Application of Optimization Techniques in Corporate Cash Management

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APPLICATION OF OPTIMIZATION TECHNIQUES

IN CORPORATE CASH MANAGEMENT

by

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Ph. D. (Operations Research), May 1986, Case Western Reserve University, Cleveland, Ohio

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ABSTRACT
APPLICATION OF OPTIMIZATION TECHNIQUES IN CORPORATE CASH MANAGEMENT

Venkateswara Reddy Dondeti
Old Dominion University, 2021
Director: Dr. Mohammad Najand

For any individual person or firm, there is a trade-off between carrying too much or too little cash on hand to meet the day-to-day transactions demand for cash. The BAT model, named after three eminent economists, Baumol, Allais, and Tobin, is the foundation for almost all cash management models in use today. The goal of the BAT model is to minimize the total costs involving the brokerage fees and the opportunity cost of interest lost on the cash held on hand. The brokerage fees are incurred in connection with the transactions for liquidating securities and converting them into cash. The opportunity cost of lost interest represents the income the firm could have earned by investing the cash in an interest-bearing asset instead of holding it on hand. In Chapter 1, a proof of the equivalency of the three seemingly different models of Baumol, Allais and Tobin is provided. The BAT model yields a square-root-formula that helps us determine the optimal level of cash to carry on hand.

In practice the square-root-formula of the BAT model often leads to fractional number of transactions involving the liquidation of securities and also fractional number of time periods (days or weeks) in the cycle-time between two consecutive transactions. Therefore, the results are not useful from a managerial or implementation point of view. Mathematical methods for obtaining integer solutions both for the number of transactions and the number of time periods between two consecutive transactions under different scenarios are described in Chapters 2 and 3.

In the basic version of the BAT model there is no provision for the use of short-term credit. However, in the case of individual persons as well as corporations, it is sometimes beneficial
to borrow funds on a short-term basis and repay the loan as soon as the funds become available. An extended version of the BAT model that not only includes the flexibility of short-term borrowing as described in the Sastry-Ogden-Sundaram (SOS) model, but also incorporates the requirement for the firm to buy insurance on the maximum amount borrowed during any time interval is discussed in Chapter 4. Further, a generalized version of the BAT and SOS models with insurance requirement is presented in Chapter 5. These cash management models are often considered as derivatives of some of the optimization models well-known in the field of Inventory Control and Production Management. The similarities and differences between the two types of models are also highlighted in this Chapter.

A single-period stochastic-demand cash management model is discussed in Chapter 6. In this model, the demand for cash is random and cannot be predicted in advance, but some past data is available. A formula is developed for the optimal amount of cash to be kept on hand at the beginning of the period with the goal of minimizing the total expected cost, given the interest rate at the beginning of the period and the interest rate that may be charged by the bank when the funds are borrowed on an emergency basis, should such a need arise.

The BAT model is static in the sense that the parameter values remain constant from one period to the next. In contrast, in a multi-period dynamic (MPD) cash management model the transaction costs, interest rates and proportional charge rates vary from one period to the next. Mixed linear-integer programming techniques for solving the multi-period dynamic (MPD) cash management model are described in Chapter 7. Conclusions and suggestions for future research are presented in Chapter 8.

Dissertation Committee Members
Dr. Kenneth Yung
Dr. David Selover
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DEDICATION

I would like to dedicate this dissertation to all my teachers from whom I have taken a variety of classes in different programs of study throughout my education. They have influenced my career and perhaps, my whole life, in many different ways. I am thankful to all of them for the lessons I have learned from them. I would like to mention the names of the following people (some of them no longer with us) with special fondness and a sense of gratitude.

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Dr. Hamilton Emmons (Doctoral Degree Program, Operations Research)
Dr. Daniel Solow (Doctoral Degree Program, Operations Research)

Finally, I would like to add the name of Mrs. Audrey Solow, wife of Dr. Daniel Solow, to the list given above, for continuing their tradition of welcoming foreign students and hosting Thanksgiving dinners over the years. I still remember very vividly some of the Thanksgiving dinners I enjoyed at their home in Cleveland Heights, Ohio.
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NOMENCLATURE

$T$: Number of time periods (or days) in the planning horizon (always an integer).

$D$: Rate of disbursement of cash per time period (i.e., per day) in a continuous stream

$P$: Rate of replenishment of cash per time period (i.e., per day) in a continuous stream

$\rho = D/P$  (Note: $D < P, \ 0 \leq \rho < 1$)

$F$: Fixed brokerage fee for one transaction, regardless of the amount liquidated or invested

$r$: Interest rate per time period (i.e., per day); rate represents lost income on the cash withdrawn

$s$: Interest rate per time period (i.e., per day); rate charged by the bank on the money borrowed

$e$: Rate of insurance charged by the bank on the maximum amount borrowed

$\alpha, \beta, \gamma, \delta$: Components of the cycle-time, assumed to be continuous variables

$g = \alpha + \beta$; $g$ is the total time in the cycle during which money is owed to the bank

$h = \gamma + \delta$; $h$ is the total time in the cycle during which cash is held on hand

$t = (\alpha + \beta) + (\gamma + \delta) = g + h$. $t$: Total cycle time (measured in days or other unit of time)

$\pi, \lambda, \tau, \theta$: Parameters derived from the values of variables or constants used in the analysis

$Q$: Amount of money received in a continuous stream from one liquidation of securities

$C$: Maximum level of cash balance on hand

$B$: Maximum amount of loan balance or money borrowed from the bank

$S$: Amount of money invested in securities

$Y = \{1, 2, \cdots, T\}$ Set of integer values from 1 to $T$

$y$: Assumed to be an integer variable, unless otherwise specified: $y \in Y$

$[x]$, $\lfloor x \rfloor$: Ceiling and Floor functions of a variable $x$ that can assume only positive values

$\psi(t)$: Total cost per cycle of length or duration $t$.

$\mu(t)$: Average cost per unit of time in a cycle of length $t$.

$E(y), W(y), R(y), Z(y)$: Functions of $y$ defined to suit the context

$\eta(\cdot), \xi(\cdot), \zeta(\cdot)$: Functions of variables such as $y$ or $t$ defined to suit the context
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CHAPTER 1
EQUIVALENCY OF BAUMOL, ALLAIS, AND TOBIN MODELS:
FOUNDATIONS OF THE BAT CASH MANAGEMENT MODEL

1.1 Introduction: Every firm needs to have some cash on hand to meet its payment obligations to its suppliers and settle the accounts with its creditors. If the firm carries too much cash on hand, it is losing income it could have earned by investing some of it in interest-bearing assets or other short-term securities traded in the financial markets. Of course, if it carries too little cash, it runs the risk of not being able to make a payment to a supplier on a due date. The latter scenario could indeed be catastrophic for some firms, if the financial markets perceive it as a default and start selling the firm’s shares at deep discounts. The critical roles of modern money transfer technologies and lending institutions in the management of the working capital needs of firms are discussed in Sanger (2014). Bates et al. (2018) have provided empirical proof that a steady increase in cash holdings adds value to the firm in the long term. The practical aspects of cash management from a corporate treasurer’s point of view are described in Bragg (2020). Zhang (2020) has presented results about the impact of the management of cash flows and accruals on the credit rating performance of a firm. Clearly, for the firm, there is a trade-off between carrying too much or too little cash on hand in meeting its transactions demand for cash. In a similar fashion, an individual person can deposit cash in a savings account and earn interest on it, instead of keeping too much cash on hand. On other hand, if a person keeps too little cash on hand, sometimes that person may not be able to pay even for day-to-day transactions. In other words, for an individual person too, there is a trade-off between carrying too much or too little cash on hand to meet the transactions demand for cash.
Trade-off problems, similar to the ones encountered in cash management by firms, arose in the management of inventories of raw materials, intermediate and finished goods in the earlier part of the 20th century itself. A planning model, involving production in lots and withdrawal of inventory in a continuous stream at a constant rate, called the Economic Order Quantity (EOQ) model was developed by F. W. Harris in 1913 and was later used extensively in the industry by R. H. Wilson under the name Wilson’s lot size formula (Hax and Candea, 1984, and Erlenkotter, 1990). In the industry the set-up cost to plan and schedule a production run for an item is often very significant, regardless of the size of the lot produced, and carrying the produced items in the inventory to meet future demand is also equally costly. The solution of the EOQ model leads to what is popularly known as the square-root-formula which helps in the estimation of the optimal lot size for each production run and also the number of production runs needed in a year. The adoption of the square-root-formula under the name “Wilson’s lot size formula” across many manufacturing industries led to the development of several related models in the field of production and inventory management (Whitin, 1957).

It is interesting to note that several of the techniques that were developed for the optimal management of physical inventories were later applied to cash management. In his classic paper entitled “The Transactions Demand for Cash: An Inventory Theoretic Approach,” Baumol (1952) presented a simple formula for estimating the optimal level of cash a firm needs to maintain in order to meet its transactions demand for cash while minimizing the related costs. Mathematically, the EOQ model and Baumol’s cash management model are identical and lead to the same square-root-formula and this fact was recognized by Whitin (1952), who also published an article dealing with the EOQ model and related topics in the same issue of The Quarterly Journal of Economics.
in which Baumol published his own article on the optimal cash balance. In his paper, Baumol (1952) acknowledged his exchange of notes with Whitin (1952) in a footnote. Subsequently, in one of his classic papers, Tobin (1956) discussed another optimization model that incorporates the partial investment of cash before disbursing funds to meet the transactions demand for cash. However, as described in Baumol and Tobin (1989), Allais, a French economist who was also awarded a Nobel Prize in 1988, seven years after Tobin, developed the same formula in 1947. For this reason, the square-root-formula is called the BAT model in honor of the three eminent economists who independently developed it. The Baumol-Tobin model has been analyzed from different perspectives and with different parameters and assumptions by several researchers including Barro (1978), Eppen and Fama (1969), Johnson (1975), Smith (1986), and Whalen (1968). Around the same time a review of the cash flow models was also published by Gregory (1976), perhaps, as aid to other researchers. It is worth noting that even today with all the advanced technologies, the results of the models developed by Baumol, Allais, and Tobin are applicable to the trade-offs in the cash management problems faced by many corporations and individual persons as well. In this chapter, we will first review the results published in the foundational papers of Baumol (1952), Allais (1947) and Tobin (1956) and prove that they are equivalent to one another through an illustrative example.

### 1.2 Baumol’s Model A:

Suppose that a retiree’s (firm’s) demand for cash transactions for a period of one year is known to be \( M \) dollars. Further, assume that the retiree will spend the money throughout the year in a continuous stream at a constant rate. In other words, there will be no ups and downs in disbursements. The retiree’s funds are already invested at an annual interest rate \( r \) in a brokerage account. Whenever the retiree withdraws funds, there is a fixed brokerage fee of \( F \).
dollars regardless of the amount withdrawn, and a proportional charge of $V$ for every dollar withdrawn. If $C$ is the amount withdrawn at one time, the cost of withdrawal will be equal to $(F + VC)$. In a year, the number of withdrawals will be equal to $M/C$. Immediately after the withdrawal, the retiree will have $C$ dollars on hand. Since the spending rate is constant, the cash balance on hand will reach zero at a point in time, when the retiree will withdraw the next installment of $C$ dollars. Therefore, the average balance on hand will be equal to $C/2$ dollars as shown in Figure 1.1 between any two consecutive withdrawals and for the whole year as well.

![Figure 1.1. Amount of Cash on Hand in Baumol’s Model](image)

Once the money is withdrawn, it will not earn any interest. As a result, the retiree is losing an amount of $rC/2$ dollars for the year. Therefore, the total cost of meeting the transactions demand for cash for the year is given by the following cost function:
\[
\psi(C) = (F + VC)(M / C) + rC / 2
\]  
(1.1)

Or, \( \psi(C) = (FM / C) + VM + rC / 2 \)  
(1.2)

Therefore, \( \frac{d\psi}{dC} = -FM / C^2 + r / 2 = 0 \)  
(1.3)

Or, the optimal amount of money to be withdrawn, \( C^* = \sqrt[2]{\frac{2FM}{r}} \)  
(1.4)

Equation (1.4) is popularly known as the square-root-formula in the parlance of cash management. Further, it is also well-known as the Economic Order Quantity formula or simply as the EOQ formula in production and inventory management.

1.3 Allais’ Model A: Suppose that an individual person can save a total of \( M \) dollars for a year. Further, the savings are accumulated in a continuous stream at a constant rate. There are no ups and downs in the rate of saving or in the rate of spending by the individual person. In other words, the amount of money available for investment grows steadily over the year. However, if it is not invested in any type of asset or security, it will not earn any interest. Obviously, it would be better to invest the money periodically, rather than wait till the end of the year. Assume that there is a fixed brokerage fee of \( F \) dollars for every transaction involving the investment of the funds in any type of securities, regardless of the amount invested. Once the money is invested, we assume that it will earn interest at an annual rate \( r \). If \( C \) is the amount invested at one time, the number of transactions will be equal to \( M / C \) and the brokerage fees for the year will be equal to \( FM / C \). Since the saving rate is constant, the cash balance on hand will be zero at the beginning of the cycle and will reach a peak value of \( C \) before it is invested. Therefore, the average balance on hand
will be equal to $C/2$ dollars as shown in Figure 1.2 between any two consecutive investment transactions and for the whole year as well. The cash on hand will not earn any interest and the interest foregone on the average balance of cash represents an opportunity cost. The amount of interest foregone on the average balance is equal to $rC/2$ dollars for the year. Assume that there is no proportional charge on the money invested. Therefore, the total of the brokerage fees for the investment transactions and the interest foregone on the cash held on hand for the year is given by the following cost function:

$$
\psi(C) = FM / C + rC / 2
$$

Therefore, \( \frac{d\psi}{dC} = -FM / C^2 + r / 2 = 0 \)

Figure 1.2. Amount of Cash on Hand in Allais’ Model
Or, the optimal amount of money to be invested, \( C^* = \sqrt{\frac{2FM}{r}} \)  (1.7)

It is clear that formulas (1.4) and (1.7) are identical except that they are presented from two different perspectives. It may be noted that the result will be the same as in equation (1.7), even if there is a proportional charge imposed by the brokerage firm at the time of the investment. In essence, Allais’ model is a mirror image of Baumol’s model.

**1.4 Tobin’s Model A:** In Baumol’s analysis, the amount of cash withdrawn \( C \) is assumed to be a continuous variable. If \( y \) represents the number of cash withdrawals in a year, then \( y = M / C \). The average amount of cash on hand will be equal to \( C/2 \) for the whole year, only if \( y \) is an integer. In general, the value of \( y \) may turn out to be any positive number, but not necessarily an integer. If we get a fractional value for the number of cash withdrawals \( y \), obviously it is not possible to implement it in practice. Further, it calls into question the accuracy of the value of \( C^* \) obtained in equation (1.4), since the average amount of cash on hand will no longer be equal to \( C/2 \). To resolve this issue, Tobin (1956) starts the analysis with the number of cash withdrawals \( y \), and restricts \( y \) to be an integer-valued variable. In Baumol’s Model A, the amount \( M \) was already invested in a brokerage account. In Tobin’s Model A, the initial amount \( M \) is received by an individual at time \( t = 0 \), and it is not yet invested in any asset such as bonds. Nonetheless, by the end of the year, when \( t = 1 \), all the money is spent and nothing is left behind. As in Baumol’s Model A, in Tobin’s Model A also, it is assumed that the disbursement is done throughout the year in a continuous stream at a constant rate.
In Tobin’s analysis, the first step is to keep enough cash on hand to make the first disbursement and invest the rest in bonds at an interest rate \( r \). Subsequently, the amount invested in bonds is liquidated periodically to make the necessary disbursements. The obvious reason for investing a portion of the funds is that there is no need to keep the entire amount \( M \) in cash, since the invested funds can be withdrawn later to make the disbursements as and when needed. Suppose that the whole process is completed in \( y \) transactions with the first transaction involving an investment in bonds and the other \((y-1)\) transactions involving liquidations (or withdrawals). At time \( t=0 \), an amount equal to \((y-1)M/y\) is invested in bonds and the remainder equal to \(M/y\) is held as cash. Further, the amount of each liquidation is equal \( M/y \). Assume that the brokerage fee is \( F \) for any transaction, regardless of whether it is an investment or liquidation.

Our objective is to find the optimal number of transactions \( y \) so that the net revenue is maximized. Before finding the optimal number of transactions, let us start with \( y = 5 \) as a trial value. The corresponding positions in cash and bonds are shown in Figure 1.3. Since \( y = 5 \), we have 5 time-intervals, each equal to 0.2 years. In each of these intervals, the largest amount of cash on hand is equal to \( M/y \) and the smallest is 0. During the five intervals, the amount invested in bonds (represented by the shaded rectangles) will be equal to \(4M/5, 3M/5, 2M/5, M/5, \) and 0. The average amount \( \bar{S} \) invested in bonds per period is given by:

\[
\bar{S} = \frac{1}{5}((4 + 3 + 2 + 1 + 0)(M/5) = \frac{1}{5}(5 \times 4/2)(M/5) = 4M/10 = 2M/5.
\]

In a similar fashion, we now derive the value of \( \bar{S} \) for any integer value of \( y \). It is assumed here that all time-intervals are equal in duration. It can be proved that it is the optimal choice based on theory of convex functions (or by contradiction). The proof is given in Appendix A.
For any integer value of $y$, the average amount held in bonds per each period and for the whole year as well is given by:

$$\bar{S} = \frac{1}{y}((y-1) + (y-2) + \cdots + 1 + 0)(M / y) = \frac{1}{y}(y(y-1)/2)(M / y)$$

Or, $$\bar{S} = (y-1)(M / 2y) \quad (1.8)$$

Since there is a fee of $F$ for every transaction, net earnings, $E = \bar{S}r - Fy = (y-1)Mr / 2y - Fy$

Or, $$E(y) = Mr / 2 - Mr / 2y - Fy \quad (1.9)$$

Ignoring the fact that $y$ is restricted to integer values, we find the derivative of the earnings function $E$ with respect to $y$:

$$\frac{dE}{dy} = Mr / 2y^2 - F = 0. \quad \text{Or, } y_A = \sqrt{Mr / 2F} \quad (1.10)$$
\[
\frac{d^2 E}{dy^2} = -Mr / y^3 < 0 \quad \text{for any } y \geq 1 \quad (1.11)
\]

According to Baumol’s formula, \( C^* = \sqrt{\frac{2FM}{r}} \) and \( \hat{y} = M / C^* = \sqrt{Mr / 2F} \).

In essence, Tobin’s formula (1.10) did not yield anything new except that we need to solve for \( y^* \) that maximizes \( E \) in equation (1.9).

1.5 Illustrative Example A

Let \( M = 100000, \ r = 0.08, \ F = 68 \)

Calculations for Baumol’s Model A:

\[ C^* = \sqrt{2 \times 68 \times 100000 / 0.08} = 13038.40 \quad \text{from formula (1.4)} \]

If we let \( V = 0 \) in (1.2), we get \( \psi(C) = (FM / C) + rC / 2 \)

Total cost, \( \psi(C^*) = 68 \times 100000 / 13038.40 + 0.08 \times 13038.40 / 2 = 1043.07 \)

Further, number of cash withdrawals, \( \hat{y} = M / C^* = 100000 / 13038.4 = 7.67 \)

Obviously, we cannot have fractional number of withdrawals in a year.

If we let \( V = 0 \) in (1.2), total cost with \( y \) transactions is given by: \( \xi(y) = Fy + rM / 2y \)

The functional relationship between \( \xi(y) \) and \( y \) is shown in Figure 1.4.
Our objective here is to minimize the total cost $\xi(y)$.

$$\frac{d \xi}{dy} = F - rM / 2y^2 = 0.$$  Again, $\hat{y} = \sqrt{rM / 2F}$

$$\frac{d^2 \xi}{dy^2} = rM / y^3 > 0, \text{ for any } y \geq 1$$

Based on the theory of convex functions, it can be proved that if the value of $\hat{y}$, at which the minimum of $\xi(y)$ occurs, is not acceptable or feasible, it can be proved that the optimal number of withdrawals $y^*$ will be the integer value obtained by either rounding down or rounding up the value of $\hat{y}$. In other words, in this case, $y^*$ will be either 7 or 8. Basic properties of convex functions are discussed in Appendix A.

$$\xi(7.67) = 7.67 \times 68 + 0.08 \times (100000 / 7.67) / 2 = 1043.07$$
\( \xi(7) = 7 \times 68 + 0.08 \times (100000 / 7) / 2 = 1047.43 \)

\( \xi(8) = 8 \times 68 + 0.08 \times (100000 / 8) / 2 = 1044.00 \)

Therefore, in case of Baumol’s Model A, the optimal number of transactions, \( y^* \) is 8 and the lowest cost for meeting the transactions demand \( \xi^*(8) \) is equal to $1044.00.

Calculations for Allais’ Model:

As before, \( M = 100000, \ r = 0.08, \ F = 68 \)

\[ C^* = \sqrt{2 \times 68 \times 100000 / 0.08} = 13038.40 \text{ from formula (1.7)} \]

we have: \( \psi(C) = (FM / C) + rC / 2 \)

Total cost, \( \psi(C^*) = 68 \times 100000 / 13038.40 + 0.08 \times 13038.40 / 2 = 1043.07 \)

Further, number of purchase transactions, \( \hat{y} = M/C^* = 100000 / 13038.4 = 7.67 \)

Obviously, we cannot have fractional number of purchase transactions in a year.

Total cost with \( y \) transactions is given by: \( \xi(y) = Fy + rM / 2y \)

The calculations are identical to those given in Baumol’ Model A and the optimal number of purchase transactions \( y^* \) is 8 and the lowest cost associated with the investment process \( \xi^*(8) \) is equal to $1044.00.

Let us now return to Tobin’s Model A: \( M = 100000, \ r = 0.08, \ F = 68 \).

\[ y_A = \sqrt{100000 \times 0.08 / 2 \times 68} = 7.67 \]
In this case also, the optimal number of transactions is not an integer and therefore, not a feasible solution. The value of earnings $E$, as a function of the number of transactions $y$, is shown in Figure 1.5. Since the second derivative is negative for any $y \geq 1$, we can conclude that $E(y)$ is a concave function of $y$, for any $y \geq 1$.

![Figure 1.5. Functional Relation between $E$ and $y$.](image)

Based on the theory of concave functions, it can be proved that if the value of $y_A$, at which the maximum of $E$ occurs, is not acceptable or feasible, then, $y^*$ will be the integer value obtained by rounding up or rounding down the value of $y_A$. Basic properties of concave functions are discussed in Appendix A.

At $y = 7$, $E(y) = (y - 1)rM / 2y - Fy = 6*100000*0.08/(2*7) - 7*68 = 2952.57$
At $y = 8$, $E(y) = (y - 1)rM / 2y - Fy = 7 \times 100000 \times 0.08 / (2 \times 8) - 8 \times 68 = 2956.00$

Therefore, in case of Tobin’s Model A, the optimal number of transactions, $y^*$ is 8 and the highest value of earnings $E^*(8)$ is equal to $2956.00$, while meeting the transactions demand.

The time interval between any two consecutive transactions, $\tau = (1 / y^*) = 1 / 8 = 0.125$

Amount of cash retained for payment during the first time-interval = $\tau * M = 12500$

Amount invested in bonds = $(y^* - 1)\tau M = 7 \times 0.125 \times 10000 = 87500$

In Tobin’s Model A, there will be one initial transaction involving the purchase of bonds for an amount of $87,500 and subsequently, seven transactions involving cash withdrawals, each for an amount of $12,500.

