscript j_0 $(2 \leq j_0 \leq m)$ such that

$$|H_j| \geq 2 \text{ if, and only if, } j \geq j_0.$$

To simplify the notation, we write

$$A = \bigcup_{i=1}^{i_0-1} F_i \quad \text{and} \quad A' = \bigcup_{j=1}^{j_0-1} H_j.$$

Now the definition of A and A' imply that

$$\text{every vertex in } A \text{ is adjacent to all the remaining vertices in } N(w) \tag{5}$$

and

$$\text{every vertex in } A' \text{ is non-adjacent to all the remaining vertices in } N'(w). \tag{6}$$

Fact 4. *Let i, j be arbitrary subscripts such that $i_0 \leq i \leq k$ and $j_0 \leq j \leq m$. Then, either every vertex in F_i is adjacent to all the vertices in H_j or no vertex in F_i is adjacent to a vertex in H_j.*

Proof of Fact 4. Since, by assumption, H_j is not homogeneous, some vertex y in $V - H_j$ is adjacent to some, but not all the vertices in H_j. By the connectedness of H_j, we find adjacent vertices h, h' in H_j such that $yh \in E$ and $yh \notin E$. Trivially, $y \in N(w)$.

We claim that $y \in A$. [If $y \in F_p$ for some $p \geq i_0$, then wy extends to a P_4, namely $wyhh'$, contradicting Fact 2.]

Next, if for some subscript i ($i_0 \leq i \leq k$), F_i contains vertices that are adjacent to all the vertices in H_j along with vertices which are adjacent to none of the vertices in H_j then by the connectedness of $\bar{F}_i$ we find vertices z, z' in F_i with $zz' \notin E$, such that $zu \in E$, and $z'u \notin E$ for all vertices u in H_j.

In particular, zh, $zh' \in E$, and $z'h$, $z'h' \neq E$; but now, in $\bar{G}$, $whz'z$ and $hwh'y$ are both P_4s, contradicting that w is a special vertex.

This completes the proof of Fact 4. □

Since by the induction hypothesis $G - w$ is perfectly orderable, we let $<$ stand for an arbitrary perfect order on $G - w$. A component F_i with ($i_0 \leq i \leq k$) is referred to as *impure* if there exist vertices u, u' in F_i and a vertex t in A' such that $tu \in E$, uu', $tu' \notin E$, and $t < u$. A component F_i ($i_0 \leq i \leq k$) that is not impure is called *pure*.

Trivially, we can write $N(w) = A \cup P \cup I$ with P and I standing for the set of all pure and impure components F_i, respectively.

Let $<'$ be the linear order on $G - w$ defined as follows:

- $x <' y$ whenever $x \in A \cup P \cup (N'(w) - A')$ and $y \in I \cup A'$;
- $x <' y$ whenever $x < y$ and $x, y \in A \cup P \cup (N'(w) - A')$, or $x, y \in I \cup A'$.

To complete the proof of Theorem 1, we use the following result that we shall prove later.

Theorem 2. *$<'$ is a perfect order on $G - w$.*

We propose to show that $<'$ extends naturally to a perfect order on G. To see this, note that the definition of $<'$ guarantees that we can enumerate the vertices of $G - w$ as

$$z_1 <' z_2 <' \cdots <' z_r <' z_{r+1} <' \cdots <' z_{|V|-1}$$

in such a way that

$$z_j \in A' \cup I \quad \text{for } j = r + 1, \ldots, |V| - 1.$$

We claim that the linear order on G defined by

$$z_1 <' z_2 <' \cdots <' z_r <' w <' z_{r+1} <' \cdots <' z_{|V|-1}$$

is a perfect order.

Consider an obstruction $x_1x_2x_3x_4$ in G with $x_1 <' x_2$ and $x_4 <' z_3$. Now Theorem 2 together with the symmetry of the P_4 allows us to assume that w coincide with x_1 or with x_2.

