Some Solutions to a Lens Model With Applications to Warm-Core Eddies

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SOME SOLUTIONS TO A LENS MODEL WITH APPLICATIONS TO WARMCORE EDDIES

by

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A Dissertation Submitted to the Faculty of
Old Dominion University in Partial Fulfillment of the
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To Professor Guan Bingxian

who

introduced and encouraged

me to the intricacies of physical oceanography

in China
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Abstract

SOME SOLUTIONS TO A LENS MODEL WITH APPLICATIONS TO WARM-CORE EDDIES

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A model of lens-shaped anticyclonic eddies based on nonlinear shallow water equations is developed. The model is a three-layer fluid and allows for one asymmetric mode as well as specified environmental flows. The solution scheme is a polynomial expansion of the field variables. When inserted into the hydrographic equations, the expansion yields eight first-order differential equations for the time dependent amplitudes. This system of ordinary differential equations is numerically tractable. As long as the initial values meet the requirement of elliptical structure and the prescribed external force is tolerable for the initial values, the numerical solutions are stable. Numerical solutions are developed which show a wide variety of characteristics. Using different assumptions, six analytical solutions are obtained and discussed. For isolated lenses, three special solutions show different oscillations of the amplitudes. One has only the inertial frequency. The other two have superinertial and subinertial frequencies, respectively. For forced lenses, three special solutions are related to different exterior prescribed flows. One is an equilibrium solution having a steady-state external flow. The two other solutions are derived from external flows with subinertial and inertial frequencies, respectively. An attempt is made to apply the special
solutions to observations of warm-core eddies in the Gulf of Mexico and other regions of the world ocean. The simulations of warm-core eddies with the special solutions are in general agreement with available data.
1 Introduction

A milestone in the investigation of ocean eddies was the international POLY-MODE experiment of 1977–1978. Since then, an increasing number of experiments have been carried out and various types of eddies have been documented in the world ocean. There is also a variety of theoretical models and laboratory simulations. One reason why eddies are so important is because they are the oceanic analogs of weather. Thus, studies of eddies contribute to mid- and short-term oceanic prediction.

Warm (or anticyclonic) eddies have been studied more than cold eddies. In the Gulf Stream region (Joyce, 1984), the Gulf of Mexico (Kirwan, Lewis, Indest, Reinersman, & Quintero, 1988) and the North Atlantic (Armi, Hebert, Oakey, Price, Richardson, Rossby, & Ruddick, 1989), warm eddies were well documented with data from hydrographic stations, direct current measurements, neutral buoyant floats and satellite-tracked drifters.


There are also some studies on the application of lens models to observations. Cushman-Roisin et al. (1985) gave a comparison of their model with an ideal warm ring. Lewis, Kirwan and Forristall, (1989) first used this lens model to estimate the geometry, divergence, vorticity and deformation of Gulf of Mexico eddies. Liu and Kirwan (1990) showed two examples of model lens application to the Gulf Stream and the Kuroshio warm rings. Compared with theoretical development, the application of these models to oceanographic data has been relatively scarce.

The purpose here is twofold. First, some general and special solutions for both isolated and forced lenses are developed. These solutions provide insight into the nonlinear flow physics of shallow water equations. Secondly, these special solutions are applied to observations of warm-core eddies in the world ocean. It is hoped that this application will strengthen the connection between theory and observation and allow for better understanding of real ocean processes.

There are two main parts to the paper: theory and application. The theory is described in sections 2 through 5, and the application in section 6, 7 and 8. In section 2, the three layer model of the shallow water equations is developed. Section 3 discusses some examples of numerical solutions and three special solutions for isolated lenses. Section 4 gives three special solutions for forced lenses. Section 5 shows five invariants to the unforced model equations and their special forms for the special solutions. The next three sections present results of applications of the theory to observations of ocean eddies. Section 6 discusses how to use the Lagrangian, hydrographic and
satellite data for application of the model to ocean eddies. In section 7, the special solutions are applied to calibration of six warm-core eddies in the Gulf of Mexico. In section 8, two more warm-core eddies are modeled with the special solutions in the Indian Ocean and the North Atlantic. The last section summarizes and discusses the results.

2 Model Equations

A three layer lens model was discussed by Kirwan and Liu (1991b). The model considers an inviscid three layer fluid with a confined middle layer lens. The upper and lower layers are prescribed flows, which serve only to force the lens layer. There is no effect of the lens layer on the upper and lower layers. The governing nondimensional equations for the middle lens layer, which are equivalent to those given by Ruddick (1987), are

\[
\frac{dU_i}{dt} + \sigma_{ij} U_j + \frac{\partial h}{\partial x_i} = W_1 \left( \frac{dU_i^{(1)}}{dt} + \sigma_{ij} U_j^{(1)} \right) + W_3 \left( \frac{dU_i^{(3)}}{dt} + \sigma_{ij} U_j^{(3)} \right) \tag{2.1.a}
\]

\[
\frac{dh}{dt} + h \frac{\partial U_j}{\partial x_j} = 0. \tag{2.1.b}
\]

Here, \( U_i \) is the \( i^{th} \) component of the velocities, superscripts (1) and (3) indicate the upper and the lower layer, \( h \) is the thickness of the middle lens layer, \( x_i \) is the \( i^{th} \) component of the coordinates, \( t \) is the time, and

\[
d/dt = \partial/\partial t + U_j \partial/\partial x_j,
\]

\[
\sigma_{ij} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},
\]

\[
W_1 = \rho_1 (\rho_3 - \rho_2)/\rho_2 (\rho_3 - \rho_1),
\]

\[
W_3 = \rho_3 (\rho_2 - \rho_1)/\rho_2 (\rho_3 - \rho_1)
\]

and \( \rho_i \) is the density of the \( i^{th} \) layer. In (2.1), \( h \) is scaled by a reference depth \( H \), such as a characteristic depth of the lens center. The coordinate \( x_i \) is scaled
by the radius of deformation \( R_d = \sqrt{g' H / f} \) where \( f \) is the Coriolis parameter and 
\[ g' = g(p_2 - p_1)(p_3 - p_2)/\rho_2(p_3 - \rho_1) \]
is the reduced gravity. The velocity \( U_i \) is scaled by the gravity wave speed \( \sqrt{g' H} \). The \( t \) is scaled by the inertial time \( 1/f \). Fig. 2.1 is the vertical section of the three layer model. The configuration of the upper and lower portion of the lens depends upon \( \rho_1, \rho_2 \) and \( \rho_3 \).

The equations (2.1) are invariant in the \( f \)-plane under the transformation \( (x,y) \rightarrow \lambda(x,y), (u,v) \rightarrow \lambda(u,v) \) and \( h \rightarrow \lambda^2 h \) with arbitrary \( \lambda \). It is therefore possible to interpret \((x, y)\) as the horizontal coordinates relative to the center of a specified water mass. According to Goldsborough (1930), an exact solution to the equations (2.1) in the lens layer can take the form

\[
\begin{align*}
u &= (G_s + G_R)x + (G/2 - G_N)y \quad (2.2.a) \\
v &= (G_s + G_R)x + (G/2 - G_N)y \quad (2.2.b) \\
h &= h_0 + (B/2 + B_N)x^2 + 2B_Sxy + (B/2 - B_N)y^2. \quad (2.2.c)
\end{align*}
\]

Here, \( G, G_R, G_N, G_s, h_0, B, B_N \) and \( B_S \) are functions of time only. Under the conditions \( B < 0 \) and \( B^2/4 - (B_N^2 + B_S^2) > 0 \), (2.2.c) describes the lens geometry. Also \( h_0 \) is the central depth of the lens while \( B, B_N \) and \( B_S \) determine the horizontal configuration of the lens. The boundary of the lens is \( h = 0 \); from (2.2.c) it is clear that this is an ellipse. Finally, \( G, 2G_R, 2G_N \) and \( 2G_S \) are the divergence, vorticity, normal and shear deformation of the velocity field within the lens.

Substituting (2.2) into (2.1) yields eight equations:

\[
\dot{h}_0 + h_0G = 0 \quad (2.3.a)
\]
\[ \dot{B} + 2[BG + 2(B_N G_N + B_S G_S)] = 0 \]  \hspace{1cm} (2.3.b)

\[ \dot{B}_S + 2B_S G + G_S B - 2B_N G_R = 0 \]  \hspace{1cm} (2.3.c)

\[ \dot{B}_N + 2B_N G + G_N B + 2B_S G_R = 0 \]  \hspace{1cm} (2.3.d)

\[ \dot{G} + G^2/2 + 2(G_N^2 + G_S^2 - G_R^2 - G_R + B) = W_1 \mathcal{L}_G^{(1)} + W_3 \mathcal{L}_G^{(3)} \]  \hspace{1cm} (2.3.e)

\[ \dot{G}_R + G G_R + G/2 = 0 \]  \hspace{1cm} (2.3.f)

\[ \dot{G}_N + G G_N - G_S + 2B_N = W_1 \mathcal{L}_N^{(1)} + W_3 \mathcal{L}_N^{(3)} \]  \hspace{1cm} (2.3.g)

\[ \dot{G}_S + G G_S + G_N + 2B_S = W_1 \mathcal{L}_S^{(1)} + W_3 \mathcal{L}_S^{(3)} \]  \hspace{1cm} (2.3.h)

where (\( \dot{ } \)) is the ordinary derivative of time. We assume that the prescribed flows in the upper and lower layers also take the form of (2.2.a) and (2.2.b). With the superscript \( i \) for the variables in the \( i^{th} \) layer, the forcing terms in the right-hand side of (2.3) are expressed as

\[ \mathcal{L}_G^{(i)} = \dot{G}^{i} + G^{2i}/2 + 2(G_N^{2i} + G_S^{2i} - G_R^{2i} - G_R^{i}) \]

\[ \mathcal{L}_N^{(i)} = G_N^{i} + G^{i}G_N^{i} - G_S^{i} \]

\[ \mathcal{L}_S^{(i)} = G_S^{i} + G^{i}G_S^{i} + G_N^{i}. \]

Here, \( i = 1 \) or 3, indicating respectively the upper and lower layers.

As a special case, a surface lens can also be described by the lens model if the upper layer is assumed to be the atmosphere. The equations governing the surface lenses are the same as (2.3) except that \( W_1 = 0, W_3 = 1 \) and the reduced gravity.
\[ g' = g(\rho_2 - \rho_1)/\rho_2 \] where the \( \rho_1 \) and \( \rho_2 \) are the densities of surface and bottom layers, respectively.

The rim of the lens (at \( h = 0 \)) is an ellipse. The ellipses have major radius \( R_a \), minor radius \( R_b \), the orientation \( \theta \) (the angle between \( R_a \) and east) and the rotation rate \( \dot{\theta} \) (the derivative of \( \theta \) with respect to time) as follows (Kirwan and Liu, 1991a):

\[ R_a = [h_0/(|B/2| - \sqrt{B_N^2 + B_S^2})]^{1/2} \quad (2.4.a) \]

\[ R_b = [h_0/(|B/2| + \sqrt{B_N^2 + B_S^2})]^{1/2} \quad (2.4.b) \]

\[ 2\theta = \tan^{-1}(B_S/B_N) \quad (2.4.c) \]

\[ \dot{\theta} = G_R - B(B_NG_S - B_SG_N)/2(B_N^2 + B_S^2). \quad (2.4.d) \]

The trajectory of the particles in the lenses can be calculated from

\[ \ddot{x} = (G/2 + G_N)x + (G_S - G_R)y \quad (2.5.a) \]

\[ \ddot{y} = (G_S + G_R)x + (G/2 - G_N)y \quad (2.5.b) \]

if the initial position is given.

If there is external flow, at least one nonzero term appears in the right-hand side of (2.3). This defines a forced lens. If there is no environmental flow, then (2.3) is homogeneous. We refer to this as an isolated lens.
Fig. 2.1. The vertical sketch of the three layer model. \( h \), \( \rho_2 \) and \( U_i \) are the thickness, density and velocity of the middle layer lens; \( \rho_1 \), \( \rho_3 \) and \( U_i^{(1)}, U_i^{(3)} \) are the densities and velocities of the upper and lower layers, respectively.
3 Solutions of Isolated Lenses

Kirwan and Liu (1991a) discussed the numerical solutions and two special solutions for isolated lenses. They showed that the numerical solutions produced a wide range of behavior of the lens evolution for different combinations of the initial values. The special solution rodon describes an elliptical lens with fixed size and rotation rate. The special solution pulson produces an axisymmetric lens with a pulsating rim. In this paper, another special solution is added.

The special solutions are discussed further and several examples of numerical solutions are given to show the effect of the orientation difference between the mass and flow field on the solutions.

3.1 Examples of General Solutions

The general solutions of (2.3) discussed here are obtained from numerical solutions performed with a standard IMSL routine for first-order ordinary differential equations. Given a group of the initial values of the eight variables, the solutions are calculated at each interval of $(20\pi)^{-1}$ nondimensional day. The precision is prescribed as $10^{-10}$. For the unforced version of (2.3), the numerical solutions are always stable if $B < 0$ and $B^2/4 - (B_N^2 + B_S^2) > 0$ initially.

Following are several examples of the numerical solutions for different initial values. All of the examples, of course, meet the above requirements. We examine the numerical solutions by varying the phase relation between the mass and flow field. As shown in (2.4.c), the major axis of the lens ellipse has an angle $\theta$ relative to east. With (2.2.a) and (2.2.b), the total angular momentum $xv - yu + (1/2)(x^2 + y^2)$ is also a conic surface with orientation angle $\kappa$ which can be expressed as

$$2\kappa = \tan^{-1}(-G_N/G_S).$$

We choose the initial values from
\[ B_S = -B_D \cos(\phi) \] (3.1.2.a)

\[ B_N = -B_D \sin(\phi) \] (3.1.2.b)

\[ G_N = G_D \cos(\phi + \Delta \phi) \] (3.1.2.c)

\[ G_S = -G_D \sin(\phi + \Delta \phi) \] (3.1.2.d)

where \( B_D, G_D, \phi \) and \( \Delta \phi \) are constants. Then, the initial phase difference of the mass and flow field is \( 2(\theta - \kappa) = \Delta \phi \).

First, we take an arbitrary group of the initial values:

\[
\begin{align*}
    h_0 &= 1 \\
    B &= -0.3 \\
    G &= 0 \\
    G_R &= -0.4 \\
    B_D &= 0.1 \\
    G_D &= 0.2 \\
    \phi &= \pi/2 \\
    \Delta \phi &= \pi/4.
\end{align*}
\]

The numerical solutions for ten inertial days are shown in Fig. 3.1.1. The eight variables seem to fall into two categories; \( h_0, B, G \) and \( G_R \) show strong inertial oscillations while \( B_S, B_N, G_N \) and \( G_S \) show strong subinertial frequencies. Both groups have superinertial frequencies as well.

To understand the effect of \( \Delta \phi \) on the behavior of these variables we compare three cases, \( \Delta \phi = \pi/2, \pi/4 \) and 0, with the same initial values above. Fig. 3.1.2 shows the three cases for \( h_0 \) and \( G_N \). In Fig. 3.1.2, the solid lines stand for the case
\[ \Delta \phi = \pi/2, \] the dot-dashed lines for \( \Delta \phi = \pi/4 \) and the dotted lines for \( \Delta \phi = 0 \). It is seen that there are about 14 peaks of oscillation in ten days for \( h_0 \) and about 16 peaks for \( G_N \). The larger \( \Delta \phi \) produces stronger superinertial oscillations but \( \Delta \phi \) does not affect the values of the superinertial frequencies. Fig. 3.1.3 shows the east (U) and north (V) components of the velocity in the three cases. Like the dynamic field, the superinertial frequency looks similar in each case. The larger \( \Delta \phi \) produces only a larger amplitude of the velocity.

