1992

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Chinh T. Hoang

Frederic Maffray

Stephan Olariu
Old Dominion University

Myriam Preissmann

Original Publication Citation

Hoang, C. T., Maffray, F., Olariu, S., & Preissmann, M. (1992). A charming class of perfectly orderable graphs. *Discrete Mathematics, 102*(1), 67-74. doi:10.1016/0012-365x(92)90348-j

Discrete Mathematics 102 (1992) 67–74
North-Holland

Communication

A charming class of perfectly orderable graphs

Chính T. Hoàng
Department of Mathematical Sciences, Lakehead University, Thunder Bay, Ont., Canada P7B 5E1

Frédéric Maffray
CNRS, Laboratoire de Structures Discrètes, IMAG, BP 53X, 38041 Grenoble Cédex, France

Stephan Olariu
Department of Computer Science, Old Dominion University, Norfolk, VA 23529-0162, USA

Myriam Preissmann
CNRS, Laboratoire ARTEMIS, IMAG, BP 53X, 38041 Grenoble Cédex, France

Communicated by V. Chvátal
Received 14 November 1991

Abstract

Hoàng, C.T., F. Maffray, S. Olariu and M. Preissmann, A charming class of perfectly orderable graphs, Discrete Mathematics 102 (1992) 67–74.

We investigate the following conjecture of Vašek Chvátal: any weakly triangulated graph containing no induced path on five vertices is perfectly orderable. In the process we define a new polynomially recognizable class of perfectly orderable graphs called charming. We show that every weakly triangulated graph not containing as an induced subgraph a path on five vertices or the complement of a path on six vertices is charming.

A classical problem in graph theory is of colouring the vertices of a graph in such a way that no two adjacent vertices receive the same colour. For this purpose a natural way consists of ordering the vertices linearly and colouring them one by one along this ordering, assigning to each vertex v the smallest colour not assigned to the neighbours of v that precede it. This method is called the *greedy algorithm*. Unfortunately it does not necessarily produce an optimal colouring of the graph (i.e., one using the smallest possible number of colours).

0012-365X/92/$05.00

Given an ordered graph $(G, <)$, the ordering $<$ is called *perfect* ([2]) if for each induced ordered subgraph $(H, <)$ the greedy algorithm produces an optimal colouring of H. The graphs admitting a perfect ordering are called *perfectly orderable.* An *obstruction* in an ordered graph is a chordless path with four vertices $abcd$ such that $a < b$ and $d < c$. It is easily seen that a perfectly ordered graph has no obstruction. Chvátal has shown that this condition is also sufficient: *a graph is perfectly orderable if and only if it admits an obstruction-free ordering* ([2]).

Recall that a graph is *perfect* if every induced subgraph H admits an optimal colouring with a number of colours equal to the largest size of a clique of H (see [7, 1]). Chvátal ([2]) has shown that perfectly orderable graphs are perfect, and that perfectly orderable graphs include two well-known classes of perfect graphs (chordal graphs and transitively orientable graphs). More generally it is natural to wonder which graphs among the important families of (perfect) graphs are also perfectly orderable. Chvátal has investigated this question for line-graphs ([5]) and for claw-free graphs ([4]). Another possible class to consider is that of *weakly triangulated graphs.* A graph G is called weakly triangulated if neither G nor its complement $\bar{G}$ contains an induced cycle of length at least five. We denote by P_k (resp. C_k) a chordless path (resp. cycle) with k vertices.

Conjecture 1 (Chvátal [3]). Every weakly triangulated graph with no induced P_5 is perfectly orderable.

The aim of this note is to examine this conjecture. Our main result is the following.

Theorem 1. *Every weakly triangulated graph with no induced P_5 and $\bar{P}_6$ is perfectly orderable.*

For reasons of convenience we will use an alternate definition of perfect orderability. One says that an orientation of a graph G is *perfect* if and only if it is acyclic and its does not contain an induced P_4 $abcd$ with arcs ab and dc. Using the natural correspondence between orderings and acyclic orientations, it is straightforward to check that a graph admits a perfect ordering if and only if it admits a perfect orientation. Without ambiguity a P_4 as in the definition of a perfect orientation will also be called an obstruction.

In a P_k with $k \geqslant 2$ the two vertices of degree 1 are called the *endpoints* of the P_k. In a P_4 the two vertices of degree 2 are called the *midpoints.* The neighbour set of a vertex x is denoted by $N(x)$, and $\bar{N}(x)$ will denote the neighbour set of x in the complement graph.