However, in case $w = x_1$, by the definition of $<'$, x_2 must belong to I and, by (6), x_3 and x_4 must belong to $N'(w) - A'$, contradicting Fact 4; in case $w = x_2$, (5) together with $x_4 <' x_3$ implies that $x_1, x_3 \in F_i \subseteq P$ and so, by Fact 4, $x_4 \in A'$, contradicting that $x_1x_2x_3x_4$ is an obstruction. □

Proof of Theorem 2. We shall inherit the notation and the entire context of the proof of Theorem 1. If $<'$ fails to be a perfect order on $G - w$, then we find an obstruction $abcd$ with $a <' b$ and $d <' c$.

Fact 5. $a \notin N(w)$.

Proof of Fact 5. To begin, we claim that

$$\text{There is no } P_4 \; xypq \text{ in } G - w \text{ with } x \in A \text{ and } y \in (N(w) - A) \cup A'. \quad (7)$$

[Suppose this is not the case; if y belongs to $N(w) - A$ then, by virtue of (5) and (6) combined, p, q belong to $N'(w)$. But now, y is adjacent to p and non-adjacent to q, contradicting Fact 4. Similarly, if y belongs to A', then by (5) and (6) lead to an immediate contradiction.]

It is easy to see that $a \notin A$. [Otherwise, by (5), $c, d \in N'(w)$; by (6), $c, d \in (N'(w) - A')$. Now, if b belongs to $N(w)$ then, by Fact 4, b belongs to A; if b belongs to $N'(w) - A'$ by (6). In both cases $abcd$ is an obstruction in $<$.]

Next, we claim that

$$\text{If an edge } xy \text{ with } x \in F_i \; (i_0 \leqslant i \leqslant k) \text{ and } y \in A \cup (N'(w) - A') \text{ extends to a } P_4 \; xyzt \text{ in } G - w, \text{ then either } z, t \in H_j \text{ for some } j \geqslant j_0, \text{ or else } z \in F_i \text{ and } t \in A'. \quad (8)$$

[First, if $y \in N'(w) - A'$, then by Fact 4 together with the definition of the F_j's ($j = 1, 2, \ldots, k$), it follows that $z \in F_i$ and $t \in A'$. Next, if $y \in A$ then either $z \in F_i$ and, by Fact 4, $t \in A'$, or else $z \in N'(w) - A'$ and, by Fact 4, (5), and (6) combined $t \in N'(w) - A'$, as claimed.]

We note that, by virtue of (8),

$$a \notin P.$$

[Suppose $a \in F_i \subset P$. If $d \in N(w)$ then since d is not adjacent to a, we have $d \in F_i$. Since there is no obstruction in $<$, we can set without loss of generality that $c \notin F_i$. Since a is not adjacent to c, $c \in N'(w)$ and by Fact 4, $c \in A'$. Now by (6), $b \in N(w)$ and, since b is not adjacent to d, $b \in F_i$. We obtain a contradiction either of the fact that $<$ has no obstruction or that F_i is pure. If $d \in N'(w)$ then we consider two cases. If $c \in N(w)$ then $c \in F_i$ and by Fact 4, $d \in A'$ and so $abcd$ is not an obstruction in $<'$. If $c \in N'(w)$ then since c and d are adjacent, we have that c and d belong to some $H_j \subset N'(w) - A'$. Now by Fact 4 and (6) combined, we have that $b \in A$ or $b \in H_j$, a contradiction to the fact that $<$ has no obstruction.]

Finally, to complete the proof of Fact 5, we need show that the assumption that a belongs to I leads to a contradiction. To see this, note that by the definition of $<'$ together with the fact that $a <' b$, it must be that $b \in I \cup A'$. By (6), at least two of the vertices a, b, c belong to some component $F_i \subseteq I$. Since $cd \in E$ and $bd \notin E$, d cannot belong to $A \cup P \cup (N'(w) - A')$. But now, $abcd$ is an obstruction in $<$. This is the desired contradiction and the proof of Fact 5 is complete. □

Observe that Fact 5 guarantees, by symmetry, that

$$d \notin N(w). \tag{9}$$

Fact 6. *One of the vertices a, d belongs to $N'(w) - A'$ and the other one to A'.*

Proof of Fact 6. By (9) and Fact 5 combined, it follows that $a, d \in N'(w)$.