For \( \Delta \phi = 0 \), a special group of initial values produces a numerical solution with only subinertial frequencies. This is the special solution rodon, which has an analytical expression obtained by Cushman-Roisin (1987). One example of this special group is

\[
\begin{align*}
h_0 &= 1 \\
B &= -0.280134 \\
G &= 0 \\
G_R &= -0.361260 \\
B_D &= 0.1 \\
G_D &= 0.222222 \\
\phi &= \pi/2 \\
\Delta \phi &= 0.
\end{align*}
\]

With this special group of initial values, the numerical solutions of \( B_S, B_N, G_N \) and \( G_S \) oscillate with one subinertial frequency (about ten inertial days) while the other four variables, \( h_0, B, G, \) and \( G_R \), are constant. In Fig. 3.1.4, the dotted lines show \( h_0 \) and \( G_N \) for the rodon special solution. The solid lines and the dot-dashed lines in Fig. 3.1.4 compare the solutions with the same initial values except \( \Delta \phi = -\pi/2 \) and \( \Delta \phi = \pi/2 \), respectively. It appears that both the superinertial signals produced with \( \Delta \phi = \pi/2 \) and \( \Delta \phi = -\pi/2 \) have the same magnitudes and frequencies. The difference between the superinertial signals is only phase lag. Fig. 3.1.5 shows the east and north components of the velocity. The velocity field shows the same effect
of $\Delta \phi$ on the superinertial frequency as in the dynamic field.

To examine the behavior of $\Delta \phi$, we plot the $\theta$ (solid lines) and $\kappa$ (dotted lines) in Fig. 3.1.6. Fig. 3.1.6(a) and (b) show the cases of $\Delta \phi = \pi/4$ and $\Delta \phi = 0$ for the arbitrary group of initial values. Fig. 3.1.6(c) shows the case of $\Delta \phi = 0$ for the special group of initial values or for rodon special solutions. In general, $\theta$ and $\kappa$ are out of phase even when $\Delta \phi$ is initially zero, as in Fig. 3.1.6(b). However, for the rodon initial values, $\Delta \phi$ is always zero or $\theta$ and $\kappa$ are always in phase as shown in Fig. 3.1.6(c). In this special case, $\Delta \phi = 0$ makes the divergence $2G$ vanish in the evolution, which leads to quasi-geostrophic solutions to (2.3).

Any change of the initial values of the rodon special solution destroys the quasi-geostrophic balance and introduces superinertial and subinertial frequencies. An example of these changes is shown in Fig. 3.1.7. Three cases are shown for $G_R = -0.2, -0.36126$ and $-0.6$ with the other initial values the same as for the rodon special solution. It is seen that a new lower (about 20 days) and a new higher (about 3 days) subinertial oscillation are introduced by $G_R = -0.2$ and $G_R = -0.6$, respectively. Fig. 3.1.7 shows that $G_R = -0.2$ and $G_R = -0.6$ also produce superinertial oscillations.

The numerical solution to (2.3) exhibits a wild behavior for different combinations of the initial values of the eight variables. However, an individual variable seems to be especially responsible for a certain range of frequencies. The phase difference between the mass and flow field is directly related to superinertial frequencies. Larger $\Delta \phi$ produces stronger superinertial oscillations. $\Delta \phi = 0$ may yield a quasi-geostrophic balance so that the solution keeps the subinertial and drops the inertial and superinertial frequencies.
Fig. 3.1.1. The numerical solutions of (2.3) for 10 days with the initial values $h_0=1$, $B=-0.3$, $G=0$, $G_R=-0.4$. The others are initialized with $B_D=0.1$, $G_D=0.2$, $\phi=\pi/2$ and $\Delta\phi=\pi/4$ in (3.1.2).
Fig. 3.1.2. The numerical solutions of $h_0$ and $G_N$ for the initial values $h_0=1$, $B=-0.3$, $G=0$, $G_R=-0.4$, $B_D=0.1$, $G_D=0.2$, $\phi=\pi/2$ and $\Delta \phi=\pi/2$ (the solid lines); $\Delta \phi=\pi/4$ (the dot-dashed lines); and $\Delta \phi=0$ (the dotted lines).
Fig. 3.1.3. The velocity components of the numerical solution for the initial values $h_0=1$, $B=-0.3$, $G=0$, $G_R=-0.4$, $B_D=0.1$, $G_D=0.2$, $\phi=\pi/2$ and $\Delta\phi=\pi/2$ (the solid lines); $\Delta\phi=\pi/4$ (the dot-dashed lines); and $\Delta\phi=0$ (the dotted lines).
Fig. 3.1.4. The numerical solutions of $h_0$ and $G_N$ for the initial values $h_0=1$, $B=-0.280134$, $G=0$, $G_R=-0.36126$, $B_D=0.1$, $G_D=0.222222$, $\phi=\pi/2$, $\Delta\phi=-\pi/2$ (the solid lines); $\Delta\phi=\pi/2$ (the dot-dashed lines); and $\Delta\phi=0$ (the dotted lines).
Fig. 3.1.5. The velocity components of the numerical solution for the initial values $k_0=1$, $B=-0.280134$, $G=0$, $G_R=-0.36126$, $B_D=0.1$, $G_D=0.222222$, $\phi=\pi/2$ and $\Delta\phi=-\pi/2$ (the solid lines); $\Delta\phi=\pi/2$ (the dot-dashed lines); and $\Delta\phi=0$ (the dotted lines).
Fig. 3.1.6. The ellipse orientation angle $\theta$ (the solid lines) and angular momentum orientation angle $\kappa$ (dotted lines) for the initial values $h_0=1$, $B=-0.3$, $G=0$, $G_R=-0.4$, $B_D=0.1$, $G_D=0.2$, $\varphi=\pi/2$ and (a): $\Delta\varphi=\pi/4$, (b): $\Delta\varphi=0$. The initial values for (c) are $h_0=1$, $B=-0.280134$, $G=0$, $G_R=-0.36126$, $B_D=0.1$, $G_D=0.222222$, $\varphi=\pi/2$ and $\Delta\varphi=0$. 

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Fig. 3.1.7. The numerical solutions of \( h_0 \) and \( G_S \) for the initial values \( h_0=1, B=-0.280134, G=0, B_D=0.1, G_D=0.222222, \phi=\pi/2, \Delta\phi=0 \) and three different \( G_R \) marked in each panel.
3.2 Solution of Superinertial Lenses

Let the total velocity be expressed as

\[ u = -\frac{\partial \Psi}{\partial y} + \frac{\partial \Phi}{\partial x} \]  
(3.2.1.a)

\[ v = \frac{\partial \Psi}{\partial x} + \frac{\partial \Phi}{\partial y}. \]  
(3.2.1.b)

Here \( \Psi \) is the stream function and \( \Phi \) is the velocity potential. The relation between \( \Psi, \Phi \) and \( G, G_R, G_N \) and \( G_S \) in (2.2) is

\[ G = 2\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) \]  
(3.2.2.a)

\[ G_R = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} \]  
(3.2.2.b)

\[ G_N = -\frac{\partial^2 \Psi}{\partial x \partial y} + \left(\frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial^2 \Phi}{\partial y^2}\right) \]  
(3.2.2.c)

\[ G_S = \frac{\partial^2 \Phi}{\partial x \partial y} + \left(\frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi}{\partial y^2}\right). \]  
(3.2.2.d)

If the total velocity is given only by \( \Phi \) with no contribution from the stream function \( \Psi \), it follows that \( G_R = 0 \). Substitution of \( G_R = 0 \) into (2.3.f) yields \( G = 0 \). Then, \( h_0 = \text{constant} \) is obtained from (2.3.a). Finally, equations (2.3) reduce to

\[ \dot{B} + 4(B_N G_N + B_S G_S) = 0 \]  
(3.2.3.a)

\[ \dot{B}_S + B G_S = 0 \]  
(3.2.3.b)

\[ \dot{B}_N + B G_N = 0 \]  
(3.2.3.c)
\[
\begin{align*}
\dot{G}_N - G_S + 2B_N &= 0 \quad (3.2.3.d) \\
\dot{G}_S + G_N + 2B_S &= 0 \quad (3.2.3.e)
\end{align*}
\]

and

\[
G_N^2 + G_S^2 + B = 0. \quad (3.2.3.f)
\]

From (3.2.3.a), (3.2.3.d) and (3.2.3.e) one obtains

\[
G_N^2 + G_S^2 - B = \text{constant}. \quad (3.2.4)
\]

It follows that both \( B \) and \( G_N^2 + G_S^2 \) are constants from (3.2.3.f) and (3.2.4). The solution to (3.2.3) is

\[
\begin{align*}
B_N &= -B_D\sin(\omega t + \phi) \quad (3.2.5.a) \\
B_S &= -B_D\cos(\omega t + \phi) \quad (3.2.5.b) \\
G_N &= G_D\cos(\omega t + \phi) \quad (3.2.5.c) \\
G_S &= -G_D\sin(\omega t + \phi) \quad (3.2.5.d)
\end{align*}
\]

with the constraints

\[
G_D^2 + B = 0 \quad (3.2.5.e)
\]

\[
\omega = B G_D / B_D \quad (3.2.5.f)
\]
Here \( B_D, G_D, \omega \) and \( \phi \) are constants. Of \( B, B_D, G_D \) and \( \omega \), only one is independent. If \( B \) is given, (3.2.5.e), (3.2.5.f) and (3.2.5.g) can be expressed as

\[
G_D = \sqrt{-B}
\]  
(3.2.6.a)

\[
\omega = \frac{(1 \pm \sqrt{1 - 8B})}{2}
\]  
(3.2.6.b)

\[
B_D = (1 - \omega)\sqrt{-B}/2.
\]  
(3.2.6.c)

Recall that \( B < 0 \) is required for the ellipse structure so that \( \omega \) is always greater than 1 with the "+" sign in (3.2.6.b). The "−" sign in (3.2.6.b) results in \( B_D^2 / 4 - B_D^2 < 0 \) which violates the elliptical constraint. Hence only the \( \omega^+ \) solution makes sense. Substitution of (3.2.5) into (2.4) yields that rotation rate of the ellipse \( \dot{\theta} = -\omega/2 \). The rotation rates of the ellipses in this solution, therefore, are always greater than \(-1/2\) (negative \( \dot{\theta} \) indicating the anticyclonic rotation). Therefore, the elliptical lenses in this solution rotate anticyclonically and always at superinertial frequencies. We call this solution the special solution for superinertial lenses.

As an example, the motions of the horizontal ellipse and a particle on the lens boundary for this special solution are shown in Fig. 3.2.1. The initial values are specified from the given constants:

\[
\begin{align*}
h_0 & = 0.5 \\
B & = -0.25 \\
G_D & = 0.5 \\
B_D & = 0.0915 \\
\omega & = 1.36
\end{align*}
\]
\( \phi = 0. \)

In this example, the rotation rate of the horizontal ellipse of the lens is \( \dot{\theta} = -0.68 \), which is higher than the inertial rotation rate \(-0.5\). Fig. 3.2.1 shows three specific times for the lens rotation. The lens rotates anticyclonically 180° in about 4.6 inertial time units (about 0.7 inertial day). The shape of the lens is unchanged during the evolution. Following the lens, the particle also moves anticyclonically but its speed is slower than that of the lens. The particle moves anticlockwise around the lens from its initial position.

There is neither divergence nor vorticity in this special solution. The motions of lenses and particles are caused by the oscillation of \( G_N \) and \( G_S \). These are given by

\[
G_N = \frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial^2 \Phi}{\partial y^2} \quad (3.2.7.a)
\]

\[
G_S = \frac{\partial^2 \Phi}{\partial x \partial y}. \quad (3.2.7.b)
\]
Fig. 3.2.1. The ellipses (dashed lines) and the trajectory (solid line) of a particle (squares) at the boundary of the lens in the case of the superinertial lenses. The t indicates the time. The numbers are nondimensional.
3.3 Solution of Subinertial Lenses

If the total velocity is composed of only the stream function $\Psi$ with no contribution from the velocity potential $\Phi$, it follows that $G = 0$ and the equations (2.3) reduce to

$$\dot{h}_0 = 0 \quad (3.3.1.a)$$

$$\dot{B} + 4(B_N G_N + B_S G_S) = 0 \quad (3.3.1.b)$$

$$\dot{B}_S + B G_S - 2B_N G_R = 0 \quad (3.3.1.c)$$

$$\dot{B}_N + B G_N + 2B_S G_R = 0 \quad (3.3.1.d)$$

$$G_N^2 + G_S^2 - G_R^2 - G_R + B = 0 \quad (3.3.1.e)$$

$$\dot{G}_R = 0 \quad (3.3.1.f)$$

$$\dot{G}_N - G_S + 2B_N = 0 \quad (3.3.1.g)$$

$$\dot{G}_S + G_N + 2B_S = 0. \quad (3.3.1.h)$$

In this case, both $h_0$ and $G_R$ are constants. From (3.3.1.b), (3.3.1.g) and (3.3.1.h), one can easily get

$$G_N^2 + G_S^2 - B = c_1 \quad (3.3.2.a)$$

and rewrite equation (3.3.1.e) into

24
\[ G_N^2 + G_S^2 + B = c_2 \]  \hspace{1cm} (3.3.2.b)

where \( c_1 \) and \( c_2 \) are constants. Then, it follows that the solution to (3.3.1) requires both \( B \) and \( G_N^2 + G_S^2 \) to be constants.

This solution (3.3.1) is called the \textit{rodon} solution, and was first obtained by Cushman-Roisin (1987). It can be written as

\[
B_N = -B_D \sin(\omega t + \phi) \quad (3.3.3.a)
\]
\[
B_S = -B_D \cos(\omega t + \phi) \quad (3.3.3.b)
\]
\[
G_N = G_D \cos(\omega t + \phi) \quad (3.3.3.c)
\]
\[
G_S = -G_D \sin(\omega t + \phi) \quad (3.3.3.d)
\]

with three constraints

\[
G_D^2 - G_R^2 - G_R + B = 0 \quad (3.3.3.e)
\]
\[
G_D(1 - \omega) - 2B_D = 0 \quad (3.3.3.f)
\]
\[
B_D(2G_R + \omega) - BG_D = 0. \quad (3.3.3.g)
\]

The solution includes eight integral constants: \( h_0, B, G_R, B_D, G_D, \omega, \phi \) and \( G = 0 \). Kirwan and Liu (1991a) discussed the details about the three constraints for specification of the lenses and showed that

\[
\omega = 1 - M/2 \quad (3.3.4.a)
\]
\[ \dot{\theta} = -\omega/2 \]  

(3.3.4.b)

where

\[ M = \sqrt{1 + \delta \pm \sqrt{1 - \delta}} \]  

(3.3.4.c)

\[ \delta = 4\sqrt{B^2 - B_D^2}. \]  

(3.3.4.d)

The condition of elliptical structure requires \( B^2/4 - B_D^2 > 0 \). Thus, equations (3.3.4.c) and (3.3.4.d) imply \( 0 < \delta \leq 1 \). For the extreme values, \( \delta = 1 \) results in \( \dot{\theta} = -1/2 + \sqrt{2}/4 \); \( \delta = 0 \) yields \( \dot{\theta} = 0 \) from the "+" sign of (3.3.4.c) and \( \dot{\theta} = -1/2 \) from the "-" sign of (3.3.4.c). Therefore, the value of the rotation rate of the ellipse is between 0 and \(-1/2\). This means that the lenses always rotate at a rate lower than \(-1/2\) in this solution. We call this solution the special solution for subinertial lenses.