1.6 Baumol’s Model B: Suppose that at the beginning of a year, a researcher has received a lumpsum amount of $M$ dollars. None of it is invested in a bank or brokerage account yet. No other remuneration will be paid to the researcher during the year and assume that the researcher’s total transactions demand for cash for the year is $M$ dollars. Further, assume that, as in the case of the retiree, the researcher will also make disbursements in a continuous stream at a constant rate throughout the year. Obviously, from a financial point of view, it is better for the researcher to retain an amount of $K$ dollars to meet the transactions demand in the near-term and deposit the remainder of $S$ dollars in a bank, that pays an annual interest rate $r$ on the amount invested. The amount of $K$ dollars will be enough to make the payments for the time period of $K / M$. The average balance during this time period will be $K / 2$. Hence the amount of lost interest is equal to $(K / 2)r(K / M)$. Suppose that the bank will charge a fixed amount of $F_d$ dollars to open the deposit account and an additional amount of $V_d$ dollars for every dollar deposited. Needless to say that $V_d$
is relatively very small compared to \( r \). The researcher has to bear the loss of interest on the retained amount \( K \) or \((M - S)\) dollars and also incur the additional cost of \((F_d + V_d S)\) dollars for depositing the amount of \( S \) dollars in the bank. Therefore, the cost to the researcher as a result of retaining \( K \) dollars and depositing \( S \) dollars is given by the following equation:

\[
\psi_1(S) = \frac{K}{2}r\left(\frac{K}{M}\right) + (F_d + V_d S) = \frac{K^2}{2M}r + (F_d + V_d S)
\]

Or, \( \psi_1(S) = r\left(\frac{(M - S)^2}{2M}\right) + (F_d + V_d S) \) \hspace{1cm} (1.12)

Immediately after the time-period \( K/M \), the researcher has to begin the first withdrawal from the deposited amount. If an amount of \( C \) dollars is withdrawn periodically to make disbursements for the rest of the year, the number of withdrawals will be equal to \((S/C)\). As in the case of the retiree, assume that there is a fixed brokerage fee of \( F_w \) dollars for each withdrawal, and a proportional charge of \( V_w \) for every dollar withdrawn. Then, the cost of withdrawals will be equal to \((F_w + V_w C)(S / C)\). Further, there would also be a loss of interest income on the average balance of \((C/2)\) during the time period \((S/M)\). Therefore, the total cost during the time period \( S/M \) is given by the following equation:

\[
\psi_2(C) = (F_w + V_w C)(S / C) + (S / M)r(C / 2) = (F_w S / C) + V_w S + r(SC / 2M) \hspace{1cm} (1.13)
\]

Or, total cost for the year, \( \psi(S, C) = \psi_1(S) + \psi_2(C) \)

Or, \( \psi(S, C) = r(M - S)^2 / 2M + (F_d + V_d S) + (F_w S / C) + V_w S + r(SC / 2M) \) \hspace{1cm} (1.14)
Partial differentiation yields the following results:

\[ \frac{d\psi}{dC} = -F_w S / C^2 + r S / 2M = 0 \]

Or, \( C^2 = 2F_w M / r \)

Or, the optimal amount of money to be withdrawn, \( C^* = \sqrt{\frac{2F_w M}{r}} \) \hspace{1cm} (1.15)

\[ \frac{\partial \psi}{\partial S} = -(M - S) r / M + V_d + V_w + F_w / C + Cr / 2M = 0 \] \hspace{1cm} (1.16)

Since \( K = M - S \), and \( C^2 = 2F_w M / r \), after simplification of the equation (1.14), we get:

\( K = C^* + M(V_d + V_w) / r \)

Let \( J = M(V_d + V_w) / r \)

Let \( N = M - J \) \hspace{1cm} (1.17)

Also, \( K = J + C^* \) \hspace{1cm} (1.18)

From (1.18) we see that \( K \) is the sum of the fixed component \( J \) and the installment payment \( C^* \). But \( C^* \) is also the first installment payment whose value depends on the total number of the installment payments made in the year from the amount \( N \) which denotes the total amount of all the payments excluding \( J \). Only when we fix the number of the installment payments \( y \), the values of \( C \) and \( K \) can be determined. In the first step, we fix the value if \( y \), and then calculate the value of \( C \). In the next step we calculate the value of \( K \).
1.7 Illustrative Example B

Let $M=100000$, $r=0.08$, $F_w=68$, $V_w=0.0075$, $F_d=68$, $V_d=0.0075$, $V_d+V_w=0.015$

In this example, we have assumed that $F_d=F_w$ and $V_d=V_w$. But even if $F_d \neq F_w$ and $V_d \neq V_w$ the basic results would be the same. We have selected equal values for them so as to maintain consistency with Tobin’s Model B.

Calculations for Baumol’s Model B:

$C^* = 13,038.40$ from formula (1.15)

Let $J = M(V_d + V_w) / r = 100000*(0.0075+0.0075)/0.08 = 18750$

$K = C^* + M(V_d + V_w) / r = C^* + J = 13038.40 + 18750 = 31788.40$

$S = M - K = 100000 - 31788.40 = 68211.60$

$N = M - J = 100000 - 18750 = 81250$

The recommendation is to retain an amount, $K=31788.40$, and invest the amount, $S=68211.60$

The amount of $K$ will be completely disbursed when $t_1 = K/M = 31788.40/100000 = 0.31788$

Number of withdrawals from the invested amount $\hat{y} = S / C^* = 68211.60/13,038.40 = 5.23$

If we withdraw $C^* = 13,038.40$ from the invested amount of 68211.60 every time we need cash, we need to make 5.23 withdrawals. Of course, this is not feasible solution. We have run into the same problem as in Baumol’s Model A. We have to round down to 5 withdrawals or round up to 6 withdrawals. If we change the number of withdrawals, obviously, we are changing the value of the amount $C$ withdrawn. Either way, this implies that we need to modify the value of $C^*$ also. In
turn, since $K = C + J$, the value of $K$ will also be affected. But $J$ will remain at its original value.

Rounding down $\hat{y}$, we get $y_D = 5$. But $y_D$ does not include the first installment payment. The revised value for the number of equal installment payments $y_R = y_D + 1 = 5 + 1 = 6$. We can now calculate revised value of $C$.

Revised value of $C$ corresponding to $y_R = 6$ is given by: $C_D = N / y_R = 81250/6 = 13541.67$

Revised value of $K$ is given by: $K_D = J + C_D = 18750 + 13541.67 = 32291.67$

Then the revised value of $S$ is given by $S_D = M - K_D = 100000 - 32291.67 = 67708.33$

Or, $S_D = M - K_D = M - J - C_D = N - C_D = y_R C_D - C_D = (y_R - 1) C_D = y_D C_D = 67708.33$

Also, $y_D = S_D / C_D$

Using the values corresponding to the rounding-down option, we get from (1.14):

$$\psi(S, C) = r(K_D^2 / 2M) + (V_d + V_w)S_D + F_d + F_w S_D / C_d + r(S_D C_D / 2M)$$

Or, $\xi(y_D) = r(K_D^2 / 2M) + (V_d + V_w)S_D + F_d + F_w y_D + r(S_D C_D / 2M)$

$$\xi(y_D) = \frac{0.08 \times (32292.67)^2}{2 \times 100000} + (0.015) \times 67708.33 + 68 + 68 \times 5 + \frac{0.08 \times 67708.33 \times 13541.67}{2 \times 100000}$$

Or, $\xi(y_D) = 417.13 + 1015.62 + 68 + 340 + 366.75 = 2207.50$

Similarly, rounding up $\hat{y}$, we get $y_U = 6$. Again, we re-calculate $C$ using the total amount of instalment payments and the revised value of $y_R = y_U + 1 = 6 + 1 = 7$.

Revised value of $C$ corresponding to $y_R = 7$ is given by: $C_U = N / y_R = 81250/7 = 11607.14$
Revised value of $K$ is given by: $K_U = C_U + J = 11607.14 + 18750 = 30357.14$

Then the revised value of $S$ is given by $S_U = M - K_U = 100000 - 30357.14 = 69642.86$

Or, $S_U = M - K_U = M - J - C_U = N - C_U = y_R C_U - C_U = (y_R - 1) C_U = y_U C_U = 69642.86$

Also, $y_U = S_U / C_U$

Using the values corresponding to the rounding-up option, we get from (1.14):

$$\xi(y_U) = r(K_U^2 / 2M) + (V_d + V_w)S_U + F_d + F_w y_U + r(S_U C_U / 2M)$$

$$\xi(y_U) = \frac{0.08 \times (30357.14)^2}{2 \times 100000} + (0.015 \times 69642.86 + 68 + 68 + 6 + \frac{0.08 \times 69642.86 \times 11607.14}{2 \times 100000}$$

Or, $\xi(y_U) = 368.62 + 1044.64 + 68 + 408 + 323.34 = 2212.60$

Comparing $\xi(y_D)$ and $\xi(y_U)$, we see that the rounding-down option is cheaper. Therefore, the recommended option to retain an amount of 32291.67 and invest the remainder of 67708.33 in bonds. The optimal number of withdrawals $y^*$ from the invested amount is equal to 5 and the amount of each withdrawal will be 13541.67.

There is no withdrawal in Allais’ Model at the end of the year and there is no proportional charge for investment either. There is no other version of Allais’ Model. But corresponding to Baumol’s Model B, there is a version B of Tobin’s Model and we will now consider that model.
1.8 Tobin’s Model B: In Tobin’s Model A, the starting assumption is that the proportional charge $V=0$ for every dollar invested or liquidated. In Tobin’s Model B, it is assumed that $V > 0$, in addition to the fixed fee of $F$ dollars per transaction regardless of whether it involves a purchase of securities or liquidation of securities. Because of the two-way proportional charge imposed on every dollar invested or withdrawn, the approach adopted by Tobin in this model is to explore a two-step strategy. First, recover the total amount of the proportional charge and then, explore the possibility that the interest income from subsequent investment will exceed the transaction fees. In other words, we try to first recover the total proportional charge on the amount invested and later withdrawn. Suppose we invest an amount $x$ in bonds. Then, the total proportional charge will be $2Vx$, since we have to pay the amount $Vx$ when we purchase the bonds and subsequently, again the amount $Vx$ when we liquidate the same bonds, but in several installments. If we invest $x$ at an interest rate $r$ from time $t=0$ to time $t=t_1$, the interest income will be $xrt_1$. Therefore, in order to break-even, the interest income $xrt_1$ must be equal to $2Vx$. In other words, we must invest $x$ from time $t=0$ to time $t=t_1$ such that $t_1 = 2V/r$. If $V > r/2$, then $t_1 \geq 1$. In such a case, there is no benefit from investing any funds at all, since we cannot even recover the proportional charges. Clearly, a condition for any investment to generate net revenue is that $t_1 = 2V/r < 1$ or $2V < r$, and we invest some funds beyond time $t_1$. Of course, the best-case scenario is that $V=0$, which is nothing but Model A discussed earlier. Here it is assumed that $0 < 2V < r$. Obviously, the lower the value of $V$, the higher the net earnings will be. Of course, we need to retain an amount of $t_1M$ in cash to meet the transactions demand from time $t=0$ to time $t=t_1$. Then, we are left with an amount equal to $(1-t_1)M$ that would help us make the disbursements from $t=t_1$ to $t=1$. In Tobin’s Model A, the time under consideration is one year and the available amount is $M$. Instead, we now have to focus
on meeting the transactions demand for cash for the period \((1-t_1)\) with the available amount of 
\((1-t_1)M\), starting at time \(t = t_1\). In place of \(M\), we now have a smaller amount, only \((1-t_1)M\).

Though, the basic procedure for determining the average amount invested in bonds is the same as in Tobin’s Model A, we now have to determine the average amount invested in bonds for only a fraction of the year. With these changes, for any integer value of \(y\), with one initial investment and \((y-1)\) liquidations, the average amount \(\bar{S}_p\) invested in bonds per each period in the time interval \((1-t_1)\) and for the fraction of the year \((1-t_1)\) is given by:

\[
\bar{S}_p = \frac{1}{y}((y-1) + (y-2) + \cdots + 1 + 0)((1-t_1)M / y) = \frac{1}{y}(y(y-1)/2)((1-t_1)M / y)
\]

Or, \(\bar{S}_p = (y-1)((1-t_1)M) / 2y\) \hspace{1cm} (1.19)

It may be noted that equation (1.19) is similar to equation (1.8) except that \((1-t_1)M\) appears in place of \(M\). The interesting point to be noted is that the interest income on any amount invested from time \(t = 0\) to time \(t = t_1\) will be enough to offset the two-way (from cash to bonds and back to cash) proportional charges on that amount, since \(t_1 = 2V/r\). The net earnings will be positive if the interest income received after time \(t_1\) exceeds the fixed brokerage fees \(Fy\), associated with the first transaction involving the purchase of bonds at time \(t = 0\), and the subsequent \((y-1)\) periodic liquidations. To find the net earnings \(E\), we focus on the interest income generated only from time \(t = t_1\) to time \(t = 1\), or, only for the period \((1-t_1)\). Also, the amount available at time \(t = t_1\) is only \((1-t_1)M\). Since there is a fee of \(F\) dollars for every transaction, the equation for the net earnings \(E\) is given by:

\[
E = \bar{S}_p r (1-t_1) - Fy = (y-1)((1-t_1)M / 2y)r(1-t_1) - Fy
\]
Or, \[ E = (y-1)(1-t_1)^2 Mr / 2y - Fy \] (1.20)

Or, \[ E = (1-t_1)^2 Mr / 2 - (1-t_1)^2 Mr / 2y - Fy \]

Therefore, \[ \frac{dE}{dy} = (1-t_1)^2 Mr / 2y^2 - F = 0. \]

Or \[ y_B = \sqrt{(1-t_1)^2 Mr / 2F} = (1-t_1)\sqrt{Mr / 2F} \] (1.21)

Comparing (1.10) and (1.21), we see that \[ y_B = (1-t_1)y_A. \] (1.22)

As in Tobin’s Model A, for Model B also let \[ M=100000, r=0.08, F=68. \]

In Tobin’s Model A, the proportional charge \( V=0. \)

In Tobin’s Model B, let \( V=0.0075. \)

Then, \( t_1 = 2V/r = 2*0.0075/0.08=0.1875, \) and \( (1-t_1) = 0.8125. \)

Using (18) we get: \[ y_B = (1-t_1)\sqrt{Mr / 2F} = (1-t_1)y_A = 0.8125*7.67 = 6.23 \]

Based on the theory of concave functions, it can be proved that \( y^* \) for this case also will be the integer value obtained by rounding up or rounding down the value of \( y_B. \) Basic properties of concave functions are discussed in Appendix A.

At \( y=6, \) \[ E(6) = (y-1)(1-t_1)^2 Mr / 2y - Fy = 5*(0.8125)^2*100000*0.08/(2*6) - 68*6 = 1792.51 \]

At \( y=7, \) \[ E(7) = (y-1)(1-t_1)^2 Mr / 2y - Fy = 6*(0.8125)^2*100000*0.08/(2*7) - 68*7 = 1787.39 \]

Since \( E(6) \) is higher than \( E(7), \) the optimal number of transactions \( y^* = 6. \)
The time interval between transactions, \( \tau = (1 - t_1) / y^* = 0.8125/6 = 0.1354167 \)

Payment during the time interval \( t_1 \) to \((t_1 + \tau)\) = \( \tau \cdot M = 0.1354167 \cdot 100000 = 13541.67 \)

Amount of cash needed for disbursement from time \( t=0 \) to time \( t=t_1 \) is given by:

\[ J = t_1 \cdot M = 0.1875 \cdot 100000 = 18750 \]

Amount available for installment payments including the first one, \( N = 100000 - 18750 = 81250 \)

Or, total amount needed for all the \( y \) installment payments after time \( t_1 \) is given by:

\[ N = (1 - t_1) \cdot 100000 = 0.8125 \cdot 100000 = 81250 \]

Should we invest all the amount of \( N=81250 \) available for installment payments in bonds at time \( t=0 \)? If we were to invest the amount of 81250 at time \( t = 0 \), we need to withdraw $13541.67 at time \( t_1 \) to make a disbursement during the time interval \( t_1 \) to \((t_1 + \tau)\) forcing us to incur the additional transaction fee of \( F \) dollars at time \( t_1 \) without the benefit of any extra interest income.

Since \( x \cdot r \cdot t_1 = 2V \cdot x \) for any amount \( x \) invested from time \( t = 0 \), to time \( t = t_1 \), there will be no net revenue regardless of the amount invested during that time interval. This implies that it is optimal not only to keep the amount of $18750, but also an additional amount of $13541.67, bringing the total amount to $32291.67 on hand at time \( t = 0 \). In other words, the initial amount of cash we should retain on hand to take care of the disbursements from time \( 0 \) to time \((t_1 + \tau)\) is given by:

\[ K = (t_1 + \tau) \cdot M = t_2 \cdot M = (0.1825 + 0.1345167) \cdot 100000 = 32291.67. \]

Amount invested in bonds, \( S_F = M - K = M(1 - t_2) = M - (t_1 + \tau) \cdot M = M(y^* - 1)(1 - t_1) / y^* \)

Or, Amount invested in bonds, \( S_F = (y^* - 1) \cdot \tau \cdot M = 5 \cdot 0.1354167 \cdot 100000 = 67708.33 \)
In summary, we keep an amount of $32291.67 in cash and disburse the amount of $18750 between time 0 and time $t_1 = 0.1875$, and the remainder of $13,541.67 during the time interval $t_1$ to $(t_1 + \tau)$. In addition, we also purchase bonds for an amount of $67708.33 at time $t = 0$, and subsequently, liquidate the bonds in five installments of $13,541.67 at times $t_2 = (t_1 + \tau)$, $t_3 = (t_1 + 2\tau)$, $t_4 = (t_1 + 3\tau)$, $t_5 = (t_1 + 4\tau)$, and $t_6 = (t_1 + 5\tau)$. Initially, we pay the fixed transaction fee $F$ when we buy the bonds and later we will incur the transaction fee of $5F$, when we make five withdrawals. Of course, there will be no money invested in bonds during the last time interval from time $t = t_6$ to time $t = t_7 = (t_1 + 6\tau) = 1$. However, the cash balance will start at $13,541.67 at time $t = t_6$ and go down to zero at the end of the year ($t = 1$), as it should be.

Let us now compare the results of Tobin’s Model B to those of Baumol’ Model B. In the illustrative example of Baumol’s Model B solved earlier, it is given that $F_d = F_w = 68$ and further, $V_d = V_w = 0.0075$. Since the parameter values are the same in both models, obviously the results are the same. However, in Baumol’s Model B, the brokerage fee $F_d$ for purchasing securities is assumed to be different from the withdrawal fee $F_w$ for liquidating the securities. Similarly, the proportional charge $V_d$ for investment can also be different from the proportional charge $V_w$ for withdrawal. To facilitate the comparison of the two models, we simply define $V = (V_d + V_w) / 2$ and $F = F_w$ in Tobin’s Model B. Even if $F_d$ is greater than $F_w$, this modification will have no effect on the final results of either model as long as the difference between $F_d$ and $F_w$ is relatively small. The only change will be in the value of the earnings given by equation (1.20).

$$E = (y - 1)(1 - t_i)^2 Mr / 2 y - F_d - F(y - 1)$$

(1.23)
From (1.23) we see that $E$ will be higher by the amount $(F_u - F_d)$ if $F_u > F_d$. Of course, it will be smaller by the amount $(F_d - F_u)$ if $F_d > F_u$. Regardless, the optimal number of transactions will remain the same. In essence both the models are equivalent.

1.9 Equality of Installment Payments

In all the five models it is assumed that it is optimal to have equal amounts of installment payments. In other words, the time intervals between any two consecutive disbursements must be the same. However, no proof is given in Baumol (1952) for either of his models about the equality of the installment payments or the corresponding time intervals. Tobin (1956) has presented such a proof for Model B in the Appendix of his paper. Obviously, it is then automatically true for Tobin’s Model A, for both models of Baumol and Allais Model as well. There seem to be two errors, perhaps typographical, in the proof given in Tobin (1956). An outline of the corrected proof is described here.

In Tobin’s Model B, it assumed that $0 < 2V < r$. Also, $t_1 = 2V/r$.

If $J$ is the amount of cash needed for payment from time $t = 0$ to time $t = t_1$, then, $J = t_1 \times M$.

Total amount needed for all the $y$ installment payments, including the first one, after time $t_1$ is given by: $N = M - J = (1 - t_1) \times M$.

Let $\tau = (1 - t_1) / y$. Then, amount invested at time $t = 0$ in bonds, $S_0 = M - K = (y - 1) \times \tau \times M$. 
Suppose that we arbitrarily decide to have eight installment payments, not necessarily equal to one another. Now we divide the time interval between $t = t_1$ and $t = 1$ into eight intervals of arbitrary lengths and number them $t_2, t_3, \ldots, t_8$, with the understanding that $t_9 = 1$.

In other words, we have: $(t_2 - t_1) + (t_3 - t_2) + (t_4 - t_3) + \cdots + (t_8 - t_7) = 1 - t_1$.

Our next step is to maximize just the interest income, ignoring the fixed transaction fees during the time interval $(1 - t_1)$, since the number of transactions is already fixed, though arbitrarily.

If $\zeta$ represents the interest income for the period $(1 - t_1)$, the equation for $\zeta$ is given by:

$$\zeta = Mr(1 - t_2)(t_2 - t_1) + Mr(1 - t_3)(t_3 - t_2) + Mr(1 - t_4)(t_4 - t_3) + Mr(1 - t_5)(t_5 - t_4) + Mr(1 - t_6)(t_6 - t_5) + Mr(1 - t_7)(t_7 - t_6) + Mr(1 - t_8)(t_8 - t_7)$$

It may be noted that the amount invested during the time interval $(t_2 - t_1)$ is $M(1 - t_2)$.

Similarly, $M(1 - t_3)$ is the amount invested during the time interval $(t_3 - t_2)$, and so on.

Further, there is no money invested during the last time interval $(t_8 - t_7)$ or $(1 - t_8)$.

We set the partial derivatives to 0 to determine the optimal values for $t_j, j = 2, \ldots, 8$.

$$t_1 - 2t_2 + t_3 = 0 \quad \text{Or,} \quad t_2 - t_1 = t_3 - t_2$$

$$t_2 - 2t_3 + t_4 = 0 \quad \text{Or,} \quad t_3 - t_2 = t_4 - t_3$$

$$t_3 - 2t_4 + t_5 = 0 \quad \text{Or,} \quad t_4 - t_3 = t_5 - t_4$$

$$t_4 - 2t_5 + t_6 = 0 \quad \text{Or,} \quad t_5 - t_4 = t_6 - t_5$$
\[ t_5 - 2t_6 + t_7 = 0 \quad \text{Or,} \quad t_6 - t_5 = t_7 - t_6 \]

\[ t_6 - 2t_7 + t_8 = 0 \quad \text{Or,} \quad t_7 - t_6 = t_8 - t_7 \]

\[ t_7 - 2t_8 + 1 = 0 \quad \text{Or,} \quad t_8 - t_7 = 1 - t_8 \] (1.24)

From all the equations on the right side, it is clear that all the time intervals must be equal to one another. Further, adding all the equations on the left side, we get: \( t_1 - t_2 - t_8 + 1 = 0 \). Then, \( (t_2 - t_1) = (1 - t_8) \). Let \( 1 - t_8 = \tau \). Since \( t_9 = 1 \), we can conclude immediately that \( \tau = (1 - t_1) / 8 \).

We now get: \( t_2 = t_1 + \tau, \ t_3 = t_1 + 2\tau, \ t_4 = t_1 + 3\tau, \) and so on: \( t_j = t_1 + (j - 1)\tau, \ j = 2, 3, \ldots, 8 \).

