1995

Error Estimates and Lipschitz Constants for Best Approximation in Continuous Function Spaces

M. Bartelt

W. Li
Old Dominion University

Original Publication Citation

Bartelt, M., & Li, W. (1995). Error-estimates and Lipschitz-constants for best approximation in continuous function spaces. *Computers & Mathematics with Applications, 30*(3-6), 255-268. doi:10.1016/0898-1221(95)00104-2

Pergamon

Computers Math. Applic. Vol. 30, No. 3–6, pp. 255–268, 1995

0898-1221/95 \$9.50 + 0.00

0898-1221(95)00104-2

Error Estimates and Lipschitz Constants for Best Approximation in Continuous Function Spaces

M. BARTELT
Department of Mathematics, Christopher Newport University
Newport News, VA 23606, U.S.A.
mbartelt@pcs.cnu.edu

W. LI
Department of Mathematics and Statistics, Old Dominion University
Norfolk, VA 23529, U.S.A.
wuli@math.odu.edu

Abstract—We use a structural characterization of the metric projection $P_G(f)$, from the continuous function space to its one-dimensional subspace G, to derive a lower bound of the Hausdorff strong unicity constant (or weak sharp minimum constant) for P_G and then show this lower bound can be attained. Then the exact value of Lipschitz constant for P_G is computed. The process is a quantitative analysis based on the Gâteaux derivative of P_G, a representation of local Lipschitz constants, the equivalence of local and global Lipschitz constants for lower semicontinuous mappings, and construction of functions.

Keywords—Error bounds, Lipschitz constants, Gâteaux derivatives, Metric projections, Strong uniqueness.

1. INTRODUCTION

Consider the following minimization problem:

$$\inf_{g\in G} \Phi(g), \tag{1}$$

where G, called the feasible set, is a subset of a normed linear space Y with norm $\|\cdot\|$, and Φ, called the objective function, is a real-valued function defined on Y. Assume that $\Phi_{\min} := \inf_{g\in G} \Phi(g)$ is finite and the optimal solution set $S := \{g \in G : \Phi(g) = \Phi_{\min}\}$ is not empty. Then there are two fundamental problems associated with (1)—error estimates and stability analysis [1–3].

Error estimates refer to estimates of the distance from an approximate solution to the optimal solution set. Error estimates are extremely important in convergence analysis of iterative algorithms for finding an optimal solution of (1), as shown in recent literature [4–21]. Another important application of error estimates is to provide *a priori* information on how far an approximate solution is from the optimal solution set [20,22–46]. Such *a priori* information can be used as a reliable termination criterion of an iterative method for solving (1). Stability analysis (or sensitivity analysis) refers to the study of the behavior of the optimal solution set under perturbation of parameters (or data) involved in the definition of Φ and/or G [1–3].

Here we are interested in the following type of error estimates:

$$\operatorname{dist}(g, S) \le \gamma\left(\Phi(g) - \Phi_{\min}\right), \qquad \text{for } g \in G, \tag{2}$$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX where γ is some positive number and $\mathrm{dist}(g, S)$ is the distance from g to the optimal solution set S defined as

$$\mathrm{dist}(g, S) := \inf_{s \in S} \|g - s\|.$$

If (2) holds, then one can say that (1) has a weak sharp minimum (cf. [5,47–50]). See [4,5,9,21,50] for applications of the weak sharp minimum property in convergence analysis of iterative methods for solving (1). The existence of γ is sufficient for qualitative applications of weak sharp minimum properties, such as in the convergence analysis of algorithms. However, in order to obtain *a priori* error estimates, one must also have a quantitative analysis of γ. For this purpose, it is important to derive an explicit expression for the smallest γ which satisfies (2):

$$\gamma_{\min} := \inf_{g \in G \setminus S} \frac{\Phi(g) - \Phi_{\min}}{\mathrm{dist}(g, S)}. \tag{3}$$

In this paper, we give a quantitative analysis of $\gamma_{\min}$ for a special optimization problem—the best approximation problem in continuous function spaces. For this special problem, $\gamma_{\min}$ is closely related to the Lipschitz constant of S with respect to perturbations of the data function involved. Therefore, we also give a quantitative analysis of the related Lipschitz constant.

Let G be a finite-dimensional subspace of the Banach space $C_0(T)$ of all real-valued continuous functions on a locally compact Hausdorff space T which vanish at infinity (i.e., $\{x \in T : |f(x)| \geq \epsilon\}$ is compact for $f \in C_0(T)$ and $\epsilon > 0$). The supremum norm of $C_0(T)$ is defined as $\|f\| := \sup_{x \in T} |f(x)|$ for $f \in C_0(T)$ and the objective function for the best approximation problem is $\Phi(g) := \|f - g\|$ which depends on a (data) function f in $C_0(T)$. In this setting, the optimal solution set is actually a set-valued mapping $P_G(\cdot)$ from $C_0(T)$ to subsets of G, called the range of the metric projection and defined as

$$P_G(f) := \{g \in G : \|f - g\| = \mathrm{dist}(f, G)\}.$$

See [51] for set-valued analysis. Note that weak sharp minimum in this case was also called Hausdorff strong uniqueness [52], because it is a set-valued version of the classical strong uniqueness property of Haar subspaces [53–55]. Here we want to find the exact values of the uniform Hausdorff strong unicity constant Γ of P_G and the Lipschitz constant Λ of P_G, respectively, where

$$\Gamma := \inf \left\{ \frac{\|f - g\| - \mathrm{dist}(f, G)}{\mathrm{dist}\,(g, P_G(f))} : f \in C_0(T),\ g \in G \text{ with } g \notin P_G(f) \right\}, \tag{4}$$

$$\Lambda := \sup \left\{ \frac{\mathrm{H}\,(P_G(f), P_G(h))}{\|f - h\|} : f, h \in C_0(T) \text{ with } f \neq h \right\}, \tag{5}$$

where $\mathrm{H}(A, B)$ is the Hausdorff distance between two sets A and B defined as

$$\mathrm{H}(A, B) := \max \left\{ \sup_{a \in A} \mathrm{dist}(a, B),\ \sup_{b \in B} \mathrm{dist}(b, A) \right\}.$$

A special case of the best approximation problem in $C_0(T)$ is data regression in $\mathbb{R}^n$ with the supremum norm [56,57]. Note that $C_0(T) \equiv (\mathbb{R}^n, \|\cdot\|_\infty)$, the n-dimensional vector space with the norm $\|y\|_\infty := \max_{1 \leq i \leq n} |y_i|$, if T consists of n isolated points. It is well known that the best approximation problem in $(\mathbb{R}^n, \|\cdot\|_\infty)$ can be reformulated as a linear programming problem (cf. [56,58]). In [29,31], sharp Lipschitz constants for (basic) optimal solutions and (basic) feasible solutions of a linear program with right-hand side perturbations are given in terms of seminorms of pseudoinverses of certain submatrices. However, we do not know whether the analysis given in [29,31] can be modified to find the exact values of Γ and Λ if G is a closed polyhedral subset of $(\mathbb{R}^n, \|\cdot\|_\infty)$. By using Hoffman's error estimate, Li proved that $\Gamma > 0$ and $\Lambda < \infty$ for any closed convex polyhedral subset G of $(\mathbb{R}^n, \|\cdot\|_\infty)$ [49]. However, for a finite-dimensional subspace of $C_0(T)$, it is not necessary that $\Gamma > 0$ or $\Lambda < \infty$. For any finite-dimensional subspace G of $C_0(T)$, Li proved that the following statements are equivalent [59]:

(a) $\Gamma > 0$.
(b) $\Lambda < \infty$.
(c) $\text{supp}(g) := \{x : g(x) \neq 0\}$ is compact for any $g \in G$.