As an example, the motions of the lens boundary and a particle on the elliptical boundary for this special solution are shown in Fig. 3.3.1. The initial values of the eight variables are specified with the given constants:

\[ h_0 = 0.5 \]
\[ B = -0.280134 \]
\[ B_D = 0.1 \]
\[ G_R = -0.361260 \]
\[ G_D = 0.222222 \]
\[ \omega = 0.1 \]
\[ \phi = \pi/2. \]

The rotation rate of the horizontal ellipse of the lens is \( \dot{\theta} = -0.05 \). The lens rotates anticyclonically 180° in 62.8 inertial time units (ten inertial days). The particle moves clockwise around the ellipse. The shape and rotation rate are unchanged during the evolution.
The motions of the elliptical lenses and the particles in this special solution are determined only by the stream function $\Psi$. This assumption makes the divergence of the velocity field vanish, leading to a quasi-geostrophic balance in (2.3). The inertial and superinertial frequencies are filtered out by the quasi-geostrophic approximation so that the motions in this special solution show only subinertial frequencies.
Fig. 3.3.1. The ellipses (dashed lines) and the trajectory (solid line) of a particle (squares) at the boundary of the lens in the case of the subinertial lenses. The $t$ indicates the time. The numbers are nondimensional.

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3.4 Solution of Inertial Lenses

For circular lenses, it follows that $B_S = B_N = 0$. Then, the equations (2.3.c) and (2.3.d) yield $G_S = G_N = 0$. The equations (2.3) reduce to

\[ \dot{h}_0 + h_0 G = 0 \]  
\[ \dot{B} + 2BG = 0 \]  
\[ \dot{G} + G^2/2 - 2G_R^2 - 2G_R + 2B = 0 \]  
\[ \dot{G}_R + GG_R + G/2 = 0. \]

The solution to (3.4.1) was first obtained by Cushman-Roisin (1987). Kirwan and Liu (1991b) gave a general form of the solution to (3.4.1) and named it the pulson solution. The general form of the solution is

\[ h_0 = H_0 p \]  
\[ B = -\left(\Lambda_B/4\right)p^2 \]  
\[ G = \gamma \cos(t)p \]  
\[ G_R = -(1/2)(1 - \Lambda_Q) \]

where

\[ p = \left[\sqrt{\gamma^2 + \Lambda_Q^2 + \Lambda_B + \gamma \sin(t)}\right]^{-1}. \]
Here $H_0$, $\Lambda_B$, $\gamma$ and $\Lambda_Q$ are constants which are determined by the initial conditions. The four constants are independent. The sole requirement is $\Lambda_B > 0$. This condition insures the circular structure and hence stable solutions.

This special solution defines a lens with circular horizontal rim. Instead of rotation, the rim expands and contracts at the rate of the inertial frequency. We call this solution the special solution for inertial lenses.

As an example, the motion of the rim of a circular lens and a particle at the rim is shown in Fig. 3.4.1. The initial values of $h_0$, $B$, $G$ and $G_R$ are determined by the given constants:

$\begin{align*}
H_0 &= 0.5 \\
\Lambda_B &= 0.39 \\
\gamma &= 0.5 \\
\Lambda_Q &= 0.6.
\end{align*}$

It is seen from Fig. 3.4.1 that the rim of the lens is a circle. The rim expands (at $t = 1$) and contracts (at $t = 4$) coupled with the increase and decrease of the central depth $h_0$ (not shown). The rate of the expansion and contraction is inertial. The particle at the rim moves clockwise around the rim of the lens.

A characteristic of this solution is that the frequency is independent of the amplitudes. This is unlike many other nonlinear problems where the frequency is strongly amplitude dependent. The assumption of axisymmetry requires that the velocity field have only divergence and vorticity. There is no normal and shear deformation in the solution to (2.3). This implies that only the deformation can produce asymmetric lenses. The sole inertial oscillation in this special solution indicates that the deformation is responsible for the superinertial and subinertial frequencies in isolated lenses.
Fig. 3.4.1. The rim of the lens (dashed lines) and the trajectory (solid line) of a particle (squares) at the boundary of the lens in the case of the inertial lenses. The $t$ indicates the time. The numbers are nondimensional.
4 Solutions of Forced Lenses

4.1 Equilibrium Solution

Here the term *forced lenses* refers to prescribed flow in the layers adjacent to the lens. With specified external motions, the lens model should be more realistic than the unforced one. There is very little work relevant to this subject. Ruddick (1987) and Brickman and Ruddick (1990) obtained an equilibrium solution to a three layer model similar to equations (2.3) with a steady external flow

\[ u^{(1)}, u^{(3)} = \alpha y \quad (4.1.1.a) \]

\[ v^{(1)}, v^{(3)} = \alpha x \quad (4.1.1.b) \]

at the upper and lower layers where \( \alpha \) is a constant. Here we develop an equivalent expression to Ruddick’s (1987) solution. The external flow like (4.1.1) has only shear deformation in its velocity field. At steady state, (2.3) simplifies to

\[ B_N G_N + B_S G_S = 0 \quad (4.1.2.a) \]

\[ G_S B - 2B_N G_R = 0 \quad (4.1.2.b) \]

\[ G_N B + 2B_S G_R = 0 \quad (4.1.2.c) \]

\[ G_N^2 + G_S^2 - G_R^2 - G_R + B = \alpha^2 \quad (4.1.2.d) \]

\[ -G_S + 2B_N = -\alpha \quad (4.1.2.e) \]
\[ G_N + 2B_S = 0 \] 

with \( G = 0 \). Comparing (4.1.2.a), (4.1.2.e) and (4.1.2.f) yields \( B_S = 0 \) and \( G_N = 0 \).

The equilibrium solution of (2.2), therefore, becomes

\[ u = (G_S - G_R)y \]  
\[ v = (G_S + G_R)x \]  
\[ h = h_0 + (B/2 + B_N)x^2 + (B/2 - B_N)y^2. \]

The constants \( B, B_N, G_S \) and \( G_R \) must satisfy the relations

\[ G_S B - 2B_N G_R = 0 \]  
\[ G_S^2 - G_R^2 - G_R + B = \alpha^2 \]  
\[ -G_S + 2B_N = -\alpha. \]

The elliptical structure and the anticyclonic motion within the ellipse require \( B < 0 \), \( B^2/4 - B_N^2 > 0 \) and \( G_R < 0 \). To meet these requirements, \( \alpha \) must be constrained to

\[ 0 < \alpha < 1. \]

The constraint (4.1.4.a) turns out to be the mass continuity equation

\[ \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} + h \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0. \]
Under the condition $B^2/4 - B_N^2 > 0$, (4.1.4.a) requires $G_R^2 - G_S^2 > 0$ which makes the velocity in (4.1.3.a) and (4.1.3.b) stable.

Fig. 4.1.1 shows the relation between $B^2/4 - B_N^2$ and $\alpha$ calculated from (4.1.4) for the case $B = -0.3$. The requirement $B^2/4 - B_N^2 > 0$ is satisfied if $|\alpha| < 1$. For any negative values of $B$, $|\alpha| > 1$ makes $B^2/4 - B_N^2$ negative, destroying the elliptical structure.

Note that the equilibrium solution can be obtained only with the constraint (4.1.4). This solution cannot be achieved for arbitrary initial values and $|\alpha| < 1$. 

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Fig. 4.1.1. The relation between $B^2/4 - B_N^2$ and the steady shear rate $\alpha$ in the case of $B = -0.3$ in the equilibrium solution of forced lenses.
4.2 Solution of a Linear System

Suppose the flow in the upper and lower layers takes the form

\[ u^{(1)} = S_1 \cos(\omega t + \phi)x - [V_1 + S_1 \sin(\omega t + \phi)]y \]  
(4.2.1.a)

\[ v^{(1)} = [V_1 - S_1 \sin(\omega t + \phi)]x - S_1 \cos(\omega t + \phi)y \]  
(4.2.1.b)

\[ u^{(3)} = S_3 \cos(\omega t + \phi)x - [V_3 + S_3 \sin(\omega t + \phi)]y \]  
(4.2.1.c)

\[ v^{(3)} = [V_3 - S_3 \sin(\omega t + \phi)]x - S_3 \cos(\omega t + \phi)y \]  
(4.2.1.d)

where \( S_i, V_i, \omega \) and \( \phi \) are constants. The velocity field exterior to the lens has constant vorticity of \( 2V_i \), normal deformation of \( 2S_i \cos(\omega + \phi) \) and shear deformation of \( -2S_i \sin(\omega + \phi) \) in the \( i^{th} \) layer. There is no divergence in the external flows. If \( G = 0 \) in the lens layer, (2.3) reduces to

\[ \dot{h}_0 = 0 \]  
(4.2.2.a)

\[ \dot{B} + 4(B_N G_N + B_S G_S) = 0 \]  
(4.2.2.b)

\[ \dot{B}_S + B G_S - 2B_N G_R = 0 \]  
(4.2.2.c)

\[ \dot{B}_N + B G_N + 2B_S G_R = 0 \]  
(4.2.2.d)

\[ G_N^2 + G_S^2 - G_R^2 - G_R + B = \gamma \]  
(4.2.2.e)
\[
\dot{G}_R = 0 \quad (4.2.2.f)
\]

\[
\dot{G}_N - G_S + 2B_N = \lambda(1 - \omega)\sin(\omega t + \phi) \quad (4.2.2.g)
\]

\[
\dot{G}_S + G_N + 2B_S = \lambda(1 - \omega)\cos(\omega t + \phi) \quad (4.2.2.h)
\]

where

\[
\gamma = W_1(S_1^2 - V_1^2 - V_1) + W_3(S_3^2 - V_3^2 - V_3)
\]

\[
\lambda = W_1S_1 + W_3S_3.
\]

Equation (4.2.2.e) can be rewritten as

\[
G_N^2 + G_S^2 + B = \text{const.} \quad (4.2.3)
\]

From (4.2.2.b), (4.2.2.g) and (4.2.2.h) one can obtain

\[
\frac{d}{dt}(G_N^2 + G_S^2 - B) = 2G_N\lambda(1 - \omega)\sin(\omega t + \phi) + 2G_S\lambda(1 - \omega)\cos(\omega t + \phi). \quad (4.2.4)
\]

If \( G_N = G_D\cos(\omega t + \phi) \) and \( G_S = -G_D\sin(\omega t + \phi) \) where \( G_D \) is a constant, then

\[
G_N^2 + G_S^2 - B = \text{const.} \quad (4.2.5)
\]

It follows that \( B \) and \( G_N^2 + G_S^2 \) are constants and (4.2.2) is a linear system. If (4.2.5) does not hold, there must be

\[
G_N^2 + G_S^2 - B = f(t) \quad (4.2.6)
\]

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where $f(t)$ is a function of time. It follows from (4.2.6) and (4.2.3) that $B$ must be a function of time. In this case, equations (4.2.2) are nonlinear. Here we only consider the linear system of (4.2.2) in which $B$ is a constant. The solution to the linear system of (4.2.2) is that $h_0$, $B$ and $G_R$ are constants and

\[ B_N = -B_D\sin(\omega t + \phi) \quad (4.2.7.a) \]

\[ B_S = -B_D\cos(\omega t + \phi) \quad (4.2.7.b) \]

\[ G_N = G_D\cos(\omega t + \phi) \quad (4.2.7.c) \]

\[ G_S = -G_D\sin(\omega t + \phi) \quad (4.2.7.d) \]

with three constraints

\[ B_D(2G_R + \omega) - BG_D = 0 \quad (4.2.7.e) \]

\[ G_D(1 - \omega) - 2B_D = \lambda(1 - \omega) \quad (4.2.7.f) \]

\[ G_D^2 - G_R^2 - G_R + B = \gamma. \quad (4.2.7.g) \]

Without external flow, the solution (4.2.7) is the same as the solution of subinertial lenses (3.3.3). This means that the external flows must oscillate at the same low frequency as the free lens to keep the system linear. Substitution of (4.2.7) into (2.4.d) shows that the rotation rate of the lenses in this solution is also

\[ \dot{\theta} = -\omega/2. \quad (4.2.8) \]
To examine the effect of external flows on the lens, the variation of $\omega$ and the ratio of minor to major radius $R_b/R_a$ are calculated with different $S_i$ and $V_i$ using (4.2.7). Given

$$B = -0.280134$$

$$B_D = 0.1$$

$$G_R = -0.361260$$

$$G_D = 0.222222,$$

the isolated lens has the frequency of $\omega = 0.1$ and $R_b/R_a = 0.41$. For a simple case, we take $W_1 = W_3 = 0.5$, $S_1 = S_3 = S$ and $V_1 = V_3 = V$. Fig. 4.2.1 shows the variation of $\omega$ with $S$ and $V$. The approach will be to examine $S$ while neglecting $V$ and vice versa. For this situation, the permissible values of $S$ have a limited range. The positive values of $S$ must be less than $G_D$. An $S$ larger than $G_D$ makes the solution of $\omega$ singular. There is also a boundary of negative values of $S$. Any $S$ less than this boundary value makes $B^2/4 - B_D^2 < 0$ which destroys the elliptical structure. In this example, the boundary of negative values for $S$ is $-0.1$. The $V$ plays a similar role for the $\omega$. The boundary of negative values for $V$ is also $-0.1$. Any $V$ less than $-0.1$ makes $B^2/4 - B_D^2 < 0$; however, there is no positive boundary of $V$. The forced lens is always stable for positive values of $V$. The effect of $S$ and $V$ on the eccentricity of the lens is shown in Fig. 4.2.2. Forced lenses become more elongated with more negative values of $S$ and $V$. On the other hand, the boundaries of negative values of $S$ and $V$ depend upon the eccentricity of the free oscillating lenses. More circular lenses permit more negative values of the boundaries. For example, given another group of parameters

$$B = -0.197167$$

$$B_D = 0.01$$

$$G_R = -0.269074$$

$$G_D = 0.022222$$

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the isolated lens also has the frequency $\omega = 0.1$, but the ratio $R_b/R_a = 0.9$. For this lens, the limit of negative values of $S$ is $-0.2$, which is related to a forced lens with $\omega = 0.48$ and $R_b/R_a = 0.39$. The limit of negative values of $V$ is $-0.2$, which corresponds to a forced lens with $\omega = 0.41$ and $R_b/R_a = 0.69$. 

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Fig. 4.2.1. The effect of the shear ($S$) and the spin ($V$) of the external flow on the frequency ($\omega$) of the lens in an example of the special solution (4.2.7). The asterisks indicate the value of $\omega$ without the external flow.
Fig. 4.2.2. The effect of the shear (S) and the spin (V) of the external flow on the eccentricity of the lens in an example of the special solution (4.2.7). The asterisks indicate the value of $R_b/R_a$ without the external flow.
4.3 Solution of a Nonlinear System

Now suppose the flow in the upper and lower layers takes the form

\[ u^{(1)} = (\alpha x + \beta y)P \]  \hspace{1cm} (4.3.1.a)

\[ v^{(1)} = (\alpha y - \beta x)P \]  \hspace{1cm} (4.3.1.b)

\[ u^{(3)} = -(\alpha x + \beta y)P \]  \hspace{1cm} (4.3.1.c)

\[ v^{(3)} = -(\alpha y - \beta x)P \]  \hspace{1cm} (4.3.1.d)

\[ P = [\lambda + \gamma \sin(t + \phi)]^{-1} \]  \hspace{1cm} (4.3.1.f)

where \( \alpha, \beta, \lambda, \gamma \) and \( \phi \) are constants. The condition \( |\lambda| > |\gamma| \) is required to avoid a singular value of \( P \). Examination of (4.3.1.a-d) shows that there is no deformation in the prescribed flows. The velocity field of the external flow in the upper layer has a divergence of \( 2\alpha P \) and a vorticity of \( -2\beta P \). In the lower layer, the divergence and vorticity of the velocity have the same magnitude as the upper layer but have the opposite sign. Using (4.3.1), the forcing terms on the right-hand side of (2.3) occur only in (2.3.e). Taking \( W_1 = W_3 = 1/2 \), (2.3.e) becomes

\[ \dot{G} + G^2/2 + 2(G_N^2 + G_S^2 - G_R^2 - G_R + B) = (2\alpha^2 - 2\beta^2)P^2. \]  \hspace{1cm} (4.3.2)

With the above assumptions, a solution to (2.3) can be written as

\[ h_0 = H_0P \]  \hspace{1cm} (4.3.3.a)

\[ B = -(\Lambda_B/4)P^2 \]  \hspace{1cm} (4.3.3.b)
\[ G = \gamma P \cos(t + \phi) \]  

(4.3.3.c)

\[ G_R = -(1/2)(1 - \Lambda_Q P) \]  

(4.3.3.d)

\[ B_N = -B_D P^2 \sin(\zeta) \]  

(4.3.3.e)

\[ B_S = -B_D P^2 \cos(\zeta) \]  

(4.3.3.f)

\[ G_N = G_D P \cos(\zeta) \]  

(4.3.3.g)

\[ G_S = -G_D P \sin(\zeta) \]  

(4.3.3.h)

where

\[ \zeta = t + \omega t' \]  

(4.3.3.i)

\[ t' = \int_0^t P(\tau) d\tau. \]  

(4.3.3.j)

Here \( H_0, \Lambda_B, \Lambda_Q, B_D \) and \( G_D \) are constants which can be determined by the initial conditions of the eight variables and the external flows. Substitution of (4.3.3) into (2.3) yields three constraints:

\[ \lambda - \gamma^2 - \Lambda_B - \Lambda_Q^2 + 4(G_D^2 - \alpha^2 + \beta^2) = 0 \]  

(4.3.4.a)

\[ B_D \omega + G_D \Lambda_B / 4 + B_D \Lambda_Q = 0 \]  

(4.3.4.b)
\[ G_D \omega + 2B_D = 0. \quad (4.3.4.c) \]

Note that \( \Lambda_B > 0 \) and \( \Lambda_B^2/16 - 4B_D^2 > 0 \) are required for the elliptical structure.