In general, if we have \( y \) equal installment payments, we get:

\( (1 - t_1) = \tau y, \) \quad \text{or,} \quad \tau = (1 - t_1) / y, \text{ and } t_j = t_1 + (j - 1)\tau, \ j = 2, 3, \ldots, y. \)

Also, \( 1 - t_j = 1 - t_1 - (j - 1)\tau = y\tau - (j - 1)\tau = (y - j + 1)\tau. \)

Then we can re-write \( \zeta \) as follows:

\[
\zeta = Mr\tau^2 \sum_{j=2}^{j=y} (y - j + 1) = Mr\tau^2 ((y - 1) + (y - 2) + \cdots + 1) = Mr\tau^2 y(y - 1) / 2
\]

Or, \( \zeta = Mr(1 - t_1) / y^2 y(y - 1) / 2 = (y - 1)Mr(1 - t_1)^2 / 2y \)

Then, the average amount invested in bonds from \( t = t_1 \) to \( t = 1 \) is given by:

\[
\bar{S}_\tau = \zeta / (r(1 - t_1)) = (y - 1)((1 - t_1)M) / 2y
\]

But we have also invested an amount \( S_0 = (y - 1)(1 - t_1)M / y \) from time \( t = 0 \) to time \( t = t_1 \),
For the whole year, the average amount in bonds is given by: 

$$ \bar{S}_{\mu} = S_0 t_1 + \bar{S}_v (1 - t_1) $$

$$ \bar{S}_{\mu} = ((y - 1)(1 - t_1)M / y)t_1 + ((y - 1)((1 - t_1)M) / 2y)(1 - t_1) $$

Or, 

$$ \bar{S}_{\mu} = ((y - 1)(1 - t_1)M / y)(t_1 + (1 - t_1) / 2) = ((y - 1)(1 - t_1)M / y)(1 + t_1) / 2 $$

Or, 

$$ \bar{S}_{\mu} = (y - 1)M (1 - t_1^2) / 2y $$

(1.25)

But equation (A8) given in the Appendix of Tobin (1956) is as follows:

$$ \bar{S}_{\mu} = (y - 1)M (1 - t_1^2) / 2y \ This\ could\ be\ a\ typographical\ error. $$

When \( y = 8 \), the last of the equations in the set (1.24) is of the form: 

$$ t_7 - 2t_8 + 1 = 0 $$

Or, for any integer value of \( n \), the last equation is of the form: 

$$ t_{n-1} - 2t_n + 1 = 0 \quad (1.26) $$

In the set of equations (A3) given in the Appendix of Tobin (1956), the last equation is given as follows: 

$$ t_{n-1} - 2t_n = 1. \ This\ equation\ is\ different\ from\ (1.26)\ in\ the\ sense\ that\ number\ 1\ appears $$

on the right-hand-side of the equation. \ This\ also\ could\ be\ a\ typographical\ error. \)

There is an extensive discussion on the impact of the fixed charge \( F \) on the value of the number of transactions \( y \) in Tobin (1956). However, no specific method is presented for finding the optimal solution yielding integer values for \( y^* \).

As before, if \( E \) is the net earnings, from (1.20) we have:

$$ E = (y - 1)(1 - t_1)^2 Mr / 2y - Fy $$
Obviously, if $F$ is relatively large, there would be no investment at all in bonds. Further, we must have at least one investment and one liquidation to generate any net earnings. This implies that we must always consider the values of $y \geq 2$.

To generate net positive earnings for any integer value of $y \geq 2$, we must have:

$$\text{Or, } F < (y-1)(1-t_1)^2 Mr / 2y^2$$

(1.27)

At $y = 2$, we must have: $F < (1-t_1)^2 Mr / 8$

As before, let $M=100000$, $r=0.08$, $F=68$, $V=0.0075$.

Then, $t_1 = 2V/r = 2*0.0075/0.08=0.1875$, and $(1-t_1) = 0.8125$.

By substituting the given values, we see that $F < 660.156$.

In other words, there is no benefit in investing any funds in bonds, if $F \geq 660.156$.

Of course, if $F < 660.156$, we may have $y^* \geq 2$.

Then, what is the value of $F$ for which it will be an advantage to switch from $y$ to $(y+1)$?

Again, to have one more transaction that will generate extra earnings, we get from (1.20):

$$((y+1)-1)(1-t_1)^2 Mr / 2(y+1) - F(y+1) \geq (y-1)(1-t_1)^2 Mr / 2y - Fy$$

$$\text{Or, } F \leq (1-t_1)^2 Mr / (2y(y+1))$$

(1.28)

From (1.28), we see that if currently $y = 2$, it would be an advantage to switch from 2 transactions to 3, only if $F \leq (1-t_1)^2 Mr / 12$, or, if $F \leq 440.10$ for the illustrative example given here.
In a similar fashion we can calculate the other values for $F$ that would warrant a switch from $y$ to $(y+1)$ transactions. The details are given below in Table 1.1 and Figure 1.6.

### Table 1.1: Brokerage Fees vs. Number of Transactions

<table>
<thead>
<tr>
<th>From Current $y$</th>
<th>To $y+1$</th>
<th>Switch at $F$ value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>660.16</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>440.10</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>220.05</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>132.03</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>88.02</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>62.87</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>47.15</td>
</tr>
<tr>
<td>8</td>
<td>9</td>
<td>36.68</td>
</tr>
<tr>
<td>9</td>
<td>10</td>
<td>29.34</td>
</tr>
</tbody>
</table>

Figure 1.6. Functional Relation between $F$ and $y$. 
As one should expect, the lower the brokerage fee for a transaction, the higher will be the number of transactions. The method used in Tobin (1956) to prove that the time intervals between two consecutive installment payments should be equal to one another is somewhat unwieldy. The theory of convex functions might not have been available that time. Based on theory of convex functions, a simpler proof can now be presented to establish the same result.

In summary, Baumol (1952) and Tobin (1956) described cash management methods in their foundational papers from two different perspectives, but in the end, they lead to the same results. In this chapter an algorithmic approach to find optimal integer solutions for the number of transactions in both their models is presented in a unified manner.
CHAPTER 2

ENHANCEMENTS TO THE BAUMOL VERSION OF THE BAT MODEL
OF CASH MANAGEMENT WITH COST MINIMIZATION OBJECTIVE

2.1 Introduction

The Baumol-Allais-Tobin model of cash management, commonly called the BAT model, is often the starting point in the discussion of cash management models in classrooms and professional training seminars. It provides a simple square-root-formula for setting a target cash balance to meet the payment obligations of a firm. Mathematically, the BAT model of cash management is identical to the Economic Order Quantity (EOQ) model which is used extensively in the management of physical inventories. In both models, it is assumed that the parameters such as the transaction (order-processing) costs, interest rates (inventory-holding costs) and demand rate (for cash or inventory) are constant over the entire planning horizon. Both these models are insightful and elegant in theory, and also easy to understand. However, in practice the results from the square-root-formula lead to fractional number of transactions and also fractional number of days (or weeks) in the cycle-time between two consecutive transactions. Therefore, the results are not useful from an implementation point of view. Since thousands of business students are exposed to both the BAT model and the EOQ model every year, it would be worthwhile to seek enhancements to these models so as to generate meaningful and implementable results. The main goal in this chapter is to present mathematical techniques that will generate integer solutions both for the number of transactions and the number of time periods between two consecutive transactions.
The transaction cost involved in liquidating and converting short-term securities into cash in the BAT model is called the set-up cost for a production run in the EOQ model (Johnson and Montgomery, 1974). The foregone interest on the liquidated securities is akin to the holding cost on the inventories, since inventories are similar to idle cash on hand. The EOQ formula (Stevenson, 2018) yields fractional values for the number of batch production runs to be scheduled and also the number of days between two consecutive production runs just as the BAT model whose results also lead to fractional number of transactions and fractional number of days between two consecutive transactions. Mathematically, the results of the square-root-formula will be accurate only if the planning horizon were long enough to yield integer values for both the number of transactions and the number of days (or weeks) in the cycle-time between two consecutive transactions. This leads to the critical assumption underlying the EOQ model that the planning horizon is infinite (Hax and Candea, 1984). Obviously, the same assumption implicitly holds for the BAT model as well. But firms do need to make plans for time horizons shorter than a year and accordingly, need to have cash on hand to meet their payment obligations. In this respect, the BAT model is different from the EOQ model and clearly, there is a need to develop techniques that would help us solve the BAT model with integer requirements for the number of transactions and the number of days between two consecutive transactions as well. The objective is to minimize the total costs associated with the lost interest income and replenishment of cash through periodic liquidation of securities for meeting the transactions demand for cash.

### 2.2 Illustrative Examples

We will look at some numerical examples before delving into the mathematical techniques.
2.2.1 An Example from a Textbook with Enhancements

We will start with the numerical example discussed in one of the standard textbooks, *Fundamentals of Corporate Finance*, 12th Edition, by Ross, Westerfield and Jordan (2019). (This textbook is adopted for the introductory finance course FIN 323 at ODU). The following problem is extracted from Chapter 19, Cash and Liquidity Management, The BAT Model: Example 19A.1 (from page 667 of the book).

“The Vulcan Corporation has cash outflows of $100 per day, seven days a week. The interest rate is 5 percent, and the fixed cost of replenishing cash balances is $10 per transaction. What is the optimal cash balance? What is the total cost?”

It is not explicitly stated in the problem but it is assumed that the time period under consideration is one year. The total cash needed for the year is $36,500. (365*100=36500).

The parameters and variables relevant to the problem are defined below.

Unit of time: one day.

Time horizon under consideration, $T = 365$ days.

Cash outflows, $D = $100 per day (Daily demand for cash)

Cost per transaction, $F = $10 (brokerage fee for liquidating short-term securities)

Annual interest rate = 5%
Daily interest rate, \( r = \frac{0.05}{365} = 0.000137 \)

The pattern of cash outflows or disbursements according to the assumptions in the Baumol (1952) version of the BAT model is displayed in Figure 2.1 below. Let \( t \) denote the \textit{cycle-time} or time interval between two consecutive transactions.

![Figure 2.1. Cash Disbursement Pattern under the Assumptions of the BAT Model](image)

Mathematical Notation:

- \( C \) = Cash balance on hand at the beginning of each cycle or immediately after a transaction.
- \( t \) = cycle-time or time interval between two consecutive transactions in days.
- \( y \) = number of transactions or cycles

If \( C \) is the amount liquidated in every transaction, the average cost per day is given by:
\[ \xi_i(C) = (D \div C)F + rC \div 2 \] 

(2.1)

Using the square-root-formula of the BAT model, we get:

Optimal starting cash balance, \[ C^* = \sqrt{2DF \div r} = \sqrt{2 \times 10 \times 100 \div 0.000137} = 3821 \] (rounded to dollars)

Cycle time between two consecutive transactions \[ t^* = C^* \div D = 3821 \div 100 = 38.21 \] days.

Number of transactions (or cycles), \[ y = \frac{T}{t^*} = \frac{365}{38.21} = 9.552 \]

Total cost for any value of \( C \) for the time horizon of \( T=365 \) days is given by the following equation:

\[ \xi_2(C) = T((D \div C)F + rC \div 2) = 365*((100/3821)*10 + 0.000137*3821/2) = \$191.05 \]

Immediately, there is one issue with the recommended solution. We cannot have 9.552 transactions with 38.21 days between two consecutive transactions in a year. We may choose to have either 9 or 10 transactions. Then, the time interval or the cycle time between two consecutive transactions will have to be either 40.56 days or 36.50 days. However, a solution that can be easily implemented in practice is as follows; let the number of transactions, \( y \) be set to 10; then there will be 5 transactions with cycle-time of 37 days and 5 more transactions with cycle-time of 36 days. The total cost of the enhanced or modified solution is \$191.27 compared to the original cost of \$191.05. Instead, if one chooses to have 9 transactions, there will be 5 transactions with cycle-times of 41 days and 4 more transactions with cycle-times of 40 days. The resulting annual cost for this alternative will be \$191.40.
The interesting fact to be noted is that the original cycle-time of 38.21 days (or the corresponding rounded-off value of 38 or 39) does not appear in either of the two alternative solutions.

The enhanced solutions have integer values for both the number of transactions and the number of days in the cycle-times between two consecutive transactions. In this particular instance, simple rounding-off may yield an acceptable or implementable solution, but it need not be the case in all situations, especially when large sums of money are involved. Also, the planning horizon need not be one year. Further, the fixed cost per transaction may also be high in some cases such as those of the construction companies which have to submit audited progress reports to the financial institutions for the release of funds before commencing the next cycle of operations.

2.2.2 An Example with a Typical Convex Cost Curve

Suppose that the Everest Construction Company has several construction projects in progress. The following data is available.

Unit of time: one day.

Time horizon under consideration, \( T = 215 \) days. (All projects have to be completed in 215 days.)

Cash outflows, \( D = $10,000 \) per day (Daily demand for cash)

Cost per transaction, \( F = $127 \) (brokerage fee for liquidating short-term securities)

Annual interest rate = 6%

Daily interest rate, \( r = \frac{0.06}{365} = 0.00016438 \)
From the mathematical notation given in the previous example we have the following result.

If $\xi_2(C)$ is the total cost for any value of $C$ for the time horizon of $T = 215$ days, we get:

$$\xi_2(C) = (DT / C)F + rTC / 2$$  \hspace{1cm} (2.2)

The relationship between the total cost for the time horizon of 215 days and the starting cash balance is shown in Figure 2.2.

![Figure 2.2 Relationship Between the Total Cost and Starting Cash Balance](image)

**Definition of a Convex Function:** The function $\xi_2(C)$ is called a convex function, since it is not only continuous and differentiable at every value of $C$, but also, it has a unique global minimum (Winston, 2004, and Bazaraa, et.al. 2013). It implies that its first derivative is zero at just one value of $C$ in the interval for which $\xi_2(C)$ is defined and its minimum occurs at just one point.
\[
\frac{d\xi_2(C)}{dC} = -FDT / C^2 + rT / 2 = 0
\]

Or, \[ C^* = \sqrt{2FD / r} = \sqrt{2 * 127 * 10000 / 0.00016438} = $124304.70 \]

Then, \[ \xi_2(C^*) = \frac{10000 * 215 * 127}{124304.70} + \frac{0.00016438 * 215 * 124304.70}{2} = 2196.62 + 2196.62 = 4393.24 \]

While the starting cash balance \( C^* \) is our primary focus, from a practical implementation point of view, we are interested in determining integer solutions for \( t \), the cycle-time or time interval (in days) between two consecutive transactions and \( y \), the number of cycles or transactions.

We are already given the values of the parameters \( T, D, F, \) and \( r \). Let \( \lambda = Dr \).

If \( t \) is the cycle time in days, the cost for any cycle can be defined as follows:

\[
\psi(t) = F + (Dt)(rt) / 2 = F + Drt^2 / 2 = F + \lambda t^2 / 2 \quad (2.3)
\]

Then, the average cost per day (or unit time) is given by:

\[
\mu(t) = \psi(t) / t = F / t + \lambda t / 2 \quad (2.4)
\]

**Proposition 2.1:** The functions \( \xi_1(C) \), \( \xi_2(C) \), \( \psi(t) \), and \( \mu(t) \) as defined in equations (2.1), (2.2), (2.3), and (2.4) respectively are convex functions. Basic properties of the convex functions and the related proofs are described in Appendix A.
The relationship between the average cost per day in a cycle and the length of the cycle-time is displayed in Figure 2.3.

![Figure 2.3. Relationship Between Average Cost per Day and Cycle Time in Days](image)

From Figure 2.3, it is clear that $\mu(t)$ is a convex function for all values of $t$, $0 < t \leq T$.

\[ \frac{d\mu(t)}{dt} = -\frac{F}{t^2} + \frac{\lambda}{2} = 0. \]  

(2.5)

Or, $t^* = \sqrt{\frac{2F}{\lambda}} = \sqrt{\frac{2F}{(Dr)}} = 12.43047$, and further, $C^* = Dt^* = 10000 \times t^* = 124304.70$

Since $t^*$ is not an integer, it may not be a feasible option. Now we will briefly digress and consider a special case in which the cycle time is not required to be an integer, and the length of the
planning horizon, denoted by $H$, may assume any positive (not necessarily integer) value. It implies that under some circumstances, the firm can adopt a fractional value for the cycle time $t^*$, provided the number of transactions $y$ is an integer. Then, the following two propositions will help us find the optimal cash balance.

We define the ceiling and floor functions used in the propositions as follows:

$$[x] = \text{The ceiling function that yields the smallest integer greater than or equal to } x$$
$$\lfloor x \rfloor = \text{The floor function that yields the largest integer smaller than or equal to } x$$

In the derivation of Baumol’s square-root-formula for estimating the optimal cash balance $C^*$, the cycle-times or the time intervals between any two consecutive liquidations are assumed to be equal to one another. It also implies that the amounts of the installment payments are equal to one another. The pattern of liquidations depicted in Figure 2.1 is also based on this assumption, though no specific proof is provided by Baumol (1952) about the equality of the amounts of the disbursements or the corresponding time intervals. Tobin (1956) not only pointed out this fact but also presented such a proof in the Appendix of his paper. The method used in Tobin (1956) to prove the equality of the cycle-times is somewhat unwieldy. A simpler proof, based on the theory of convex functions, is presented in Appendix A to establish the same result which is stated in Proposition 2.2 given below.
Let $Z(y)$ denote the total cost involving $y$ transactions for the planning horizon of $T$ time periods. If the cycle-times or time intervals between two consecutive transactions are different from one another, let $t_j, j=1,2,\ldots,y,$ denote each of the cycle-times. Obviously, $t_1 + t_2 + \cdots + t_y = T$. Further, if we let $\lambda = D r$, from equation (2.3), we get:

$$Z(y) = \sum_{j=1}^{y} \psi(t_j) = \sum_{j=1}^{y} (F + \lambda t_j^2 / 2) = F y + (\lambda / 2) \sum_{j=1}^{y} t_j^2$$

(2.6)

**Proposition 2.2:** For any fixed integer value of $y$, the total cost function $Z(y)$ is minimized at $t_j^* = T / y, j=1,2,\ldots,y$, and the minimum value of the cost function $Z(y) = F y + \lambda T^2 / 2y$. The proof is given in Appendix A.

We will now state another proposition that addresses one of the questions related to the optimal value of the number of transactions.

**Proposition 2.3:** Suppose that in a special case of the BAT model, only the number of transactions is required to be an integer, and the planning horizon $H$ and the cycle time $t$ are allowed to assume any (positive) continuous values. Define the functions: $\psi(t) = F + \lambda t^2 / 2; \mu(t) = F / t + \lambda t / 2; \mu(t)$ is minimized at $t = t^*$; and, $R(y) = F y + (\lambda / 2)(H^2 / y)$; further, let $y^* = H / t^*$. If $y^*$ turns out to be an integer, then, let $\pi = y^*$. Otherwise, if $R([y^*]) \leq R([y^*])$, let $\pi = [y^*]$; or else, $\pi = [y^*]$. Then, $\pi$ is the optimal number of transactions, and for the entire planning horizon, the minimum cost $R^\pi(\pi) = F \pi + (\lambda / 2)(H^3 / \pi)$. The proof is given in Appendix A.
For the problem at hand, we have: \( H = T = 215 \). Since \( t^* = 12.43 \), \( y^* = \frac{H}{t^*} = \frac{215}{12.43} = 17.297 \).

Clearly, \( y^* \) is not an integer. Then, we find the values of \( R(\lfloor y^* \rfloor) \), and \( R(\lceil y^* \rceil) \).

\[
R(\lfloor y^* \rfloor) = 127 \times 17 + (1.6438/2) (215 \times 215/17) = 4393.89 \\
R(\lceil y^* \rceil) = 127 \times 18 + (1.6438/2) (215 \times 215/18) = 4396.73
\]

Since \( R(\lfloor y^* \rfloor) \leq R(\lceil y^* \rceil) \), we let \( \pi = \lfloor y^* \rfloor = 17 \). Then, the cycle time, \( \tau = \frac{215}{17} = 12.647 \).

Target cash balance, \( C^* = \tau D = 12.647 \times 10000 = 126,470.00 \); \( \pi = 17 \), and \( R^*(\pi) = 4393.89 \)

Our primary goal is to have integer solutions for both the number of transactions and the cycle time. If we round up \( t^* \), we get 13 days for the cycle time and rounding it down yields a cycle time of 12 days. Then, the number of cycles, \( y = \frac{T}{13} = \frac{215}{13} = 16.54 \) or \( y = \frac{T}{12} = \frac{215}{12} = 17.92 \). The number of cycles \( y \) also turns out to be a non-integer in both cases and therefore, neither one meets our requirement. Of course, our requirement is only that all cycles must have integer lengths; it is clear that it is not possible to have all cycles of equal lengths, with integer values. In other words, we need to find a way to partition the planning horizon \( T \) which is an integer into an integer number of \( y \) cycles with integer lengths, but not necessarily equal to one another. The following two propositions help us accomplish this task.

**Proposition 2.4**: Define the set, \( Y = \{1, 2, \ldots, T\} \). In the BAT model, suppose that the number of cycles is fixed at some \( y \in Y \) and further, the cycle length is restricted to only integer values. Then, in the corresponding optimal solution, no two cycles can differ by more than one period in length.
We will call the cycles that differ in length by just one unit of time *near-equal-length* cycles. The proof is given in Appendix A.

**Proposition 2.5:** Suppose that in the BAT model, $Z(y)$ denotes the minimum cost for the time horizon of integer length $T$, when the number of cycles is fixed at some $y \in Y$ and the cycle length is restricted to only integer values. Define $U = \lceil T / y \rceil$, $L = U - 1$, $m = T - Ly$, and $n = y - m = Uy - T$. Then, $Z(y) = m\psi(U) + n\psi(L)$. Or, $Z(y) = Fy + (\lambda / 2)(mU^2 + nL^2)$. The proof is given in the Appendix A (Papachristos and Ganas, 1998, and Ganas and Papachristos, 2005).

The relationship between the total cost $Z(y)$ and $y$, the number of transactions is shown in Figure 2.4.

![Total Cost vs. Number of Transactions](image_url)

*Figure 2.4. Relationship Between the Total Cost and the Number of Transactions*
Since $y$ is an integer, the function $Z(y)$ is not continuous and therefore not differentiable. Nonetheless, it is a *piecewise linear convex function* and has a global minimum at some integer value of $y$. Define $y^* = T / t^*$, where $y^*$ is not necessarily an integer. To find the integer value of $y$ that minimizes $Z(y)$, we need to search in the neighborhood of $y^*$.

Continuing with the illustrative example 2 we solved earlier in this Section, we have: $T = 215$.

Since $t^* = 12.43$, $y^* = T / t^* = 215 / 12.43 = 17.297$. Clearly, $y^*$ is not an integer. Since $Z(y)$ is a piecewise linear convex function, the minimum of $Z(y)$ occurs in the neighborhood of $y^* = 17.297$.

We search for the minimum of $Z(y)$ at $y_1 = \lceil y \rceil = 17$ or $y_2 = \lfloor y \rfloor = 18$.

**Case (1):** $y_1 = 17$. $U_1 = \lceil T / y_1 \rceil = 13$, $L_1 = U_1 - 1 = 12$, $m_1 = T - L_1 y_1 = 215 - 12 \times 17 = 11$, $n_1 = y_1 - m_1 = 6$.

$$Z(y_1) = F y_1 + (\lambda / 2)(m_1 U_1^2 + n_1 L_1^2) = 127 \times 17 + (1.6438/2) \times (11 \times 13 + 6 \times 12 \times 12) = 4397.08$$

**Case (2):** $y_2 = 18$. $U_2 = \lceil T / y_2 \rceil = 12$, $L_2 = U_2 - 1 = 11$, $m_2 = T - L_2 y_2 = 215 - 11 \times 18 = 17$, $n_2 = y_2 - m_2 = 1$.

$$Z(y_2) = F y_2 + (\lambda / 2)(m_2 U_2^2 + n_2 L_2^2) = 127 \times 18 + (1.6438/2) \times (17 \times 12 + 1 \times 11 \times 11) = 4397.50$$

Since $Z(y_1) < Z(y_2)$, the optimal number of cycles or transactions, $\pi = 17$ and $Z(\pi) = 4397.08$.