Therefore, we should only consider a finite-dimensional subspace G whose elements have compact supports. Due to difficulty of the problem, we will only treat the one-dimensional case in the present paper. Therefore, we make the following assumption throughout this paper, unless stated otherwise.

ASSUMPTION 1. *Let* $G := \text{span}\{g_1\}$ *be a one-dimensional subspace of* $C_0(T)$ *such that* $\{x : g_1(x) \neq 0\}$ *is compact.*

The paper is organized as follows. In Section 2, we use a structural characterization of $P_G(f)$ to derive a lower bound of Γ and then show this lower bound can be attained by constructing a function. Section 3 is devoted to finding the exact value of Λ. The process is a quantitative analysis based on the Gâteaux derivative of P_G, a representation of local Lipschitz constants, the equivalence of local and global Lipschitz constants for lower semicontinuous mappings, and construction of functions. In order to give a clean presentation, we put the complicated construction of functions with certain desirable properties in Section 4.

2. HAUSDORFF STRONG UNICITY

In this section, we first give a structural characterization of $P_G(f)$. Using this characterization, we can derive a lower bound for Γ. Then, by constructing a function, we show that this lower bound can be attained. Thus, we obtain the exact value of Γ.

First, we establish a structural characterization of P_G.

LEMMA 2. *Let* $l \leq u$. *Then* $P_G(f) = \{cg_1 : l \leq c \leq u\}$ *if and only if there exist two points* x_l *and* x_u *such that* $g_1(x_l) \neq 0$, $g_1(x_u) \neq 0$, *and*

$$\text{dist}(f, G) = \|f - lg_1\| = \text{sgn}\,(g_1(x_l))\,(f\,(x_l) - lg_1\,(x_l))\,, \tag{6}$$

$$\text{dist}(f, G) = \|f - ug_1\| = -\text{sgn}\,(g_1\,(x_u))\,(f\,(x_u) - ug_1\,(x_u))\,, \tag{7}$$

where $\text{sgn}(a)$ *denote the sign of a number* a.

PROOF. First assume $P_G(f) = \{cg_1 : l \leq c \leq u\}$. Since $\text{supp}(g_1)$ is compact, $\text{sgn}(g_1(x))$ is a continuous function on $\text{supp}(g_1)$. Therefore, there exists $x_l \in \text{supp}(g_1)$ such that

$$\text{sgn}\,(g_1\,(x_l))\,(f\,(x_l) - lg_1\,(x_l)) = \max_{x \in \text{supp}(g_1)} \text{sgn}\,(g_1(x))\,(f(x) - lg_1(x))\,.$$

We claim that

$$\text{dist}(f, G) = \text{sgn}\,(g_1\,(x_l))\,(f\,(x_l) - lg_1\,(x_l))\,. \tag{8}$$

If (8) does not hold, then

$$\delta := \text{dist}(f, G) - \text{sgn}\,(g_1\,(x_l))\,(f\,(x_l) - lg_1\,(x_l)) > 0.$$

Let $0 < l - l_\delta < \delta\|g_1\|$. Then

$$\begin{aligned}\text{sgn}\,(g_1(x))\,(f(x) - l_\delta g_1(x)) &\leq \text{sgn}\,(g_1(x))\,(f(x) - lg_1(x)) + (l - l_\delta)\,|g(x)| \\ &\leq \text{sgn}\,(g_1\,(x_l))\,(f\,(x_l) - lg_1\,(x_l)) + \delta \leq \text{dist}(f, G)\end{aligned}$$

and

$$\begin{aligned}-\text{sgn}\,(g_1(x))\,(f(x) - l_\delta g_1(x)) &\leq -\text{sgn}\,(g_1(x))\,(f(x) - lg_1(x)) + (l - l_\delta)\,|g(x)| \\ &\leq \text{sgn}\,(g_1(x))\,(f(x) - lg_1(x)) \leq \|f - lg_1(x)\| = \text{dist}(f, G).\end{aligned}$$

As a consequence, $\|f - l_\delta g_1(x)\| \le \operatorname{dist}(f,G)$ and $l_\delta g_1 \in P_G(f)$, a contradiction to the assumption that $P_G(f) = \{cg_1 : l \le c \le u\}$. Therefore, (8) holds.

Similarly, we can prove that there exists a point x_u such that $g_1(x_u) \ne 0$ and

$$\operatorname{dist}(f,G) = -\operatorname{sgn}(g_1(x_u))(f(x_u) - ug_1(x_u)).$$

Obviously, we also have

$$\|f - lg_1\| = \|f - ug_1\| = \operatorname{dist}(f,G).$$

On the other hand, if (6) and (7) hold, then, by convexity of $P_G(f)$, we get

$$P_G(f) \supset \{cg_1 : l \le c \le u\}.$$

Since $g_1(x_l) \ne 0$ and $g_1(x_u) \ne 0$, the two equations (6) and (7) imply that $cg_1 \notin P_G(f)$ if $c < l$ or $c > u$. Thus,

$$P_G(f) = \{cg_1 : l \le c \le u\}.$$

∎

Now we can derive the exact value of Γ.

THEOREM 3.

$$\Gamma = \inf\left\{\frac{|g_1(x)|}{\|g_1\|} : x \in T \text{ with } g_1(x) \ne 0\right\}. \tag{9}$$

PROOF. First we show that, if $g_1(x_1) \ne 0$, then

$$\Gamma \le \frac{|g_1(x_1)|}{\|g_1\|}. \tag{10}$$

In fact, if $|g_1(x_1)| = \|g_1\|$, then (10) holds, since Γ always satisfies $\Gamma \le 1$ (cf. [53, page 83]). Otherwise, by Proposition (13), there exists a function $f(x)$ in $C_0(T)$ such that $P_G(f) = \{0\}$ and

$$\|f - g_1\| \le \operatorname{dist}(f,G) + |g_1(x_1)|. \tag{11}$$

Since

$$\|f - g_1\| \ge \operatorname{dist}(f,G) + \Gamma \cdot \operatorname{dist}(g_1, P_G(f)) = \operatorname{dist}(f,G) + \Gamma \cdot \|g_1\|, \tag{12}$$

inequality (10) follows from (11) and (12).