Substitution of (4.3.3) into (2.5) yields the ratio of the minor radius to the major radius and the rotation rate of the ellipse \( \dot{\theta} \) as

\[
R_b/R_a = \left[ (\Lambda_B/8 - |B_D|)/(\Lambda_B/8 + |B_D|) \right] P \quad (4.3.5.a)
\]

\[
\dot{\theta} = -1/2 + (1/2)(\Lambda Q + \Lambda_B G_D/4B_D) P. \quad (4.3.5.b)
\]

As defined in (4.3.1.f), \( P \) is time dependent and varies at the inertial frequency. Therefore, (4.3.5.a) and (4.3.5.b) indicate that the eccentricity and the rotation rate of the lens ellipse in the solution (4.3.3) varies at the inertial frequency in the evolution of the lenses. The values of the rotation rate depend on the initial conditions.

To examine the frequencies occurring in the solution (4.3.3), we take an example of the solution. Given

\[ H_0 = 0.5 \]
\[ \Lambda_Q = 0.3 \]
\[ G_D = 0.1 \]
\[ \lambda = 1 \]
\[ \gamma = 0.1 \]
\[ \phi = 0 \]
\[ \alpha = 0.05 \]
\[ \beta = 0.1, \]

then \( \Lambda_B \) and \( B_D \) can be calculated from

\[
\Lambda_B = \lambda - \gamma^2 - \Lambda_Q^2 + 4(G_D^2 - \alpha^2 + \beta^2) \quad (4.3.6.a)
\]
\[ \omega = (-\Lambda_Q \pm \sqrt{\Lambda_Q^2 + 2\Lambda_B})/2 \quad (4.3.6.b) \]

\[ B_D = -G D \omega/2. \quad (4.3.6.c) \]

Two solutions of (4.3.6) are

\[ \Lambda_B = 0.97 \]
\[ \omega = -0.86 (\omega^-) \]
\[ B_D = 0.043 \]

and

\[ \Lambda_B = 0.97 \]
\[ \omega = 0.56 (\omega^+) \]
\[ B_D = -0.028. \]

With these values the rotation rates of the ellipse are calculated from (4.3.5.b). The "+" sign solution of (4.3.6.b) leads to superinertial frequencies, shown by the dashed line in Fig. 4.3.1. The "−" sign solution of (4.3.6.b) results in subinertial frequencies, shown by the solid line in Fig. 4.3.1. Both the solid and dashed curves in Fig. 4.3.1 oscillate at the inertial frequency, which in turn produces the inertial oscillation in the motion of the lenses.

Solution (4.3.3) with subinertial frequencies is shown in Fig. 4.3.2 in which the eight variables are calculated for 15 inertial days. It is seen that \( h_0, B, G \) and \( G_R \) oscillate only at the inertial frequency while \( B_S, B_N, G_N \) and \( G_S \) have both sub and inertial frequencies. The low frequency is about 0.13 cpd (≈ period of 7.5 inertial days) which corresponds to an average \( \dot{\theta} = -0.065 \) cpd (see the solid line of Fig. 4.3.1). The evolution of the ellipse is shown in Fig. 4.3.3 for six specific times. It is seen that the ellipse completes a cycle in about 15 inertial days, which is in agreement with the average rotation rate of \(-0.065\) cpd. Fig. 4.3.4 shows the trajectory of a particle which was initially located at the north end of the ellipse.
The solution of (4.3.3) with superinertial frequencies is shown in Fig. 4.3.5. It is seen that $h_0$, $B$, $G$ and $G_R$ have the same pattern as in Fig. 4.3.2, but $B_S$, $B_N$, $G_N$ and $G_S$ oscillate at superinertial frequencies with a subinertial modulation. The average of the superinertial frequencies can be recognized from the ellipse revolution in Fig. 4.3.6. The ellipse rotates anticyclonically for 180° in 4 inertial time units ($\sim 0.63$ inertial day) so that the rotation rate of the ellipse is about $-0.79$ cpd which is close to the mean of the dashed line in Fig. 4.3.1. The trajectory of a particle at the rim of the lens is shown in Fig. 4.3.7 for 30 inertial time units.

When $\gamma = 0$ in (4.3.1), the environmental flow is in steady state. Then, $h_0$, $B$ and $G_R$ become constants and $G = 0$. Moreover, size, rotation rate and vorticity of velocity field are all constants. Solution (4.3.3), then, becomes a solution for a linear system of (2.3).
Fig. 4.3.1. The rotation rates of the ellipse of the lens in the special solution (4.3.3). The solid line is $\omega^-$ and the dashed line is $\omega^+$. The negative values indicate anticyclonic rotation.
Fig. 4.3.2. The solution of (4.3.3) for the subinertial frequency.
Fig. 4.3.3. The evolution of the ellipse of the solution of (4.3.3) for the subinertial frequency.
Fig. 4.3.4. The trajectory of a particle at the rim of the lens in the solution (4.3.3) for the subinertial frequency. The arrows denote every 10 inertial time units. The square indicates the initial position of the particle.
Fig. 4.3.5. The solution of (4.3.3) for the superinertial frequency.
Fig. 4.3.6. The evolution of the ellipse of the solution of (4.3.3) for the superinertial frequency.
Fig. 4.3.7. The trajectory of a particle on the rim of the lens in the solution (4.3.3) for the superinertial frequency. The arrows denote every 10 inertial time units. The square indicates the initial position of the particle.
5 Invariants

5.1 General Invariants for Isolated Lenses

The five well-known integral invariants to the shallow water equations were first established by Ball (1963). They are potential vorticity, volume, angular momentum, total energy, and a fifth, which lacks a familiar name. Cushman-Roisin et al. (1985) and Young (1986) further studied these invariants with Goldsbrough's (1930) reduction, which is equivalent to (2.2). The five invariants to the unforced version of (2.3) can be written as

\[ I_1 = Q_R/h_0 \]
\[ I_2 = (G_D^2 - B)/h_0^2 \]
\[ I_3 = [(B_N G_S - B_S G_N - (1/2)BQ_R)/h_0^3 \]
\[ I_4 = \Delta_B/h_0^4 \]
\[ I_5 = [4\Delta_B + 4G_R(B_N G_S - B_S G_N) + 2G(B_S G_S + B_N G_N) - B(G^2/4 + G_D^2 + G_{R}^2)]/h_0^3 \]

where

\[ Q_R = G_R + 1/2 \]
\[ B_D^2 = B_N^2 + B_S^2 \]
\[ G_D^2 = G_N^2 + G_S^2 \]

\[ \Delta_B = B^2/4 - B_0^2. \]

Compared with the invariants given by Cushman-Roisin, et al. (1985), \( I_1, I_3, I_4 \) and \( I_5 \) are related to the potential vorticity, angular momentum, volume and total energy of lenses, respectively. Let \( Q \) be the volume of the lens, \( Z \) the potential vorticity, \( J \) the angular momentum and \( E \) the total energy. The integral forms of these physical quantities are (see Cushman-Roisin, et al., 1985)

\[ Q = \int \int h \, dx \, dy \quad (5.1.2.a) \]

\[ Z = \int \int \left( \frac{v_z - u_z + 1}{h} \right) h \, dx \, dy \quad (5.1.2.b) \]

\[ J = \int \int [(xv - yu) + \frac{1}{2}(x^2 + y^2)] h \, dx \, dy \quad (5.1.2.c) \]

\[ E = \int \int \left[ \frac{1}{2} h + \frac{1}{2}(u^2 + v^2) \right] h \, dx \, dy. \quad (5.1.2.d) \]

Neglecting some constant coefficients, the proportions of the physical quantities to the invariants \( I_1, I_3, I_4, \) and \( I_5 \) are

\[ Q \sim \frac{1}{\sqrt{I_4}} \quad (5.1.3.a) \]

\[ Z \sim Q I_1 \quad (5.1.3.b) \]

\[ J \sim Q^3 I_3 \quad (5.1.3.d) \]
There is no popular name for the physical meaning of $I_2$. $G_N^2 + G_D^2$ is the square of the total deformation of motion and $B$ is the Laplacian of the thickness of the lens ($\nabla^2 h$). $B$ is required to be negative for the elliptical lenses, so that $\nabla^2 h$ must be negative. Therefore, $[G_N^2 + G_D^2 + |\nabla^2 h|]/h_0^3$ is conserved in lenses. From this point of view, we shall call $I_2$ potential deformation.

Some specific forms of the invariants for the special solutions are now given. For the special solution for the superinertial lens, the invariants simplify to

\begin{align}
I_1 &= (1/2)/h_0 \\
I_2 &= (G_D^2 - B)/h_0^2 \\
I_3 &= (B_DG_D - B/4)/h_0^3 \\
I_4 &= (B^2/4 - B_D^2)/h_0^4 \\
I_5 &= [4(B^2/4 - B_D^2) - BG_D]/h_0^3.
\end{align}

For the special solution for the subinertial lens, the invariants become

\begin{align}
I_1 &= Q_R/h_0 \\
I_2 &= (G_D^2 - B)/h_0^2 \\
I_3 &= [(B_DG_D - (1/2)BQ_R)/h_0^3.
\end{align}
\[ I_4 = \frac{(B^2/4 - B_D^2)}{h_0^4} \]  
\[ I_5 = \frac{[4(B^2/4 - B_D^2) + 4G_RB_DG_D - B(G_D^2 + G_R^2)]}{h_0^3} \]  
\[ I_1 = \frac{Q_R}{h_0} \]  
\[ I_2 = -2\sqrt{I_4} \]  
\[ I_3 = \frac{I_1 I_2}{2} \]  
\[ I_4 = \frac{B^2}{4h_0^4} \]  
\[ I_5 = \frac{[B^2 - B(G^2/4 + G_R^2)]}{h_0^3}. \]

For the special solution for the inertial lens, the five invariants are

The five invariants in general are independent for the isolated elliptical lenses. However, for the pulson special solution \( I_2 \) and \( I_3 \) can be expressed in terms of \( I_1 \) and \( I_4 \). This is because of the two simplifications for this lens: symmetry in the mass field and zero deformation in the flow field. With these assumptions, \( I_2 \) reduces to \(-B/h_0^2\) and \( I_3 \) reduces to \(-(1/2)BQ_R/h_0^3\).

### 5.2 “Equivalent” Lenses

Equations (2.3) have eight dependent variables and five invariants. This prompts the question as to whether the invariants uniquely specify the evolution of the system, i.e. can different solutions have the same invariants? This issue is particularly relevant to the special solutions discussed above where the invariants are not all independent.
For example, the *pulson* solution can be specified by just three independent invariants which, of course, are determined by the initial conditions. Since there are eight initial values in general and some restraints in the special solutions, this implies that not all of the initial values are arbitrary. It would seem, then, that there are several possibilities for specifying the invariants and their interrelations from arbitrary specifications of eight initial values. Such solutions, if indeed they exist, are called here "equivalent" lenses.

First, we examine the *rodon* and *pulson* special solutions. To search for more general solutions that could produce a lens equivalent to the special solutions, the model (3.1.2) is again used to specify the initial values. The specified initial values of $B_S$, $B_N$, $G_N$ and $G_S$ are

\[
B_S = -B_D \cos(\phi) \tag{5.2.1.a}
\]

\[
B_N = -B_D \sin(\phi) \tag{5.2.1.b}
\]

\[
G_N = G_D \cos(\phi + \Delta \phi) \tag{5.2.1.c}
\]

\[
G_S = -G_D \sin(\phi + \Delta \phi). \tag{5.2.1.d}
\]

If $\Delta \phi = 0$, the *rodon* solution may result. Nonzero $\Delta \phi$ produces general solutions.

Consider first the *rodon* special solution. The only invariant in which $\Delta \phi$ enters is $I_5$. For the *rodon* solution, this is

\[
I_{SR} = \left[4\Delta B + 4G_R B_D G_D - B(G_D^2 + G_R^2)\right]/\hbar_0^3 \tag{5.2.2.a}
\]

while the invariant $I_{SC}$ for the general solution is
\[ I_{5G} = [4\Delta B + 4G_R B_D G_D \cos \Delta \phi + 2G_B G_D \sin \Delta \phi ] \\
\]
\[-B(G^2/4 + G_D^2 + G_R^2)]/h_0^3. \quad (5.2.2.b)\]

The difference between the rodon solution and the general solution is that divergence occurs in the latter. Equating \( I_{5R} = I_{5G} \) gives a quadratic equation in \( G \), the solution of which is

\[ G = \left( -B/4 \right)[-B_D G_D \sin \Delta \phi \\
\pm \sqrt{B_D G_D (B_D G_D \sin^2 \Delta \phi + B G_R (\cos \Delta \phi - 1))}] \quad (5.2.3) \]

From (5.2.3) it appears that a series of \( G \) could be obtained for different \( \Delta \phi \). Unfortunately, nonzero values of \( G \) are not found because any \( \Delta \phi \neq 0 \) makes the square root in (5.2.3) imaginary (see Appendix B for details). The only possible value of \( G \) is \( G = 0 \) which arises only when \( \Delta \phi = 0 \). This, of course, is the rodon special solution. This means that there are no equivalent lenses for the rodon special solution from the model (5.2.1). Note that this does not preclude other models for equivalent lenses.

For the pulson special solution, the invariants \( I_{2p} \) and \( I_{4p} \) are

\[ I_{2p} = -B \quad (5.2.4.a) \]
\[ I_{4p} = B^2/4. \quad (5.2.4.b) \]

For convenience \( h_0 \) is set at 1 since it does not affect the final result. For the general solution from (5.2.1), the invariants \( I_{2G} \) and \( I_{4G} \) are
\[ I_{2G} = \frac{(G_D^2 - B')}{h_0^2} \]  
(5.2.5.a)

\[ I_{4G} = \frac{(B'^2/4 - B_D^2)}{h_0^4} \]  
(5.2.5.b)

where the "'" denotes a different initial value of \( B \) from that in (5.2.4). From the equalities \( I_{2p} = I_{2G} \) and \( I_{4p} = I_{4G} \), \( G_D \) and \( B_D \) are

\[ G_D = \sqrt{-B h_0^2 + B'} \]  
(5.2.6.a)

\[ B_D = \frac{(\sqrt{B'^2 - B^2 h_0^2})}{2}. \]  
(5.2.6.b)

Recall that the elliptical structure in (2.2.c) requires \( B < 0 \). Hence, (5.2.6.a) implies \( |B'| < |B h_0^2| \) and (5.2.6.b) implies \( |B'| > |B h_0^2| \). The only solution of these inequalities is \( B_D = G_D = 0 \) which in turn defines the pulson solution. This means that the pulson special solution also has no equivalent lenses under the assumption (5.2.1). As before, this does not preclude other approaches for determining equivalent lenses.