We claim that

$$\text{at least one of the vertices } a \text{ and } d \text{ does not belong to } A'. \tag{10}$$

[To justify (10), note that if both a, d belong to A' then, the definition of $<'$ together with the assumption that $a <' b$ and $d <' c$ imply that $b, c \in I$, and so $abcd$ must be an obstruction in $<$, a contradiction.]

Next, we claim that

$$\text{at least one of the vertices } a, d \text{ does not belong } N'(w) - A'. \tag{11}$$

Our justification of (11) relies on the following simple observation.

Observation 1. *Let j be a subscript such that $F_j \subset I$, and let x be a vertex in $N'(w) - A'$ adjacent to some vertex in F_j. Then, for a suitably chosen vertex y in F_j, $xy \in E$ and $x < y$.*

[By Fact 4, x is adjacent to all the vertices in F_j. Since F_j is impure, we find a vertex t in A' and non-adjacent vertices u, u' in F_j such that $tu \in E'$, $tu' \notin E$, and $t < u$. Now the conclusion follows from the P_4 $tuxu'$.]

To justify (11), assume that both a, d belong to $N'(w) - A'$. By (6), it follows that neither of b, c belongs to A'.

Note, further, that the assumption that $a <' b$ and $d <' c$ together with the fact that $abcd$ is not an obstruction in $<$, guarantees that

at least one of the vertices b, c belongs to I.

Symmetry allows us to assume that $b \in F_j \subseteq I$.

Now Observation 1 guarantees the existence of a vertex b' in F_j such that $ab' \in E$ and $a < b'$. Since $ab \in E$ and $ac \notin E$, Fact 4 guarantees that $c \notin F_j$, and so we must have $b'c \in E$.

By virtue of Fact 4, again, $db' \notin E$ implying that $ab'cd$ is a P_4 in $G - w$. Observe that c must belong to I: otherwise, the definition of $<'$ would imply that $d < c$ and $ab'cd$ would be an obstruction in $<$. Hence, we find a subscript k distinct from j such that $c \in F_k \subseteq I$. By Observation 1, we find a vertex c' in F_k with $dc' \in E$ and $d < c'$.

Trivially, $b'c' \in E$ and, by Fact 4, $ac' \notin E$. Consequently, $ab'c'd$ is an obstruction in $<$, a contradiction. Thus, (11) must hold true.

Finally, we note that the conclusion of Fact 6 follows directly from (10) and (11), combined. □

Symmetry, together with Fact 6 allow us to assume that $a \in N'(w) - A'$ and $d \in A'$.

We claim that $b, c \in I$, and $d < c$. [To see this, note that since $d \in A'$, the definition of $<'$ implies that $c \in I$ and, consequently, $d < c$. Since *abcd* cannot be an obstruction in $<$, we must have $b < a$, and so $b \in I \cup A'$. Now the conclusion follows directly from (6).]

Consequently, we find distinct subscripts i, j $(i, j \geq i_0)$ such that $b \in F_i \subset I$ and $c \in F_j \subset I$.

Since F_i is impure, the set T of all the vertices t of A' for which there exist non-adjacent vertices u, u' in F_i such that $tu \in E$, $tu' \notin E$, and $t < u$ is non-empty.

Let t be an arbitrary vertex in T. Observe that *tuau'* is a P_4 in $G - w$. Since $<$ is perfect, it follows that

$$a < u'. \tag{12}$$

We claim that

$$du' \in E. \tag{13}$$

[Otherwise, since $cu' \in E$, *au'cd* would be an obstruction in $<$.]

Further, note that by (13), the vertices t and d are distinct, having distinct neighbourhoods; since t was an arbitrary vertex in T, it follows that

$$d \text{ is distinct from all the vertices in } T. \tag{14}$$

Next, we note that

$$du \in E. \tag{15}$$

[If $du \notin E$, then since $t < u$ and $d < c$, we must have $tc \in E$, or else *tucd* would be an obstruction in $<$. The P_4 *au'ct* implies that $c < t$; the P_4 *dcua* implies that $u < a$; the P_4 *uau'd* implies that $u' < d$. However, now $\{a, u', d, c, t, u\}$ induces a directed cycle in $<$, a contradiction.]