Then, we have 11 cycles, each with a length of 13 days, 6 cycles, each with a length of 12 days.

Target cash balances: $C_v^* = U_1 D = 13 \times 10000 = 130,000$ and $C_l^* = L_1 D = 12 \times 10000 = 120,000$.

Searching for $\pi$, the optimal number of transactions may be very simple, or sometimes it may involve considerable effort, depending the values of the parameters.
2.2.3 An Example with Suitably Adjusted Data

Let us consider a problem for which we can find integer solutions easily.

Unit of time: one day.

Time horizon under consideration, \( T = 300 \) days.

Cash outflows, \( D = $1000 \) per day

Cost per transaction, \( F = $22.50 \) (brokerage fee for liquidating short-term securities)

Annual interest rate = 7.3%

Daily interest rate, \( r = \frac{0.073}{365} = 0.0002 \)

Then, \( Dr \lambda = 1000 \times 0.0002 = 0.2 \),
\[ \frac{2}{2} \times \frac{22.5}{0.2} = 15, \]
\[ \frac{T \lambda}{t} = \frac{300}{15} = 20. \]

Since \( t^* \) and \( y^* \) are both integers, we have the optimal solution.

\( \pi = 20, \) and \( Z(\pi) = \pi(F + (\lambda / 2)y^2) = 20(22.5 + (0.2/2) \times 15 \times 15) = 900.00 \)

2.2.4 An Example with a Flat-Bottom V-shaped Cost Curve

Let us consider a problem which requires considerable work to generate the complete set of integer solutions.

Unit of time: one day.

Time horizon under consideration, \( T = 180 \) days.

Cash outflows, \( D = $120,000 \) per day (Daily demand for cash)
Cost per transaction, \( F = \$288.00 \) (brokerage fee for liquidating short-term securities)

Annual interest rate = 5.84%

Daily interest rate, \( r = \frac{0.0584}{365} = 0.00016 \)

The relationship between the total cost \( Z(y) \) and the number of transactions \( y \) is given in Figure 2.5.

As mentioned earlier, the total cost \( Z(y) \) is a piecewise linear convex function of \( y \). From Figure 2.5, we see that there are several values of \( y \) that yield the minimum (because of the flat-bottom-V shape).

To find them all, we need to adopt a procedure different from the ones we discussed earlier.

As before, let \( \lambda = Dr = 120000 \times 0.00016 = 19.2 \)

\[
\begin{align*}
\hat{t} &= \sqrt{\frac{2F}{\lambda}} = \sqrt{\frac{2 \times 288}{19.2}} = 5.4772 \\
\hat{y} &= T / \hat{t} = 180 / 5.4772 = 32.863
\end{align*}
\]

Since both \( \hat{t} \) and \( \hat{y} \) are not integers, we need to do additional work to find the optimal solution.

![Total Cost vs. Number of Transactions](image)

**Figure 2.5. Flat-Bottom V-Shaped Piecewise Linear Convex Cost Curve**
If we let \( y_1 = \lceil y^* \rceil = 32 \), and \( y_2 = \lceil y^* \rceil = 33 \), we get two possible solutions, one of which could be the optimal solution.

But, the flat-bottom-V shape of the cost curve in Figure 2.5 indicates that there could be more than two solutions, optimal or otherwise. This implies that more work needs to be done to identify one or more of the optimal solutions.

Let \( \tau_1 = \lceil t^* \rceil = 6 \), and \( \tau_2 = \lceil t^* \rceil = 5 \).

Then, the average cost per period \( \mu(\tau_1) = F / \tau_1 + \lambda \tau_1 / 2 = 288/6 + 19.2*6/2 = 105.60 \)

Similarly, the average cost per period \( \mu(\tau_2) = F / \tau_2 + \lambda \tau_2 / 2 = 288/5 + 19.2*5/2 = 105.60 \)

We have two cycles of different lengths, but the average cost per period is the same in both cases. This points to the possibility of multiple optimal solutions. Let \( y_1 = T / \tau_1 = 180 / 6 = 30 \), and further, \( y_{l1} = T / \tau_2 = 180 / 5 = 36 \). Now calculate the total cost, \( Z(y) \) at \( y = 30, 31, 32, 33, 34, 35, 36 \). In each case, we find the corresponding values of \( U, L, m, \) and \( n \). Then, we find the value of \( Z(y) \).

\[
Z(y) = Fy + (\lambda / 2)(mU^2 + nL^2) \tag{2.7}
\]

\[
Z(30) = 288*30 + (19.2/2)*(30*6*6+0*5*5)= 19008
\]

\[
Z(31) = 288*31 + (19.2/2)*(25*6*6+6*5*5)= 19008
\]

\[
Z(32) = 288*32 + (19.2/2)*(20*6*6+12*5*5)= 19008
\]

\[
Z(33) = 288*33 + (19.2/2)*(15*6*6+18*5*5)= 19008
\]
\[ Z(34) = 288 \times 34 + \frac{19.2}{2} \times (10 \times 6 \times 6 + 24 \times 5 \times 5) = 19008 \]

\[ Z(35) = 288 \times 35 + \frac{19.2}{2} \times (5 \times 6 \times 6 + 30 \times 5 \times 5) = 19008 \]

\[ Z(36) = 288 \times 36 + \frac{19.2}{2} \times (36 \times 5 \times 5 + 0 \times 4 \times 4) = 19008 \]

Clearly, there are seven different optimal solutions, but each one has cycle lengths of 6 or 5 days. Initially, if we rounded up or down \( y^* \), we would have generated only two optimal solutions with \( \pi = 32 \), and \( \pi = 33 \). This example illustrates the need to explore the possible values of \( y \) corresponding to the cycle lengths of \( \tau_1 = \left\lceil t^* \right\rceil \) and \( \tau_2 = \left\lfloor t^* \right\rfloor \).

This example also illustrates the need to consider several possible scenarios in developing the algorithms for finding the optimal solutions. (One of the contributions of this dissertation is the development of the methodology for identifying the existence of multiple optimal solutions like the ones described here and the corresponding flat-bottom-V shaped graph)
CHAPTER 3

ENHANCEMENTS TO THE TOBIN VERSION OF THE BAT MODEL

OF CASH MANAGEMENT WITH PROFIT MAXIMIZATION OBJECTIVE

3.1 Introduction

There are two different perspectives regarding the objectives of the Baumol-Allais-Tobin (BAT) model of cash management. The most common objective is the minimization of total costs associated with the lost interest and the brokerage fees related to the liquidation of securities as described in Baumol (1952). Baumol’s approach provides a simple square-root-formula for determining the target cash balance to meet the payment obligations of a firm. In contrast, Tobin (1956) presented another optimization model in which the objective is to maximize the net earnings from investing a portion of the available funds. Though both models are mathematically equivalent, analysis of the models from the two different perspectives provides additional insights that could help corporate executives make better decisions about their cash management operations. In both models the time horizon under consideration is one year. The square-root-formula developed by Baumol (1952) leads to fractional number of transactions in a year. To resolve this issue, Tobin (1956) starts the analysis with the number of cash withdrawals $y$, and restricts $y$ to be an integer-valued variable. Regarding the availability of funds, two scenarios, one in which the funds are already invested in a brokerage account and the other in which funds become available for investment and disbursement at the beginning of the year, are studied in Baumol (1952). Only Baumol’s second scenario in which funds become available for investment and disbursement at the beginning of the year is considered in Tobin (1956). In this Chapter we focus on Tobin’s model whose objective is to maximize the
net revenue from investment after accounting for the lost interest on the cash balance held on hand and the transaction costs related to the investment and liquidation of funds.

3.2 Equality of Time Intervals between Two Consecutive Withdrawals

In the derivation of the square-root-formula that helps us determine the target cash balance, Baumol (1952) assumed that the cycle-times or time intervals between any two consecutive withdrawals should be the same. In other words, it is assumed that the liquidation of securities at unequal time intervals would not be optimal. Of course, intuitively everyone would come to the same conclusion. However, no specific proof is provided by Baumol (1952) about the optimality of making cash withdrawals at equal time intervals. Tobin (1956) not only pointed out this fact, but also provided a proof. Here we will present an alternative proof based on theory of convex functions that is somewhat simpler than the proof given in Tobin (1956).

We will start with an example: Suppose the planning horizon, $T = 120$ days. The daily demand for cash, $D = 1000$. Further, the total amount of $M = DT = 120,000$ needed to meet the transactions demand for the entire horizon of $T$ days is available at the beginning itself. Obviously, it would be advantageous to invest a part of the available funds in interest-bearing securities. Without loss of generality, as in Tobin’s model, we can assume that the brokerage fee for the initial investment transaction and for each subsequent liquidation is $F$, regardless of the amount invested or withdrawn. We may also assume that the proportional charge for every dollar invested or withdrawn $V = 0$. At this juncture, it necessary to determine the number of liquidation transactions. Suppose that at time $t = 0$, we decide to keep $25,000$ on hand and invest the remainder of
$95,000 in short-term securities yielding a daily interest rate of $r$. Clearly, our first withdrawal will have to take place at the end of the day when $t = 25$, since by that time the initial amount of cash held on hand is reduced to zero. Arbitrarily assume that we decide to have three other liquidations at the end of the following time intervals, measured in days from each of the previous withdrawal: $t_2 = 13, t_3 = 33, t_4 = 18$. The pattern of investment and withdrawal is displayed in Figure 3.1. The shaded area in Figure 3.1 shows the amount invested during different time intervals.

![Figure 3.1 Pattern of Withdrawals at Time Intervals of Different Lengths](image)

The amount of cash on hand at the beginning of each of the time intervals and also the amount invested during each of the time intervals and related details are given in Table 3.1A. If we let $C(t), S(t)$, and $M(t)$ denote the amount of cash on hand, the amount of money invested in interest-bearing securities and the total amount of the cash on hand and the funds invested at any time $t$,
0 \leq t \leq T$, the relationship between the three variables can be represented by the following equation (Tobin, 1956).

\[ M(t) = S(t) + C(t), \quad 0 \leq t \leq T. \]

Further, if \( \bar{M}, \bar{S}, \) and \( \bar{C} \) represent the corresponding average values per day of the planning horizon, we get: \( \bar{M} = \bar{S} + \bar{C}. \)

Or, \( \bar{S} = \bar{M} - \bar{C}. \) (3.1)

Table 3.1A: Investment and Withdrawal Pattern A

<table>
<thead>
<tr>
<th>Time Interval Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ending time of Interval</td>
<td>25</td>
<td>38</td>
<td>71</td>
<td>89</td>
<td>120</td>
</tr>
<tr>
<td>Length of Interval</td>
<td>( t_1 = 25 )</td>
<td>( t_2 = 13 )</td>
<td>( t_3 = 33 )</td>
<td>( t_4 = 18 )</td>
<td>( t_5 = 31 )</td>
</tr>
<tr>
<td>Cash on Hand at the Beginning of the Interval</td>
<td>25000</td>
<td>13000</td>
<td>33000</td>
<td>18000</td>
<td>31000</td>
</tr>
<tr>
<td>Average amount of cash per day during the Interval</td>
<td>12500</td>
<td>6500</td>
<td>16500</td>
<td>9000</td>
<td>15500</td>
</tr>
<tr>
<td>Amount Invested during the Interval</td>
<td>95000</td>
<td>82000</td>
<td>49000</td>
<td>31000</td>
<td>0</td>
</tr>
</tbody>
</table>

Since the lengths of the time intervals or cycle-times are different from one another, to determine the average amount of cash on hand per day of the planning horizon \( \bar{C} \), we need calculate the weighted average as given below.

\[ \bar{C} = \frac{(12500 \times 25 + 6500 \times 13 + 16500 \times 33 + 9000 \times 18 + 15500 \times 31)}{120} = \frac{1584000}{120} = 13200 \]
Since we start with $M = DT = 120000$, and end with zero, the average of the total funds is given by:

$$\bar{M} = \frac{(120000 + 0)}{2} = \frac{DT}{2} = 60000.$$ Therefore, $\bar{S} = \bar{M} - \bar{C} = 60000 - 13200 = 46800$.

Now, suppose we keep the lengths of the time intervals between two withdrawals the same as in Table 3.1A, but change the sequence as given in Table 3.1B below.

<table>
<thead>
<tr>
<th>Table 3.1B: Investment and Withdrawal Pattern B</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Time Interval Number</strong></td>
</tr>
<tr>
<td>Ending time of Interval</td>
</tr>
<tr>
<td>Length of Interval</td>
</tr>
<tr>
<td>Cash on Hand at the Beginning of the Interval</td>
</tr>
<tr>
<td>Average amount of cash per day during the Interval</td>
</tr>
<tr>
<td>Amount Invested during the Interval</td>
</tr>
</tbody>
</table>

As before, we calculate the average amount of cash on hand and the average amount of funds invested in securities:

$$\bar{C} = \frac{(9000*18 + 16500*33 + 6500*13 + 15500*31 + 12500*25)}{120} = \frac{1584000}{120} = 13200$$

Therefore, $\bar{S} = \bar{M} - \bar{C} = 60000 - 13200 = 46800$.

From Tables 3.1A and 3.1B, it is clear that as long as the lengths of time intervals are kept constant, the average amount of cash on hand $\bar{C}$, and the average amount invested $\bar{S}$ do not change regardless of the sequence in which the withdrawals are scheduled. In fact, with 5 different time intervals as
given in Table 3.1A or 3.1B, we have 120 possible sequences to withdraw the funds. Nonetheless, the values of $\bar{C}$ and $\bar{S}$ remain the same in all cases.

**Proposition 3.1:** Suppose the demand for cash is $D$ per time period (or per unit of time such as a day or week) and we have a planning horizon of $T$ time periods (days or weeks), where $T$ can assume only positive integer values. Further, $T$ is divided into $y$ intervals, where $y$ is any positive integer, and $t_1 + t_2 + \cdots + t_y = T$, with $t_j, j = 1, 2, \ldots, y$, denoting any one of the time intervals of positive integer length. Then, if the lengths (durations) of time intervals $t_j, j = 1, 2, \ldots, y$, are held constant, the average values of $\bar{C}$ and $\bar{S}$ remain the same regardless of the sequence in which the withdrawals are made after the re-arrangement of the time intervals in each of the sequences.

**Proof:** Consider a time interval $t_j$ in any sequence (or permutation of the integers $1, 2, \ldots, y$). The amount of cash on hand at the beginning of the time interval would be $Dt_j$ and at the end, the cash balance will reach zero, with the average equal to $(Dt_j / 2)$ during the time interval. Since the time intervals could be different from each other, the (weighted) average value of $\bar{C}$ is given by:

$$\bar{C} = \frac{(Dt_1 / 2)t_1 + (Dt_2 / 2)t_2 + \cdots + (Dt_y / 2)t_y)}{T}$$

Or, $\bar{C} = (D / 2T) \sum_{j=1}^{j=y} t_j^2$

Or, $\sum_{j=1}^{j=y} t_j^2 = 2\bar{C}T / D$ \hspace{1cm} (3.2)
Since the summation involves all time intervals from \( j = 1 \) to \( j = y \), the result will be same regardless of the sequence in which they are added.

Let \( F \) be the fixed brokerage fee for a withdrawal done at the beginning of \( t_j, j = 2, 3, \ldots, y \). In Tobin’s model, there is a brokerage fee \( F \) for the investment at time \( t = 0 \). There is no withdrawal fee at time \( t = 0 \), since we keep enough cash needed for disbursement during the first time-interval. Let \( r \) be the interest rate per time period. Since the average balance during the time interval \( t_j \) is equal to \( Dt_j / 2 \), the interest income lost during this time interval will be \( (rt_j)Dt_j / 2 \).

Further, the total cost for any one of the time intervals \( t_j, j = 2, 3, \ldots, y \), is given by:

\[
\psi(t_j) = F + \left( \frac{Dr t_j^2}{2} \right)
\]  

(3.3)

If we include the brokerage fee for the initial investment, total cost for any given value of \( y \) for the entire planning horizon is given by:

\[
Z(y) = F + \left( \frac{Dr t_1^2}{2} \right) + \sum_{j=2}^{j=y} \psi(t_j) = F + \left( \frac{Dr}{2} \right) \sum_{j=1}^{j=y} t_j^2 = \sum_{j=1}^{j=y} \psi(t_j) = Fy + \bar{C}Tr
\]  

(3.4)

For any fixed integer value of \( y \), we see from (3.4) that the value of \( Z(y) \) depends on \( \bar{C} \), the average level of cash on hand. Our goal is to minimize \( Z(y) \), which in turn implies that we need to minimize \( \bar{C} \), the average level of cash on hand. We have already encountered this problem in Chapter 2 and addressed it through Propositions 2.2. However, for the sake of continuity in this Chapter, we will re-state Proposition 2.2 as Proposition 3.2. Let \( \lambda = Dr \).
**Proposition 3.2:** For any fixed integer value of $y$, the total cost function $Z(y)$ is minimized at $t_j^* = T / y$, $j = 1, 2, \ldots, y$, and the minimum value of the cost function $Z(y) = Fy + \lambda T^2 / 2y$.

**Proof:** Mathematically, we can state the optimization problem as given below:

Minimize $Z(y) = \sum_{j=1}^{y} \psi(t_j)$

Subject to: $t_1 + t_2 + \cdots + t_y = T$  \hspace{1cm} (3.5)

The proof is given in Appendix A. Since $\psi(t_j)$ and $Z(y)$ are convex functions, to minimize $Z(y)$, we must have: $t_1^* = t_2^* = \cdots = t_y^* = T / y$. In effect, from Proposition 3.2 (or Proposition 2.2) we see that the cycle-times or time intervals between any two consecutive withdrawals should be equal to one another in the optimal solution.

We will now state another proposition that addresses one of the questions related to the optimal value of the number of transactions.

**Proposition 3.3:** Suppose that in a special case of the BAT model, only the number of transactions is required to be an integer, and the planning horizon $H$ and the cycle time $t$ are allowed to assume any (positive) continuous values. Define the functions: $\psi(t) = F + \lambda t^2 / 2$; $\mu(t) = F / t + \lambda t / 2$; $\mu(t)$ is minimized at $t^* = \sqrt{2F / \lambda}$; further, let $y^* = H / t^* = H(\sqrt{\lambda / 2F})$, $R(y) = Fy + (\lambda / 2)(H^2 / y)$. If $y^*$ turns out to be an integer, then, let $\pi = y^*$. Otherwise, if $R([y^*]) \leq R([y^*])$, let $\pi = [y^*]$; or
else, \( \pi = \lceil y^* \rceil \). Then, \( \pi \) is the optimal number of transactions, and for the entire planning horizon, the minimum cost \( R^*(\pi) = F \pi + (\lambda / 2)(H^2 / \pi) \). The proof is given in Appendix A.

### 3.3 Partitioning the Planning Horizon into Cycles of Near-equal-length

Our goal is to have not only an integer number of transactions but also an integer number of time periods in every cycle of the optimal solution. Depending on the values of \( T \) and \( y \), the cycle times \( t_j^* = T / y, j = 1, 2, \ldots, y \), may have integer values in a few instances, but most often the solution could lead to fractional values. Our primary requirement is that the cycle times should be integers, but not necessarily equal, in the optimal solution. If the cycle times differ by more than one unit of time, they cannot be in the optimal solution as stated below. We need to find a mechanism to partition the time horizon \( T \) into intervals of integer lengths that will be also be a pathway to identifying the optimal solution. The following two propositions help us accomplish this task.

**Proposition 3.4**: Define the set, \( Y = \{1, 2, \ldots, T\} \). In the BAT model, suppose that the number of cycles is fixed at some \( y \in Y \) and further, the cycle length is restricted to only integer values. Then, in the corresponding optimal solution, no two cycles can differ by more than one period in length. We will call the cycles that differ in length by just one unit of time *near-equal-length* cycles. The proof is given in Appendix A (Chand, 1982).
Proposition 3.5: Suppose that in the BAT model, \( Z(y) \) denotes the minimum cost for the time horizon of integer length \( T \), when the number of cycles is fixed at some \( y \in Y \) and the cycle length is restricted to only integer values. Define \( U = \lceil T / y \rceil \), \( L = U - 1 \), \( m = T - Ly \), and \( n = y - m = Uy - T \). Then, \( Z(y) = m\psi(U) + m\psi(L) \). Or, \( Z(y) = Fy + (\lambda / 2)(mU^2 + nL^2) \). The proof is given in the Appendix A (Papachristos and Ganas, 1998, and Ganas and Papachristos, 2005).

The partitioning of the planning horizon of \( T \) periods leads to the following results:

\[
m + n = y, \tag{3.6}
\]

\[
mU + nL = T. \tag{3.7}
\]

\[
Z(y) = Fy + (Dr / 2)(mU^2 + nL^2) = Fy + (\lambda / 2)(mU^2 + nL^2) \tag{3.8}
\]

Since funds are available at the beginning of the year in Tobin’s model, any investment decision should necessarily take this fact into consideration in order to maximize the net revenue.

It may be noted that Propositions 3.2 through 3.5 are merely re-statements of Propositions 2.2 through 2.5 given in Chapter 2. They are given here to serve as ready references in deriving the results related to the optimality of the solutions.
**Proposition 3.6:** Suppose that in the BAT model, \( \zeta(y) \) denotes the maximum profit for the time horizon of integer length \( T \), when the number of cycles is fixed at some \( y \in Y \) and the cycle length is restricted to only integer values. Define \( U = \lceil T / y \rceil \), \( L = U - 1 \), \( m = T - Ly \), and \( n = y - m = Uy - T \).

Then, \( \zeta(y) = (DT)(Tr) / 2 - m\psi(U) - n\psi(L) \).

Or, \( \zeta(y) = (DT)(Tr) / 2 - Fy - (Dr / 2)(mU^2 + nL^2) \) \hspace{1cm} (3.9)

Or, \( \zeta(y) = (DT)(Tr) / 2 - Z(y) \) \hspace{1cm} (3.10)

**Proof:** From equation (3.1), we have \( \bar{S} = \bar{M} - \bar{C} \).

Since \( \bar{S} \) is the average amount invested per time period, the interest income from it for the entire planning horizon of \( T \) time periods will be equal to \( \bar{S}Tr \). Further, we need to incur the total brokerage fees of \( Fy \) for the initial investment and the subsequent \( (y-1) \) withdrawal transactions. No other costs are involved in making the disbursements. Therefore, \( \zeta(y) = \bar{S}Tr - Fy \). From (3.1) we get:

\[
\bar{S}Tr - Fy = (\bar{M} - \bar{C})Tr - Fy = \bar{M}Tr - \bar{C}Tr - Fy
\]

Or, from (3.4) we get: \( \zeta(y) = \bar{M}Tr - \bar{C}Tr - Fy = (DT / 2)Tr - Z(y) \)
Or, \( \zeta(y) = (DT)(Tr) / 2 - Z(y) \) \hfill (3.10)

Here, equation (3.10) is derived in a somewhat indirect way. We can arrive at the same equation for determining the maximum possible interest income in a direct way by examining the partitioning of the planning horizon itself into \( m \) intervals of length \( U \) and \( n \) intervals of length \( L \).