For the numerical solutions to (2.3), an arbitrary group of initial values of \( h_0, B, B_S, B_N, G_R, G_N \) and \( G_S \) corresponds to two values of \( G \), which results in a pair of lenses with the same invariants. An example of a pair of equivalent lenses is shown in Fig. 5.2.1 and Fig. 5.2.2. Fig. 5.2.1 shows the numerical solutions of the three variables \( h_0, G \) and \( G_S \). Fig. 5.2.2 shows the major and minor radii of the two lenses. The solid lines stand for the solutions of one lens with the initial values

\[
\begin{align*}
h_0 &= 1 \\
B &= -0.3 \\
B_S &= 0 \\
B_N &= -0.1
\end{align*}
\]

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\[ G = 0 \]
\[ G_R = -0.36 \]
\[ G_N = 0.27 \]
\[ G_S = 0 \]

while the dashed lines stand for the solutions of its partner, which has the same initial values except \( G = 0.72 \). It is seen that the evolution of the variables appears to be different for these two cases. But careful inspection of the solution shows that the amplitude and frequency of each pair of variables are the same. Fig. 5.2.3 shows the spectra of \( G_S \) in both cases. It indicates that the pair of equivalent lenses oscillates at exactly the same frequencies and that the energy at each frequency is the same. The difference between these equivalent lenses is nothing but a phase shift.

From the above examples it seems that the invariants uniquely specify the isolated lenses. There is only a phase shift between the equivalent lenses. This conclusion comes from model (5.2.1) and associated assumptions; however, the question is still open as to the existence of different solutions which have the same invariants using different approaches.

### 5.3 Invariants for Forced Lenses

For forced lenses, only \( I_1 \) and \( I_4 \), in general, remain constant. This means that the potential vorticity and volume are still conserved while the other physical quantities vary because of the external forces. For some special external flows, the situation is not so complicated. For example, in the case of the special solution of a linear system (4.2.7), the five invariants are still constants and have the same form as in the rodon case.

In the case of the special solution of the nonlinear system (4.3.3), \( I_1, I_2, I_3 \) and \( I_4 \) remain constant as
\[ I_1 = \Lambda Q / 2H_0 \]  \hspace{1cm} (5.3.1.a)

\[ I_2 = (4G_D^2 + \Lambda B) / 4H_0^2 \]  \hspace{1cm} (5.3.1.b)

\[ I_3 = (16B_DG_D + \Lambda B \Lambda Q) / 16H_0^3 \]  \hspace{1cm} (5.3.1.c)

\[ I_4 = (\Lambda_B^2 - 64B_D^2) / 64H_0^4. \]  \hspace{1cm} (5.3.1.d)

But \( I_5 \) is not an invariant. It becomes

\[ I_5 = (\lambda \Lambda_B - \Lambda_B \Lambda Q - 16B_DG_D) / (8H_0^3) + \frac{a^2 - b^2}{4H_0^2} P \]  \hspace{1cm} (5.3.1.e)

where \( P \) is a time-dependent function occurring in the external flow. \( P \) was defined in (4.3.1.f) as

\[ P = \frac{1}{\lambda + \gamma \sin(t + \phi)}. \]  \hspace{1cm} (5.3.1.f)

This means that the energy of the middle layer lens changes with time in this case. Note that the variation of energy does not evoke instability as long as the initial condition satisfies the elliptical constraint \( B < 0 \) and \( B^2/4 - B_D^2 > 0 \).

The \( I_5 \) will become a constant only if \( \gamma = 0 \). The solid line in Fig. 5.3.1 shows invariant \( I_5 \) for the solution of the nonlinear system discussed in section 4.3. It is seen that the invariant \( I_5 \) varies with the inertial frequency, which is the same as that in the external flows. If \( \gamma = 0 \), the solution reduces to the special rodon solution and the \( I_5 \) becomes a constant as shown by the dotted line in Fig. 5.3.1.
Fig. 5.2.1. The numerical solutions of $h_0$, $G$, and $G_s$ for the equivalent lenses with the initial values (in solid lines): $h_0=1$, $B=-0.3$, $B_S=0$, $B_N=-0.1$, $G=0$, $G_R=-0.36$, $G_N=0.27$, $G_S=0$; (in dashed lines): $h_0=1$, $B=-0.3$, $B_S=0$, $B_N=-0.1$, $G=0.72$, $G_R=-0.36$, $G_N=0.27$, $G_S=0$.

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Fig. 5.2.2. The major and minor radii of the equivalent lenses with the initial values (in solid lines): \( h_0=1, B=-0.3, B_N=-0.1, G=0, G_R=-0.36, G_N=0.27, G_S=0 \); (in dashed lines): \( h_0=1, B=-0.3, B_N=0, B_A=-0.1, G=0.72, G_R=-0.36, G_N=0.27, G_S=0 \).
Fig. 5.2.3. The spectra of $G_S$ in the numerical solution with the initial values $h_0=1$, $B=-0.3$, $B_S=0$, $B_N=-0.1$, $G_R=-0.36$, $G_N=0.27$, $G_S=0$ and different $G$ indicated in each panel.
Fig. 5.3.1. The invariant $I_5$ for the solution of the nonlinear system with $\gamma=0.1$ (solid line) and $\gamma=0$ (dotted line). The other constants are $H_0=0.5$, $A_g=0.3$, $G_D=0.1$, $A_B=0.97$, $B_D=0.043$, $\alpha=0.05$, $\beta=0.1$ and $\lambda=1$. 
6 Application to Warm Eddies

6.1 Data Suitable for Model

Warm eddies are usually observed with three kinds of data: hydrographic, Lagrangian (drifters and floats) and satellite IR images.

Hydrographic data such as temperature or density profiles show warm eddies as a pool of warm water in a surface layer surrounded by cooler water or as a lens of warm and salty water in a subsurface layer. The hydrographic surveys can also show the horizontal extent and shape of warm eddies. Surveys usually show these eddies as elliptical. So far, the data from hydrographic stations are not synchronous. Therefore, vertical and horizontal sections obtained from hydrographic data should be considered at best as an average state. For simulation of warm eddies, equation (2.2.c) describes an elliptical lens which has an elliptical rim and an ellipsoidal bottom. The bottom of real anticyclonic eddies, of course, is not exactly ellipsoidal, but some of the data do show the similarity between observations and the model (see Lewis & Kirwan, 1987; Lewis et al. 1989).

Lagrangian data give the time series of positions of a particle usually as latitude and longitude. From Lagrangian data, one can obtain the swirl velocity of particles in eddies and also infer the geometry of eddies (Kirwan, et al. 1988). The velocity of a drifter includes two kinds of motion: swirl velocity, which is the flow about the center of the eddy, and the translation velocity, which is the motion of the eddy center. In (2.2), the $u$ and $v$ are the swirl velocity. The displacement calculated from (2.5) is the trajectory of a particle in a coordinate system moving with the center of the eddy. A particle at the boundary always stays at the boundary in the lens model. In analyzing observations from drifters, it is often assumed that the drifters are located near the eddy boundaries. However, in a long migration, the drifters may not always stay near the boundaries of eddies. It is therefore reasonable to choose data covering minimal translation and relatively short periods.
Satellite IR data give surface temperature gradients, which often are the signatures of eddies. With the surface signatures, one can obtain the scale and the eccentricity of the horizontal ellipse of eddies. The advantage of satellite IR data is that it is easy to determine shape changes of eddies.

With hydrographic, drifter and IR data, one can estimate the average size of eddies, the trajectory and the rotating period of the particles in eddies. Based on these features, it is possible to specify the initial values of the variables in (2.3) to produce numerical or analytical solutions. The next section discusses the approach taken here to specify the initial condition from observations.

### 6.2 Estimate of Initial Values

The key to application of the lens model is to obtain appropriate estimates of the initial values of $h_0$, $B$, $B_S$, $B_N$, $G$, $G_R$, $G_N$ and $G_S$ from observations. Once the initial values are known, (2.3) can be solved numerically, the constants in the special solutions are specified and the invariants are determined. In the special solutions, the initial values of $B_S$, $B_N$, $G_N$ and $G_S$ can be specified if $B_D$, $G_D$ and $\phi$ are given. Since $\phi$ is related to the initial orientation of the horizontal ellipse of lenses and does not affect the invariants, the characteristics of a lens are determined by $h_0$, $B$, $B_D$, $G$, $G_R$ and $G_D$ for the special solutions. With the constraints of the special solutions, $B$, $B_D$, $G$, $G_R$ and $G_D$ are not independent. For example, only two are needed for the special solution (3.3.3); the others can be calculated from the constraints.

To apply the nondimensional model to observations, the typical scales for warm eddies are taken as

- $H = 500m$
- $R_d = 30km$
- $f = 10^{-4}s^{-1}$.
With these values, the reduced gravity \( g' = 1.8 \cdot 10^{-2} \text{ms}^{-2} \).

The initial values of \( B \) and \( B_D \) are associated with the size and the shape of lenses. The smaller the \( B_D \), the more circular the lens. If the size and shape of an eddy are known from hydrographic surveys, satellite data or drifter trajectories, \( B \) and \( B_D \) can be estimated with the equations (2.4.a) and (2.4.b).

\( G \), \( G_R \) and \( G_D \) can be estimated from drifter trajectories using the inverse technique developed by Kirwan et al. (1988) and Kirwan, Indest, Liu, and Clark (1990). The algorithm of the inverse technique is briefly reviewed in Appendix A. The inverse technique is examined with the special solution (4.2.7). Let \( d \), \( a \), \( b \) and \( c \) in the inverse technique (see Appendix A) correspond to \( G \), \( G_N \), \( G_S \) and \( G_R \) in the model. Fig. 6.2.1 gives the ratios of \( c \) to \( G_R \) and the magnitude \( \sqrt{a^2 + b^2} (D) \) to that of \( \sqrt{G_N^2 + G_S^2} (G_D) \) versus different rotation rates of an ellipse with the major radius twice the minor radius. The test shows that the output of \( G \), \( G_R \), \( G_N \) and \( G_S \) is very close to the input for the slowly rotating lenses but decreases substantially as the lens rotation rate increases. The accuracy of the inversion also increases with decreasing eccentricity. For the case of \( R_b/R_a = 0.9 \), the test shows (figure not shown) that \( c/G_R \) is greater than 0.95 as the rotation rate reaches \(-0.02 \). However, \( D/G_D \) only increases to 0.7. Apparently, \( G_D \) is underestimated by this approach; nevertheless, it is possible to calibrate for the rotation rate and eccentricity so that the initial condition of \( G_R \) and \( G_D \) can be adequately estimated.

\( G \) is associated with the inertial frequency of the eight variables and the mean value of \( G \) is always near zero. For the mesoscale eddies, the inertial frequency in observations is usually filtered out in the data processing. Since \( G \) has a strong inertial component its initial value can be taken to zero in simulation of the mesoscale eddies.

The initial value of \( h_0 \) can be estimated from the vertical structure of hydrographic data. For example, the depth of the thermoclines in temperature profiles can be used to define \( h_0 \). Note that \( h_0 \) is a stratified depth, which is less than the true depth.
(see LeBlond and Mysak, 1978, p.136). There are no data available at present to
determine the stratified depth. Therefore, the specification of the initial value of $h_0$
from observations has a high degree of uncertainty. In practice, the estimation of
$h_0$ must be consistent with the initial values of $B$, $B_D$, $G_R$ and $G_D$ specified from
above procedures. Adjustment of $h_0$ is often needed to match the constraints and the
velocities of the drifters. For example, if the velocity of model lenses is larger than
that measured by the drifters, $h_0$ should be smaller, and vice versa. Failure of the
adjustment suggests that the model may not be appropriate.

6.3 Drifter Trajectory Simulation

During the past decade, most of the observations of warm eddies were obtained
by satellite-tracked drifters. The positions of Agros drifters are determined by the
satellite approximately four times a day. A set of the position data shows a rather
continuous trajectory. These drifter data have stimulated a number of eddy models
(Kirwan, Merrell, Lewis, Whitaker, & Legeckis, 1984; Kirwan et al. 1988; Glenn,

The model given by Kirwan et al. (1984, 1988) describes the drifter motion with a
translation and a solid body rotation. The six model parameters specify the transla­
tion of the eddy center, divergence, vorticity and deformation of the swirl velocity of
eddies. The parameters can be derived from time derivatives of the drifter positions.
The calculation of the time derivatives to fourth-order requires at least eight adjacent
drifter positions. In the 1988 version the approach was modified so that the inversion
was obtained as a running filter. This allowed for “slow time” variation of the model
parameters.

Glenn et al. (1990) developed another kinematic model called the feature model.
Like the last one, the model assumes the drifter trajectory is composed of a solid
body rotation about a translating center. There are ten parameters in the model
which describe the translation of the eddy center, the elliptical structure of the eddy and the magnitude and frequency of the swirl velocity. Without calculation of time derivatives, the ten parameters are determined by a least squares fit to the drifter positions. To obtain a unique fit, at least ten drifter positions are required.

Since the parameters in both model are constants, the required numbers of the drifter positions give a constraint which requires a steady state of the dynamic field in the period covered by these positions. However, the available drifter position data have an average time interval of 0.25 day so that 8 and 10 drifter positions cover at least 2 and 2.5 days, respectively. If the dynamic field of warm eddies varies at a frequency comparable to these periods, the parameters obtained from these drifter positions are not reliable.

Fortunately, the model results show that the rotation rate of the eddy ellipse was about 3°/day for a Gulf of Mexico eddy (Kirwan et al., 1989) and 13°/day for a Gulf Stream eddy (Glenn et al., 1990) so that the dynamic field of these eddies varied at periods of about 1 to 4 months. Probably this is one of the reasons for the successful applications of the two models to warm eddies.

The present model also assumes that the drifters have solid body rotation about the eddy center. But the substantial difference with the previous models is that the eight parameters are time-dependent. Under a quasi-geostrophic assumption $G = 0$, the rodon special solution has four variable parameters $B_N$, $B_S$, $G_N$ and $G_S$, which describe the ellipse orientation and the deformation of the dynamic field of eddies.

The simulation of drifter trajectories in the next two sections is performed with the rodon special solution and the solution of a linear system which allows a prescribed external flow. There are seven constants in the rodon special solution: $h_0$, $B$, $B_D$, $G_R$, $G_D$, $\omega$ and $\phi$. As mentioned before, an arbitrary $\phi$ does not affect the invariants of lenses. Therefore, with three solution constraints, only three constants are independent. We prefer to utilize $h_0$, $B$ and $B_D$ because these three constants

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are related with the geometry of lenses and can be easily specified from profiles of hydrographic data and drifter trajectories.

The simulation procedure is as follows: (a) remove the translation of the eddy center thus obtaining the displacements of the drifter relative to the center; (b) specify $h_0$, $B$ and $B_D$ based on the displacements and profiles of the relevant hydrographic data such as temperature profiles; (c) calculate $G_R$, $G_D$ and $\omega$ with the solution constraints; (d) produce the time series of $B_N$, $B_S$, $G_N$ and $G_S$ with the rodon special solution and calculate the trajectory of a particle at the rim of the lens; (e) compare the model and the drifter trajectories; (f) try external forces and simulation with the solution of a linear system if the model trajectory does not fit that of the drifter.

