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Abstract

This paper presents an alternate proof of the divergence of the unique maximizer sequence $\{x_n^*\}$ of a function sequence $\{F_n(x)\}$ that is derived from an adaptive algorithm based on the now-classic optimal stopping problem, known by many names but here 'the secretary problem'. The alternate proof uses a result established by (Nguyen, Xu, & Zhao, n.d.) regarding the uniqueness of maximizer points of a generalized function sequence $\{S_n^{\mu,\sigma}\}$ and relies on the strict monotonicity of $F_n(x)$ as n increases in order to show divergence of $\{x_n^*\}$. Towards this, limits of the exponentiated Gaussian CDF are established as well as a closed form of $F_n'(x)$, the derivative of the sequence's function. The proof is elementary but nontrivial. The result in (Nguyen et al., n.d.) relies heavily on a technical lemma but, here, the proof is more transparent and relies solely on fundamentals.

1. Introduction

The secretary problem (or the dowry problem, or the beauty contest problem, or Googol, or others) and its solutions (and those of some of its variants) are concisely outlined in (Gilbert & Mosteller, 1966) and it is also well-summarized for the purposes of this paper in (Nguyen et al., n.d.). Briefly, the game consists of determining the optimal number of elements of an n-length sequence to reject before choosing a probable sequence maximum. The solution algorithm in (Gilbert & Mosteller, 1966) is to reject a ratio of the total elements equal to n/e before choosing the potential maximum with success probability $1/e$.

A unique, adaptive algorithm is studied in (Zhou, An, Fan, Zhao, & Arora, n.d.), enabling flexible utility in applications by further parameterizing the problem using the length n of the sequence. In (Zhou et al., n.d.), the authors study a function sequence $\{S_n^{\mu,\sigma}(x)\}$ s.t.

$$S_n^{\mu,\sigma}(x) := [1 - F(x)^{n-1}]\frac{\mu - E(x)}{1 - F(x)} + F(x)^{n-1}\mu$$

is the expected score of an element in a 'candidate' sequence—in contrast to its value rank, as is exclusively-used in (Gilbert & Mosteller, 1966)—where $x \in \mathbb{R}$ is a benchmark score and $F(x)$ is the normal cumulative distribution function. The conclusion in (Zhou et al., n.d.) is that the optimal strategy is to maximize expected score on x. Also in (Zhou et al., n.d.), an alternate form of $S_n^{\mu,\sigma}$ is shown to be

$$S_n^{\mu,\sigma}(x) = \mu + \sigma p(x) \cdot \sum_{j=0}^{n-2} F(x)^j.$$

In that paper further studying this expected score sequence (Nguyen et al., n.d.), the authors examine the standardized version of this alternate form, prove the uniqueness of its maximum point in $\mathbb{R}$, and prove the divergence of its maximizer sequence as n goes to infinity.

In this paper, an alternate proof of this divergence is presented. After brief note of the seeming yet illusory convergence of the maximizer sequence in small computational experiments, the main result is proved assuming the sufficient condition of the standardized score function's monotonicity as n goes to infinity. Limits of the exponentiated standard normal cumulative distribution function are shown and utilized in demonstrating a closed form of the standardized score function's derivative. This is developed in order to show the sufficient condition. Throughout the paper, well-known properties of the standard normal probability density function and cumulative distribution function are assumed.

2. Preliminaries and Statement of Main Result

2.1. Preliminaries

Let $p : \mathbb{R} \to \mathbb{R}^+$ s.t. $p(x)$ is the probability density function for the standard normal distribution, namely

$$p(x) := \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-x^2}{2}\right\},\ x \in \mathbb{R},$$

and let $f : \mathbb{R} \to \mathbb{R}^+$ s.t. $f(x)$ is the cumulative distribution function for the standard normal distribution, namely

$$f(x) := \int_{-\infty}^{x} p(s)ds,\ x \in \mathbb{R}.$$

Note that both functions are always positive and specifically

$$0 < f(x) < 1,\ \forall x \in \mathbb{R}.$$

Also note their derivatives:

$$f'(x) = p(x) \tag{1}$$

$$p'(x) = -xp(x). \tag{2}$$

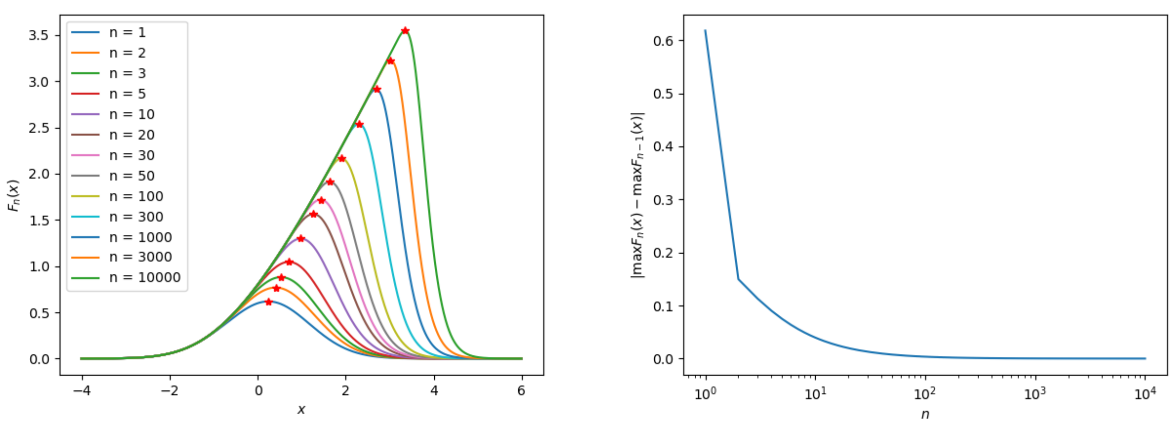
Consider the function sequence $\{F_n(x)\}$ where $F_n : \mathbb{R} \to \mathbb{R}^+$ is defined by

$$F_n(x) := p(x) \sum_{j=0}^{n} f(x)^j,\; n \in \mathbb{N}, \tag{3}$$

which represents the expected value of a normal random variable that has been conditioned on

$$\{\, x_n \mid x_n > x_{n-m},\; \forall m, n \in \mathbb{N} :\; m < n \,\},$$

that is, the condition that the nth term of a sequence is greater than all of the previous terms. Note that, by nature of the Gaussian distribution, the function sequence $\{F_n(x)\}$ is monotone increasing for any given $x \in \mathbb{R}^+$. Consider also the maximizer sequence $\{x_n^*\}$ of $\{F_n(x)\}$. It is assumed that a unique maximizer x_n^* of $F_n(x)$ exists for all n in $\mathbb{N}$ based on the work in (Nguyen et al., n.d.). Interestingly, in graphs of computational experiments (Figure 1), $\{F_n(x)\}$ appears to begin converging as n increases; the maximizer sequence appears to begin converging to a true global maximizer. But this is not the case as will be demonstrated through proof of the main result.



(a) $F_n(x)$ and x_n^* for increasing n (b) $|F_n(x_n^*) - F_{n-1}(x_{n-1}^*)|$ vs. n

Figure 1: Behavior of $\{F_n(x)\}$ and its maximizers

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2.2. Main Result

In (Nguyen et al., n.d.), the authors show proof of the divergence of the maximizer sequence. In this paper, an alternate version of that proof is motivated by neglecting the use of a technical lemma in favor of an approach using fundamentals. The common result is summarized in the following theorem:

Theorem 1. *Denote* $\{x_n^*\}$*, a sequence of unique maximizers in* $\mathbb{R}^+$ *of the function sequence* $\{F_n(x)\}$*. Then*

$$\lim_{n\to\infty} \{x_n^*\} \to +\infty,\ \forall n \in \mathbb{N}. \tag{4}$$

3. Proof of Main Result

3.1. A Key Observation to Support Main Result

The proof of the main result relies on this key observation:

Proposition 1: Denote x_n^*, a unique maximizer of the function $F_n(x)$, $\forall n \in \mathbb{N}$. Then $\lim_{n\to\infty} x_n^* = +\infty$ if

$$\lim_{n\to\infty} F_n'(x) > 0,\ \forall x \in \mathbb{R}^+.$$

Proof. Suppose toward contradiction that $\lim_{n\to\infty} x_n^* \neq \infty$. Then $\exists K > 0$, and a subsequence $\{n_\ell\}$ s.t. $x_{n_\ell}^* \leq K$. Given that $x_{n_\ell}^*$ is a maximizer, then the slope of F_{n_ℓ} at or to the right of K would be negative. This contradicts the hypothesis since $\lim_{n\to\infty} F_n'(K) > 0$. So

$$\lim_{n\to\infty} x_n^* = +\infty. \tag{5}$$

□

It only remains to show that $\lim_{n\to\infty} F_n'(x) > 0$, $\forall x \in \mathbb{R}^+$. To do so, it will be helpful to have some limits and a closed form of F_n's derivative.

3.2. Useful Limits

To evaluate the derivative of F_n, the following limits will be needed:

$$\begin{aligned} \lim_{n\to\infty} f(x)^n &= 0, \\ \lim_{n\to\infty} nf(x)^n &= 0. \end{aligned} \tag{L}$$

They are evaluated by examining their sequences for a fixed x:

Proof. Choose $x_0 \in \mathbb{R}$ and denote $f_0 = f(x_0)$. Let the sequence

$$\{f_0{}^n\} := \left\{ f_0 \in (0,1) \mid f_0{}^n = \left[\int_{-\infty}^{x_0} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{s^2}{2}\right\} ds\right]^n ;\ n \in \mathbb{N} \right\}.$$

Then it is elementary to show that $\lim_{n\to\infty} \{f_0{}^n\} = 0$ and one proof is explained in (3.1.11 (b)) from (Bartle & Sherbert, 2010, p. 60).

Again choosing $x_0 \in \mathbb{R}$ and defining $\{f_0{}^n\}$ as above, the second limit can be determined using the Stolz-Cezàro theorem (Stolz, 1885, pp. 173-175) and the previous limit. Breaking the sequence into two subsequences, let a_m be the m^{th} term of $\{n\}$ and b_m be the m^{th} term of $\{1/f_0{}^n\}$.[1] Then

$$\begin{aligned} \lim_{n\to\infty} \{nf_0{}^n\} &= \lim_{m\to\infty} \frac{a_m}{b_m} \\ \textbf{Stolz-Cezàro} \implies &= \lim_{m\to\infty} \frac{a_{m+1} - a_m}{b_{m+1} - b_m} \\ &= \lim_{m\to\infty} f_0{}^m \cdot (1/f_0 - 1)^{-1} = 0. \end{aligned}$$

Therefore

$$\lim_{n\to\infty} nf(x)^n = 0,\ \forall x \in \mathbb{R}.$$

$\square$

[1] Given the result of the first limit, b_n satisfies the condition of being increasing and unbounded. The proof is akin to that of the first limit and thus trivial. It is also worthwhile to point out that individual terms of b_n are well defined for real x.

3.3. The Derivative of $F_n(x)$

Consider that, $\forall n \in \mathbb{N}$, the derivative with respect to x of $F_n(x)$ is given by

$$F_n'(x) = -xp(x) \cdot \sum_{j=0}^{n} f(x)^j + p(x)^2 \cdot \sum_{j=1}^{n} j \cdot f(x)^{j-1},$$

and that the two summations contained within are geometric series.