**Proposition 3.7:** Suppose the planning horizon \( T \) is partitioned into a fixed number of \( y \) cycles such that we have \( m \) cycle of length \( U \) and \( n \) cycles of length \( L \), where \( m + n = y \), and \( mU + nL = T \). Further, all the \( m \) cycles each with a length of \( U \) appear consecutively in the partition from time \( t = 0 \) to time \( t = mU \), followed by the \( n \) cycles of length \( L \). Then, the value of net earnings \( \eta(y) \) is given by the following equation:

\[
\eta(y) = TDmU - Dr(m + 1)mU^2 / 2 + Drn(n - 1)L^2 / 2 - Fy
\]

**Proof:** The planning horizon \( T \) is divided into \( y \) time intervals such that each of the first \( m \) intervals has a length of \( U \) time periods, and each of the next \( n \) intervals has a length of \( L \) time periods as shown in Figure 3.2. We will refer to this type of partition as the \( U-L \) pattern. Then, the amount invested in the first \( m \) intervals will be equal to \( (T-U)D \), \( (T-2U)D \), \( (T-3U)D \), and so on. The amount invested in the last interval \( m \) will be equal to \( nLD \), or \( (T-mU)D \), since \( mU + nL = T \). Further, the interest income on every dollar invested during any time interval \( U \) is equal to \( rU \), since \( r \) is the interest rate per time period. Similarly, the amount invested in the next \( n \) intervals will be equal to \( (n-1)LD \), \( (n-2L)D \), \( (n-3L)D \), and so on. The amount invested in the last interval \( n \) will
be zero. The interest income on every dollar invested during any time interval $L$ is equal to $rL$, since $r$ is the interest rate per time period. Let $\zeta(y)$ represent the net earnings for the time horizon $T$ divided into $y$ time intervals.

$$
\eta_1(y) = (T - U)DrU + (T - 2U)DrU + \cdots + (T - mU)DrU \\
+ ((n - 1)L)DrL + ((n - 2)L)DrL + \cdots + (2L)DrL + (L)DrL - Fy
$$

Or, $\eta_1(y) = TDr mU - Dr(m + 1)mU^2 / 2 + Dr(n - 1)L^2 / 2 - Fy$ \hspace{1cm} (3.11)

Figure 3.2 Partitioning of the Planning Horizon under the $U$-$L$ Pattern

As an alternative, suppose we divide the planning horizon $T$ into $y$ time intervals such that each of the first $n$ intervals has a length of $L$ time periods, and each of the next $m$ intervals has a length of $U$ time periods. We will refer to this partition as the $L$-$U$ pattern. Now let $\eta_2(y)$ represent the net
earnings from this new configuration shown in Figure 3.3. Using the same arguments as before, we get for the following expression for $\eta_2(y)$.

$$
\eta_2(y) = TDrLn - Dr(n+1)nL^2 / 2 + Drm(m-1)U^2 / 2 - Fy
$$

(3.12)

Figure 3.3 Partitioning of the Planning Horizon under the $L-U$ Pattern

Is the value of $\eta_2(y)$ different from $\eta_1(y)$

Let $\Delta = \eta_1(y) - \eta_2(y)$

Or, $\Delta = TDrmU - Dr(m+1)mU^2 / 2 + Drn(n-1)L^2 / 2 - Fy$

$$
-(TDrLn - Dr(n+1)nL^2 / 2 + Drm(m-1)U^2 / 2 - Fy)
$$

Or $\Delta = TDr(mU - nL) - Dr(m+1)mU^2 / 2 - Drm(m-1)U^2 / 2$
\[ +Drn(n-1)L^2 / 2 + Dr(n+1)nL^2 / 2 \]

\[ \Delta = TD(mU - nL) - Drn^2U^2 + Drn^2L^2 = TD(mU - nL) - (Drnm^2U^2 - Drn^2L^2) \]

\[ \Delta = TD(mU - nL) - Drm^2U^2 + Drn^2L^2 = TD(mU - nL) - (Drnm^2U^2 - Drn^2L^2) \]

\[ \Delta = TD(mU - nL) - (Drm^2U^2 - Drn^2L^2) = TD(mU - nL) - Dr(mU + nL)(mU - nL) \]

But \((mU + nL) = T\).

Therefore, \[\Delta = TD(mU - nL) - TD(mU - nL) = 0\]

From equation (3.10), we have value of the earnings expressed in a slightly different way as given below:

\[ \zeta(y) = (DT)(Tr) / 2 - Z(y) \]  \hspace{1cm} (3.10)

We have proved above that \(\eta_1(y) = \eta_2(y)\). But \(\zeta(y)\) represents the value of earnings just as \(\eta_1(y)\) and \(\eta_2(y)\). We now prove that \(\zeta(y) = \eta_1(y)\). The result is derived from the fact that \((mU + nL) = T\).

Let \(\Delta = \zeta(y) - \eta_1(y)\)

Or, \(\Delta = (DT)(Tr) / 2 - Z(y)\)

\[-TDrmU + Dr(m + 1)mU^2 / 2 - Drn(n - 1)L^2 / 2 + Fy\]

Or \(\Delta = TD(mU + nL) / 2 - (Dr / 2)(mU^2 + nL^2)\)
It may be noted that if the planning horizon $T$ is partitioned into $m$ cycles of length $U$ and $n$ cycles of length $L$, there are many possible sequences for withdrawal of the funds. But we have already proved in Proposition 3.1 that as long as the cycle-times remain constant, the sequence has no effect on the total costs. Therefore, from now onwards, we will use $\zeta(y)$ to represent the earnings that can be realized when the number of cycles is fixed at $y$. From (3.10) we see that the value of the earnings $\zeta(y)$ will be maximum, if $Z(y)$ is the minimum. If $\pi$ denotes the optimal value of $y$, the optimal value of $\zeta(y)$ is obtained in two steps: (a) First determine $\pi = y^*$ at which $Z^*(\pi)$ is minimized. (b) Next, find $\zeta^*(\pi) = DT^2r / 2 - Z^*(\pi)$. Then, $\zeta^*(\pi)$ will be the maximum value of the earnings that can be realized through the partial investment of the funds.

3.4 Illustrative Examples

We will look at some numerical examples that will help us better understand the solution process for estimating the net revenue.
3.4.1 An Example from a Textbook with Enhancements

We will start with the numerical example discussed in one of the standard textbooks, \textit{Fundamentals of Corporate Finance}, 12$^{\text{th}}$ Edition, by Ross, Westerfield and Jordan (2019). (This textbook is adopted for the introductory finance course FIN 323 at ODU). The following problem is extracted from Chapter 19, Cash and Liquidity Management, The BAT Model: Example 19A.1 (from page 667 of the book of 12$^{\text{th}}$ Edition)

“The Vulcan Corporation has cash outflows of $100 per day, seven days a week. The interest rate is 5 percent, and the fixed cost of replenishing cash balances is $10 per transaction. What is the optimal cash balance? What is the total cost?”

It is not explicitly stated in the problem but it is assumed that the time period under consideration is one year. The total cash needed for the year is $36,500. (365*100=36500).

Instead of finding the lowest cost, in this section we ask: what is the maximum amount of earnings possible if a portion of the available is invested initially in securities at an annual interest of 5%?

The parameters and variables relevant to the problem are defined below.

Unit of time: one day

Planning horizon under consideration: $T = 365$

Daily demand, $D = 100$  (the book uses the symbol $T$ instead of $D$)

Cost per transaction, $F = $10
Annual interest rate = 5%

Daily interest rate, \( r = \frac{0.05}{365} = 0.000137 \)

\( t \) = cycle-time or time interval between two consecutive transactions in days.

Using the square-root-formula of the BAT model, we get:

Optimal starting cash balance, \( C^* = \sqrt{2FD} / r = \frac{\sqrt{2 \times 10 \times 100}}{0.000137} = 3821 \) (rounded to dollars)

Cycle time, \( t^* = \sqrt{\frac{2F}{Dr}} = 38.21, \)

Number of transactions (or cycles), \( \bar{y} = \frac{T}{t^*} = \frac{365}{38.21} = 9.552 \)

Total annual cost (for any value of \( C \)) is given by the following equation:

\[ \xi(C) = \left(\frac{D}{C}\right)F + rC/2 = \left(\frac{36500}{3821}\right)\times10 + 0.05\times3821/2 = $191.05 \]

Rounding down the value of \( \bar{y} \), we have \( y_1 = 9. \) Further, \( m=5, \ U=41, \ n=4, \) and \( L=40. \)

\[ Z(y_1) = Fy + (Dr/2)(mU^2 + nL^2) = 191.40 \]

Rounding up the value of \( \bar{y} \), we have \( y_2 = 10. \) Further, \( m=5, \ U=37, \ n=5, \) and \( L=36. \)

\[ Z(y_2) = Fy + (Dr/2)(mU^2 + nL^2) = 191.27 \]

Since \( Z(y_2) < Z(y_1) \) in this case, \( \pi = y_2 = 10. \) Also, \( \zeta^*(\pi) = DT^2r/2 - Z^*(\pi) = 721.23 \)

The Vulcan Corporation can have net earnings of $721.23 by having one initial investment and nine subsequent withdrawals. The relationship between earnings and the number of transactions is shown in Figure 3.4.
3.4.2 An Example with a Typical Concave Profit Curve

Suppose that the Everest Construction Company has several construction projects in progress. The following data is available.

Unit of time: one day.

Time horizon under consideration, \( T = 215 \) days. (All projects have to be completed in 215 days.)

Cash outflows, \( D =$10,000 per day

Cost per transaction, \( F =$127 \) (brokerage fee for liquidating short-term securities)

Annual interest rate = 6%

Daily interest rate, \( r = \frac{0.06}{365} = 0.00016438 \)
Mathematical Notation:

$C =$ Cash balance on hand at the beginning of each cycle or immediately after a transaction.

t = cycle-time or time interval between two consecutive transactions in days.

y = number of transactions or cycles

Total cost for any value of $C$ for the time horizon of $T = 215$ days is given by the following equation:

$$\xi_2(C) = (DT / C)F + rTC / 2$$

Or, $t^* = \sqrt{2F / (DR)} = 12.43047$, and further, $C^* = Dt^* = 10000 \times t^* = 124304.70$

Since $t^* = 12.43$, $\bar{y} = T / t^* = 215 / 12.43 = 17.297$. Clearly, $\bar{y}$ is not an integer. Since $Z(y)$ is a piecewise linear convex function, the minimum of $Z(y)$ occurs in the neighborhood of $\bar{y} = 17.297$.

We search for the minimum of $Z(y)$ at $y_1 = \lfloor y \rfloor = 17$ or $y_2 = \lceil y \rceil = 18$.

Case (1): $y_1 = 17$. $U_1 = \left\lceil T / y_1 \right\rceil = 13$, $L_1 = U_1 - 1 = 12$, $m_1 = T - L_1 y_1 = 215 - 12 \times 17 = 11$, $n_1 = y_1 - m_1 = 6$.

$$Z(y_1) = Fy_1 + (Dr / 2)(m_1 U_1^2 + n_1 L_1^2) = 127 \times 17 + (1.6438 / 2) \times (11 \times 13 \times 13 + 6 \times 12 \times 12) = 4397.08$$

Case (2): $y_2 = 18$. $U_2 = \left\lceil T / y_2 \right\rceil = 12$, $L_2 = U_2 - 1 = 11$, $m_2 = T - L_2 y_2 = 215 - 11 \times 18 = 17$, $n_2 = y_2 - m_2 = 1$.

$$Z(y_2) = Fy_2 + (Dr / 2)(m_2 U_2^2 + n_2 L_2^2) = 127 \times 18 + (1.6438 / 2) \times (17 \times 12 \times 12 + 1 \times 11 \times 11) = 4397.50$$

Since $Z(y_1) < Z(y_2)$, the optimal number of cycles or transactions, $\pi = 17$ and $Z(\pi) = 4397.08$

Then, we have 11 cycles, each with a length of 13 days, 6 cycles, each with a length of 12 days.

Target cash balances: $C'_{U} = U_1 D = 13 \times 10000 = 130,000$ and $C'_{L} = L_1 D = 12 \times 10000 = 120,000$. 
In this example, $\pi = y_1 = 17$. Also, $\zeta^*(\pi) = DT^2 r / 2 - Z^*(\pi) = 33596.07$

The Everest Construction Company can have net earnings of $33596.07$ by having one initial investment and 16 subsequent withdrawals. The corresponding relationship between earnings and number of transactions is shown in Figure 3.5.

![Earnings vs. number of transactions](image)

Figure 3.5 Functional Relationship between and $\zeta(y)$ and $y$ for the Example 3.4.2.

Searching for $\pi$, the optimal number of transactions may be very simple, or sometimes it may involve considerable effort, depending the values of the parameters.

### 3.4.3 An Example with Suitably Adjusted Data

Let us consider a problem for which we can find integer solutions easily.

Unit of time: one day.

Time horizon under consideration, $T = 300$ days.

Cash outflows, $D = $1000 per day
Cost per transaction, $F = $22.50 (brokerage fee for liquidating short-term securities)

Annual interest rate = 7.3%

Daily interest rate, $r = 0.073/365 = 0.0002$

Then, $\lambda = Dr = 1000 \times 0.0002 = 0.2$, $t^* = \sqrt{2F/\lambda} = \sqrt{2 \times 22.5 / 0.2} = 15$, $y^* = T / t^* = 300 / 15 = 20$.

Since $t^*$ and $y^*$ are both integers, we have the optimal solution.

$\pi = 20$, and $Z(\pi) = \pi (F + (\lambda / 2)t^2) = 20 \times (22.5 + (0.2/2) \times 15 \times 15) = 900.00$

In this example, $\pi = y^* = 20$. Also, $\zeta^*(\pi) = DT^2r / 2 - Z^*(\pi) = 8100.00$

The corresponding relationship between the earnings (or profit) function and number of transactions is shown in Figure 3.6.

![Earnings vs. number of transactions](image)

Figure 3.6 Functional Relationship between and $\zeta(y)$ and $y$ for the Example 3.4.3.

### 3.4.4 An Example with a Frustum-shaped Profit Curve

Let us consider a problem which requires considerable work to generate the complete set of integer solutions.
Unit of time: one day.

Time horizon under consideration, $T = 180$ days.

Cash outflows, $D =$120,000 per day

Cost per transaction, $F =$288.00  (brokerage fee for liquidating short-term securities)

Annual interest rate = 5.84%

Daily interest rate, $r = 0.0584/365=0.00016$

The relationship between the total cost $Z(y)$ and the number of transactions $y$ is given in Figure 5.

As mentioned earlier, the total cost $Z(y)$ is a piecewise linear convex function of $y$. From Figure 5, we see that there are several values of $y$ that yield the minimum (because of the flat-bottom-V shape).

To find them all, we need to adopt a procedure different from the ones we discussed earlier.

As before, let $\lambda = D r = 120000*0.00016= 19.2$

$$t^* = \sqrt{2F/\lambda} = \sqrt{2*288/19.2} = 5.4772 \quad \bar{y} = T/t^* = 180 / 5.4772 = 32.863$$

Since both $t^*$ and $\bar{y}$ are not integers, we need to do additional work to find the optimal solution.

If we let $y_1 = \lfloor \bar{y} \rfloor = 32$, and $y_2 = \lceil \bar{y} \rceil = 33$, we get two possible solutions, one of which could be the optimal solution. The corresponding relationship between earnings and number of transactions is shown in Figure 3.7.
But, the frustum shape of the earnings curve in Figure 3.7 indicates that there could be more than two solutions, optimal or otherwise. This implies that more work needs to be done to identify one or more of the optimal solutions.

Let $\tau_1 = \lceil T^* \rceil = 6$, and $\tau_2 = \lfloor T^* \rfloor = 5$.

Then, the average cost per period $\mu(\tau_1) = F / \tau_1 + \lambda \tau_1 / 2 = 288/6 + 19.2*6/2 = 105.60$

Similarly, the average cost per period $\mu(\tau_2) = F / \tau_2 + \lambda \tau_2 / 2 = 288/5 + 19.2*5/2 = 105.60$

We have two cycles of different lengths, but the average cost per period is the same in both cases.

This points to the possibility of multiple optimal solutions. Let $y_1 = T / \tau_1 = 180 / 6 = 30$, and further, $y_r = T / \tau_2 = 180 / 5 = 36$. Now calculate the total cost, $Z(y)$ at $y = 30, 31, 32, 33, 34, 35, 36$. Using this value of $Z(y)$, find the value of the earnings $\zeta^*(\pi) = DT^2 r / 2 - Z^*(\pi)$.

We get: $\zeta^*(\pi) = DT^2 r / 2 - Z^*(\pi) = 292032$ for $\pi = 30, 31, 32, 33, 34, 35, 36$. 
Clearly, there are seven different optimal solutions, but each one has cycle lengths of 6 or 5 days. Initially, if we rounded up or down \( \bar{y} \), we would have generated only two optimal solutions with \( \pi = 32 \), and \( \pi = 33 \). This example illustrates the need to explore the possible values of \( y \) corresponding to the cycle lengths of \( \tau_1 = \lceil t^* \rceil \) and \( \tau_2 = \lfloor t^* \rfloor \).

This example also illustrates the need to consider several possible scenarios in developing the algorithms for finding the optimal solutions. (One of the contributions of this dissertation is the development of the methodology for identifying the existence of multiple optimal solutions like the ones described here and the corresponding frustum-shaped graph.)
CHAPTER 4
SASTRY-OGDEN-SUNDARAM MODEL: AN EXTENSION OF THE BAT CASH MANAGEMENT MODEL WITH SHORT-TERM BORROWING

4.1 Introduction: In the basic version of the BAT cash management model there is no provision for short-term borrowing. However, in the case of individual persons as well as corporations, it is sometimes beneficial to borrow funds on a short-term basis and repay the loan as soon as the funds become available. An analytical model related to the effect of credit on the transactions demand for cash is described in Sastry (1970), but the focus of Sastry’s model was on estimating the elasticity of the demand for cash as a function of the interest rate and the total amount of the disbursements made by an individual in a year. Further analysis of the income and interest rate elasticities of the demand for cash can be found in Akerlof and Milbourne (1978). There was no discussion about the possible savings a firm could realize by using credit either in Sastry (1970) or Akerlof and Milbourne (1978). However, some results from the studies whose focus is on short-term financing and cash reserves have appeared in Robichek et al. (1965), and White and Norman (1965). The square-root-formula used in Baumol’s basic cash management model is mathematically identical to the well-known EOQ formula in inventory control theory (Baumol, 1952). In a similar way, Sastry’s model, as acknowledged in Sastry (1970), is also mathematically identical to another inventory management model, often called the backlogging or shortage-cost model which is presented in Churchman et al. (1957). In the inventory backlogging model, if a firm cannot meet the demand from its inventory, it may backlog the demand and ship the back-ordered quantity to the customers as soon as the production starts and the inventory is replenished. Of course, the firm will incur additional costs or penalties because of the shortage or it may even
lose some customers. Nonetheless, a firm may prefer to backlog some orders, if that policy leads to a net reduction in costs.

The analogous cash management model, dealing with the potential savings a corporation could realize through the use of short-term credit, is described in Ogden and Sundaram (1998). Mathematically, the models of Sastry (1970) and Ogden and Sundaram (1998) are identical. However, through an illustrative example, Ogden and Sundaram (1998) demonstrated that a firm could reduce the costs of meeting its payment obligations by using credit, whereas Sastry (1970) provided no such example. Since the model was originally published by Sastry in 1970, about three decades before the publication of the same model in 1998 by Ogden and Sundaram, in recognition of this fact, we will refer to the short-term borrowing model as the Sastry-Ogden-Sundaram (SOS) model from now onwards. In this Chapter, we will consider an enhancement to the Sastry-Ogden-Sundaram model. In the enhanced version, we assume that the firm is required to buy insurance on the maximum amount borrowed during any time interval, in addition to paying interest on the borrowed money. The corresponding inventory theoretic model is described in Hadley and Whitin (1963) and Johnson and Montgomery (1974). A penalty rate similar to the insurance rate is included in the theoretical models of Hadley and Whitin (1963) and Johnson and Montgomery (1974), but its impact on the final results of the model was not discussed in any detail, since the costs related to inventory maintenance account for only a small fraction of the total productions costs. In contrast, banks and other lending institutions often do require that the borrowing firms buy insurance on the maximum amount of money borrowed. The obvious reason is that the lenders want to get back all the money they loaned, should any of the borrowing firms default. For this reason, it is worthwhile to analyze in some reasonable detail the effect of the
insurance rate on the amount of the money a firm can borrow and the savings that may accrue to
the firm.

4.2 Sastry-Ogden-Sundaram Model: As a variant of basic BAT model, consider the case of a
firm that has invested all of its cash in short-term securities. Suppose the rate of interest per time
period (i.e., per day or another unit of time) paid by these short-term securities is \( r \) and there is a
fixed brokerage fee \( F \) for each liquidation regardless of the amount of securities liquidated. Also
assume that the firm has established a credit facility with a bank from which it can borrow at an
interest rate of \( s \) per time period (i.e., per day or another unit of time) at any time. Further, the firm
has to pay insurance at a rate equal to \( e \) on the maximum amount borrowed at the time of repayment
of the borrowed money. As in the basic BAT model, suppose that the firm has to make
disbursements at the constant rate of \( D \) dollars per time period (i.e., per day or other unit of time)
during a planning horizon of \( T \) time periods. How often should the firm liquidate the securities and
what is the amount of each liquidation? Also, what is the amount, if any, the firm should borrow
from the bank in every cycle? Obviously, the goal is to meet the transactions demand for cash
during the planning horizon at the lowest cost. Sastry (1970) and Ogden and Sundaram (1998)
presented a solution to a simplified version of this cash management problem in which the firm is
not required to buy any insurance. Here we consider an extension of the Sastry-Ogden-Sundaram
model in which a firm is required to buy insurance at a rate \( e \) on the maximum amount borrowed
in every cycle and explore the impact of this requirement on the short-term borrowing decisions
of the firm.

4.3 Solution of the SOS Model: We will now define the parameters and variables used in the
model and describe a solution procedure.
Mathematical Notation:

Parameters with constant (or stationary) values:

\( T \): Number of time periods (or days) in the planning horizon (always an integer).

\( D \): Rate of disbursement of cash per time period (i.e., per day) in a continuous stream

\( M \): Total amount of disbursements in the planning horizon (\( M = TD \))

\( r \): Interest rate per time period (i.e., per day); rate represents lost income on the cash withdrawn

\( s \): Interest rate per time period (i.e., per day); rate charged by the bank on the money borrowed

\( e \): rate of insurance charged on the maximum amount borrowed from the bank

\( F \): Fixed brokerage fee for one transaction for liquidating securities, regardless of the amount

\( Q \): Amount of money received from one liquidation of securities

\( C \): Maximum amount of cash on hand

\( B \): Maximum amount of money borrowed from the bank (Line of Credit)

\( t \): cycle time between two consecutive liquidations

\( h \): time interval in a cycle during which payments are made from the cash held on hand

\( g \): time interval in a cycle during which money is borrowed to pay for the transactions demand

\( \psi(t) \): Total cost per cycle of length \( t \).

\( \mu(t) \): Average cost per period in a cycle of length \( t \).

\( Y = \{1, 2, \ldots, T\} \)
\(y\): Assumed to be an *integer* variable, unless otherwise specified: \(y \in Y\)

\(W(y)\): Total cost for the planning horizon with \(y\) cycles of (positive) *arbitrary* lengths.

\(R(y)\): Total cost for the horizon with \(y\) cycles of *equal* (but not necessarily integer) lengths.

\(Z(y)\): Total cost for the horizon with \(y\) cycles of *integer* lengths

---

Let us consider Figure 4.1. In order to meet the disbursement obligations, first, the firm borrows money from a bank at a continuous rate of \(D\) per day (or other unit of time) during the time interval \(g\). Then, it sells securities worth \(Q\) dollars at the end of the time interval \(g\), paying a brokerage fee of \(F\) dollars. Immediately, it pays off the amount of \(B\) dollars borrowed from the bank. The firm now has an amount of cash \(C = Q - B\) on hand. The cash on hand is used to make disbursements at the rate of \(D\) per day (or other unit of time) during the time interval \(h\). Then, the cycle of...
borrowing, liquidation and cash payment is repeated. Let $t$ represent the cycle time between two consecutive liquidations of securities. Then, $t = h + g$.