Definition 1. We will say that a vertex v of a graph G is *charming* if it satisfies the

following three properties:
(c1) v is not the endpoint of a P_5 in G;
(c2) v is not the endpoint of a P_5 in $\bar{G}$;
(c3) v does not lie on a C_5 of G;

Lemma 2. *Let $G = (V, E)$ be a graph with a charming vertex v. Then G is perfectly orderable if and only if $G - v$ is.*

Proof of Lemma 2. The 'only if' part is trivial, so we only need to prove the 'if' part. We suppose that $G - v$ is perfectly orderable; so there exists a perfect orientation $(V - v, A)$ of $G - v$. We define an orientation $\vec{G} = (V, A')$ of G as follows: for every edge with an endpoint x in $N(v)$ and the other endpoint y in $\bar{N}(v) \cup \{v\}$, we put the arc xy in A'; for any other edge we put in A' the orientation which the edge has in A. We are going to prove that $\vec{G}$ is a perfect orientation of G. It is clear that it has no circuits. Let us suppose that $\vec{G}$ has an obstruction $abcd$ (with arcs ab and dc). Note that $v \neq a$ and $v \neq d$ since v has no successor in $\vec{G}$. If $v = b$, then we must have $c \in N(v)$ and $d \in \bar{N}(v)$ and thus $cd \in A'$, a contradiction. Therefore $v \neq b$ and, by symmetry, $v \neq c$. Hence a, b, c, d are all in $V - v$. Since there is no obstruction in (V, A), at least one of the arcs ab and dc is not in A. So we may assume without loss of generality that $a \in N(v)$ and $b \in \bar{N}(v)$. Since v is charming we must have $c \in N(v)$ and $d \in \bar{N}(v)$, for otherwise one of (c1), (c2), (c3) is violated by v in the subgraph induced by v, a, b, c, d. But then $abcd$ is not an obstruction because $cd \in A'$. Consequently $\vec{G}$ is a perfect orientation of G. □

We call *charming* any graph in which every induced subgraph has a charming vertex. It follows from Lemma 2 that every charming graph is perfectly orderable. In particular, this yields a new and shorter proof of the fact that every graph containing no induced P_5, $\bar{P}_5$ and C_5 is perfectly orderable (see [6]), for in such a graph every vertex is charming. We can also remark that a vertex is charming in a graph G if and only if it is charming in the complement of G. Hence a graph is charming if and only if its complement graph is charming.

An ordering $x_1, \ldots, x_n$ of the vertices of a graph G is called *charming* if for each i (with $1 \leqslant i \leqslant n$) x_i is a charming vertex in the subgraph of G induced by $x_1, \ldots, x_i$. (In particular x_n is a charming vertex of G.) The following points are easily seen:

- A graph is charming if and only if it admits a charming ordering, and a charming ordering for G is also a charming ordering for its complement $\bar{G}$.
- The existence of a charming ordering (and its construction, if one exists) can be determined in time polynomial in the size of the input graph. (Recall that in general the recognition of perfectly orderable graphs is an NP-complete problem, as shown in [10].)

• Given a charming ordering of a graph G, one can determine in polynomial time a perfect ordering of G, as in the proof of Lemma 2. However these orderings may be different. Fig. 1 shows a charming graph in which no charming ordering is perfect.

Recall that a graph is *brittle* (see [9]) if every induced subgraph H has a vertex which either is not the midpoint of any P_4 or is not the endpoint of any P_4 in H. Let us name 'domino' the bipartite graph consisting of a cycle with six vertices and with exactly one chord. Then the graph made up of a domino in which each vertex of degree 3 is substituted by the complement of a domino is charming and not brittle. On the other hand P_8 is brittle and not charming. Hence brittle graphs and charming graphs form two incomparable classes of perfectly orderable graphs.

Incidentally, we can ask the following question: is it true that a minimal imperfect graph cannot contain a charming vertex?

Since there exist P_5-free weakly triangulated graphs that are not charming (e.g. $\bar{P}_8$), Lemma 2 does not imply Chvátal's conjecture. Nonetheless we will now see that it implies the validity of a special case of the conjecture.

Definition 2. A P_4 of a graph G is *bad* if there exists a minimal cutset C of G such that the P_4 has one midpoint in $G - C$ and all three other vertices in C.