As will be seen, the results of the simulation show that the rodon special solution is suitable for warm eddies in the middle of the Gulf of Mexico. By means of external forces, the solution of a linear system is also successful for a drifter near the west coast of the Gulf of Mexico, the Great Whirl, and a Meddy.
Fig. 6.2.1. The ratios of the parameters calculated with the inverse technique (c, D) to their input values ($G_R$, $G_D$) versus rotation rates of the ellipse. The lens is fixed with $R_b/R_a = 0.5$. Here $D = \sqrt{a^2 + b^2}$ and $G_D = \sqrt{G_N^2 + G_S^2}$. 

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7 Gulf of Mexico Eddies

Warm-core eddies in the Gulf of Mexico are well documented (Lewis and Kirwan, 1987; Kirwan et al., 1988; Lewis et al., 1989). According to these references, anticyclonic eddies are pinched off from the Loop Current of the Gulf of Mexico. The eddies then migrate westward across the Gulf basin and ultimately impact the continental slope of the west coast. Fig. 7.0.1, taken from Lewis and Kirwan (1987), shows a typical temperature profile of an eddy in the middle of the Gulf of Mexico. It is seen that there is a lens-shaped pool of uniformly warm water with a depth of about 200 m and a horizontal length of about 300 km. This pool corresponds to the core of a warm eddy. Some Gulf of Mexico eddies are monitored by Agros drifters. Six segments of the drifter trajectories are taken from the four drifters 1598, 1599, 3378 and 3379 listed in Table 1.

Table 1 Observations of warm eddies in the Gulf of Mexico

<table>
<thead>
<tr>
<th>Eddy</th>
<th>Center</th>
<th>Date</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_1$</td>
<td>$25.6N - 90.7W$</td>
<td>Aug. 20 - Sep. 29, 1985</td>
<td>3378</td>
</tr>
<tr>
<td>$E_2$</td>
<td>$24.8N - 91.2W$</td>
<td>May 2 - Jun. 5, 1986</td>
<td>3379</td>
</tr>
<tr>
<td>$E_3$</td>
<td>$23.2N - 92.8W$</td>
<td>Jan. 1 - Feb. 1, 1981</td>
<td>1598</td>
</tr>
<tr>
<td>$E_5$</td>
<td>$22.6N - 95.1W$</td>
<td>Mar. 7 - Apr. 21, 1981</td>
<td>1598</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$23.4N - 96.3W$</td>
<td>Mar. 22 - May 1, 1981</td>
<td>1599</td>
</tr>
</tbody>
</table>

Fig. 7.0.2 shows positions of the eddy centers defined by these trajectories. The six segments of the drifter trajectories in Table 1 will be simulated by the special
In the simulation, the drifters are modeled by the particles located at the rim of the lenses. A particle on the lens rim always stays on the boundary during its evolution while a real drifter near the boundary of an eddy may take excursions for various reasons. For example, Lewis et al. (1989) reported that drifter 3378 took normal revolutions around an eddy called “Fast Eddy” in August and September of 1985, then abruptly moved towards the eddy center and was moving close to the center in October 1985.

The special solution used in the application is not able to simulate a drifter with a variable orbit. Hence, the drifter trajectories in Table 1 are chosen by this rule: the period is long enough to identify the frequencies of the eddies and short enough so that the drifters remain near their original orbits. In addition, since the model does not include the translation of the eddy center, the trajectories are chosen for periods where the eddy centers have nearly constant speeds of translation, thus making translation removal easier.
Fig. 7.0.1. Typical temperature profile in the middle of the Gulf of Mexico (from Lewis and Kirwan, 1987).
Fig. 7.0.2. The center positions to which the warm eddies are calibrated in the Gulf of Mexico.
7.1 Eddy $E_1$

Eddy $E_1$ is defined by the drifter trajectory in Fig. 7.1.1. The trajectory was made by drifter 3388 from Aug. 20 to Sep. 29, 1985, in the middle of the Gulf of Mexico. We assume the drifter moves near the edge of eddy $E_1$, which is centered at 25.6N-90.7W on Aug. 20, 1985. Fig. 7.1.2 shows the latitudinal and longitudinal components of the trajectory. The translation speed of the center is estimated as $u = -3.75 \text{ km/day}$ and $v = 0.5 \text{ km/day}$. After removing the translation, the displacement of the drifter relative to the center is shown in Fig. 7.1.3 (dotted lines).

It is seen that the drifter locally moves along an elliptical orbit with radii of about 80 to 90 km in the first 20 days. The period of rotation of the drifter around the center is about eight days. During the later 20 days, the displacement slightly decreases. Since the period of rotation of the drifter does not change, it is reasonable to assume that the drifter migrates slightly towards the eddy center.

There is no corresponding hydrographic profile for each of the drifter trajectories to estimate the thickness of the eddies. The vertical scales of the Gulf of Mexico eddies are taken based on the typical temperature profile in Fig. 7.0.1. Here, the central thickness of the eddies, which is related to $h_0$ in the model, is taken as 200 m. The practical test showed that this depth matched the swirl velocities for most of the drifters. More discussion about the determination of $h_0$ are in the last section.

From these observations, the geometry of eddy $E_1$ is estimated as a lens with a thickness of about 200 m and horizontal radii of about 80 to 100 km. The simulation of eddy $E_1$ is performed with the rodon special solution. The constants $h_0$, $B$ and $B_D$ can be specified by the geometry of the eddy while $G_R$, $G_D$ and $\omega$ can be calculated with the solution constraints. They are estimated as

$$h_0 = 0.4$$
$$B = -0.106888$$
$$B_D = 0.01$$
\[ G_R = -0.121144 \]
\[ G_D = 0.0204791 \]
\[ \omega = 0.023. \]

With these constants, a set of characteristics of a lens are derived. The model lens has a thickness of 200 m \( (h_0 = 0.4) \), a major radius of 90 km \( (R_a = 3) \) and a minor radius of 75 km \( (R_b = 2.5) \). The horizontal ellipse anticyclonically rotates at a rate of about 4° per day \( (\dot{\theta} = -0.0115) \). The evolution of a particle at the rim of the lens is shown in Fig. 7.1.3 (solid lines). The trajectory of the particle after adding the translation of the eddy center is shown in Fig. 7.1.4. Fig. 7.1.5 shows the swirl velocities of the particle (solid lines) and drifter 3378 (dotted lines).

It is seen from Fig. 7.1.3 that the model and the drifter trajectories have an overall similar pattern. The particle, which is the model drifter, moves along an elliptical orbit with a major radius of 90 km and a minor radius of 75 km and completes a cycle in about eight days. In Fig. 7.1.5, the model swirl velocity compares well to the observations at the low frequency. The jiggles in the drifter velocity are probably observational noise; the velocity is calculated from the raw data of the drifter positions, which are not processed with any interpolation and filtering.

Lewis et al. (1989) specified drifter 3378 for its whole life. The segment of the trajectory chosen here corresponds to days 234–275 of 1985 in their paper. They found that in this period “Fast Eddy” had a major axis of about 175 km and a minor axis of about 150 km. The minor axis anticyclonically rotated at a rate of about 3° per day. They also found that the magnitude of deformation rate was about \( 0.2 \times 10^{-5}s^{-1} \) and the spin rate was a constant of about \( -1.0 \times 10^{-5}s^{-1} \). The corresponding parameters from the present model are a major axis of 180 km, a minor axis of 150 km and a rotation rate of 4° per day. Applying the scaling \( f = 10^{-4}s^{-1} \), the magnitude of the deformation rate \( (G_D) \) is \( 0.204791 \times 10^{-5}s^{-1} \) and the spin rate \( (G_R) \) is \( -1.21144 \times 10^{-5}s^{-1} \). It is seen that the eddy parameters from two different
models are very close.

7.2 Eddy $E_2$

Eddy $E_2$ is defined by the drifter trajectory in Fig. 7.2.1. The trajectory is measured by drifter 3379 from May 2 to June 5, 1986, in the middle of the Gulf of Mexico. During this period, the center of eddy $E_2$ seems to stay at the same place. Fig. 7.2.2 shows the latitudinal and longitudinal components of the trajectory. The center of the eddy is estimated at 24.8N–91.2W and the displacement of the drifter relative to the center is shown in Fig. 7.2.3 (dotted lines). It is seen that the drifter rotates around the center at a period of about 10 days and moves along an orbit 80 to 100 km from the center.

Like eddy $E_1$, eddy $E_2$ is considered as a surface lens which has a horizontal ellipse with a major radius of about 100 km, a minor radius of about 80 km and a thickness of about 200 m. The particle at the edge of the lens rotates around the lens center at a period of about 10 days.

The simulation of drifter 3379 is performed using the rodon special solution (3.3.3). Based on the geometry of eddy $E_2$, the solution constants are estimated as

\[
\begin{align*}
  k_0 &= 0.4 \\
  B &= -0.0946375 \\
  B_D &= 0.01 \\
  G_R &= -0.105314 \\
  G_D &= 0.0203646 \\
  \omega &= 0.0179.
\end{align*}
\]

The solution with these constants defines a lens with a thickness of 0.4 (~ 200 m), a major radius of 3.27 (~ 98 km) and a minor radius of 2.64 (~ 79 km). The model lens has a fixed shape and constant rotation rate $\dot{\theta} = -0.009$ (~ 3° per day). The displacement relative to the lens center of a particle at the edge of the lens is shown...
in Fig. 7.2.3 (solid lines). Fig. 7.2.4 shows the trajectory of the particle.

It is seen from Fig. 7.2.3 that the model displacements have the same magnitude and period as that of drifter 3379. The difference between the model and the drifter trajectories at the beginning and end is accounted for in the center translation of eddy $E_2$. Drifter 3379 moved in this region on May 2, 1986, and revolved there for a month, then left for another region on June 5, 1986. The simulation of eddy $E_2$ does not include the center translation at both ends of the period. The model lens for eddy $E_2$ is very similar to that for eddy $E_1$ in size and rotation rate. Applying the scaling $f = 10^{-4} \text{s}^{-1}$, eddy $E_2$ has a magnitude of deformation rate ($G_D$) of $0.203646 \times 10^{-5} \text{s}^{-1}$ and a constant spin rate ($G_R$) of $-1.05314 \times 10^{-5} \text{s}^{-1}$.

### 7.3 Eddies $E_3$ and $E_4$

Eddies $E_3$ and $E_4$ are defined by the trajectory of drifter 1598 from Jan. 1 to Feb. 1, 1981 (Fig. 7.3.1) and the trajectory of drifter 1599 from Jan. 1 to Jan. 25, 1981 (Fig. 7.3.3). Fig. 7.3.2 and Fig. 7.3.4 show the latitudinal and longitudinal components of the trajectories of 1598 and 1599, respectively. It is seen that the two drifters show the same qualitative behavior. Hence, we select eddy $E_4$ as an example to be modeled. The center of $E_4$ is estimated at 23.1N–92.7W. The displacement relative to the center of 1599 is shown in Fig. 7.3.5 (dotted lines) after removing a center translation with a speed of $u = -6 \text{ km/day}$ and $v = 0.2 \text{ km/day}$. The displacement shows that the drifter moves around an ellipse with a major radius of about 120 km and a minor radius of about 70 km. The drifter rotates around the center of eddy $E_4$ at a period of about 12 days.

To model eddies $E_3$ and $E_4$, the rodon special solution (3.3.3) is appropriate. The thickness of this eddy is also 200 m. From the observed geometry of the eddy, the solution constants are estimated as

$$ h_0 = 0.4 $$
$B = -0.085$

$B_D = 0.01$

$G_R = -0.0917872$

$G_D = 0.0404687$

$\omega = 0.01.$

The solution describes a model lens with a major radius of 4.2 ($\sim 126$ km), a minor radius of 2.5 ($\sim 75$ km), and a thickness of 0.4 ($\sim 200$ m). The lens has a fixed shape and a constant rotation rate of $\dot{\theta} = -0.005$ ($\sim 1.8^\circ$ per day). The evolution of a particle at the edge of the lens is shown in Fig. 7.3.5 (solid lines). The model trajectory of the particle after adding the translation of the eddy center is shown in Fig. 7.3.6.

A comparison of the displacements and trajectories between the model and the observations shows that the simulation is in close agreement with the above analysis. Kirwan et al. (1984) analyzed the trajectories of drifters 1598 and 1599 with an earlier version of the 1988 model. The segments of the trajectories modeled here correspond to days 42-68 in their paper. They found that the characteristics of the eddies were relatively stationary in this period. Magnitude of deformation was about $0.4 \times 10^{-5}\text{s}^{-1}$ and vorticity about $-1.2 \times 10^{-5}\text{s}^{-1}$. The simulation here results in magnitude of deformation ($2G_D$) of about $0.8 \times 10^{-5}\text{s}^{-1}$ and vorticity ($2G_R$) of about $-1.8 \times 10^{-5}\text{s}^{-1}$. The parameters of the two different models are comparable.

7.4 Eddy $E_5$

Eddy $E_5$ is defined by the drifter trajectory in Fig. 7.4.1, which is measured by drifter 1598 at the west of the Gulf of Mexico basin from Mar. 7 to Apr. 21, 1981. Fig. 7.4.2 shows the latitudinal and longitudinal components of the trajectory. The center of eddy $E_5$ is estimated at 22.6N-95.1W with a translation at a speed of $u = -2.8$ km/day and $v = 1.1$ km/day. After removing the translation, the displacement of
the drifter relative to the center is shown in Fig. 7.4.3 (dotted lines). The amplitude of the displacement is about 100 km and the rotation period of the drifter around the center is about 15 days.

To model eddy $E_5$, the *rodon* special solution (3.3.3) is again used. The geometry of eddy $E_5$ is considered as a near circular lens with a radius of about 100 km and a thickness of 200 m. The constants in (3.3.3) are then estimated as

- $h_0 = 0.4$
- $B = -0.0626279$
- $B_D = 0.002$
- $G_R = -0.0671162$
- $G_D = 0.00403214$
- $\omega = 0.008$.

The solution describes a lens with a major radius of 3.69 (~ 111 km), a minor radius of 3.47 (~ 104 km) and a thickness of 0.4 (~ 200 m). The lens has a fixed shape and a constant rotation rate of $\dot{\theta} = -0.004$ (~ 1.4° per day). The displacement of a particle at the edge of the lens is shown in Fig. 7.4.3 (solid lines). Fig. 7.4.4 shows the model trajectory of the particle after adding the translation of the center of eddy $E_5$.

It is seen from Fig. 7.4.3 and Fig. 7.4.4 that the model trajectory compares well to the observations. According to the simulation, eddy $E_5$ has some similarities to eddies $E_3$ and $E_4$ in size, rotation rate and vorticity. From Kirwan et al. (1984), drifter 1598 took a path similar to drifter 1599 before late January 1981. In February 1981, drifter 1598 had a long trip towards the northwest then came back in early March 1981. Eddy $E_5$, followed by drifter 1598 in March 1981, seems to be relevant to eddy $E_4$. But the deformation of eddy $E_5$ is much less than that of $E_4$ because the model lens for $E_5$ is nearly circular.
7.5 Eddy $E_6$

Eddy $E_6$ is located near the western continental slope of the Gulf of Mexico. It is defined by the drifter trajectory in Fig. 7.5.1, which is measured by drifter 1599 from Mar. 22 to May 1, 1981. Fig. 7.5.2 shows the latitudinal and longitudinal components of the trajectory. The center of eddy $E_6$ is estimated at 23.4N-96.3W on Mar. 22, 1981, and the translation speed of the center is calculated as $u = 1.9$ km/day and $v = 2.6$ km/day. The displacement of the drifter relative to the center of eddy $E_6$ is shown in Fig. 7.5.3 (dotted lines). During the 40 days, drifter 1599 moves around the center for 4 cycles and keeps about 60 km from the center most of the time. There is no data available for the vertical structure of eddy $E_6$. Nevertheless, we shall use the same thickness for $E_6$ as that of $E_5$. Therefore, eddy $E_6$ should be specified as a lens with a horizontal ellipse, radii of about 60 km and a thickness of about 200 m. The particle at the edge of the lens moves around the lens center at a period of about 10 days.