Note that by (15) we have

$$d < u'. \tag{16}$$

[For otherwise, *tudu'* would be an obstruction in $<$.]

Consider the shortest path

$$\text{(P)} \quad u' = z_0, z_1, \ldots, z_p = b \quad (p \geq 1)$$

in $\bar{F}_i$ joining u' and b. Let r stand for the least subscript for which $dz_r \notin E$: since $db \notin E$, such a subscript must exist. We note that since $d < c$, the P_4 az_rcd implies that

$$z_r < a. \tag{17}$$

Furthermore, $r \geq 2$, for otherwise by (16) and (17), z_raz_0d would be an obstruction in $<$. Note that

$$z_{r-1} < d. \tag{18}$$

[Else, d could play the role of t, contradicting (14).]

By virtue of (18), we have

$$z_{r-2} < z_r. \tag{19}$$

[Otherwise, $z_{r-1}dz_{r-2}z_r$ would be an obstruction in $<$.]

By (19), it must be the case that

$$r \geq 3.$$

[Else $\{a, z_0, z_r\}$ would induce a directed cycle in $<$.]

We claim that

$$z_i < z_{i+2}, \quad \text{for all } i = 0, 1, \ldots, r-2. \tag{20}$$

[To see that this is the case, note that $r \geq 3$ guarantees that $z_{i+1}z_{i+3}z_iz_{i+2}$ is a P_4 for all $i = 0, 1, 2, \ldots, r-3$. By (19), $z_{r-2} < z_r$. Now the conclusion follows by a trivial inductive argument.]

But now, we have reached a contradiction: by (12), (16), (17), (18), and (20) combined, either $\{z_0, z_2, \ldots, z_{r-1}, d\}$ or $\{z_0, z_2, \ldots, z_r, a\}$ induces a directed cycle in $<$, depending on whether or not r is odd. With this, the proof of Theorem 2 is complete. □

Finally, we note that a set $\{x, y, z, t\}$ induces a P_4 in a subgraph H of G only if it induces a P_4 in G itself; in addition a graph $G = (V, E)$ has at most $O(|V|^4)$ distinct P_4s. Consequently, recognizing membership in the class of quasi-brittle graphs can be done in polynomial time in the size of the graph.

Acknowledgement

The author would like to thank Myriam Preissmann for her constructive comments. I would also like to thank the referees for suggestions that improved the presentation.

References

[1] C. Berge, Graphes et Hypergraphes (Dunod, Paris, 1970).
[2] C. Berge and V. Chvátal, Topics on Perfect Graphs, Ann. Discrete Math. 21 (North Holland, Amsterdam, 1984).
[3] V. Chvátal, Perfect graph seminar, McGill University, Montreal, 1983.
[4] V. Chvátal, Perfectly ordered graphs, in: Berge and Chvátal [2].
[5] A. Hertz and D. de Werra, Les Graphes Bipolarisable, Département de Mathmatiques, Ecole Polytechnique Fédérale de Lausanne, Suisse, O.R.W.P. 85/13.

[6] C. Hoang, On brittle graphs: II, manuscript.
[7] C. Hoang and N. Khouzam, On brittle graphs, J. Graph Theory 12 (1988) 391–404.
[8] B. Jamison and S. Olariu, A new class of brittle graphs, Studies Appl. Math. 81 (1989) 89–92.
[9] N. Khouzam, Masters Thesis, McGill University, Montreal, 1986.
[10] M. Middendorf and F. Pfeiffer, On the complexity of recognizing perfectly orderable graphs, Report # 89594-OR, Forschungsinstitut fur Diskrete Mathematik, Universität Bonn, 1989.
[11] S. Olariu, Weak Bipolarizable Graphs, Discrete Math. 74 (1989) 159–171.
[12] M. Preissmann, D. de Werra and N. V. R. Mahadev, A note on superbrittle graphs, Discrete Math. 61 (1986) 259–267.