Proposition 2: An equivalent form of the derivative of $F_n(x)$ is

$$F_n'(x) = p(x)\left[-x\left(\frac{1-f(x)^{n+1}}{1-f(x)}\right) + p(x)\left(\frac{1-f(x)^n}{(1-f(x))^2} - \frac{nf(x)^n}{1-f(x)}\right)\right].$$

Proof. Denote $f_0 = f(x_0)$ for any chosen $x_0 \in \mathbb{R}$. By the various k-dependant evaluations of the general finite arithmetico-geometric series,

$$A_m^k(r) := \sum_{j=1}^{m} j^k r^{i-1},\ r \in (0,1),$$

the two summations evaluate to

$$\sum_{j=1}^{n} j \cdot {f_0}^{j-1} = A_n^1(f_0) = \frac{1-{f_0}^n}{(1-f_0)^2} - \frac{n{f_0}^n}{1-f_0}, \tag{6}$$

and

$$\sum_{j=0}^{n} {f_0}^j = {f_0}^n + A_n^0(f_0) = \frac{1-{f_0}^{n+1}}{1-f_0}. \tag{7}$$

Then the derivative of $F_n(x)$ evaluated at any given $x \in \mathbb{R}$ can be expressed as

$$F_n'(x) = p(x)\left[-x\left(\frac{1-f(x)^{n+1}}{1-f(x)}\right) + p(x)\left(\frac{1-f(x)^n}{(1-f(x))^2} - \frac{nf(x)^n}{1-f(x)}\right)\right]. \tag{8}$$

□

3.4. Proof of the Condition in Proposition 1

(8) the closed form of F_n' and (L) the previous limits can be used to demonstrate the sufficient condition for proving that the maximizer sequence $\{x_n^*\}$ diverges.

Proposition 3: $\lim_{n\to\infty} F_n'(x)$ is strictly greater than 0 for all positive real x.

Proof.

$$\lim_{n\to\infty} F_n'(x) = p(x)\left[-x\left(\frac{1-\lim f(x)^{n+1}}{1-f(x)}\right)+p(x)\left(\frac{1-\lim f(x)^n}{(1-f(x))^2}-\frac{\lim nf(x)^n}{1-f(x)}\right)\right]$$

which can be evaluated using (L):

$$\lim_{n\to\infty} F_n'(x) = \frac{xp(x)}{(1-f(x))^2}\cdot\left(\frac{p(x)}{x}+f(x)-1\right).$$

Since $xp(x)/(1-f(x))^2$ is positive for $x \in \mathbb{R}^+$ then it is sufficient to examine the multiplied expression to show a positive limit:

$$\frac{p(x)}{x}+f(x)-1>0 \implies \lim_{n\to\infty} F_n'(x)>0.$$

To show that this expression is positive, consider it as a function $g:\mathbb{R}^+\to\mathbb{R}$ defined by

$$g(x) := \frac{p(x)}{x}+f(x)-1.$$

Using the first derivative test, it is seen to be monotone decreasing:

$$g'(x) = -\frac{p(x)}{x^2}<0,\ \forall x\in\mathbb{R}^+.$$

Also note that

$$\lim_{x\to\infty} g(x) = 0.$$

Assume toward contradiction that $\exists x_0 > 0$ s.t. $g(x_0) \leq 0$. Since g is decreasing, $g(x)$ will be negative for any x to the right of x_0. Equivalently, for any x to the right of $2x_0$,

$$g(x) < g(2x_0) < 0, \ \forall x > 2x_0. \tag{$*$}$$

Choose $\varepsilon_0 = g(2x_0)/2$. Since $\lim_{x\to\infty} g(x) = 0$, $\exists K$ s.t. $|g(x) - 0| < \varepsilon_0$, $\forall x > K$. This implies that

$$-\varepsilon_0 < g(x), \ \forall x > K. \tag{$**$}$$

Then by $(*)$, $(**)$,

$$-\frac{g(2x_0)}{2} < 0, \ \forall x > \max(K, x_0)$$

which is a contradiction since $-g(2x_0)/2$ is positive. Then

$$g(x) > 0, \ \forall x \in \mathbb{R}^+. \tag{9}$$

Therefore,

$$\lim_{n\to\infty} F_n'(x) > 0, \ \forall x \in \mathbb{R}^+. \tag{10}$$

$\square$

This confirms the sufficient condition for (5) the divergence of the maximizer

$$\lim_{n\to\infty} x_n^* = +\infty.$$

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