The costs incurred during one cycle time $t = h + g$ are:

(a) Fixed brokerage fee for one liquidation, $F$.

(b) Interest paid to the bank on the borrowed money: $sBg / 2$

(c) Interest lost on the cash held: $rCh / 2$

(d) Insurance paid to the bank on the borrowed money: $eB$.

Total cost per cycle, $\psi(t) = F + Bgs / 2 + Chr / 2 + eB$ \hspace{1cm} (4.1)

From (1) the average cost per day (per one time period) is given by:

$$
\mu = \psi(t) / t = F / t + (Bs / 2)(g / t) + (Cr / 2)(h / t) + eB / t
$$

(4.2)

From Figure 1, we have: $B = gD$; $C = hD$; $B + C = (g + h)D$; $Q = B + C$; $Q = tD$; $1 / t = D / Q$.

Or, $B / g = C / h = B + C / g + h = Q / t = D$

Or, $g / t = B / Q$ and $h / t = C / Q = Q - B / Q$

From (2) $\mu = \psi(t) / t = F / t + B^2s / (2Q) + C^2r / (2Q) + eB / t$

Or, $\mu = FD / Q + B^2s / (2Q) + r(Q - B)^2 / (2Q) + eBD / Q$ \hspace{1cm} (4.3)

Partial differentiation of $\mu$ with respect to $B$ and $Q$ yields the following results.

$$
\frac{\partial \mu}{dB} = Bs / Q - r(Q - B) / Q + eD / Q = 0
$$
Or, \( B = \frac{(rQ - eD)}{(r + s)} \)  

(4.4)

Also, \( \frac{d\mu}{dQ} = \frac{-FD}{Q^2} - B^2 s / (2Q^2) + r / 2 - rB^2 / (2Q^2) - eBD / Q^2 = 0 \)

Or, \( rQ^2 = 2FD + (r + s)B^2 + 2eBD \)  

(4.5)

Substituting the value of \( B \) from (4.4) in (4.5), we get:

\[
\begin{align*}
 rQ^2 &= 2FD + (r + s)(rQ - eD)^2 / (r + s) + 2eD(rQ - eD) / (r + s) \\
 &= 2FD + \frac{(rQ - eD)^2}{(r + s)} + 2eD(rQ - eD) / (r + s)
\end{align*}
\]

Or, \( rQ^2 = 2FD + (rQ - eD)^2 / (r + s) + 2eD(rQ - eD) / (r + s) \)

Or, \( rQ^2 = 2FD + (r^2Q^2 - e^2D^2) / (r + s) \)

Or, \( rQ^2 = 2FD + (r^2Q^2 - e^2D^2) / (r + s) \)

Or, \( rsQ^2 = 2(r + s)FD - e^2D^2 \)

Or, \( Q^* = \frac{\sqrt{2(r + s)FD - e^2D^2}}{(rs)} \)  

(4.6)

Then from (4.4), \( B^* = \frac{(rQ^* - eD)}{(r + s)} \)  

(4.7)

In Sastry-Ogden-Sundaram model, it is assumed that the insurance rate \( e = 0 \).

Then, \( Q^* = \frac{\sqrt{2(r + s)FD}}{(rs)} \)

Or, \( Q^* = \frac{(\sqrt{2FD / r})(\sqrt{(r + s) / s})}{(rs)} \)  

(4.8)

Also, \( B^* = \frac{Q^* r}{(r + s)} \)  

(4.9)

Then, \( C^* = Q^* - B^* = \frac{Q^* s}{(r + s)} \)  

(4.10)
First, we note that Baumol’s model is a special case of Sastry-Ogden-Sundaram model, in which borrowing from the bank is not allowed. It implies that mathematically, the interest rate on borrowed money, \( s = \infty \). Then, \( Q' = \sqrt{2FD / r} \), \( B' = 0 \), and \( C' = Q' \). Obviously, for the time horizon of length \( T \), the minimum cost according to Baumol’s model is given by:

\[
\psi(C') = (FD / C' + C'r / 2)T \quad \text{Or,} \quad \psi(Q') = (FD / Q' + Q'r / 2)T
\]

(4.11)

If the interest rate \( s \) charged by the bank on borrowed money is finite and insurance rate \( e \) is zero, from equations (4.3), (4.8), (4.9) and (4.10), we get the expression for the minimum cost for the entire planning horizon of length \( T \) as given below:

\[
\psi(Q') = (FD / Q' + C'^2r / 2Q' + B^2s / 2Q')T
\]

(4.12)

### 4.4 Data from Ogden-Sundaram (1998) Paper

We will first solve the problem using data extracted from the Ogden-Sundaram (1998) paper in which the insurance rate \( e = 0 \).

Daily demand for cash \( D = 4000 \); \( T = 365 \) days; \( M = 4000 \times 365 = 1,460,000 \).

\( F = 50 \) (Fixed brokerage fee for a liquidation transaction regardless of the amount)

Annual interest rate (opportunity cost) earned when cash is invested = 5%

Daily interest rate on cash, \( r = 0.05 / 365 = 0.000137 \) (opportunity cost or lost interest income)

Annual interest rate charged by the bank on borrowed money = 8%

Daily interest rate on money borrowed, \( s = 0.08 / 365 = 0.0002192 \) (rate charged by bank)
First, we start with Baumol’s basic model which does not allow any borrowing ($s = \infty$).

Solution based on Baumol’s model: $C^* = \sqrt{2FD / r} = \sqrt{2 \times 50 \times 4000 / 0.000137} = 54037.02$

Annual cost $= \psi(C^*) = (FD / C^* + C^*r / 2)T = 2702$ (there is no borrowing from the bank: $s = \infty$)

Solution based on Ogden-Sundaram’s model: Money is borrowed from the bank, but $e = 0$.

$Q^* = (\sqrt{2FD / r})(\sqrt{(r + s) / s}) = (\sqrt{2 \times 50 \times 4000 / 0.000137})(\sqrt{(0.000137 + 0.0002192) / 0.0002192})$

Or, $Q^* = 54037 \times \sqrt{0.0003562 / 0.0002192} = 68884$

Maximum amount of money borrowed from bank, $B^* = Q^* r / (r + s) = 26494$.

Maximum amount of cash on hand, $C^* = Q^* - B^* = Q^* s / (r + s) = 42390$.

Annual cost $= \psi(Q^*) = (FD / Q^* + C^2r / 2Q^* + B^2s / 2Q^*)T = 2120$

Annual savings because of temporary borrowing from the bank $= 2702 - 2120 = 582$.

**4.5 Effect of Insurance Rates on the Total Cost**

It is clear from equations (4.6) and (4.7) that both $Q^*$ and $B^*$ are decreasing functions of the insurance rate $e$. The values of $Q^*$, $B^*$, and $C^*$ are given in Table 4.1 as a function of the insurance rate. They are also shown graphically in Figures 4.2 and 4.3. We see from Table 4.1, that the value of $B^*$ continues to go down from a high of 26493.83 to a low of 915.39 as the value of $e$ goes up from 0 to 0.0018. Our next step is to find the value of $\hat{e}$ at which $B^*$ will be zero. We can use equations (4.6) and (4.7) to determine the value of $\hat{e}$. If $B^*$ is zero, from (4.7) we get: $Q^* = \hat{e}D / r$. 
Then, from (4.6) we must have:  

\[ eD / r = \sqrt{(2(r + s)FD - e^2 D^2) / (rs)} \]

Or,  

\[ e^2 D^2 / r^2 = (2(r + s)FD - e^2 D^2) / (rs) \]

Or,  

\[ se^2 D^2 / r + e^2 D^2 = 2(r + s)FD \]

Or,  

\[ se^2 D^2 + re^2 D^2 = 2r(r + s)FD \]

Or,  

\[ (r + s)e^2 D = 2r(r + s)F \]

After simplification, we get:  

\[ \hat{e} = \sqrt{(2rF / D)} \]  \hspace{1cm} (4.13)

From the data:  \( F=50, \ r=0.000137, \) and  \( D=4000. \)

Substituting the values of  \( r, \ F, \) and  \( D \) in (4.13), we get  \( \hat{e} = 0.001851 \)

In other words, if the insurance rate is equal to 0.00185 (i.e. 0.185%) or higher, it is no longer advantageous to borrow any money from the bank. It is also clear from Table 4.2 and Figure 4.4, that the total cost goes up as the insurance rate goes up and its value reaches the original cost of $2702, when the insurance rate is close to  \( \hat{e} \). This implies that if the insurance rate is equal to  \( \hat{e} \) or higher, it is better not to borrow any funds at all from the bank. For this reason, we will refer to the  \( \hat{e} \) as the critical value of the insurance rate. If the insurance rate  \( e \) charged by the bank is less than the critical rate  \( \hat{e} \), it follows from equation (4.3) that the minimum cost for the entire planning horizon of length  \( T \) can be expressed as given below:

\[ \psi(Q) = (FD / Q^2) + B^2 s / (2Q^2) + rC^2 / (2Q^* + eDB / Q*)T \]  \hspace{1cm} (4.14)

Revised Solution of the problem discussed in Ogden-Sundaram (1998) paper:
Money is borrowed from the bank, but \( e = \hat{e} / 2 = 0.001851 / 2 = 0.0009255 \)

From (4.6), \( Q^* = \sqrt{(2(r + s)FD - e^2D^2) / (rs)} = 65487 \)

Then, \( B^* = (rQ^* - eD) / (r + s) = 14793 \)

Also, \( C^* = Q^* - B^* = 50694 \)

Revised annual cost = \( \psi(Q^*) = FD / Q^* + C^2 r / 2Q^* + B^2 s / 2Q^* + eBD / Q^* T = 2535 \)

Annual savings because of temporary borrowing from the bank = 2702 — 2535 = 167.

Even if the bank requires the firm to buy insurance at half of the critical rate, the firm will still save $167 or about 6.2% of the original cost for the year.

<table>
<thead>
<tr>
<th>Insurance Rate</th>
<th>Value of ( Q^* )</th>
<th>Value of ( B^* )</th>
<th>Value of ( C^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000000</td>
<td>68883.96</td>
<td>26493.83</td>
<td>42390.13</td>
</tr>
<tr>
<td>0.000100</td>
<td>68845.27</td>
<td>25355.87</td>
<td>43489.40</td>
</tr>
<tr>
<td>0.000200</td>
<td>68729.06</td>
<td>24188.10</td>
<td>44540.96</td>
</tr>
<tr>
<td>0.000300</td>
<td>68534.95</td>
<td>22990.36</td>
<td>45544.58</td>
</tr>
<tr>
<td>0.000400</td>
<td>68262.26</td>
<td>21762.41</td>
<td>46499.85</td>
</tr>
<tr>
<td>0.000500</td>
<td>67910.05</td>
<td>20503.87</td>
<td>47406.19</td>
</tr>
<tr>
<td>0.000600</td>
<td>67477.08</td>
<td>19214.26</td>
<td>48262.82</td>
</tr>
<tr>
<td>0.000700</td>
<td>66961.77</td>
<td>17892.99</td>
<td>49068.78</td>
</tr>
<tr>
<td>0.000800</td>
<td>66362.22</td>
<td>16539.31</td>
<td>49822.90</td>
</tr>
<tr>
<td>0.000900</td>
<td>65676.11</td>
<td>15152.35</td>
<td>50523.76</td>
</tr>
<tr>
<td>0.001000</td>
<td>64900.69</td>
<td>13731.04</td>
<td>51169.66</td>
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<tr>
<td>0.001100</td>
<td>64032.73</td>
<td>12274.13</td>
<td>51758.61</td>
</tr>
<tr>
<td>0.001200</td>
<td>63068.41</td>
<td>10780.16</td>
<td>52288.25</td>
</tr>
<tr>
<td>0.001300</td>
<td>62003.22</td>
<td>9247.39</td>
<td>52755.83</td>
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<tr>
<td>0.001400</td>
<td>60831.87</td>
<td>7673.79</td>
<td>53158.07</td>
</tr>
<tr>
<td>0.001500</td>
<td>59548.09</td>
<td>6056.96</td>
<td>53491.13</td>
</tr>
<tr>
<td>0.001600</td>
<td>58144.44</td>
<td>4394.02</td>
<td>53750.43</td>
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<tr>
<td>0.001700</td>
<td>56612.00</td>
<td>2681.54</td>
<td>53930.46</td>
</tr>
<tr>
<td>0.001800</td>
<td>54940.00</td>
<td>915.39</td>
<td>54024.62</td>
</tr>
</tbody>
</table>
Figure 4.2: Values of $Q^*$ as a Function of the Insurance Rate

Figure 4.3: Values of $C^*$ and $B^*$ as Functions of the Insurance Rate
Table 4.2. Total Cost vs. Insurance Rate

<table>
<thead>
<tr>
<th>Insurance Rate</th>
<th>Total Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0000</td>
<td>2120</td>
</tr>
<tr>
<td>0.0001</td>
<td>2174</td>
</tr>
<tr>
<td>0.0002</td>
<td>2227</td>
</tr>
<tr>
<td>0.0003</td>
<td>2277</td>
</tr>
<tr>
<td>0.0004</td>
<td>2325</td>
</tr>
<tr>
<td>0.0005</td>
<td>2370</td>
</tr>
<tr>
<td>0.0006</td>
<td>2413</td>
</tr>
<tr>
<td>0.0007</td>
<td>2453</td>
</tr>
<tr>
<td>0.0008</td>
<td>2491</td>
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<tr>
<td>0.0009</td>
<td>2526</td>
</tr>
<tr>
<td>0.0010</td>
<td>2558</td>
</tr>
<tr>
<td>0.0011</td>
<td>2588</td>
</tr>
<tr>
<td>0.0012</td>
<td>2614</td>
</tr>
<tr>
<td>0.0013</td>
<td>2638</td>
</tr>
<tr>
<td>0.0014</td>
<td>2658</td>
</tr>
<tr>
<td>0.0015</td>
<td>2675</td>
</tr>
<tr>
<td>0.0016</td>
<td>2688</td>
</tr>
<tr>
<td>0.0017</td>
<td>2697</td>
</tr>
<tr>
<td>0.0018</td>
<td>2701</td>
</tr>
</tbody>
</table>

Figure 4.4: Total Cost as a Function of the Insurance Rate
4.6 Illustrative Examples

We will now solve some example problems to illustrate the application of SOS model.

4.6.1 An Example from a Textbook with Enhancements

We will start with the numerical example discussed in one of the standard textbooks, *Fundamentals of Corporate Finance*, 12th Edition, by Ross, Westerfield and Jordan (2019).

(This textbook is adopted for the introductory finance course FIN 323 at ODU). The following problem is extracted from Chapter 19, Cash and Liquidity Management, The BAT Model: Example 19A.1 (from page 667 of the book of 12th Edition)

“The Vulcan Corporation has cash outflows of $100 per day, seven days a week. The interest rate is 5 percent, and the fixed cost of replenishing cash balances is $10 per transaction. What is the optimal cash balance? What is the total cost?”

It is not explicitly stated in the problem but it is assumed that the time period under consideration is one year. The total cash needed for the year is $36,500. (365*100=36500).

The parameters and variables relevant to the problem are defined below.

The parameters and variables relevant to the problem are defined below.

Unit of time: one day.

Time horizon under consideration, \(T\) = 365 days.

Cash outflows, \(D\) =$100 per day (Daily demand for cash)

Cost per transaction, \(F\) = $10 (brokerage fee for liquidating short-term securities)

Annual interest rate (opportunity cost) on the cash withdrawn = 5%
Daily interest rate, \( r = \frac{0.05}{365} = 0.000137 \) (opportunity cost or lost interest income)

Annual interest rate charged by the bank on borrowed money = 8%

Daily interest rate on money borrowed, \( s = \frac{0.08}{365} = 0.0002192 \)

Critical insurance rate, \( \hat{e} = \sqrt{\frac{2rF}{D}} = \sqrt{\frac{2 \times 0.000137 \times 10}{100}} = 0.005234 \) (or, 0.5234%)

Assume that the insurance rate of the bank \( e = \frac{\hat{e}}{2} = \frac{0.005234}{2} = 0.002617 \) (or, 0.2617%)

Scenario 1: Optimal solution without short-term borrowing

Number of transactions \( y^* = 10, m=5, U=37, n=5, L=36 \)

Total Cost = 191.27

Scenario 2: Optimal solution with short-term borrowing, but insurance rate \( e=0 \).

Number of transactions \( y^* = 8, m=5, U=46, n=3, L=45 \)

Total Cost = 150.20

Scenario 3: Optimal solution with short-term borrowing and with insurance rate \( e=0.002617 \)

Number of transactions \( y^* = 8, m=5, U=46, n=3, L=45 \)

Total Cost = 179.25

4.6.2 An Example with a Typical Convex Cost Curve

Suppose that the Everest Construction Company has several construction projects in progress. The following data is available.

Unit of time: one day.

Time horizon under consideration, \( T = 215 \) days. (All projects have to be completed in 215 days.)

Cash outflows, \( D =$10,000 per day (Daily demand for cash)
Cost per transaction, $F = $127 (brokerage fee for liquidating short-term securities)

Annual interest rate (opportunity cost) on the cash withdrawn = 6%

Daily interest rate, $r = \frac{0.06}{365} = 0.00016438$ (opportunity cost or lost interest income)

Annual interest rate charged by the bank on borrowed money = 9.5%

Daily interest rate on money borrowed, $s = \frac{0.095}{365} = 0.0002603$ (rate charged by bank)

Critical insurance rate, $\hat{e} = \sqrt{\frac{2rF}{D}} = \sqrt{\frac{2 \times 0.0001648 \times 127}{10000}} = 0.002043$

Assume that the insurance rate of the bank $e = \frac{\hat{e}}{2} = 0.002043/2 = 0.001022$ (or, 0.1022%)

Scenario 1: Optimal solution without short-term borrowing

Number of transactions $y^* = 17$, $m = 11$, $U = 13$, $n = 6$, $L = 12$

Total Cost = 4397.08

Scenario 2: Optimal solution with short-term borrowing, but insurance rate $e=0$.

Number of transactions $y^* = 14$, $m = 5$, $U = 16$, $n = 9$, $L = 15$

Total Cost = 3442.91

Scenario 3: Optimal solution with short-term borrowing, and with insurance rate $e=0.001022$

Number of transactions $y^* = 14$, $m = 5$, $U = 16$, $n = 9$, $L = 15$

Total Cost = 4121.15

4.6.3 An Example with Suitably Adjusted Data

Let us consider a problem for which we can find integer solutions easily.

Unit of time: one day.

Time horizon under consideration, $T = 300$ days.

Cash outflows, $D = $1000 per day (Daily demand for cash)
Cost per transaction, \( F = $22.50 \) (brokerage fee for liquidating short-term securities)

Annual interest rate (opportunity cost) on the cash withdrawn = 7.3%

Daily interest rate, \( r = 0.073/365 = 0.0002 \) (opportunity cost or lost interest income)

Annual interest rate charged by the bank on borrowed money = 11%

Daily interest rate on money borrowed, \( s = 0.11/365 = 0.0003014 \) (rate charged by bank)

Critical insurance rate, \( \hat{e} = \sqrt{\frac{2rF}{D}} = \sqrt{\frac{2 \times 0.0002 \times 22.5}{1000}} = 0.003 \)

Assume that the insurance rate of the bank \( e = \hat{e}/2 = 0.003/2 = 0.0015 \) (or, 0.15%)

Scenario 1: Optimal solution without short-term borrowing

Number of transactions \( y^* = 20, m=20, U=15, n=0, L=14 \)

Total Cost = 900.00

Scenario 2: Optimal solution with short-term borrowing, but insurance rate \( e = 0 \).

Number of transactions \( y^* = 15, m=15, U=20, n=0, L=19 \)

Total Cost = 698.16

Scenario 3: Optimal solution with short-term borrowing, and with insurance rate \( e = 0.0015 \)

Number of transactions \( y^* = 16, m=12, U=19, n=4, L=18 \)

Total Cost = 841.90

4.6.4 An Example with a Flat-Bottom V-shaped Cost Curve

Let us consider a problem which requires considerable work to generate the complete set of integer solutions.

Unit of time: one day.

Time horizon under consideration, \( T = 180 \) days.
Cash outflows, \( D = $120,000 \) per day   (Daily demand for cash)

Cost per transaction, \( F = $288.00 \) (brokerage fee for liquidating short-term securities)

Annual interest rate = 5.84

Daily interest rate, \( r = \frac{0.0584}{365} = 0.00016 \% \) (opportunity cost or lost interest income)

Annual interest rate charged by the bank on borrowed funds = 8.91%

Daily interest rate on money borrowed, \( s = \frac{0.0891}{365} = 0.0002441 \) (rate charged by bank)

Critical insurance rate, \( e = \sqrt{\frac{2rF}{D}} = \sqrt{\frac{2 \times 0.00016 \times 288}{120000}} = 0.0008764 \)

Assume that the insurance rate by the bank \( e = \frac{e}{2} = 0.0004382 \) (or, 0.04382%)

Scenario 1: Optimal solution without short-term borrowing  (Many alternative solutions)

Number of transactions \( y^* = 30, m=30, U=6, n=0, L=5 \)

Number of transactions \( y^* = 31, m=25, U=6, n=6, L=5 \)

Number of transactions \( y^* = 32, m=20, U=6, n=12, L=5 \)

Number of transactions \( y^* = 33, m=15, U=6, n=18, L=5 \)

Number of transactions \( y^* = 34, m=10, U=6, n=24, L=5 \)

Number of transactions \( y^* = 35, m=5, U=6, n=30, L=5 \)

Number of transactions \( y^* = 36, m=0, U=6, n=36, L=5 \)

Total Cost = 19008.00

Scenario 2: Optimal solution with short-term borrowing, but insurance rate \( e=0 \).

Number of transactions \( y^* = 26, m=24, U=7, n=2, L=6 \)

Total Cost = 14725.22

Scenario 3: Optimal solution with short-term borrowing and with insurance rate \( e=0.0004382 \)

Number of transactions \( y^* = 26, m=24, U=7, n=2, L=6 \)

Total Cost = 17731.39
The cost curve corresponding to scenario 3 of the current example is displayed in Figure 4.5.

If the bank requires insurance on the amount borrowed, the cost function is still piecewise linear and convex but no longer a flat-bottom V-shaped curve. In other words, we no longer have multiple optimal solutions. There is just one unique optimal solution in this scenario.
CHAPTER 5
INTEGRATION OF THE ALLAIS VERSION OF THE BAT CASH MANAGEMENT MODEL WITH SHORT-TERM BORROWING

5.1 Introduction

In the Baumol (1952) version of the BAT cash management model, funds are withdrawn from a brokerage account to meet the transactions demand for cash. The objective in Baumol’s model is to minimize the total costs related to the liquidation of the securities and the interest lost on the cash held on hand while making the disbursements. In the Tobin (1956) version of the BAT cash management model, a portion of the total funds are invested first and later withdrawn to meet the transactions demand for cash. The objective in Tobin’s model is to maximize the net earnings after paying the investment and liquidation fees and the proportional charges, if any, on the funds invested and later withdrawn in periodic installments to make the disbursements. In contrast to both these models which primarily focus an individual person’s consumption spending and the related transactions demand for cash, Allais model (Baumol and Tobin, 1989) deals with an individual person’s saving and investment decision. In Baumol and Tobin models it is assumed that the disbursement of funds occurs in a continuous stream and at a constant rate. In a similar fashion, from a mathematical point of view, in Allais model also it is assumed that the accumulation of savings or net cash inflow occurs in a continuous stream and at a constant rate. However, none of these models examined the benefits from the use of credit in meeting the transactions demand for cash by individual persons or corporations. Two analytical models related to the effect of credit on the transactions demand for cash and the potential savings a corporation could realize through short-term borrowing, are presented in Sastry (1970) and in Ogden and
Sundaram (1998). Use of credit is akin to backlogging of demand in inventory management and comprehensive mathematical models that take into consideration the continuous accumulation of inventory through production, continuous withdrawal of inventory and backlogging of demand with penalties are discussed in Hadley and Whitin (1963) and Johnson and Montgomery (1974). In this Chapter we describe an analogous cash management model that deals with the integration of the Allais version of the BAT model with the short-term borrowing model of Sastry-Ogden-Sundaram. We also augment the model with the additional feature that the firm is required by the bank to buy insurance on the maximum amount borrowed, apart from paying interest on the borrowed funds. The Baumol-Allais-Tobin (BAT) model and the Sastry-Ogden-Sundaram (SOS) model can be considered special cases of this integrated model. From now onwards, we will refer to this version of the model as the Integrated Cash Management (ICM) model.