The rodon special solution (3.3.3) fails to model eddy $E_6$ as specified above. Either the period of the particle obtained from (3.3.3) is much less than 10 days or the model velocity is much larger than that of the observations. One remedy for applying (3.3.3) to $E_6$ is to reduce the lens thickness to match the particle velocity, but there are no available hydrographic data to support this. As an alternative, we employ the special solution (4.2.7), the solution of a linear system, to model $E_6$. Consider eddy $E_6$ as a surface lens with a moving lower layer. The flow in the lower layer is prescribed as

\[ u = -0.1y \]
\[ v = 0.1x. \]

Then, based on the geometry of eddy $E_6$, the constants in (4.2.7) are estimated as

\[ h_0 = 0.4 \]
\[ B = -0.2004 \]
\[ B_D = 0.01149 \]
\[ G_R = -0.1 \]
\[ G_D = 0.02 \]
\[ \omega = -0.15. \]

The solution defines a lens with a major radius of 2.1 (~ 63 km), a minor radius of 1.9 (~ 57 km) and a thickness of 0.4 (~ 200 m). The model displacement of a particle at the edge of the lens is shown in Fig. 7.5.3 (solid lines). Fig. 7.5.4 shows the model trajectory of the particle after adding the translation of the center of eddy \( E_6 \). The swirl velocities about the eddy center for the drifter (dotted lines) and the particle (solid lines) are shown in Fig. 7.5.5.

It is seen from Fig. 7.5.3 and Fig. 7.5.4 that the model trajectory is comparable to that of the drifter. The velocity is also in agreement between the model and the observations. Unlike the case of eddy \( E_1 \), there are no jiggles in the observed velocity because the trajectory of drifter 1599 is constructed from a piecewise spline interpolation (see Kirwan et al., 1984). The model lens has a deformation with magnitude \((2G_D)\) of \(0.4 \times 10^{-5}\) s\(^{-1}\) and vorticity \((2G_R)\) of \(-2.0 \times 10^{-5}\) s\(^{-1}\). Both the deformation and vorticity are comparable to those obtained by Kirwan et al. (1984). Eddy \( E_6 \) is measured by the same drifter, 1599, as eddy \( E_4 \), but the two eddies look different. The horizontal scale of \( E_4 \) is almost twice that of \( E_6 \). The rotation period of \( E_6 \) is about 10 days while it is about 12 days in \( E_4 \). Both the vorticity and deformation of \( E_6 \) are greater than those of \( E_4 \). Lewis and Kirwan (1985) found that there was a ring topography interaction as drifter 1599 moved over the continental slope on the west coast of the Gulf of Mexico. This is probably one of the reasons for the difference between eddies \( E_4 \) and \( E_6 \).

The prescribed external flow, which is equivalent to an external vorticity of \(2.0 \times 10^{-5}\) s\(^{-1}\), has no observational support. As mentioned before, without this external flow the isolated lens with the radii of about 60 km and the thickness of about 200 m does not match eddy \( E_6 \). In other words, eddy \( E_6 \) cannot be modeled by a quasi-
geostrophic lens. This implies that the dynamics of eddies on the west coast of the Gulf of Mexico are more complicated than in the middle of the basin.

### 7.6 Invariants of Gulf of Mexico Eddies

Eddies $E_1$ to $E_5$ are modeled by the special solution (3.3.3) and eddy $E_6$ by the special solution (4.2.7). As mentioned in section 5, the five invariants for both solutions of (3.3.3) and (4.2.7) are constant. The five invariants of eddies $E_1$ to $E_6$ are listed in Table 2.

<table>
<thead>
<tr>
<th>Table 2. Invariants of Gulf of Mexico eddies</th>
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<tbody>
<tr>
<td>$I_1$</td>
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<tr>
<td>-------</td>
</tr>
<tr>
<td>$E_1$</td>
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<td>$E_2$</td>
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<td>$E_3$</td>
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<tr>
<td>$E_5$</td>
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<tr>
<td>$E_6$</td>
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</tbody>
</table>

This table indicates that for unforced lenses ($E_1$ to $E_5$), a larger volume (smaller $I_4$) corresponds to a larger $I_1$ and smaller $I_2$, $I_3$ and $I_5$, if the thickness of lenses is fixed. This implies that lenses with larger volumes have weaker relative vorticities and smaller energies since $I_1$ increases with decreasing $|G_R|$. In the case of eddy $E_6$, however, the volume of the lens is the smallest of the six cases, but $I_1$ of $E_6$ is intermediate to the other values. The deviation of the characteristics of $E_6$ from $E_1$ through $E_5$ is caused by the external flow.
From the simulation of eddies $E_1$ to $E_5$, it seems that the special solution (3.3.3), the subinertial solution, is especially applicable to warm-core eddies in the basin of the Gulf of Mexico. As noticed from the constraints (3.3.3.e), (3.3.3.f) and (3.3.3.g), the lens geometry determines the characteristics of the special solution (3.3.3). In other words, the invariants of the lenses are fixed if the thickness and the horizontal ellipse are given in the special solution (3.3.3). Based on the simulation of eddies $E_1$ to $E_5$, we assume that the typical warm eddies in the Gulf of Mexico have cores with

$h_0 = 0.4(\sim 200 m)$

$R_b / R_a = 0.8$.

Under this assumption, $R_b$ has a minimum value of 2, which corresponds to $R_b = 1.6$, $B = -0.25625$, $B_D = 0.028125$ and $\delta = 1$. If $R_a < 2$, then $\delta > 1$, which violates (3.3.4.c). On the other hand, $R_a$ should have a maximum value of 10 ($\sim 300$ km). $G_D$ and $G_R$ are determined by (3.3.3.e), (3.3.3.f) and (3.3.3.g) if $B$, $B_D$ and $\delta$ are given. Therefore, the five invariants are determined if $R_a$ is given. Fig. 7.6.1 shows the constants $B$, $G_R$, $B_D$ and $G_D$ in the special solution (3.3.3) versus $R_a$ for the lenses with $h_0 = 0.4$ and $R_b / R_a = 0.8$. The range of the five invariants for the typical eddies in the Gulf of Mexico are shown in Fig. 7.6.2. The range of the invariants in Fig. 7.6.2 applies only to isolated lenses. All of the invariants of $E_1$ through $E_5$ fall in these ranges. The invariants of $E_6$ do not fit Fig. 7.6.2 because $E_6$ is not an isolated lens.

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Fig. 7.1.1. The trajectory of drifter 3378 from Aug. 20 to Sep. 29, 1985. The square indicates the initial position of the drifter.
Fig. 7.1.2. The latitudinal and longitudinal components of the trajectory of drifter 3378.
Fig. 7.1.3. The displacements relative to the center of $E_1$ of drifter 3378 (dotted lines) and the modeling particle on the rim of the lens (solid lines).
Fig. 7.1.4. The model trajectory of a particle at the edge of the lens for eddy $E_1$. The square indicates the initial position of the particle.
Fig. 7.1.5. The swirl velocities about the center of $E_1$ of drifter 3378 (dotted lines) and the modeling particle on the rim of the lens (solid lines).

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Fig. 7.2.1. The trajectory of drifter 3379 from May 2 to June 5, 1986. The square indicates the initial position of the drifter.
Fig. 7.2.2. The latitudinal and longitudinal components of the trajectory of drifter 3379.
Fig. 7.2.3. The displacements relative to the center of $E_2$ of drifter 3379 (dotted lines) and the modeling particle on the rim of the lens (solid lines).
Fig. 7.2.4. The model trajectory of a particle at the edge of the lens for eddy $E_2$. The square indicates the initial position of the particle.
Fig. 7.3.1. The trajectory of drifter 1598 from Jan. 1 to Feb. 1, 1981. The square indicates the initial position of the drifter.
Fig. 7.3.2. The latitudinal and longitudinal components of the trajectory of drifter 1598 in Fig. 7.3.1.
Fig. 7.3.3. The trajectory of drifter 1599 from Jan. 1 to Jan. 25, 1981. The square indicates the initial position of the drifter.
Fig. 7.3.4. The latitudinal and longitudinal components of the trajectory of drifter 1599 in Fig. 7.3.3.
Fig. 7.3.5. The displacements relative to the center of $E_4$ of drifter 1599 (dotted lines) and the modeling particle on the rim of the lens (solid lines).
Fig. 7.3.6. The model trajectory of a particle at the edge of the lens for eddy $E_4$. The square indicates the initial position of the particle.
Fig. 7.4.1. The trajectory of drifter 1598 from Mar. 7 to Apr. 21, 1981. The square indicates the initial position of the drifter.
Fig. 7.4.2. The latitudinal and longitudinal components of trajectory of drifter 1598 in Fig. 7.4.1.
Fig. 7.4.3. The displacements relative to the center of $E_5$ of drifter 1598 (dotted lines) and the modeling particle on the rim of the lens (solid lines).
Fig. 7.4.4. The model trajectory of a particle at the edge of the lens for eddy $E_3$. The square indicates the initial position of the particle.
Fig. 7.5.1. The trajectory of drifter 1599 from Mar. 22 to May 1, 1981. The square indicates the initial position of the drifter.
Fig. 7.5.2. The latitude and longitude components of the trajectory of drifter 1599 in Fig. 7.5.1.
Fig. 7.5.3. The displacements relative to the center of $E_6$ of drifter 1599 (dotted lines) and the modeling particle on the rim of the lens (solid lines).
Fig. 7.5.4. The model trajectory of a particle at the edge of the lens for eddy $E_6$. The square indicates the initial position of the particle.
Fig. 7.5.5. The swirl velocities about the center of $E_9$ of drifter 1599 (dotted lines) and the modeling particle on the rim of the lens (solid lines).
Fig. 7.6.1. The constants in the special solution (3.11) versus different $R_a$ for lenses with $h_b = 0.4$ and $R_b/R_a = 0.8$. 
Fig. 7.6.2. The invariants of the typical warm eddies with $h_0 = 0.4$ and $R_b/R_a = 0.8$ in the Gulf of Mexico.
8 Warm-Core Eddies in Other Regions

8.1 The Great Whirl

A well-known anticyclonic eddy, the Great Whirl, is located off Somali in the Indian Ocean. The formation of the eddy is mainly related to the southwest monsoon (Duiring, 1977; Bruce, 1979; Luther and O’Brien, 1985). Fig. 8.1.1(a) shows a trajectory of the satellite-tracked drifter 2288 deployed by NOAA March to April 1986 off Somalia. The eddy appears to be associated with the distribution of the wind stress curl given by Schott and Euadfasel (1982) in Fig. 8.1.1(b). Fig. 8.1.2 shows the temperature profile through the Great Whirl on August 18–25, 1976, after the southwest monsoon starts. Like the Gulf of Mexico, the bottom of the warm water pool appears to be at 200 m. The center of the Great Whirl is located at about 7.5N–52W. The displacement of drifter 2288 relative to the center is shown in Fig. 8.1.3(a).

From Fig. 8.1.2 and Fig. 8.1.3(a), the eddy is considered as a lens with a thickness of about 200 m and a horizontal scale of 400 to 500 km. The particle at the rim of the lens completes a cycle in about 25 days. The special solution (3.3.3) also fails to model the Great Whirl as scaled above because the velocity of the particle obtained by (3.3.3) is much slower than that of drifter 2288. Again, external forces are used to match the size and velocity of the eddy. Considering the Great Whirl as a surface lens with a moving lower layer, the external flow is prescribed as

\[
\begin{align*}
    u &= S \cos \omega t \ x - (V + S \sin \omega t) \ y \\
    v &= (V - S \sin \omega t) \ x - S \cos \omega t \ y
\end{align*}
\]

where \( S = 0.003 \), \( V = -0.024 \) and \( \omega = 0.004687 \). The external flow, then, has a vorticity of \(-0.048\) and an amplitude of deformation of 0.003. The simulation of the Great Whirl is performed with the special solution (4.2.7), the solution of a linear system. With the observed geometry of the Great Whirl, the constants in the special solution (4.2.7) are estimated as
\[ h_0 = 0.4 \]
\[ B = -0.014992 \]
\[ B_D = 0.000995313 \]
\[ G_R = -0.04 \]
\[ G_D = 0.005 \]
\[ \omega = 0.004687. \]

With these parameters, the model lens has a major radius of 7.84 \((\sim 235 \text{ km})\), a minor radius of 6.86 \((\sim 206 \text{ km})\), and a thickness of 200 m. Fig. 8.1.3(b) shows the displacement of a particle at the edge of the model lens and Fig. 8.1.4 shows the trajectory of the particle. The swirl velocities about the center of the Great Whirl for drifter 2288 (dotted lines) and the particle (solid lines) are shown in Fig. 8.1.5.

The model trajectory is in general agreement with the drifter trajectory in Fig. 8.1.3 and Fig. 8.1.4. The model velocity also compares well to the observations in the low frequency oscillation. The jiggles in the observed velocity can be explained by the use of unfiltered raw data of the drifter positions. Unlike the case of eddy E6, the simulation of the Great Whirl with the rodon special solution results in a model velocity much slower than the observations. The prescribed external flow, which has a negative vorticity, accelerates the rotation of the lens.
Fig. 8.1.1. (a) Trajectory of drifter 2288, March to April 1986, off Somalia. The arrows denote every five days; (b) Wind stress curl of Somalia at time of final onset of summer monsoon, 9-14 June 1979 (from Schott et al., 1982).
Fig. 8.1.2. Temperature profile through the Great Whirl August 1976 after the southwest monsoon onsets (from Bruce, 1979).
Fig. 8.1.3. The displacements relative to the center of the Great Whirl of drifter 2288 (dotted lines) and the particle on the rim of the lens (solid lines).
Fig. 8.1.4. The model trajectory of a particle at the edge of the lens for the Great Whirl. The square indicates the initial position of the particle.
Fig. 8.1.5. The swirl velocities about the center of the Great Whirl of drifter 2288 (dotted lines) and the particle on the rim of the lens (solid lines).
8.2 The Meddy

Meddies are small anticyclonic eddies with cores of warm, salty water. They were first observed in the subsurface layers of the North Atlantic. Their water mass properties were of Mediterranean origin. Recently, similar eddies have been found in the Arctic Ocean and other parts of the world ocean. The name Meddy is often used to categorize the phenomenon regardless of its origin.

Fig. 8.2.1 shows the trajectories of three Meddies measured with the neutrally buoyant SOFAR floats which are ballasted in a certain range of pressures (Richardson, Walsh, Armi, Schroder, & Price, 1989). It is seen from Fig. 8.2.1(b) that float 128, which follows Meddy 1, locally moves along an average circle with a radius of about 20 km during the first several months. The small circles in the middle of Fig. 8.2.1(b) indicate the decay of Meddy 1 after June 1985. Fig. 8.2.2 shows the radial distribution of the swirl velocity of Meddy 1 measured by four floats. Obviously, the water within 20 km from the center of Meddy 1 moves in solid body rotation. The period of the rotation is six days. Fig. 8.2.3 shows the salinity profiles through the center of Meddy 1 from the survey of October 1984 (a) and June 1985 (b). The profiles indicate that Meddy 1 has average core thickness of about 400 m.