Now let $P_G(f) = \{cg_1 : l \le c \le u\}$. By Lemma (2), there exist two points x_l and x_u such that $g_1(x_l) \ne 0$, $g_1(x_u) \ne 0$, and equations (6) and (7) hold. Let $g = \alpha g_1 \notin P_G(f)$. Assume $\alpha < l$. Then

$$\begin{aligned}
\|f - g\| &\ge |f(x_l) - g(x_l)| \\
&\ge \operatorname{sgn}(g_1(x_l))(f(x_l) - g(x_l)) \\
&= \operatorname{sgn}(g_1(x_l))((f(x_l) - lg_1(x_l)) + (l-\alpha)g_1(x_l)) \\
&= \operatorname{dist}(f,G) + (l-\alpha)|g_1(x_l)| \\
&= \operatorname{dist}(f,G) + \frac{|g_1(x_l)|}{\|g_1\|}\operatorname{dist}(g, P_G(f)),
\end{aligned} \tag{13}$$

where the second equality follows from (7).

Similarly, when $\alpha > u$, we can prove that

$$\|f - g\| \ge \operatorname{dist}(f,G) + \frac{|g_1(x_u)|}{\|g_1\|}\operatorname{dist}(g, P_G(f)). \tag{14}$$

It follows from (13) and (14) that

$$\Gamma \geq \inf\left\{\frac{|g_1(x)|}{\|g_1\|} : x \in T \text{ with } g_1(x) \neq 0\right\}. \tag{15}$$

It is easy to see that (9) follows from (15) and (10) and the proof is complete. ∎

If $G = \operatorname{span}\{g_1\}$ is a Haar space and T is a compact Hausdorff space, then $\operatorname{supp}(g_1) = T$ is compact. Therefore, the following result is a special case of Theorem 3.

COROLLARY 4. *Let T be a compact Hausdorff space and $G = \operatorname{span}\{g_1\}$ be a one-dimensional Haar space in $C(T)$. Then*

$$\Gamma = \inf\left\{\frac{|g_1(x)|}{\|g_1\|} : x \in T\right\}.$$

In particular, $\Gamma = 1$ when $G = \operatorname{span}\{1\}$.

The result in Theorem 3 holds for a line segment in $C_0(T)$ with a similar proof.

COROLLARY 5. *Let $G = \{\alpha g_1 : A \leq \alpha \leq B\}$ be a line segment in $C_0(T)$ with $\{x : g_1(x) \neq 0\}$ compact. Then*

$$\Gamma = \inf\left\{\frac{|g_1(x)|}{\|g_1\|} : x \in T \textit{ with } g_1(x) \neq 0\right\}.$$

PROOF. Let $P_G(f) = \{\alpha g_1 : l \leq \alpha \leq u\}$. Then it follows as in Lemma (2) that if $A < l$ then (6) holds, and if $u < B$ then (7) holds. Now Γ is invariant under translation, i.e., if $G_\beta = \{\alpha g_1 : A - \beta \leq \alpha \leq B - \beta\}$, then $\Gamma_G = \Gamma_{G_\beta}$. Thus, we may assume that $A \leq 0 \leq B$ so that $0 \in G$. Then the conclusion of Proposition (13) holds. Now the proof of Theorem 3 holds, where, to verify (13) and (14), it is only required that we consider $g = \alpha g_1 \notin P_G(f)$ when $A \leq \alpha < l$ and when $u < \alpha \leq \beta$ so that Lemma (2) in this case can be applied.

3. LIPSCHITZ CONSTANTS

It is well known that the uniform Hausdorff strong unicity constant Γ provides an upper bound $2/\Gamma$ for the uniform Lipschitz constant Λ. That is,

$$\Lambda \leq \frac{2}{\Gamma}. \tag{16}$$

The above inequality was first established by Cheney [53] for a Haar space G and then extended by Park [60] to general cases. However, it was not clear whether the estimate (16) was sharp or not. Our first main result in this section is to show that the equality holds in (16) if G is not a Haar space (i.e., $\operatorname{supp}(g_1) \neq T$). In this case, we prove $\Lambda = 2/\Gamma$ by constructing two functions f, h in $C_0(T)$ such that

$$\mathrm{H}\left(P_G(f), P_G(h)\right) \geq \frac{2}{\Gamma}\|f - h\| > 0.$$

However, if G is a Haar space and T is not a singleton, then we always have $\Lambda \leq 1/\Gamma$ and it is not easy to find the exact value of Λ. Fortunately, the Gâteaux derivative formula of Kolushov [61] provides some information on the exact value of Λ. The existence of the Gâteaux derivative of P_G for a Haar space G was first discovered by Kroó [62]. Later, Kolushov derived a formula for the Gâteaux derivative of P_G [61]:

$$\lim_{t \to 0+} \frac{P_G(f + t\phi) - P_G(f)}{t} = p(f, \phi), \tag{17}$$

where $p(f, \phi)$ is the unique solution of the following minimax problem:

$$\min_{g \in G} \max_{x \in E(f - P_G(f))} (\phi(x) - g(x)) \cdot \operatorname{sgn}(f(x) - P_G(f)(x)), \tag{18}$$

where $E(f - P_G(f)) := \{x \in T : |(f - P_G(f))(x)| = \|f - P_G(f)\|\}$. Note that, by (17),

$$\frac{\|p(f,\phi)\|}{\|\phi\|} = \lim_{t\to 0+} \frac{\|P_G(f + t\phi) - P_G(f)\|}{t\|\phi\|} \le \Lambda.$$

Therefore,

$$\sup\left\{\frac{\|p(f,\phi)\|}{\|\phi\|} : f,\phi \in C_0(T)\right\} \tag{19}$$

provides a seemly tight lower bound for Λ. It turns out that the expression (19) is the so-called uniform local Lipschitz constant of P_G [63]:

$$\Lambda^l = \sup\left\{\frac{\|p(f,\phi)\|}{\|\phi\|} : f,\phi \in C_0(T) \text{ with } \phi \neq 0\right\}, \tag{20}$$

where

$$\Lambda^l := \sup_{f\in C_0(T)} \inf_{\delta>0} \sup\left\{\frac{\mathrm{H}\,(P_G(f), P_G(h))}{\|f - h\|} : h \in C_0(T) \text{ with } 0 < \|f - h\| \le \delta\right\}.$$

Even though for a specific function the local Lipschitz constant need not equal the (global) Lipschitz constant, it is known that the uniform local Lipschitz constant of any Lipschitz continuous mapping is the same as the uniform Lipschitz constant of the mapping (cf. [31, Theorem 2.1] or Lemma (7). As a consequence, $\Lambda^l = \Lambda$. Therefore, in order to get the exact value of Λ, we only need to compute the norm of $p(f,\phi)$ and to do this, we will use the following explicit representation of $p(f,\phi)$:

$$p(f,\phi)(x) = \frac{\operatorname{sgn}(g_1(x_1))\cdot\phi(x_1) + \operatorname{sgn}(g_1(x_2))\cdot\phi(x_2)}{|g_1(x_1)| + |g_1(x_2)|} g_1(x), \tag{21}$$

where x_1 and x_2 are two distinct points in $E(f - P_G(f))$.