Lemma 3. *Let G be a weakly triangulated graph. Then G has an induced subgraph isomorphic to one of $\bar{P}_6$, F_1, F_2, or F_3 (see Fig. 2) if and only if there exists an induced subgraph of G that has a bad P_4.*

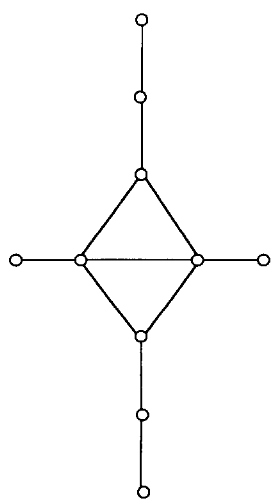

Fig. 1.

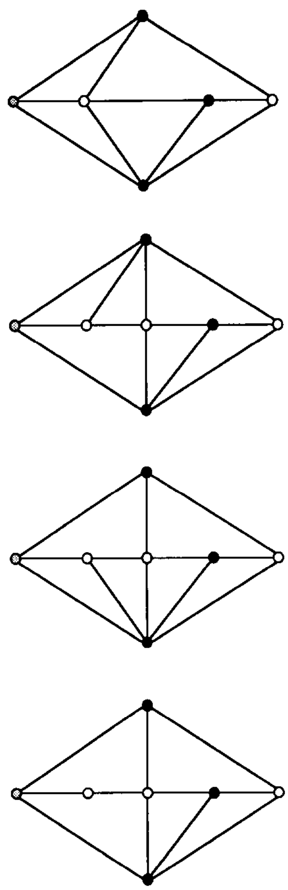

Fig. 2. The graphs $\bar{P}_6$, F_1, F_2, F_3.

Remark. Clearly, a P_5-free graph contains none of F_1, F_2, F_3 as an induced subgraph.

Lemma 4 (Hayward [8]). *Let G be a weakly triangulated graph. Let C be a minimal cutset of G, and D be any connected component of the graph $\bar{G}[C]$. Then every connected component of $G - C$ contains a vertex that is adjacent to all vertices of D.*

Proof of Lemma 3. It is easy to check on Fig. 2 that, for each of the graphs $\bar{P}_6$, F_1, F_2, F_3, the black vertices form a minimal cutset and that the subset of black or grey vertices forms a bad P_4 with respect to that minimal cutset. Hence the 'only if' part of the lemma holds true. Now we will prove 'if' part.

Let G be a weakly triangulated graph having a bad P_4 $abcd$. Let C be a minimal cutset such that $a, c, d \in C$ and $b \notin C$. Let B be the connected component of

$G - C$ that contains b, and B' be another component of $G - C$. Clearly, a, c, d belong to the same connected component of $\bar{G}[C]$. Therefore and by Hayward's Lemma, B (respectively B') contains a vertex x (respectively y) that is adjacent to all three vertices a, c, d. Note that b and x are different since b is not adjacent to d and x is. Since B is connected, there exists a chordless path Q from b to x lying entirely in B. Without loss of generality, we may choose the vertices b and x (with the property that a, c, d are neighbours of x, that a, c are neighbours of b, and that d is not a neighbour of b) in such a way that this path is as short as possible. We now examine the length of Q.

If Q is of length 1, (i.e., b and x are adjacent), then a, b, c, d, x, y induce a $\bar{P}_6$.

Observation: If Q is of length at least 2, an interior vertex v of Q cannot be adjacent to both a and c. Indeed, if v is adjacent to both a and c, consider the pair v, x if v is not adjacent to d, or the pair b, v if v is adjacent to d: in either case the new pair is connected by a subpath of Q shorter than Q, and the choice of b, x is contradicted.

If Q is of length exactly 2, let v be the vertex between b and x along Q. By the observation, v is not adjacent to both a and c. If v is adjacent to a and not to c, then v must not be adjacent to d, for otherwise v, y, b, d, a, c induce a C_6 in $\bar{G}$; now a, b, c, d, v, x, y induce an F_1 and G. If v is not adjacent to a, then v must not be adjacent to d, for otherwise v, b, a, y, d induce a C_5; now a, b, c, d, v, x, y induce an F_2 or an F_3 in G.

If Q is of length at least 3, write $Q = bv_1v_2 \cdots v_k$ with $v_k = x$ and $k \geq 3$. Remark that a must be adjacent to at least one of v_1, v_2, for otherwise we can find an induced cycle $abv_1 v_2 \cdots v_i$ of length at least 5 (where i is the smallest integer such that $v_i \in N(a)$), contradicting the fact that G is weakly triangulated. The same argument holds for c instead of a. However, by the observation above, no interior vertex of Q can be adjacent to both a and c. It follows that the edges between $\{a, c\}$ and $\{v_1, v_2\}$ are either av_1 and cv_2 or av_2 and cv_1; in either case y, a, c, v_1, v_2 induce a C_5 in G, a contradiction. This completes the proof. □

Lemma 5. *A graph G such that no induced subgraph of G has a bad P_4 contains a vertex satisfying* (c2).