To model Meddy 1, we make some changes of the scaling parameters. The reduced gravity in the layer of Meddy 1 should be smaller than that in the Gulf of Mexico eddies. The scalings are taken as

\[ H = 1000m \]
\[ R_d = 10km \]
\[ f = 10^{-4}s^{-1} \]

so that the reduced gravity \( g' = 10^{-2}ms^{-2} \). This value of \( g' \) is consistent with the observations of density \( \sigma_t \) of about 32.0 in the lens layer, 31.5 in the upper layer and 32.3 in the lower layer (Armi, et al., 1989). Consider Meddy 1 as a subsurface lens with the thickness of 400 m and a horizontal scale of 40 km. The external flows in
both the upper and lower layers are prescribed as

\[ u = -0.035y \]
\[ v = 0.035x. \]

The special solution (4.2.7), the solution of a linear system, is applied to Meddy 1. Based on the observed geometry of Meddy 1, the constants in (4.2.7) are estimated as

\[ h_0 = 0.4 \]
\[ B = -0.17736381 \]
\[ B_D = 0.01001782 \]
\[ G_R = -0.169465 \]
\[ G_D = 0.019804 \]
\[ \omega = -0.01. \]

This solution defines a lens with a thickness of 0.4 (~ 400 m), a major radius of 2.25 (~ 22.5 km) and a minor radius of 2.0 (~ 20 km). Fig. 8.2.4 shows the trajectory and its components of a particle at the edge of the model lens. The swirl velocity about the center of Meddy 1 is shown in Fig. 8.2.5.

The model trajectory compares well to observations in both displacement and frequency. The model velocity is in agreement with Fig. 8.2.2. The prescribed external flow has a weak positive vorticity (0.07). Without the external flow, the model velocity cannot match the observations. Fig. 8.2.2 shows that beyond 20 km the swirl velocity seems to decrease with increasing distance from the center of Meddy 1. This suggests a flow with a positive vorticity just outside the model lens.
Fig. 8.2.1. (a) Trajectories of three neutrally bouyant floats (at a depth of about 1000 meters) which follow Meddy 1, Meddy 2, and Meddy 3 from October 1984 to October 1986; (b) Movement of float 128 around the center of Meddy 1 from October 1984 to October 1986. This trajectory was obtained by subtracting the Meddy 1 translation in (a) (from Richardson et al., 1989).
Figure 8.2.1
Fig. 8.2.2. Rotation velocities versus radius for Meddy 1 measured with four floats. Negative values indicate anticyclonic rotation. The line shows solid body rotation at a period of 6 days (from Richardson et al., 1989).
Fig. 8.2.3. Salinity profiles through Meddy 1, (a) the survey of October 1984; (b) the survey of June 1985 (from Armi et al., 1989).
Fig. 8.2.4. (a) The path of a particle at the boundary of the modeling of Meddy 1. The square is the starting position and the arrows indicate the everyday evolution; (b) The east (x) and the north (y) components of the particle path in (a).
Fig. 8.2.5. The swirl velocity about the center of Meddy 1 of the particle on the rim of the lens.
9 Conclusion and Discussion

The full nonlinear shallow water equations for three fluid layers with a lens in the middle layer are decomposed into eight first order ordinary differential equations. The eight variables $h_0, B, B_S, B_N, G, G_R, G_N$ and $G_S$ in the equations are functions of time only. These variables specify the shape, divergence, vorticity and deformation of the lenses produced by the model. The general solution for isolated lenses can be easily obtained by numerical approximation if the initial values satisfy $B < 0$ and $B^2/4 - (B_S^2 + B_N^2) > 0$.

Different combinations of the initial values of the variables produce a wide range of complicated behavior. However, some unambiguous results are obtained from the numerical solutions. Changing the initial value of $h_0$ does not affect the frequencies of lenses. Fluctuation of $G$ always occurs at or near the inertial frequency. $G_R$ is directly related to the subinertial and superinertial frequencies of the lenses. Smaller amplitudes of $G_R(0)$ produce lower frequencies, and vice versa. The phase difference $\Delta \phi$ between the mass and flow field defined in (3.1.2) is responsible for the superinertial frequencies. In the case of $\Delta \phi = 0$, the superinertial frequencies are depressed.

Under certain assumptions, some special solutions have analytical expressions. For isolated lenses, three such special solutions are found. The *pulson* special solution is obtained for axisymmetric lenses. There is only inertial frequency in the *pulson* special solution. If velocity is determined only by a stream function, the *rodon* special solution is found. It has only subinertial frequencies. When velocity is determined by just the velocity potential, a special solution is found which exhibits only superinertial frequencies.

For forced lenses, two special solutions are found corresponding to two different external flows. A special solution to a linear version of (2.3) is found that has only subinertial frequencies, provided that the external flow also has only subinertial frequency motion. If the external flows oscillate at the inertial frequency, a special
solution to the nonlinear system of (2.3) is found with inertial, subinertial or superinertial frequencies.

There are five invariants to the unforced version of the model equations (2.3). The invariants depict the conservation of the volume, potential vorticity, angular momentum, total energy and potential deformation. For forced lenses, only volume and potential vorticity are conserved in general. However, for the two special solutions of forced lenses, only total energy is variable in the special solution (4.2.7).

Three types of data can be used to apply the lens model to warm-core eddies. They are hydrographic, Lagrangian and satellite IR data. With these data, it is possible to specify the initial values of the eight variables and then obtain numerical solutions to the model. The inverse technique is a useful tool for specifying the initial values. For mesoscale eddies, the special solution rodon and the solution of a linear system are most applicable. One advantage of these two solutions is that only three initial values such as $h_0$, $B$ and $B_D$ need to be specified; the others can be calculated from the constraints. If the size and geometry of a mesoscale eddy are known, $h_0$, $B$ and $B_D$ can be estimated.

Six mesoscale warm-core eddies in the Gulf of Mexico and two in other regions are well modeled by the two special solutions. For typical isolated eddies in the Gulf of Mexico, a range of invariants is obtained by the rodon special solution. The other three special solutions have not been used in the application here. One reason for this is because the available data for eddies are usually documented after low-pass filtering. This removes any evidence of inertial and superinertial frequencies. Some reports (D'Asaro, 1988; Tokos and Rossby, 1991) do show that inertial frequencies exist in the subsurface anticyclonic eddies. This means that the special solution of the nonlinear system (4.3.3) may be applicable to the Meddies.

It is probably most interesting when doing simulations to determine an eddy with the rodon special solution only by three parameters: $h_0$, $B$ and $B_D$. As mentioned be-
fore, the feature model given by Glenn et al. (1990) has 10 parameters for determining the propagating velocity, size, shape, orientation and swirl velocity of eddies. These parameters are obtained by fitting the observations with the least squares method. Including two parameters for the translation of the lens center, the present lens model also has 10 parameters to determine the same features of eddies as above. However, with a quasi-geostrophic approximation \( G = 0 \), which is suitable for mesoscale eddies in the ocean, the rodon special solution actually has three independent parameters. The others can be calculated by the solution constraints. Of course, the less the parameters, the less the uncertainties. Moreover, the available observations favor an accurate estimate of the geometry of eddies, which is related to \( h_0 \), \( B \) and \( B_D \). This makes the simulation more reliable.

In the application of the lens model to warm-core eddies in the Gulf of Mexico, the thickness \( h_0 \) of the model lenses is taken as 0.4 (~ 200 m) in all of the cases. This vertical scale cannot be directly compared to the temperature profile because \( h_0 \) is a stratified depth. To understand this, a rough estimation of \( h_0 \) is discussed for a two layer fluid. In this case, there is a barotropic mode and a baroclinic mode. Suppose the bottom is flat and \( H_1 \), \( H_2 \) and \( H \) are the upper, lower and total depth. An equivalent depth for the first baroclinic mode in the two layer model is

\[
h_2 = \delta H_1 H_2 / H
\]

while in the continuous stratification model the equivalent depth is

\[
h_c = H (N^2 H / g \pi^2).
\]

Here \( \delta \) is density contrast \( \delta = \Delta \rho / \rho_2 \), \( N \) is Brunt-Vaisala frequency, and \( g \) is local gravity (see LeBlond and Mysak, 1978, p. 126-136 for details). Assuming \( h_2 = h_c \) and \( H_1 = rH \), it follows that
Taking $N^2 = 10^{-5}s^{-2}$, $g = 10ms^{-2}$ and $\delta = 2 \times 10^{-3}$, one solution of $r$ is

$$r = (1 - \sqrt{1 - 0.2 \times 10^{-3}H})/2$$

where $H$ must have units of meter. For the Gulf of the Mexico eddies, $h_0$ corresponds to $rH$. If $H = 2000m$, then $r \approx 0.1$ and $rH = 200m$. Therefore, in a two layer model, $h_0 = 0.4$ is relative to the total depth of 2000 m. This is acceptable for simulation of the Gulf of Mexico eddies.

There is still an unsolved problem as to whether the invariants uniquely define a lens. We showed only that there are no equivalent lenses to the pulson and rodon special solutions from the model (5.2.1), with which $\Delta \phi \neq 0$ produces more general solutions. What about changes of $B_D$ and $G_D$ in (5.2.1)? Are there other general solutions equivalent to the pulson and rodon special solutions? These questions are still unresolved.

Another limitation of the lens model is its inability to simulate the decay of eddies. While observations of real eddies often indicate decay, the model lens never decays in size or rotation rate during the evolution. This was especially notable for Meddy 1 in Fig. 8.1.1.

There are several aspects of this lens model worthy of further investigation. A systematic study of the effect of forcing on lenses is clearly the most obvious aspect. This should include an effort to find out how long a lens survives in a forcing environment and what causes instability. The two special solutions for forced lenses discussed in section 4 may provide a starting point for such analysis. Secondly, the lens model should be extended to allow for a finite boundary with velocity decay. This will make the lens model more realistic. Thirdly, extension of the model to allow for mass exchanges between the lens and the environment is necessary for the...
model to be applied to shelf/open ocean exchanges. Also, this may be one of the solutions to the decay problem. Up to now, the lens model has been applied only to anticyclonic eddies. It would be important if an equivalent solution for cyclones could be developed.

On applications of this lens model, further simulation of ocean eddies should include inertial and superinertial motions. The special solution of the nonlinear system (4.3.3) may be useful in this aspect. But the essential point to be made is that observational programs need to be planned to account for inertial and superinertial frequencies.
Appendix A

Consider a time series of the position \((x,y)\) of a particle in an eddy. The swirl velocity component \(u\) and \(v\) are given by

\[
\begin{align*}
\dot{u} &= \dot{x} = (d/2 + a)x + (b - c)y \quad (A.1.a) \\
\dot{v} &= \dot{y} = (b + c)x + (d/2 - a)y. \quad (A.1.b)
\end{align*}
\]

\(\dot{\cdot}\) is the ordinary derivative to time and \(a, b, c\) and \(d\) are constants. These parameters can be calculated as (for details, see Kirwan, et al. 1988):

\[
\begin{align*}
d &= (g_2h_4 - g_4h_2)/(g_2h_3 - g_3h_2) \quad (A.2.a) \\
a &= -2(K_1D - K2/4 + K_3d)/LD^2 \quad (A.2.b) \\
b &= (H_1D - H_2/4 + H_3d)/LD^2 \quad (A.2.c) \\
c &= -(G_1D - G_2/4 + G_3d)/LD^2 \quad (A.2.d)
\end{align*}
\]

where \(g_2, g_3, g_4\) and \(h_2, h_3, h_4\) are second, third and fourth derivatives of \(x\) and \(y\) with respect to time, and

\[
\begin{align*}
G_1 &= 4(g_2^2 + h_2^2) \quad (A.3.a) \\
G_2 &= 64(g_3^2 + h_3^2) \quad (A.3.b)
\end{align*}
\]
\[ G_3 = 16(g_2g_3 + h_2h_3) \]  
\[ H_1 = 4(g_2^2 - h_1^2) \]  
\[ H_2 = 64(g_3^2 - h_2^2) \]  
\[ H_3 = 16(g_2g_3 - h_2h_3) \]  
\[ K_1 = 4g_2h_2 \]  
\[ K_2 = 64g_3h_3 \]  
\[ K_3 = 8(g_2h_3 + g_3h_2) \]  
\[ D = -4(g_3h_4 - g_4h_3)/(g_2h_3 - g_3h_2) \]  
\[ L = 2(g_2h_3 - g_3h_2)^3/(g_3h_4 - g_4h_3)^2 \]
Appendix B

As shown in (5.1.1.e), the general invariant $I_5$ for an isolated lens is

$$I_5 = \left[ 4 \left( \frac{B^2}{4} - (B_S^2 + B_N^2) \right) + 4G_R(B_NG_S - B_SG_N) \right]$$

$$+ 2G(B_SG_S + B_NG_N) - B\left( G^2/4 + G_N^2 + G_S^2 + G_R^2\right) / h_0^3. \quad (B.1)$$

For the rodon solution, the invariant $I_5$ becomes

$$I_5 = \left[ 4 \left( \frac{B^2}{4} - B_D^2 \right) + 4G_RG_DG_D - B\left( G_D^2 + G_R^2\right) \right] / h_0^3. \quad (B.2)$$

Suppose the initial values of $B_S$, $B_N$, $G_N$ and $G_S$ take the form

$$B_S = -B_D\cos\Delta\phi \quad (B.3.a)$$

$$B_N = -B_D\sin\Delta\phi \quad (B.3.b)$$

$$G_N = G_D\cos(\phi + \Delta\phi) \quad (B.3.c)$$

$$G_S = -G_D\sin(\phi + \Delta\phi). \quad (B.3.d)$$

This solution should be more general than the rodon. With (B.3), the general invariant $I_5$ reduces to

$$I_5 = \left[ 4 \left( \frac{B^2}{4} - B_D^2 \right) + 4G_RB_DG_D\cos\Delta\phi + 2G_BG_DG_D\sin\Delta\phi \right.$$

$$\left. - B\left( G_D^2 + G_R^2\right) \right] / h_0^3. \quad (B.4)$$
The purpose here is to find a nonzero $G$ with which the invariant $I_5$ in (B.4) equals the $I_5$ in (B.2). Assuming $h_0$, $B$, $G_R$, $B_D$ and $G_D$ in (B.4) are the same as those in (B.2), equating $I_5$ in (B.2) to $I_5$ in (B.4) yields

$$G = (-B/4)[-B_D G_D \sin \Delta \phi]$$

$$\pm \sqrt{B_D G_D (B_D G_D \sin^2 \Delta \phi + B G_R (\cos \Delta \phi - 1))}.$$  (B.5)

Real values of $G$ require $B_D G_D \sin^2 \Delta \phi + B G_R (\cos \Delta \phi - 1) \geq 0$ or

$$-B_D G_D \cos^2 \Delta \phi + B G_R \cos \Delta \phi + B_D G_D - B G_R \geq 0.$$  (B.6)

The function

$$F(\cos \Delta \phi) = -B_D G_D \sin^2 \Delta \phi + B G_R \cos \Delta \phi + B_D G_D - B G_R$$

has two zeros. These are

$$\cos \Delta \phi = \begin{cases} 1 & \frac{B G_R}{B_D G_D} - 1. \\ \end{cases}$$  (B.7)

As shown in (3.3.3.g),

$$B_D (2G_R + \omega) B G_D = 0$$  (B.8)

or

$$2B_D G_R = B G_D - B_D \omega.$$  (B.9)

For anticyclonic lenses, $B$ and $G_R$ are negative and $\omega$ is positive. In the rodon solution, $B_D$ and $G_D$ were assumed positive. Hence, $G_R$ in (B.9) has a minimum amplitude with $\omega = 0$, i.e.
\[ G_R = \frac{B G_D}{2B_D}. \] \hfill (B.10)

Substitution of (B.10) into (B.7) yields

\[ \cos \Delta \phi = \begin{cases} 1 \\ \frac{B^2}{2B_D^2} - 1. \end{cases} \] \hfill (B.11)

The condition \( B^2/4 - B_D^2 > 0 \) is required for elliptical structure in the lens model. It follows that \( B^2/2B_D^2 > 2 \). Therefore, \( \cos \Delta \phi \) has two roots: 1, and the other greater than 1. This means that only \( \cos \Delta \phi = 1 \) or \( \Delta \phi = 0 \) corresponds to the real value of \( G \) in (B.5), i.e. \( G = 0 \). This, in turn, is the case of the rodon solution. The conclusion is that the general solution based on (B.3) cannot produce the same invariant \( I_5 \) as that from the rodon special solution.
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