In short, when G is a one-dimensional Haar space, by using (20), (21), and $\Lambda^l = \Lambda$, we are able to prove that

$$\Lambda = \frac{2\,\|g_1\|}{\inf\{|g_1(x_1)| + |g_1(x_2)| : x_1, x_2 \in T \text{ with } x_1 \neq x_2\}}. \tag{22}$$

The first main result of this section shows that $\Lambda = 2/\Gamma$ if G is not a Haar space.

THEOREM 6. *Suppose that $G = \operatorname{span}\{g_1\}$, $\operatorname{supp}(g_1)$ is compact, and $Z(g_1) := \{x : g_1(x) = 0\}$ is not empty. Then*

$$\Lambda = \frac{2\,\|g_1\|}{\Gamma} = \frac{2\,\|g_1\|}{\inf\{|g_1(x)| : g_1(x) \neq 0\}}.$$

PROOF. By inequality (16), we have

$$\Lambda \le \frac{2}{\Gamma}.$$

Thus, by Theorem 3, it only remains to show that there exist $f, h \in C_0(T)$ such that

$$\mathrm{H}\,(P_G(f), P_G(h)) \ge \frac{2}{\Gamma}\|f - h\| > 0.$$

By the assumption, there exists a point x_0 such that $g_1(x_0) = 0$. Let $x_1 \in T$ such that

$$\Gamma = \frac{|g_1(x_1)|}{\|g_1\|}.$$

Then, by Proposition (14), there exist $f, h \in C_0(T)$ such that

$$\mathrm{H}\,(P_G(f), P_G(h)) \ge \frac{2}{\Gamma}\|f - h\| > 0.$$ ∎

From now on, we proceed to establish the identity (22). First we show that $\Lambda^l = \Lambda$. For a finite-dimensional subspace G of $C_0(T)$, we say that P_G is locally upper Lipschitz continuous with modulo λ, denoted by $P_G \in UL(\lambda)$ (cf. [64]), if, for any $f \in C_0(T)$, there exists a positive constant $\delta > 0$ such that

$$\operatorname{dist}(P_G(h), P_G(f)) \le \lambda \|f - h\|, \qquad \text{for } h \in C_0(T) \text{ with } \|f - h\| \le \delta,$$

where $\operatorname{dist}(\cdot, \cdot)$ is defined as

$$\operatorname{dist}(P_G(h), P_G(f)) := \sup_{p \in P_G(h)} \inf_{g \in P_G(f)} \|p - g\|.$$

Note that, if P_G is Lipschitz continuous, then $P_G \in UL(\Lambda)$. However, the converse is also true if P_G is also Hausdorff lower semicontinuous, i.e.,

$$\lim_{h \to f} \operatorname{dist}(P_G(f), P_G(h)) = 0, \qquad \text{for every } f \in C_0(T).$$

Note [52] that P_G is Hausdorff lower semicontinuous if and only if, for any nonzero function $g \in G$,

$$\operatorname{card}(\operatorname{bd}Z(g)) \le \dim\{p \in G : \operatorname{int}Z(g) \subset Z(p)\} - 1, \tag{23}$$

where $Z(g) := \{x \in T : g(x) = 0\}$, $\operatorname{bd}Z(g)$ and $\operatorname{int}Z(g)$ are the boundary and the interior of $Z(g)$, respectively, $\operatorname{card}(K)$ denotes the number of points in a set K. Therefore, the following result is a consequence of [31, Theorem 2.1].

LEMMA 7. *Suppose that G is a finite-dimensional subspace of $C_0(T)$ such that* (23) *holds for every nonzero function g in G. Then $P_G \in UL(\lambda)$ if and only if $\Lambda \le \lambda$.*

Using Lemma 7, we can easily show that $\Lambda = \Lambda^l$. In fact, we can prove the following more general result.

LEMMA 8. *Suppose that G is a finite-dimensional subspace of $C_0(T)$ such that* (23) *holds for every nonzero function g in G. Then*

$$\Lambda^u = \Lambda^l = \Lambda,$$

where

$$\Lambda^u := \sup_{f \in C_0(T)} \inf_{\delta > 0} \sup_h \left\{ \frac{\operatorname{dist}(P_G(h), P_G(f))}{\|f - h\|} : 0 < \|f - h\| \le \delta \right\}. \tag{24}$$

PROOF. It is easy to see that $\Lambda^u \le \Lambda^l \le \Lambda$. On the other hand, let $\epsilon > 0$. Then, by the definition of Λ^u, $P_G \in UL(\Lambda^u + \epsilon)$. By Lemma 7, $\Lambda \le \Lambda^u + \epsilon$. Since $\epsilon > 0$ is arbitrary, we have $\Lambda \le \Lambda^u$. ∎

Next we give a representation of Λ by using the norms of the Gâteaux derivatives of P_G.

THEOREM 9. *Let T be a compact Hausdorff space and $G = \operatorname{span}\{g_1\}$ a one-dimensional Haar subspace of $C(T)$. Then*

$$\Lambda = \sup\{\|p(f, \phi)\| : f, \phi \in C(T) \text{ with } \|\phi\| \le 1\}. \tag{25}$$

PROOF. In fact, for any finite-dimensional Haar subspace G of $C(T)$, Bartelt and Swetits [63] proved that, for any $f \in C(T)$,

$$\inf_{\delta > 0} \sup \left\{ \frac{\mathrm{H}(P_G(h), P_G(f))}{\|f - h\|} : h \in C(T) \text{ with } h \ne f \right\}$$
$$= \sup\{\|p(f, \phi)\| : \phi \in C(T) \text{ with } \|\phi\| \le 1\}. \tag{26}$$

Thus, equation (25) follows from (26) and Lemma 8. ∎

LEMMA 10. *Suppose that T is a compact Hausdorff space and $G = \text{span}\{g_1\}$ is a one-dimensional Haar subspace of $C(T)$. Let f in $C(T) \setminus G$ and ϕ in $C(T)$. Then there are two distinct points x_1 and x_2 in $E(f - P_G(f))$ such that*

$$p(f,\phi)(x) = \frac{\text{sgn}\,(g_1(x_1)) \cdot \phi(x_1) + \text{sgn}\,(g_1(x_2)) \cdot \phi(x_2)}{|g_1(x_1)| + |g_1(x_2)|} g_1(x)$$

and

$$(f - P_G(f))(x_1) \cdot g_1(x_1) \cdot (f - P_G(f))(x_2) \cdot g_1(x_2) < 0.$$

PROOF. By Kolushov's representation of the Gâteaux derivative of P_G (cf. (17) and (18) or [61]), one can easily verify that there exist two distinct points x_1 and x_2 in $E(f - P_G(f))$ such that

$$-\text{sgn}\,(g_1(x_1))\,(\phi(x_1) - p(f,\phi)(x_1)) = \text{sgn}\,(g_1(x_2))\,(\phi(x_2) - p(f,\phi)(x_2)) \tag{27}$$

and

$$(f - P_G(f))(x_1) \cdot g_1(x_1) \cdot (f - P_G(f))(x_2) \cdot g_1(x_2) < 0.$$

Substituting $p(f,\phi) := \lambda g_1$ into (27), we have a linear equation in λ with the solution

$$\lambda = \frac{\text{sgn}\,(g_1(x_1)) \cdot \phi(x_1) + \text{sgn}\,(g_1(x_2)) \cdot \phi(x_2)}{|g_1(x_1)| + |g_1(x_2)|}.$$