Proof. We will prove the lemma by induction on the order of G. The lemma is true when G has one vertex. We now assume that it is proved for all graphs with strictly less vertices than G.

We call *side* of G any set $B \subset V$ for which there exists a minimal cutset C of G such that B is a connected component of $G - C$. We will show that:

$$\text{Every side of } G \text{ contains a vertex satisfying (c2).} \qquad (1)$$

It is easy to see that every graph that is not complete has at least two non-empty sides, and that every vertex of a complete graph is charming. Thus (1) implies the lemma.

Assume that (1) is false: there exists a side B of G that contains no vertex satisfying (c2). We choose B of minimum size with this property, and we denote by C a minimal cutset of G such that B is a component of $G - C$.

We first suppose that B is of size 1, and write $B = \{b\}$. Note that $C = N(b)$ by the minimality of C. If b is the endpoint of a P_5 *bstuv* in $\bar{G}$, then *usvt* is a bad P_4 (with respect to C) in G, contradicting the hypothesis; thus b satisfies (c2).

We now suppose that B is of size at least 2. We call *homogeneous* any set S of vertices such that every vertex in $V - S$ is adjacent to either all or none of the vertices of S. We distinguish between two cases.

Case 1: *B is a homogeneous set of G.*

By the induction hypothesis the graph $G[B]$ has a vertex b that satisfies (c2) in $G[B]$. Suppose that b is the endpoint of a P_5 *bstuv* in $\bar{G}$. Since B is homogeneous, the vertices s, t, u, v are either all in B or all in $V - B$. If they are in B, then b violates (c2) in $G[B]$, a contradiction. If they are in $V - B$, then *usvt* is a bad P_4 (with respect to C) in G, contradicting the hypothesis of the lemma.

Case 2: *B is not a homogeneous set of G.*

Since B is not homogeneous, there are two non adjacent vertices b and c with $b \in B$ and $c \in C$. The set $N(b)$ is a cutset separating b and c; so it contains a minimal cutset C' of G. Clearly $C' \subseteq C \cup B$ and $c \in C - C'$. Since C is a minimal cutset of G, every vertex in C, and in particular c, has at least one neighbour in each component of $G - C$. It follows that the set $(C - C') \cup (V - C - B)$ induces a connected subgraph of $G - C'$, and so it must be contained in one connected component of $G - C'$. Hence any other connected component of $G - C'$ is included in $B - C'$. Since $c \notin C'$ and C is a minimal cutset of G, we have $C' \cap B \neq \emptyset$. We conclude that there exists a connected component B' of $G - C'$ that is strictly included in B. By the minimality of B, B' must contain a vertex that satisfies (c2) in G.

In both cases B contains a vertex satisfying (c2) in G, and the proof is complete. □

Theorem 6. *Every weakly triangulated graph with no induced P_5 and $\bar{P}_6$* is charming.

Proof. Let G be a weakly triangulated graph with no induced P_5 or $\bar{P}_6$. Note that every vertex of G satisfies conditions (c1) and (c3); thus a given vertex of G is charming if and only if it satisfies (c2). The existence of such a vertex is a consequence of Lemma 3, the remark following it, and Lemma 5. □

Now Theorem 1 follows as a simple corollary of the above.

Note that the proof above actually yields that every weakly triangulated graph with no induced P_5 and $\bar{P}_6$ either is a clique or possesses two non-adjacent charming vertices. This is not true for all charming graphs: for example P_7 is charming and has just one charming vertex.

Finally, since the complement of a charming graph is also charming and hence perfectly orderable, we obtain as a corollary of Theorem 6 that every weakly triangulated graph with no induced $\bar{P}_5$ or P_6 is perfectly orderable. This parallels a result of Hoàng and Khouzam ([9]) which states that a weakly triangulated graph with no induced $\bar{P}_5$ or domino is perfectly orderable.

Acknowledgements

This work was done while the first author was at the Institut für Ökonometrie und Operations Research and the Forschungsinstitut für Diskrete Mathematik, University of Bonn, Germany—work supported by the Alexander von Humbolt Foundation and Sonderforschungsbereich 303 (DFG). The second author was at the Department of Computer Science of the University of Toronto, Ontario. Part of this work was done while the first and third authors were invited visitors at the Laboratoire Artémis in Grenoble. Partial support by the National Science Foundation under grant CCR-8909996 is gratefully acknowledged by the third author.

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