### 5.2 Mathematical Notation for the ICM Model:

- **$T$**: Number of time periods (or days) in the planning horizon (always an integer).

- **$D$**: Rate of disbursement of cash per time period (i.e., per day) in a continuous stream

- **$M$**: Total amount of disbursements in the planning horizon ($M = TD$)

- **$P$**: Rate of replenishment of cash per time period (i.e., per day) in a continuous stream

- **$\rho = \frac{D}{P}$** (Note: $D < P$, and $0 \leq \rho < 1$)

- **$F$**: Fixed brokerage fee for one transaction for liquidating securities, regardless of the amount

- **$r$**: Interest rate per time period (i.e., per day); rate represents lost income on the cash withdrawn

- **$s$**: Interest rate per time period (i.e., per day); rate charged by the bank on the money borrowed
\( e \): Rate of insurance charged by the bank on the maximum amount borrowed

\( \alpha \): Time interval during which money is borrowed from the bank; no repayment takes place during this time interval; a liquidation transaction is completed and withdrawal of funds is initiated at the end of this time interval.

\( \beta \): Time interval during which the borrowed funds are reduced to zero through repayment from the steady inflow of cash after a liquidation transaction, while meeting the disbursement needs.

\( \gamma \): Time interval during which cash is accumulated after making the regular payments

\( \delta \): Time interval during which the accumulated cash is used to make disbursements

\( g = \alpha + \beta \); \( g \) is the total time in the cycle during which money is owed to the bank

\( h = \gamma + \delta \); \( h \) is the total time in the cycle during which cash is held on hand

\( t \): Total cycle time (measured in days or other unit of time) is the time interval between two consecutive liquidations; it is also the time interval between two points in time when the cash on hand reaches zero level and borrowing begins. The variable \( t \) may be allowed to assume continuous values or restricted to integer values.

\( t = (\alpha + \beta) + (\gamma + \delta) = g + h \) \( \alpha, \beta, \gamma, \delta, g, h \): Assumed to be continuous variables.

\( Q \): Amount of money received in a continuous stream from one liquidation of securities

\( C \): Maximum level of cash balance on hand

\( B \): Maximum amount of loan balance or money borrowed from the bank
\( \psi(t) \): Total cost per cycle of length \( t \).

\( \mu(t) \): Average cost per period in a cycle of length \( t \).

\( Y = \{1, 2, \ldots, T\} \)

\( y \): Assumed to be an integer variable, unless otherwise specified: \( y \in Y \)

\( W(y) \): Total cost for the planning horizon with \( y \) cycles of (positive) arbitrary lengths.

\( R(y) \): Total cost for the horizon with \( y \) cycles of equal (but not necessarily integer) lengths.

\( Z(y) \): Total cost for the horizon with \( y \) cycles of integer lengths

Other functional notations:

\( [x] \) = The ceiling function that yields the smallest integer greater than or equal to \( x \)

\( \lfloor x \rfloor \) = The floor function that yields the largest integer smaller than or equal to \( x \)

The ICM model discussed here has several similarities to the inventory model discussed in Chen, Dondeti, and Mohanty (2019). Therefore Figure 5.1 given here and also some of the algebraic symbols used here are similar to the ones used in Chen, Dondeti, and Mohanty (2019).

5.3 Effect of Finite Cash Replenishment Rate on the Levels of Cash and Credit

When the cash replenishment rate is finite and obviously greater than the demand rate, the maximum levels of the cash accumulated or the amount borrowed will be less than the levels corresponding to those when the replenishment is done in a lumpsum as a result of a one-time liquidation. Let us consider Figure 5.1 wherein the components of the cycle are displayed in relation to each other.
In order to meet the disbursement obligations, first, the firm borrows money from a bank at a continuous rate of \( D \) dollars per day (or other unit of time) during the time interval \( \alpha \). The maximum amount borrowed \( B \) is equal to \( \alpha D \). Then, it initiates the transaction for the sale of securities worth \( Q \) dollars at the end of the time interval \( \alpha \), after paying a brokerage fee of \( F \) dollars. But the securities are not immediately converted into cash and it does not withdraw the whole amount of \( Q \) dollars at the end of the time interval \( \alpha \). Instead, it withdraws the funds in a continuous stream at the rate of \( P \) dollars, and starts the repayment process, while making the disbursement at the rate of \( D \). It is assumed that the firm loses interest only on the funds withdrawn.

With this assumption, this model closely tracks the Allais version of the BAT cash management model. The rate of repayment is \( (P-D) \) and the loan balance is reduced to zero at the end of the time interval \( \beta \). Obviously, \( B = \beta(P-D) \). The firm continues to withdraw funds at the rate of \( P \) during the time interval \( \gamma \), while making the disbursement at the rate of \( D \). At the end of the time interval \( \gamma \), the withdrawal process is completed and the firm accumulates a cash balance of \( C = \gamma(P-D) \). The amount of cash on hand \( C \) is used to make disbursements at the rate of \( D \) during
the time interval $\delta$, and at the end, the cash balance is reduced to zero. Then, the cycle of borrowing, liquidation, loan payment, cash accumulation and final disbursement is repeated.

Define: $\rho = D / P$, $D < P$, $0 \leq \rho < 1$.

From Figure 1, we have: $B = \alpha D$; $C = \beta (P - D)$; $C = \delta D$; $Q = (\beta + \gamma) P$.

Or, $B + C = \beta (P - D) + \gamma (P - D) = (\beta + \gamma) (P - D) = (\beta + \gamma) P (1 - D / P) = Q (1 - \rho)$

Or $B + C = Q (1 - \rho)$,

Or, $C = Q (1 - \rho) - B$

If $P = \infty$, it implies that as a result of the liquidation transaction, the whole amount of $Q$ is delivered to the firm as a lumpsum instantaneously. Then, $\rho = 0$, and $B + C = Q$. But, if $P$ is finite, we see that $B + C = Q (1 - \rho)$. In other words, the total of the maximum levels if the cash on hand and the amount borrowed will be lower when $\rho$ is greater than 0. Clearly, if $\rho = 1$, both $B$ and $C$ will be zero. In the parlance of inventory management, this scenario is called the Just-in-Time (JIT) strategy, since both the levels of backlog and inventory are zero. While, this strategy may work reasonably well under some circumstances in the production and manufacturing industries, it can lead to a catastrophic outcome in case of corporations. Any delay or missed payments may be perceived as a default by the financial markets resulting in a sell-off of the shares of the company by the market participants.

5.4 Derivation of the Cost Function per Unit Time

The firm owes money to the bank during the time interval $g = \alpha + \beta$, and the maximum amount borrowed is $B$. Similarly, the firm holds cash on hand during the time interval $h = \gamma + \delta$, and the
maximum cash balance is equal to \( C \). Further, the cycle time \( t = (\alpha + \beta) + (\gamma + \delta) = g + h \). The interest on the loan for the whole cycle is equal to \( B_{gs} / 2 \) and the interest income lost on the cash balance is equal to \( Chr / 2 \).

The costs incurred during one cycle time \( t = h + g \) are:

(a) Fixed brokerage fee for one liquidation, \( F \).

(b) Interest paid to the bank on the borrowed money: \( B_{gs} / 2 \).

(c) Interest income lost on the cash held: \( Chr / 2 \).

(d) Insurance paid to the bank on the borrowed money: \( eB \).

Total cost per cycle, \( \psi(t) = F + B_{gs} / 2 + Chr / 2 + eB \) \hspace{1cm} (5.1)

The withdrawal of cash occurs only during the time interval \( (\beta + \gamma) \) at the rate \( P \). Therefore, the total amount received from one liquidation of the securities, \( Q = (\beta + \gamma)P \). Also, \( Q = tD \), since \( Q \) must be equal to the total disbursement during the cycle time \( t \).

From (5.1) the average cost per day (per one time period) is given by:

\[
\mu(t) = \psi(t) / t = F / t + (B_{s} / 2)(g / t) + (C_{r} / 2)(h / t) + eB / t \hspace{1cm} (5.2)
\]

Also, \( g = \alpha + \beta \); \( h = \gamma + \delta \); \( t = \alpha + \beta + \gamma + \delta = g + h \); \( Q = tD \); \( t = Q / D \); \( 1 / t = D / Q \).

But, \( \alpha = B / D \); \( \beta = B / (P - D) \); \( \gamma = C / (P - D) \), and \( \delta = C / D \);

Therefore, \( g / t = (\alpha + \beta) / t = (\alpha + \beta) / (\alpha + \beta + \gamma + \delta) \)

\[
= \frac{B / D + B / (P - D)}{B / D + B / (P - D) + C / (P - D) + C / D} = B / (B + C) = B / Q(1 - \rho)
\]
Similarly, \( h / t = (\gamma + \delta) / (\gamma + \delta) / (\alpha + \beta + \gamma + \delta) \)

\[
\frac{C}{D} + \frac{C}{(P - D)} \frac{B}{D + B / (P - D)} + C / (P - D) + C / D = C / (B + C) = C / Q(1 - \rho)
\]

Alternatively, from the properties of similar triangles in Figure 1, we get:

\[
\frac{B}{\alpha + \beta} = \frac{C}{\gamma + \delta} = \frac{C}{B + C} = \frac{B + Q(1 - \rho) - B}{t} = \frac{Q(1 - \rho)}{t}
\]

Or, \( g / t = \frac{B}{Q(1 - \rho)} \) and \( h / t = \frac{C}{Q(1 - \rho)} \)

Either way, after substituting the values of \((g / t), (h / t), \) and \((1 / t)\) in (5.2) we get:

\[
\mu = \psi(t) / t = F / t + B^2 s / (2Q(1 - \rho)) + C^2 r / (2Q(1 - \rho)) + eB / t
\]

Or, \( \mu = FD / Q + B^2 s / (2Q(1 - \rho)) + r(Q(1 - \rho) - B)^2 / (2Q(1 - \rho)) + eBD / Q \) \quad (5.3)

Partial differentiation of \( \mu \) with respect to \( B \) and \( Q \) yields the following results.

\[
\frac{\partial \mu}{dB} = Bs / (Q(1 - \rho)) - r(Q(1 - \rho) - B) / (Q(1 - \rho)) + eD / Q = 0
\]

Or, \( B = (1 - \rho)(rQ - eD) / (r + s) \) \quad (5.4)

Also, \( \frac{d \mu}{dQ} = -FD / Q^2 - B^2 s / (2Q^2 (1 - \rho)) + r(1 - \rho) / 2 - rB^2 / (2Q^2 (1 - \rho)) - eBD / Q^2 = 0 \)

Or, \( rQ^2 (1 - \rho)^2 = 2FD(1 - \rho) + (r + s)B^2 + 2eBD(1 - \rho) \) \quad (5.5)

Substituting the value of \( B \) from (5.4) in (5.5), we get:

\[
rQ^2 (1 - \rho)^2 = 2FD(1 - \rho) + (1 - \rho)^2 (rQ - eD)^2 (r + s)^2 / (r + s)^2 + 2eD(rQ - eD) / (r + s)
\]
Or, \( rQ^2(1 - \rho)^2 = 2FD(1 - \rho) + (1 - \rho)^2(rQ - eD)^2 + 2eD(rQ - eD) / (r + s) \)

Or, \( rQ^2(1 - \rho)^2 = 2FD(1 - \rho) + (1 - \rho)^2(r^2Q^2 - e^2D^2) / (r + s) \)

Or, \( rQ^2 = 2FD / (1 - \rho) + (r^2Q^2 - e^2M^2) / (r + s) \)

Or, \( rsQ^2 = 2(r + s)FD / (1 - \rho) - e^2D^2 \)

Or, \( Q^* = \sqrt{(2(r + s)FD / (1 - \rho) - e^2D^2) / (rs)} \) \hspace{1cm} (5.6)

Or, \( Q^* = \sqrt{\frac{2FD}{r(1 - \rho)} - \frac{e^2D^2}{r(r + s)}} \sqrt{\frac{r + s}{s}} \)

Then from (5.4), \( B^* = (1 - \rho)(rQ^* - eD) / (r + s) \) \hspace{1cm} (5.7)

5.5 Special Cases of the Integrated Cash Management Model

We now consider the special case in which the insurance rate \( e = 0 \).

Then, \( Q^* = \sqrt{2(r + s)FD / ((1 - \rho)(rs))} \)

Or, \( Q^* = (\sqrt{2FD / ((1 - \rho)r})(\sqrt{r + s}) / s) \) \hspace{1cm} (5.8)

Then, \( B^* = (1 - \rho)Q^*r / (r + s) \) \hspace{1cm} (5.9)

Further, from the various results derived earlier, linking the different variables and parameters, we have: \( C^* = Q^*(1 - \rho) - B^* = (1 - \rho)Q^*s / (r + s) \) \hspace{1cm} (5.10)

As before, we note that Baumol’s model is a special case of the Integrated Cash Management (ICM) model; simply let the interest rate on borrowed money, \( s = \infty \), and also the cash replenishment rate \( P = \infty \). Then, \( \rho = 0 \), and we get: \( Q^* = \sqrt{2FD / r} \), \( B^* = 0 \), and \( C^* = Q^* \). The
obvious result is that if the length of the planning horizon is $T$ periods (days or weeks or other unit of time), the minimum cost according to Baumol’s model is given by:

$$
\psi(C^*) = (FD / C^* + C^* r / 2) T \quad \text{or} \quad \psi(Q^*) = (FD / Q^* + Q^* r / 2) T
$$

(5.11)

If the interest rate $s$ charged by the bank on borrowed money is finite, but the insurance rate $e=0$, and the cash replenishment rate $p = \infty$, the ICM model is reduced to the Sastry-Ogden-Sundaram model. From equations (5.3), (5.8), (5.9) and (5.10), we get the expression for the minimum cost for the entire planning horizon of length $T$ as given below:

$$
\psi(Q^*) = (FD / Q^* + C^2 r / 2Q^* + B^2 s / 2Q^*) T
$$

(5.12)

If the insurance rate $e=0$, we can re-write equation (5.1) as follows:

Total cost per cycle, $\psi(t) = F + Bgs / 2 + Chr / 2$

(5.13)

But, $g = \alpha + \beta$; $h = \gamma + \delta$; $t = (\alpha + \beta) + (\gamma + \delta) = g + h$; $h = (t - g)$;

Also, $\alpha = B / D$; $\beta = B / (P - D)$; $\gamma = C / (P - D)$; $\delta = C / D$;

Or, $B = \alpha D$; $B = \beta(P - D)$; $(\alpha + \beta)D = \beta P$; $\beta = (D / P)(\alpha + \beta) = \rho g$; $\alpha = (1 - \rho)g$

$$
C = \delta D; \quad C = \gamma / (P - D); \quad (\gamma + \delta)D = \gamma P; \quad \gamma = (D / P)(\gamma + \delta) = \rho h; \quad \delta = (1 - \rho)h;
$$

Substituting the values of $B$, $C$, and $h$ from the above equations, we get:

$$
\psi(t) = F + \alpha Dgs / 2 + \delta D(t - g)r / 2
$$

(5.14)

Further, $\psi(t) = F + (1 - \rho)gDgs / 2 + (1 - \rho)(t - g)D(t - g)r / 2$

(5.15)

For the moment suppose that the cycle length $t$ is a constant and only $g$ is allowed to vary.
Then, \[ \psi(g) = F + (1 - \rho)Dsg^2 / 2 + (1 - \rho)Dr(t - g)^2 / 2 \] (5.16)

Cleary, \( \psi(g) \) is a convex function and it has a global minimum.

Differentiating \( \psi(g) \) with respect to \( g \), we get:

\[
\frac{d\psi}{dg} = (1 - \rho)Dsg - (1 - \rho)Dr(t - g) = 0
\]

(5.17)

From (5.15) we have: \( g^* = rt / (r + s) \); Then, \( h^* = t - g^* = st / (r + s) \);

Therefore, from (5.16), we get:

\[
\psi(t) = F + (1 - \rho)Dsr^2t^2 / 2(r + s)^2 + (1 - \rho)Dr^2t^2 / 2(r + s)^2
\]

Or, \( \psi(t) = F + (1 - \rho)Dsr^2t^2 / 2(r + s) \)

Let \( \lambda = (1 - \rho)Dr / (r + s) \) (5.18)

Then, \( \psi(t) = F + \lambda t^2 / 2 \) (5.19)

Also, the average cost per unit time \( \mu(t) \) is given by:

\[ \mu(t) = F / t + \lambda t / 2 \] (5.20)

\[
\frac{d\mu}{dt} = -F / t^2 + \lambda / 2 = 0
\]

(5.21)

Or, the optimal cycle time, \( t^* = \sqrt{2F / \lambda} \) (5.22)

Then, the optimal number of cycles or transactions, \( y^* = T / t^* \) (5.23)
Of course, the values of $t^*$ and $y^*$ may not turn out to be integers. Then, we need to do more work to find integer solutions for the number of transactions or the cycle times as described in Proposition 2.5 (in Chapter 2) or Proposition 3.5 (in Chapter 3). It may be noted that these two propositions are essentially the same and $\lambda = Dr$ in both the Propositions. But here in (5.18) we define $\lambda = (1 - \rho)Dr / (r + s)$. However, if we let $P = \infty$, and $s = \infty$, we get $\rho = 0$ and $\lambda = Dr$. In other words, the models and problems discussed in Chapters 2 and 3 are special cases of the model discussed in this Section with the insurance rate $e = 0$. Further, if we let $P = \infty$, and $e = 0$, we get $\rho = 0$ and $\lambda = Dr / (r + s)$. This special case is nothing but the model of Sastry-Ogden-Sundaram or the BAT model with short-term borrowing. In essence the Integrated Cash Management Model (ICM) discussed in this Chapter is a generalization of the models discussed in Chapters 2 and 4.

5.6 Determination of the Critical Insurance Rate

It is clear from equations (5.6) and (5.7) that both $Q^*$ and $B^*$ are decreasing functions of the insurance rate $e$. Our next step is to find the critical value of $\bar{e}$ at which $B^*$ will be zero. We can use equations (5.6) and (5.7) to determine the value of $\bar{e}$. If $B^*$ is zero, from (5.7) we get:

\[ Q^* = \bar{e}D / r. \]

Then, from (5.6) we must have: $eD / r = \sqrt{(2(r + s)FD / (1 - \rho) - e^2D^2) / (rs)}$

Or, $e^2D^2 / r^2 = (2(r + s)FD / (1 - \rho) - e^2D^2) / (rs)$

Or, $se^2D^2 / r + e^2D^2 = 2(r + s)FD / (1 - \rho)$

Or, $se^2D^2 + re^2D^2 = 2r(r + s)FD / (1 - \rho)$

Or, $(r + s)e^2D = 2r(r + s)F / (1 - \rho)$
After simplification, we get: \( \bar{e} = \sqrt{\frac{2rF}{(1 - \rho)D}} \) \hspace{1cm} (5.24)

In Chapter 4, we have defined the critical insurance rate \( \hat{e} = \sqrt{\frac{2rF}{D}} \)

Therefore, \( \bar{e} = \sqrt{\frac{2rF}{(1 - \rho)D}} = \hat{e} / \sqrt{1 - \rho} \) \hspace{1cm} (5.25)

We will refer to \( \bar{e} \) also as the critical value of the insurance rate.

From (5.14) we see that if \( \rho \) is less than 1, \( \bar{e} \) will greater than \( \hat{e} \). In turn, this implies that it may be beneficial for a firm to borrow from a bank, even if the bank requires a slightly higher insurance rate, if the cash replenishment rate \( P \) is finite.

If \( \rho = 0 \), but \( e \) is greater than 0 and less than the critical value \( \bar{e} \), we have:

\[
\psi(Q') = \left( FD / Q + B^2s / (2Q^*) + rC^2 / (2Q^*) + eDB / Q^* \right)T
\]
\hspace{1cm} (5.26)

If \( 0 \leq \rho < 1 \), and \( 0 < e < \bar{e} \), we have:

\[
\psi(Q') = \left( FD / Q + B^2s / (2Q(1 - \rho)) + r(Q(1 - \rho) - B)^2 / (2Q(1 - \rho)) + eBD / Q \right)T
\]
\hspace{1cm} (5.27)

In obtaining \( \psi(Q') \), we first calculate \( Q^* \) using equation (5.6), and then calculate the corresponding values of \( B \) or \( B^* \) and \( C \) or \( C^* \) using equations (5.9) and (5.10) respectively.

We will now re-visit the illustrative example discussed in Ogden and Sundaram (1998) paper:

\( F = 50, r = 0.000137, \) and \( D = 4000 \). We will let \( P = 8000 \); then, \( \rho = 0.5 \), unless otherwise specified.

Time horizon under consideration, \( T = 365 \) days.

Substituting the values of \( r, F, \rho \) and \( D \) in equation (5.14), we get \( \bar{e} = 0.002617 \)
In other words, if the insurance rate is equal to 0.002617 (i.e. 0.2617%) or higher, it is no longer advantageous to borrow any money from the bank. Assume that the bank requires the firm to buy insurance at the rate equal to half of \( \bar{e} \). Or, \( e = \bar{e} / 2 = 0.001309 \)

For purposes of comparison, we will provide the solutions for three scenarios.

Scenario 1: Optimal solution without short-term borrowing

Number of transactions \( y^* = 19, m=4, U=20, n=15, L=19 \)

Total Cost = 1910.27

Scenario 2: Optimal solution with short-term borrowing, but insurance rate \( e=0 \).

Number of transactions \( y^* = 15, m=5, U=25, n=10, L=24 \)

Total Cost = 1499.00

Scenario 3: Optimal solution with short-term borrowing and with insurance rate \( e=0.001309 \)

Number of transactions \( y^* = 16, m=13, U=23, n=3, L=22 \)

Total Cost = 1792.61

5.7 Illustrative Examples

We will solve the same problems that were solved in the previous Chapters so that we can see the differences in the total costs.

5.7.1 An Example from a Textbook with Enhancements

We will start with the numerical example discussed in one of the standard textbooks, *Fundamentals of Corporate Finance*, 12th Edition, by Ross, Westerfield and Jordan (2019).

(This textbook is adopted for the introductory finance course FIN 323 at ODU). The following problem is extracted from Chapter 19, Cash and Liquidity Management, The BAT Model: Example 19A.1 (from page 667 of the book of 12th Edition)
“The Vulcan Corporation has cash outflows of $100 per day, seven days a week. The interest rate is 5 percent, and the fixed cost of replenishing cash balances is $10 per transaction. What is the optimal cash balance? What is the total cost?”

It is not explicitly stated in the problem but it is assumed that the time period under consideration is one year. The total cash needed for the year is $36,500. (365*100=36500).

The parameters and variables relevant to the problem are defined below.

Unit of time: one day.

Time horizon under consideration, $T = 365$ days.