∎

THEOREM 11. *Suppose that T is a compact Hausdorff space and $G = \text{span}\{g_1\}$ is a one-dimensional Haar subspace of $C(T)$. If $G = C(T)$, then $\Lambda = 1$; otherwise,*

$$\Lambda = \frac{2\,\|g_1\|}{\inf\{|g_1(x_1)| + |g_1(x_2)| : x_1, x_2 \in T \text{ with } x_1 \neq x_2\}} \leq \frac{1}{\Gamma}. \tag{28}$$

PROOF. If $G = C(T)$, then P_G is the identity mapping and, obviously, $\Lambda = 1$. Otherwise, it follows from Theorem 9, and Lemma 10 that

$$\begin{aligned}
\Lambda &= \sup\{\|p(\phi,f)\| : f,\phi \in C_0(T) \text{ with } \|\phi\| \leq 1\} \\
&\leq \sup_{\|\phi\|\leq 1, x_1 \neq x_2} \left\| \frac{\text{sgn}\,(g_1(x_1)) \cdot \phi(x_1) + \text{sgn}\,(g_1(x_2)) \cdot \phi(x_2)}{|g_1(x_1)| + |g_1(x_2)|} \right\| \cdot \|g_1\| \\
&\leq \sup\left\{ \frac{2\,\|g_1\|}{|g_1(x_1)| + |g_1(x_2)|} : x_1, x_2 \in T,\, x_1 \neq x_2 \right\}.
\end{aligned}$$

Now, let $x_1, x_2 \in T$ with $x_1 \neq x_2$. By Proposition (15), there exist functions f, ϕ in $C(T)$ such that $\|\phi\| = 1$ and

$$\|p(f,\phi)\| \geq \frac{2\,\|g_1\|}{|g_1(x_1)| + |g_1(x_2)|}. \tag{29}$$

By Theorem 9 and (29), we get

$$\Lambda \geq \sup\left\{ \frac{2\,\|g_1\|}{|g_1(x_1)| + |g_1(x_2)|} : x_1, x_2 \in T \text{ with } x_1 \neq x_2 \right\}.$$

This completes the proof. ∎

COROLLARY 12. *Suppose that T is a compact Hausdorff space with no isolated points and $G = \text{span}\{g_1\}$ is a one-dimensional Haar subspace of $C(T)$. Then*

$$\Lambda = \frac{1}{\Gamma} = \frac{\|g_1\|}{\inf\{|g_1(x)| : x \in T\}}.$$

4. CONSTRUCTION OF FUNCTIONS

In this section, we construct several functions with certain desirable properties. Let

$$T_1 := \{x \in T : g_1(x) \neq 0\}.$$

For convenience, we use the following notation:

$$[f_1(x)]_a^b := \begin{cases} 0, & \text{if } x \notin T_1, \\ a, & \text{if } x \in T_1 \text{ and } f_1(x) < a, \\ b, & \text{if } x \in T_1 \text{ and } f_1(x) > b, \\ f_1(x), & \text{if } x \in T_1 \text{ and } a \leq f_1(x) \leq b \end{cases}$$

for scalars a, b with $a < b$ and a function f_1 defined on T_1. Note that $[f_1(x)]_a^b$ is actually the truncation of $f_1(x)$ on T_1 by the lower bound a and the upper bound b which is naturally extended to a function on T with values 0 outside T_1. If $f_1(x)$ is continuous on T_1, then $[f_1(x)]_a^b$ is in $C_0(T)$ for any $a < b$, due to the fact that T_1 is both open and compact.

PROPOSITION 13. *For any $x_1 \in T$ with $g_1(x_1) \neq 0$, there exists a function $f(x)$ in $C_0(T)$ such that $P_G(f) = \{0\}$ and*

$$\|f - g_1\| \leq \operatorname{dist}(f, G) + |g_1(x_1)|. \tag{30}$$

PROOF. If $g_1(x) = 0$ for all $x \neq x_1$, then $f = 0$ is the required function; otherwise, let $x_2 \in T_1 \setminus \{x_1\}$. Without loss of generality, we may assume $\|g_1\| = 1$ and $g_1(x_1) > 0$.

Define a function $\hat{f}_1$ on $\{x_1, x_2\}$ by $\hat{f}_1(x_1) := -1$ and $\hat{f}_1(x_2) := \operatorname{sgn}(g_1(x_2))$. Since $|g_1(x_2)| \leq 1$, it follows that

$$\left|\hat{f}_1(x_i) - g_1(x_i)\right| \leq 1 + |g_1(x_1)|, \qquad \text{for } i = 1, 2.$$

By the Tietze Extension Theorem, there exists a continuous extension $(f_1 - g_1)$ of $(\hat{f}_1 - g_1)$ on T_1 such that

$$|f_1(x) - g_1(x)| \leq 1 + |g_1(x_1)|, \qquad \text{for } x \in T_1. \tag{31}$$

Let $f(x) := [f_1(x)]_{-1}^1$. Then $f(x) \in C_0(T)$ and $\|f\| \leq 1$. Since $f_1(x_1) = \hat{f}_1(x_1) = -1$ and $f_1(x_2) = \hat{f}_1(x_2) = \operatorname{sgn}(g_1(x_2))$, we have $f(x_1) = -1$ and $f(x_2) = \operatorname{sgn}(g_1(x_2))$. Therefore, for $\alpha > 0$,

$$\|f - \alpha g_1\| \geq |(f - \alpha g_1)(x_1)| = |1 + \alpha g_1(x_1)| > 1,$$

and, for $\alpha < 0$,

$$\|f - \alpha g_1\| \geq |(f - \alpha g_1)(x_2)| = |\operatorname{sgn}(g_1(x_2)) - \alpha g_1(x_2)| > 1.$$

As a consequence, $\|f\| = 1 < \|f - \alpha g_1\|$ for $\alpha \neq 0$ and $P_G(f) = \{0\}$.

Now we claim that

$$|f(x) - g_1(x)| \leq 1 + |g_1(x_1)|, \qquad \text{for } x \in T. \tag{32}$$

In fact, (32) is trivially true if $x \notin T_1$. If $x \in T_1$ and $|f_1(x)| \leq 1$, then (32) follows from (31). For $x \in T_1$ with $f_1(x) > 1$, by (31) and $|g_1(x)| \leq 1$,

$$0 \leq 1 - g_1(x) = f(x) - g_1(x) \leq f_1(x) - g_1(x) \leq 1 + |g_1(x_1)|.$$

For $x \in T_1$ with $f_1(x) < -1$, by (31) and $|g_1(x)| \leq 1$,

$$-(1 + |g_1(x_1)|) \leq f_1(x) - g_1(x) \leq f(x) - g_1(x) = -1 - g_1(x) \leq 0.$$

Thus, (32) holds. The inequality (30) follows from (32) and $\operatorname{dist}(f, G) = 1$. ∎

PROPOSITION 14. *Let x_0 and x_1 be two distinct points in T such that $g_1(x_0) = 0$ and $g_1(x_1) \neq 0$. Then there exist $f, h \in C_0(T)$ such that*

$$\mathrm{H}\left(P_G(f), P_G(h)\right) \geq \frac{2\|g_1\|}{|g_1(x_1)|}\|f-h\| > 0.$$

PROOF. Suppose first that $g_1(x) = 0$ for $x \neq x_1$. Let $f_1(x)$ be a nonzero function in $C_0(T \setminus \{x_1\})$ and define

$$f(x) := \begin{cases} f_1(x), & \text{if } x \neq x_1, \\ \max_{x \neq x_1} |f_1(x)|, & \text{if } x = x_1. \end{cases}$$

Then it is easy to verify that $P_G(f) = \{cg_1(x) : 0 \leq c \leq 2f(x_1)\}$. Let $h(x) = 0$. Then $P_G(h) = \{0\}$ and

$$\mathrm{H}\left(P_G(f), P_G(h)\right) = 2|f(x_1)| = 2\|f-h\| = \frac{2\|g_1\|}{|g_1(x_1)|}\|f-h\| > 0.$$

Now suppose that $g_1(x_2) \neq 0$ for some $x_2 \in T \setminus \{x_0, x_1\}$. Without loss of generality, we may assume that $\|g_1\| = 1$ and $g_1(x_1) > 0$. Let $\eta := |g_1(x_1)|$ and $\beta := (1/\eta) + 1$. Define

$$\hat{f}_1(x) = \begin{cases} \beta + 1, & x = x_0, \\ -\beta, & x = x_1, \\ \beta \cdot \operatorname{sgn}(g_1(x_2)), & x = x_2. \end{cases} \tag{33}$$

Then it is easy to verify that

$$\left|\hat{f}_1(x_i) - \frac{g_1(x_i)}{\eta}\right| \leq \beta + 1, \qquad \text{for } 0 \leq i \leq 2.$$

According to the Tietze Extension Theorem, there exists a continuous extension $(f_1 - (g_1/\eta))$ of $(\hat{f}_1 - (g_1/\eta))$ on T_1 such that

$$\left|f_1(x) - \frac{g_1}{\eta}(x)\right| \leq \beta + 1, \qquad \text{for } x \in T_1. \tag{34}$$

Let $f(x) := [f_1(x)]_{-\beta}^{\beta+1}$. Then $f \in C_0(T)$ and $f(x_i) = \hat{f}_1(x_i)$ for $0 \leq i \leq 2$. Obviously, for any scalar α,

$$\|f - \alpha g_1\| \geq |f(x_0) - \alpha g_1(x_0)| = \left|\hat{f}_1(x_0) - \alpha g_1(x_0)\right| = \beta + 1. \tag{35}$$

We claim that

$$\left|f(x) - \frac{g_1}{\eta}(x)\right| \leq \beta + 1, \qquad \text{for } x \in T_1. \tag{36}$$

In fact, (36) follows from (34) if $-\beta \leq f_1(x) \leq \beta + 1$. For $f_1(x) > \beta + 1$, by (34) and $|g_1(x)| \leq 1$, we have

$$0 \leq \beta + 1 - \frac{g_1(x)}{\eta} = f(x) - \frac{g_1(x)}{\eta} \leq f_1(x) - \frac{g_1(x)}{\eta} \leq \beta + 1.$$

If $f_1(x) < -\beta$, it follows from (34) and $|g_1(x)| \leq 1$ that

$$-(\beta + 1) \leq f_1(x) - \frac{g_1(x)}{\eta} \leq -\beta - \frac{g_1(x)}{\eta} = f(x) - \frac{g_1(x)}{\eta} \leq 0.$$

Thus, (36) holds. Since $f(x) = g_1(x) = 0$ for $x \notin T_1$, by (35) and (36), we get

$$\left\|f - \frac{g_1}{\eta}\right\| = \beta + 1,$$

which, along with (35), implies

$$\frac{g_1}{\eta} \in P_G(f). \tag{37}$$

This completes the analysis of $f(x)$. Next we construct a function $h(x)$.

Let $0 < \epsilon < 1/2$ and $h(x) := [f_1(x) - \epsilon]_{-\beta-\epsilon}^{\beta+1-\epsilon}$. Since

$$h(x) = [f_1(x) - \epsilon]_{-\beta-\epsilon}^{\beta+1-\epsilon} = [f_1(x)]_{-\beta}^{\beta+1} - \epsilon = f(x) - \epsilon$$

for all $x \in T_1$, we have

$$\|f - h\| = \epsilon$$

and

$$h(x_1) = f(x_1) - \epsilon = -\beta - \epsilon. \tag{38}$$

Now we show that

$$P_G(h) \subset \left\{ \alpha g_1 : \alpha \leq \frac{1-2\epsilon}{\eta} \right\}. \tag{39}$$

By the definition of $h(x)$,

$$\operatorname{dist}(h, G) \leq \|h\| \leq \beta + 1 - \epsilon. \tag{40}$$

If $\alpha > (1-2\epsilon)/\eta$, by (38), we get

$$h(x_1) - \alpha g_1(x_1) = -\beta - \epsilon - \alpha\eta < -\beta - \epsilon - (1-2\epsilon) = -\beta - 1 + \epsilon.$$

Thus, for $\alpha > (1-2\epsilon)/\eta$, $\|h - \alpha g_1\| > \beta + 1 - \epsilon$ and, by (40), $\alpha g_1 \notin P_G(h)$. This proves (39).

By (37) and (39), we see that

$$\begin{aligned}
\mathrm{H}(P_G(f), P_G(h)) &\geq \operatorname{dist}\left(\frac{g_1}{\eta}, P_G(h)\right) \\
&\geq \min\left\{\left\|\frac{g_1}{\eta} - \alpha g_1\right\| : \alpha \leq \frac{1-2\epsilon}{\eta}\right\} \\
&= \left\|\frac{g_1}{\eta} - \frac{1-2\epsilon}{\eta} g_1\right\| = \frac{2\epsilon}{\eta} \\
&= \frac{2\|g_1\|}{|g_1(x_1)|} \cdot \|f - h\| > 0.
\end{aligned}$$

This completes the proof of Proposition 14. ∎

PROPOSITION 15. *Let T be a compact Hausdorff space and $G = \operatorname{span}\{g_1\}$ a one-dimensional Haar subspace of $C(T)$. Then, for any two distinct points x_1 and x_2 in T, there exist functions f, ϕ in $C(T)$ such that $\|\phi\| = 1$ and*

$$\|p(f, \phi)\| \geq \frac{2\|g_1\|}{|g_1(x_1)| + |g_1(x_2)|}.$$

PROOF. Without loss of generality, we may assume that $g_1(x_1) > 0$. Let V_1 and V_2 be disjoint neighborhoods of x_1 and x_2. By Urysohn's Lemma, there is a function h_i such that $h_i(x_i) = 1$, $0 \leq h \leq 1$, and $h_i(x) = 0$ for $x \notin V_i$. Let

$$f_i(x) := \max\{h_i(x) - |g_1(x_i) - g_1(x)|, 0\}, \qquad \text{for } i = 1, 2.$$

Then $0 \leq f_i(x) \leq h_i(x) \leq 1$. If $f_i(x) = 1$, then

$$1 = f_i(x) = h_i(x) - |g_1(x_i) - g_1(x)| \leq 1 - |g_1(x_i) - g_1(x)|,$$

which implies $h_i(x) = 1$ and $g_1(x) = g_1(x_i)$. Thus,

$$\{x : f_i(x) = 1\} \subset \{x \in V_i : g_1(x) = g_1(x_i)\}. \tag{41}$$

Define

$$f(x) := \operatorname{sgn}(g_1(x_2)) \cdot f_2(x) - f_1(x).$$

Then $-1 \leq f(x) \leq 1$, since the supports of f_1 and f_2 are disjoint. It is easy to verify that $P_G(f) = \{0\}$. Moreover, if $x \in V_i$ and $|f(x)| = 1$, then $f_i(x) = 1$ and it follows from (41) that $g_1(x) = g_1(x_i)$. Therefore,

$$(-1)^i f(x) \cdot g_1(x) > 0 \tag{42}$$

and

$$g_1(x) = g_1(x_i) \tag{43}$$

for $x \in V_i$ with $|f(x)| = 1$.

Let $\phi(x) = \operatorname{sgn}(g_1(x_2)) \cdot f_2(x) + f_1(x)$. Then $\|\phi\| = 1$. By Lemma 10, there exist two points x_1^* and x_2^* such that $|f(x_i^*)| = 1$,

$$p(f,\phi)(x) = \frac{\operatorname{sgn}(g_1(x_1^*)) \cdot \phi(x_1^*) + \operatorname{sgn}(g_1(x_2^*)) \cdot \phi(x_2^*)}{|g_1(x_1^*)| + |g_1(x_2^*)|} g_1(x), \tag{44}$$

$$f(x_1^*) \cdot g_1(x_1^*) \cdot f(x_2^*) \cdot g_1(x_2^*) < 0. \tag{45}$$

If both x_1^* and x_2^* are in the same V_i, then (45) contradicts (42). Without loss of generality, we may assume that $x_i^* \in V_i$. Since $f(x) = (-1)^i\phi(x)$ for $x \in V_i$, it follows from (45) and $|f(x_i^*)| = 1$ that

$$|\operatorname{sgn}(g_1(x_1^*)) \cdot \phi(x_1^*) + \operatorname{sgn}(g_1(x_2^*)) \cdot \phi(x_2^*)| = 2.$$

Thus, by (44) and (43), we get

$$\|p(f,\phi)\| = \frac{2\,\|g_1(x)\|}{|g_1(x_1)| + |g_1(x_2)|}. \qquad ∎$$

REFERENCES

1. B. Bank, J. Guddat, D. Klatte, B. Kummer and K. Tammer, *Nonlinear Parametric Optimization*, Birkhäuser Verlag, Basel, (1983).
2. A.V. Fiacco, *Introduction to Sensitivity and Stability Analysis in Nonlinear Programming*, Academic Press, New York, (1983).
3. T. Gal, *Postoptimal Analyses, Parametric Programming and Related Topics*, McGraw-Hill, (1979).
4. L. Cromme, Strong uniqueness: A far reaching criterion for convergence analysis of iterative processes, *Numer. Math.* **29**, 179–193 (1978).
5. M.C. Ferris, Finite termination of the proximal point algorithm, *Math. Programming* **50**, 359–366 (1990).
6. M.C. Ferris and O.L. Mangasarian, Error bounds and strong upper semicontinuity for monotone affine variational inequalities, Technical Report: TR 1056a, Computer Sciences Department, University of Wisconsin-Madison, (November 1991) (Revised February 1992).
7. P.T. Harker and J.-S. Pang, Finite-dimensional variational inequality and nonlinear complementarity problems: A survey of theory, algorithms and applications, *Math. Programming Series B* **48**, 161–220 (1991).
8. J.L. Goffin, The relaxation method for solving systems of linear inequalities, *Math. Oper. Res.* **5**, 388–414 (1980).
9. K. Jittorntrum and M.R. Osborne, Strong uniqueness and second order convergence in nonlinear discrete approximation, *Numer. Math.* **34**, 439–455 (1980).
10. W. Li, Error bounds for piecewise convex quadratic programs and applications, Technical Report TR93-1, Department of Mathematics and Statistics, Old Dominion University, Norfolk, VA, Jan., 1993, *SIAM J. Control Optim.* (to appear).
11. W. Li, Linearly convergent descent methods for unconstrained minimization of a convex quadratic spline, Department of Mathematics and Statistics, Old Dominion University, Norfolk, VA, March 1993, *J. Optim. Theory Appl.* (to appear).

12. W. Li, P. Pardalos and C.-G. Han, Gauss-Seidel method for least distance problems, *J. Optim. Theory Appl.* **75**, 487–500 (1992).
13. W. Li and J. Swetits, A new algorithm for solving strictly convex quadratic programs, Technical Report TR92-14, Department of Mathematics and Statistics, Old Dominion University, Norfolk, VA, 1992, *SIAM J. Optim.* (to appear).
14. Z.-Q. Luo, Convergence analysis of primal-dual interior point algorithms for convex quadratic programs, Communications Research Laboratory, McMaster University, Hamilton, Ontario, Canada, Manuscript dated June 20, 1992.
15. Z.-Q. Luo and P. Tseng, On the convergence rate of dual ascent methods for strictly convex minimization, *Math. Oper. Res.* (to appear).
16. Z.-Q. Luo and P. Tseng, On the convergence of the coordinate descent method for convex differentiable minimization, *J. Optim. Theory Appl.* **72**, 7–36 (1992).
17. Z.-Q. Luo and P. Tseng, Error bound and convergence analysis of matrix splitting algorithms for the affine variational inequality problem, *SIAM J. Optimiz.* **2**, 43–54 (1992).
18. Z.-Q. Luo and P. Tseng, On the linear convergence of descent methods for convex essentially smooth minimization, *SIAM J. Control Optim.* **30**, 408–425 (1992).
19. Z.-Q. Luo and P. Tseng, Error bound and reduced gradient projection algorithms for convex minimization over a polyhedral, *SIAM J. Optim.* **3** (1) (1993).
20. O.L. Mangasarian and R. De Leone, Error bounds for strongly convex programs and (super)linearly convergent iterative schemes for the least 2-norm solution of linear programs, *Ann. Math. Oper.* **17**, 1–14 (1988).
21. M.R. Osborne and R.S. Womersley, Strong uniqueness in sequential linear programming, *J. Austral. Math. Soc. Ser. B* **31**, 379–384 (1990).
22. A. Auslender and J.-P. Crouzeix, Global regularity theorems, *Math. Oper. Res.* **13**, 243–253 (1988).
23. C. Bergthaller and I. Singer, The distance to a polyhedron, *Linear Alg. Appl.* **169**, 111–129 (1992).
24. J.V. Burke and P. Tseng, A unified analysis of Hoffman's bound via Fenchel duality, Department of Mathematics, University of Washington, March 1993, *SIAM J. Control Optim.* (to appear).
25. W. Cook, A.M.H. Gerards, A. Schrijver and É. Tardos, Sensitivity theorems in integer linear programming, *Math. Programming* **34**, 251–264 (1986).
26. A.J. Hoffman, Approximate solutions of systems of linear inequalities, *J. Res. Nat. Bur. Standards* **49**, 263–265 (1952).
27. O. Güler, Distance to a convex polyhedron and Hoffman's lemma, Working Paper 91-3, University of Iowa, Department of Management Sciences, (January 1991).
28. O. Güler, A.J. Hoffman and U.R. Rothblum, Approximations to solutions to systems of linear inequalities, Working Paper, University of Maryland at Baltimore County, Department of Mathematics and Statistics, (September 1992).
29. W. Li, The sharp Lipschitz constants for feasible and optimal solutions of a perturbed linear program, *Linear Algebra and Applications* **187**, 15–40 (1993).
30. W. Li, Error bounds for solutions of parametric convex-concave minimax problems, In *Parametric Optimization and Related Topics III, Approximation & Optimization*, (Edited by J. Guddat, H.Th. Jongen, B. Kummer and F. Nozicka), Vol. 3, pp. 373–394, Verlag Peter Lang, (1993).
31. W. Li, Sharp Lipschitz constants for basic optimal solutions and basic feasible solutions of linear programs, *SIAM J. Control Optim.* **32**, 140–153 (1994).
32. X.-D. Luo and Z.-Q. Luo, Extension of Hoffman's error bound to polynomial systems, Communication Research Laboratory, McMaster University, Hamilton, Ontario, Canada, May 15, 1992 (preprint).
33. Z.-Q. Luo and J.-S. Pang, Error bounds for analytic systems and their applications, Department of Mathematical Sciences, The Johns Hopkins University, Baltimore, MD, January 1993 (preprint).
34. Z.-Q. Luo and P. Tseng, Perturbation analysis of a condition number for linear systems, Department of Mathematics, University of Washington, Seattle, WA, December 27, 1991.
35. Z.-Q. Luo and P. Tseng, On global error bound for a class of monotone AVI problems, *Operations Research Letters* **11**, 159–165 (1992).
36. O.L. Mangasarian, A condition number of linear inequalities and equalities, In *Methods of Operations Research 43, Proceedings of 6^{th} Symposium über Operations Research, Universität Augsburg*, (Edited by G. Bamberg and O. Opitz), September 7–9, 1981, pp. 3–15, Verlagsgruppe Athennäum/Hain/Scriptor/Hanstein, Königstein, (1981).
37. O.L. Mangasarian, Error bounds for nondegenerate monotone linear complementarity problems, *Math. Programming* **48**, 437–445 (1990).
38. O.L. Mangasarian, A condition number for differentiable convex inequalities, *Math. Oper. Res.* **10**, 175–179 (1985).
39. O.L. Mangasarian, Global error bounds for monotone affine variational inequality problems, *Linear Alg. Appl.* **174**, 153–164 (1992).
40. O.L. Mangasarian and R.R. Meyer, Nonlinear perturbation of linear programs, *SIAM J. Control Optim.* **17**, 745–752 (1979).
41. O.L. Mangasarian and T.-H. Shiau, Error bounds for monotone linear complementarity problems, *Math. Programming* **36**, 81–89 (1986).

42. O.L. Mangasarian and T.-H. Shiau, Lipschitz continuity of solutions of linear inequalities, programs and complementarity problems, *SIAM J. Control Optim.* **25**, 583–595 (1987).
43. R. Mathias and J.S. Pang, Error bounds for the linear complementarity problem with a P-matrix, *Linear Alg. Appl.* **132**, 123–136 (1990).
44. S.M. Robinson, Bounds for error in the solution set of a perturbed linear program, *Linear Alg. Appl.* **6**, 69–81 (1973).
45. S.M. Robinson, An application of error bounds for convex programming in a linear space, *SIAM J. Control Optim.* **13**, 271–273 (1975).
46. J.W. Demmel and N.J. Higham, Improved error bounds for underdetermined system solvers, *SIAM J. Matrix Anal. Appl.* **14** (1) (1993).
47. J.V. Burke and M.C. Ferris, Weak sharp minima in mathematical programming, *SIAM J. Control Optim.* **31** (5) (1993).
48. M.C. Ferris and O.L. Mangasarian, Minimum principle sufficiency, *Math. Programming* **57**, 1–14 (1992).
49. W. Li, A.J. Hoffman's Theorem and metric projections in polyhedral spaces, *J. Approx. Theory* **75**, 107–111 (1993).
50. B.T. Polyak, *Introduction to Optimization*, Optimization Software, Inc., New York, (1987).
51. J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, (1990).
52. W. Li, Strong uniqueness and Lipschitz continuity of metric projections: A generalization of the classical Haar Theory, *J. Approx. Theory* **56**, 164–184 (1989).
53. E.W. Cheney, *Introduction to Approximation Theory*, 2nd edition, Chelsea Pub. Co., New York, (1982).
54. D.J. Newman and H.S. Shapiro, Some theorems on Chebyshev approximation, *Duke Math. J.* **30**, 673–684 (1963).
55. G. Nürnberger, Strong unicity constants in Chebyshev approximation, In *Numerical Methods of Approximation Theory*, (Edited by L. Collatz, G. Meinardus and G. Nürnberger), Vol. 8, ISNM 81, pages 144–168, Birkhäuser Verlag, Basel, (1987).
56. M.R. Osborne, *Finite Algorithms in Optimization and Data Analysis*, Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons, New York, (1985).
57. H. Späth, *Mathematical Algorithms for Linear Regression*, Academic Press, New York, (1992).
58. W. Li, Best approximations in polyhedral spaces and linear programs, In *Approximation Theory: Proceedings of the Sixth Southeastern International Conference*, (Edited by G. Anastassiou), pages 393–400, Marcel Dekker, New York.
59. W. Li, Lipschitz continuous metric selections in $C_0(T)$, *SIAM J. Math Anal.* **21**, 205–220 (1990).
60. S.-H. Park, Uniform Hausdorff strong uniqueness, *J. Approx. Theory* **58**, 78–89 (1989).
61. A.V. Kolushov, Differentiability of the best approximation operator, *Math. Notes Acad. Sci. USSR* **29**, 295–306 (1981).
62. A. Kroó, Differential properties of the operator of best approximation, *Acta Math. Hungar.* **34**, 185–203 (1977).
63. M.W. Bartelt and J.J. Swetits, Characterizations of the local Lipschitz constant, *J. Approx. Theory* **77**, 249–265 (1994).
64. S.M. Robinson, Some continuity properties of polyhedral multifunctions, *Math. Programming Study* **14**, 206–214 (1981).