Cash outflows, $D =$100 per day (Daily demand for cash)

Cost per transaction, $F =$10  (brokerage fee for liquidating short-term securities)

Annual interest rate (opportunity cost) on the cash withdrawn = 5%

Daily interest rate, $r = 0.05/365=0.000137$ (opportunity cost or lost interest income)

Annual interest rate charged by the bank on borrowed money = 8%

Daily interest rate on money borrowed, $s = 0.08/365=0.0002192$

Assume that $\rho = 0.5$, in all three scenarios unless otherwise specified (i.e., $P=2D$).

Critical insurance rate, $\tilde{e} = \sqrt{(2rF / (1-\rho)D)} = \sqrt{2 * 0.000137 * 10 / (0.5 * 100)} = 0.007402$

Assume that the insurance rate of the bank $e = \tilde{e} / 2 = 0.007402 / 2 = 0.003701$ (or, 0.3701%)

Scenario 1: Optimal solution without short-term borrowing
Number of transactions $y^* = 7, m=1, U=53, n=6, L=52$

Total Cost = 135.18

Scenario 2: Optimal solution with short-term borrowing, but insurance rate $e=0$.

Number of transactions $y^* = 5, m=5, U=73, n=0, L=72$

Total Cost = 106.15

Scenario 3: Optimal solution with short-term borrowing and with insurance rate $e=0.003701$

Number of transactions $y^* = 6, m=5, U=61, n=1, L=60$

Total Cost = 127.01

5.7.2 An Example with a Typical Convex Cost Curve

Suppose that the Everest Construction Company has several construction projects in progress. The following data is available.

Unit of time: one day.

Time horizon under consideration, $T = 215$ days. (All projects have to be completed in 215 days.)

Cash outflows, $D =$10,000 per day

Cost per transaction, $F =$127 (brokerage fee for liquidating short-term securities)

Annual interest rate (opportunity cost) on the cash withdrawn = 6%

Daily interest rate, $r = 0.06/365=0.00016438$ (opportunity cost or lost interest income)

Annual interest rate charged by the bank on borrowed money = 9.5%

Daily interest rate on money borrowed, $s=0.095/365=0.0002603$ (rate charged by bank)

Assume that $\rho = 0.5$, in all three scenarios unless otherwise specified (i.e., $P=2D$).
Critical insurance rate, \( \tilde{e} = \sqrt{\frac{2rF}{(1-\rho)D}} = \sqrt{\frac{2 \times 0.0001648 \times 127}{0.5 \times 10000}} = 0.002889 \)

Assume that the insurance rate of the bank \( e = \frac{\tilde{e}}{2} = 0.002889/2 = 0.001445 \) (or, 0.1445%)

Scenario 1: Optimal solution without short-term borrowing

Number of transactions \( y^* = 12, m=11, U=18, n=1, L=17 \)

Total Cost = 3107.42

Scenario 2: Optimal solution with short-term borrowing, but insurance rate \( e=0 \).

Number of transactions \( y^* = 10, m=5, U=22, n=5, L=21 \)

Total Cost = 2434.94

Scenario 3: Optimal solution with short-term borrowing, and with insurance rate \( e=0.001445 \)

Number of transactions \( y^* = 10, m=5, U=22, n=5, L=21 \)

Total Cost = 2931.29

5.7.3 An Example with Suitably Adjusted Data

Let us consider a problem for which we can find integer solutions easily.

Unit of time: one day.

Time horizon under consideration, \( T = 300 \) days.

Cash outflows, \( D = \$1000 \) per day

Cost per transaction, \( F = \$22.50 \) (brokerage fee for liquidating short-term securities)

Annual interest rate (opportunity cost) on the cash withdrawn = 7.3%

Daily interest rate, \( r = 0.073/365 = 0.0002 \)

Annual interest rate charged by the bank on borrowed money = 11.0%

Daily interest rate on money borrowed, \( s = 0.11/365 = 0.0003014 \) (rate charged by bank)
Assume that $\rho = 0.5$, in all three scenarios unless otherwise specified (i.e., $P=2D$).

Critical insurance rate, $\tilde{e} = \sqrt{\frac{2rF}{(1-\rho)D}} = \sqrt{\frac{2\times0.0002\times22.5}{(0.5\times1000)}} = 0.004243$

Assume that the insurance rate by the bank $e = \tilde{e}/2 = 0.004243/2 = 0.002122$ (or, 0.2122%)

Scenario 1: Optimal solution without short-term borrowing

Number of transactions $y^* = 14, m=6, U=22, n=8, L=21$

Total Cost = 636.60

Scenario 2: Optimal solution with short-term borrowing, but insurance rate $e=0$.

Number of transactions $y^* = 11, m=3, U=28, n=8, L=27$

Total Cost = 493.47

Scenario 3: Optimal solution with short-term borrowing, and with insurance rate $e=0.002122$

Number of transactions $y^* = 12, m=12, U=25, n=0, L=24$

Total Cost = 595.42

5.7.4 An Example with a Flat-Bottom V-shaped Cost Curve

Let us consider a problem which requires considerable work to generate the complete set of integer solutions.

Unit of time: one day.

Time horizon under consideration, $T = 180$ days.

Cash outflows, $D = $120,000 per day

Cost per transaction, $F = $288.00 (brokerage fee for liquidating short-term securities)

Annual interest rate = 5.84

Daily interest rate, $r = 0.0584/365=0.00016\%$ (opportunity cost or lost interest income)

Annual interest rate charged by the bank on borrowed finds = 8.91%

Assume that $\rho = 0.5$, in all three scenarios unless otherwise specified (i.e., $P=2D$).
Daily interest rate on money borrowed, \( s = \frac{0.0891}{365} = 0.0002441 \) (rate charged by bank)

Critical insurance rate, \( \bar{\varepsilon} = \sqrt{\left(2RF / (1 - \rho)D\right)} = \sqrt{\left(2 \times 0.00016 \times 288 / (0.5 \times 120000)\right)} = 0.000124 \)

Assume that the insurance rate by the bank \( e = \frac{\bar{\varepsilon}}{2} = \frac{0.000124}{2} = 0.000062 \) (or, 0.062\%)

Scenario 1: Optimal solution without short-term borrowing

Number of transactions \( y^* = 23, m=19, U=8, n=4, L=7 \)

Total Cost = 13402.60

Scenario 2: Optimal solution with short-term borrowing, but insurance rate \( e=0 \).

Number of transactions \( y^* = 18, m=18, U=10, n=0, L=9 \)

Total Cost = 10403.15

Scenario 3: Optimal solution with short-term borrowing and with insurance rate \( e=0.00062 \)

Number of transactions \( y^* = 20, m=20, U=7, n=0, L=8 \)

Total Cost = 12536.87

The total cost as a function of the number of transactions is displayed in Figure 5.2. The cost curve does no longer appear to have a flat bottom; but at \( y = 18, 19, \) and 20, the costs are almost equal.

Figure 5.2: Diminished Flat Bottom V-Shaped Piecewise Linear Convex Cost Function
It may be noted that $\rho = \frac{D}{P}$ where $D$ is rate of disbursement of cash per time period (i.e., per day) in a continuous stream and $P$ is rate of replenishment of cash per time period (i.e., per day) in a continuous stream. Obviously, there is a need to maintain that $0 \leq \rho < 1$. In the parlance of inventory management, if the value of $P$ is maintained slightly higher but close to $D$ (or $\rho$ is close 1), as mentioned earlier the corresponding policy is called the Just-In-Time (JIT) Strategy. Of course, such a strategy can sometimes lead to severe disruption in the supply chain in the production and manufacturing industries. Because of the severe adverse impact a delay or default in payments may have on the share price of a corporation, from a cash management point of view, the ICM model presented here is more of a theoretical concept than a practical feasibility.
CHAPTER 6

SINGLE PERIOD STOCHASTIC DEMAND CASH MANAGEMENT MODELS WITH BETA PROBABILITY DISTRIBUTION

6.1 Introduction

All the cash management models analyzed in the previous Chapters are deterministic in the sense that the values of the parameters are known with certainty. Especially, if the demand rate for cash is constant and known in advance, it will be relatively easy for a firm to make arrangements with a bank to borrow the necessary funds to meet its transactions demand for cash. In contrast, there are situations wherein the demand for cash is unknown in advance and the firm has to make some type of contingency plans. In other words, the demand for cash is a random variable. There are inventory problems in which the demand for an item is not only random but also transitory in the sense that the item itself is in demand for a short interval of time such as a day or a season. At the end of the time interval, be it a day or a season, the item loses almost all of its value. Two common examples, often cited in the literature, that fit this scenario are the Newsvendor Problem and the Christmas Tree Problem (Hadley and Whitin, 1963, and Winston, 2004). A newspaper loses its value after 24 hours, and similarly, nobody will buy a (natural) Christmas Tree in the month of January after the Christmas season has ended. In inventory theory, problems of this type are called Single Period Stochastic Demand models. Similar problems are encountered in cash management also. Since the demand is a random variable, the results of the analysis depend on the assumed probability distribution for the demand. Some results involving stochastic or probabilistic demand for cash with shortage costs or proportional charges are described in Girgis
(1968) and Neave (1970). In several of the stochastic demand models (Gregory, 1976) the results are derived based on the assumption that the demand follows uniform or exponential distribution. But in cash management, a beta distribution may be a better fit. In this Chapter we will consider the hypothetical case of the JFS Company and find the optimal cash balance for the single period stochastic demand cash management model using beta distribution for the demand.

6.2 Problem Definition

The JFS Company, whose motto is “Just-For-the-Season,” is a midsize company that specializes in supplying various seasonal items to several retailers, big and small. The company prepares its cash management budgets on a quarterly basis. Since people buy different items during different seasons, the company’s cash disbursement needs differ significantly from one quarter to the next. The fourth quarter of every year covering the months of October, November and December, often referred to as the Holiday Season, is the most profitable quarter for the company. While it is a profitable quarter from an accounting point of view, the company’s finance department has to make arrangements to meet the company’s payment obligations to its suppliers for the quarter, since it receives payments from its own customers much later, often in the next quarter.

Suppose that the finance director of the company is about to embark on the task of preparing the cash budget for the fourth quarter. *It is the company’s policy to have the entire amount of the cash needed to meet the payment obligations for the quarter on hand at the beginning of the quarter.* An analysis of the past data reveals that the total of the cash disbursements was never less than 20 million dollars in the fourth quarter of any year and also never higher than 80
116 million dollars. However, depending on the economic conditions in the past the total of disbursements was sometimes slightly higher than the minimum of 20 million and, of course, at other times it was closer to the maximum of 80 million. There were also times when the total of the disbursements was closer to the average of 50 million, a few times higher and a few times lower than the average. For the coming fourth quarter, the finance director believes that economic conditions are very favorable and the amount needed would be higher than the average, perhaps closer to the maximum of 80 million. Currently, the company earns a quarterly interest rate of 0.5% (nominal annual rate of 2%) on its short-term securities and the bank charges 1.25% per quarter (nominal annual rate of 5%) on the money borrowed from the line of credit. What is the amount of short-term securities that should be redeemed and converted into cash at the beginning of the fourth quarter? Obviously, the objective is to minimize the total expected cost of meeting the disbursement obligations for the fourth quarter without depending on any other cash inflows or receipts.

6.3 Probability Distribution of Demand

The cash management problem described here is similar to the single-period stochastic inventory model in terms of its mathematical structure. Of course, several articles related to the solution of the single-period stochastic inventory model have appeared in the literature. There have been a few articles specifically addressing the single-period cash management problem itself. However, in almost all cases, the probability distribution describing the demand for the inventory item or cash is assumed to be either uniform or exponential. From a theoretical point of view, both these distributions are easy to understand and analyze. However, from a practical point of view their
usefulness is limited, since the exponential distribution assigns higher probabilities to smaller values and smaller probabilities to higher values. The uniform distribution ignores the fact that the values of many random variables cluster around the mean or have a mode different from the mean. The exponential distribution also has the same weakness. Based on the description of the cash management problem, it is reasonable to consider the beta distribution as a viable alternative. The parameters of the beta distribution can be so chosen that it can be a good approximation for the distribution of the amount of cash needed on hand at the beginning of the quarter. Two of the advantages of the beta distribution are that the random variable can have a specific lower boundary and a specific upper boundary and also a mode same as the mean or different from the mean. Further, depending on the values chosen for the parameters, it is possible to generate several different shapes for the probability density function. Generally, the beta distribution is presented in two different forms. The probability density function (pdf) corresponding to the Beta distribution with parameters $\alpha$ and $\beta$ is defined as follows:

$$\xi(u) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1}(1-u)^{\beta-1}, \quad 0 \leq u \leq 1$$

Also, the expected value of $u$, $E(u) = \alpha / (\alpha + \beta)$

Alternatively, If the random variable can assume values in the interval $[a,b]$ is given below:

$$\xi(w) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \left( \frac{w-a}{b-a} \right)^{\alpha-1} \left( \frac{b-w}{b-a} \right)^{\beta-1} \left( \frac{1}{b-a} \right), \quad 0 < a \leq w \leq b$$

Let $\alpha = 4, \beta = 2$, and $\alpha + \beta = 6$.

Then, $\xi(w) = 20 \left( \frac{w-a}{b-a} \right)^3 \left( \frac{b-w}{b-a} \right) \left( \frac{1}{b-a} \right), \quad 0 < a \leq w \leq b$
Further, it may also be noted that the uniform distribution is a special case of the beta distribution with parameters $\alpha = 1, \beta = 1$, and $\alpha + \beta = 2$.

When $\alpha = 1, \beta = 1$, and $\alpha + \beta = 2$, the corresponding probability density functions will simply be in the form given below:

$$
\xi(u) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1}(1-u)^{\beta-1} = \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)} u^0(1-u)^0 = 1, \quad 0 \leq u \leq 1
$$

Or, $\xi(u) = 1, \quad 0 \leq u \leq 1$

Similarly, $\xi(w) = 1/(b-a), \quad 0 < a \leq w \leq b$

We will now describe three different shapes of the beta distribution.

1a) Beta(4,2, (0,1)): $\xi(u) = 20u^3(1-u)^2, \quad 0 \leq u \leq 1$

The pdf of the beta distribution with $\alpha = 4, \beta = 2$, and $0 \leq u \leq 1$ is shown in Figure 6.1a.

![Beta pdf α=4, β=2](image)

Figure 6.1a. pdf of the beta distribution with $\alpha = 4, \beta = 2$, and $0 \leq u \leq 1$

1b) Beta(3,3, (0,1)): $\xi(u) = 20u^2(1-u)^2, \quad 0 \leq u \leq 1$
The pdf of the beta distribution with $\alpha = 3, \beta = 3$, and $0 \leq u \leq 1$ is shown in Figure 6.1b.

![Beta pdf $\alpha=3, \beta=3$](image)

**Figure 6.1b.** pdf of the beta distribution with $\alpha = 3, \beta = 3$, and $0 \leq u \leq 1$

(1c) Beta(2,4, (0,1)): $\xi(u) = 20u(1-u)^3$, $0 \leq u \leq 1$

The pdf of the beta distribution with $\alpha = 2, \beta = 4$, and $0 \leq u \leq 1$ is shown in Figure 6.1c.

![Beta pdf $\alpha=2, \beta=4$](image)

**Figure 6.1c.** pdf of the beta distribution with $\alpha = 2, \beta = 4$, and $0 \leq u \leq 1
The values of the means and modes are given below for the three cases:

1a) Beta(4,2, (0,1)): \( \xi(u) = 20u^3(1-u)^2, \quad 0 \leq u \leq 1 \)

Mean = 2/3 and mode= 3/4

1b) Beta(3,3, (0,1)): \( \xi(u) = 20u^2(1-u)^2, \quad 0 \leq u \leq 1 \)

Mean = 1/2 and mode=1/2

1c) Beta(2,4, (0,1)): \( \xi(u) = 20u(1-u)^3, \quad 0 \leq u \leq 1 \)

Mean = 1/3 and mode=1/4

6.4 An Example of Empirical Data

For the current cash management problem, the finance director believes that economic conditions are very favorable and the amount needed would be higher than the average, perhaps closer to the maximum of 80 million for the next fourth quarter. The empirical distribution (Relative Frequency) of the amount of cash required for disbursement at the beginning of the fourth quarter of each year, based on past data collected by the company, is given in Table 6.1, and also shown in Figure 6.2 below.

| Table 6.1. Past Cash Requirements in the Fourth Quarter |
|----------------|----------------|----------------|----------------|
| Index \( j \) | Amount in millions | Relative Frequency | Cumulative Frequency |
| 1 | 20 | 0.03 | 0.03 |
| 2 | 31 | 0.04 | 0.07 |
| 3 | 44 | 0.08 | 0.15 |
| 4 | 53 | 0.21 | 0.36 |
| 5 | 65 | 0.29 | 0.65 |
| 6 | 69 | 0.24 | 0.89 |
| 7 | 71 | 0.07 | 0.96 |
| 8 | 80 | 0.04 | 1.00 |
6.5 Results under the Assumption of a Beta Distribution

A comparison of Figures 6.1a and 6.2 shows that perhaps, the beta distribution with $\alpha = 4, \beta = 2$, can be used as an approximation to the given empirical distribution. Let $r$ be the quarterly interest rate on the short-term securities that will be foregone, if converted into cash. Let $s$ denote the quarterly interest rate on the amount borrowed from the line of credit. Then, we have: $r = 0.005$ and $s = 0.0125$. Further, let $a = 20, b = 80$, and $w$ denote the demand for the amount of cash needed on hand at the beginning of the fourth quarter. We can represent the corresponding beta density function as given below:

$$\xi(w) = 20 \left( \frac{w-a}{b-a} \right)^3 \left( \frac{b-w}{b-a} \right) \left( \frac{1}{b-a} \right), \quad 0 < a \leq w \leq b \quad (6.1)$$

Further, let $Q$ be the amount of cash obtained from the liquidation of short-term securities at the beginning of the fourth quarter. It is assumed that there is no carry-over of funds from the previous (third) quarter into the current (fourth) quarter and therefore, the liquidation charges cannot be avoided.
and also, it is assumed the brokerage fees or redemption charges are fixed for each transaction and do not depend on the amount of securities liquidated. For this reason, the cost of converting short-term securities into cash is not included in the expected cost.

Then the expected interest cost (total of the interest foregone because of the redemption of short-term securities and the interest paid on borrowed funds) for the quarter is given by the following expression. The details are given in the Appendix. The expected cost is plotted as a function of $Q$ in Figure 6.3 and also given in Table 6.2.

$$ E[\psi(Q)] = \int_{Q}^{a} (Q - w) \xi(w) dw + \int_{b}^{Q} (w - Q) \xi(w) dw \quad \text{for any } Q, a < Q < b. \quad (6.2) $$

![Figure 6.3. Expected Cost vs. $Q$, Initial Amount of Cash on Hand – Case A](image)

From Table 6.2, it can be observed that the optimal amount of securities to be converted into cash, $Q^* = 67.40$ million dollars and the corresponding expected cost, $E[\psi(Q^*)] = \$60,430$. 


From Figure 6.3 also, it can be seen that the minimum occurs at the value of $Q$ between 60 and 70 millions. The values of the horizontal and vertical axes are in millions. The curve representing the expected cost is a convex function and has a global or unique minimum. It is proved in the appendix that there is no need for Table 6.3 at all. The cost function $E[\psi(Q)]$ is minimized at the value of $Q^*$ when $F(Q^*) = s / (r + s)$, where $F(Q^*)$ is the cumulative probability, $P(w \leq Q^*)$.

Using the transformation or scaling, $w = a + (b - a)u$, we must have: $F(u^*) = s / (r + s)$.

Since $r = 0.005$, and $s = 0.0125$, we get $F(u^*) = 0.7143$.

The cumulative distribution function, $F(u) = 5u^4 - 4u^5$.

We have to solve for $u$ such that $5u^4 - 4u^5 = 0.7143$.

Using any one of the several methods (including Newton’s Method), we get $u^* = 0.7889$

Then, $Q^* = a + (b - a)u^* = 20 + 60 \times 0.7889 = 67.334$

From Appendix B, $E[\psi(Q)] = 20(b - a)\left[(r + s)(x^5 / 20 - x^6 / 30) + s(1 / 30 - x / 20)\right]$.

The variable $x$ serves the same purpose as $u$.

Hence, $E[\psi(Q)] = (b - a)\left[(r + s)(u^5 - 2u^6 / 3) + s(2 / 3 - u)\right] = 0.060430$ million = $60,430$

The recommendation is to have an amount of 67.334 million dollars on hand at the beginning of the fourth quarter.
Table 6.2. Expected Cost as a Function of $Q$
Under the Assumption of Beta (4,2) Distribution

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<th>Expected Cost</th>
<th>Amount $Q$</th>
<th>Expected Cost</th>
<th>Amount $Q$</th>
<th>Expected Cost</th>
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6.6 Results under the Assumption of a Discrete Distribution

We have so far proceeded with the assumption that the given empirical distribution can be approximated by a beta distribution. Of course, we can try to arrive at the optimal value of $Q$, using the discrete distribution itself. It is proved in the Appendix that in that case the optimal solution would be the smallest value of $Q$ such that $F(Q^*) \geq s / (r + s)$.

Since $s / (r + s) = 0.7143$, from Table 6.1 we see that for $Q^* = w_e = 69$, $F(Q^*) \geq s / (r + s)$.

The expected cost, $E[\psi(Q)] = r \sum_{j=1}^{j=k} (Q - w_j) \xi_j + s \sum_{j=k+1}^{j=n} (w_j - Q) \xi_j$

Since $j = 6$, we get: $E[\psi(Q)] = r \sum_{j=1}^{j=5} (Q - w_j) \xi_j + s \sum_{j=6}^{j=n} (w_j - Q) \xi_j = 0.0548$

The values of the expected cost $E[\psi(Q)]$ corresponding to the different cash requirements given in Table 6.1 are listed in Table 6.3. It can be seen from Table 6.3 that at $Q^* = w_e = 69$, $E[\psi(Q)] = $54,800. It is also the minimum compared to the other values. The relationship between $Q$ and $E[\psi(Q)]$ under the assumption of discrete distribution is also shown graphically in Figure 6.4.

![Figure 6.4: Expected Cost vs. $Q$, Initial Amount of Cash on Hand – Case B](image-url)
Table 6.3. Expected Cost as a Function of $Q$
Under the Assumption of the Discrete Distribution

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CHAPTER 7
AN EXTENSION OF BAUMOL’S BASIC CASH MANAGEMENT MODEL:
DIFFERENT PARAMETER VALUES IN DIFFERENT PERIODS

7.1 Introduction

In the basic version of Baumol’s cash management model, it is assumed that the values of the parameters such as the demand rate for cash, the brokerage fee for the liquidation of any arbitrary amount of securities, the proportional charge for every dollar withdrawn and the interest rate remain constant from one year to the next. Further, another implicit assumption underlying the model is that the planning horizon is infinite. In other words, it is a static model in which the values of the parameters never change, regardless of the length of the time horizon. Of course, in practice the longer the time horizon, the higher are the odds that the parameter values will change significantly. Further, the model’s square-root-formula may, more often than not, lead to fractional number of transactions and also fractional number of days (or weeks) in the cycle-times between two consecutive transactions. We have already seen the enhancements to this model that help us deal with shorter time horizons and the integer requirements for the number of transactions and the number of days in the cycle-times between two consecutive transactions. At the other end of the spectrum, we frequently encounter situations in which the parameter values change from one month to the next or from one quarter to the next, if not on a daily or weekly basis. We may call this type of model that involves different parameter values in different periods a Multi-Period Dynamic (MPD) cash management model. Just as the EOQ model (Whitin, 1952) and the basic model of Baumol (1952) are considered to be mathematically equivalent, so are the Wagner and Whitin (1958) model in inventory management theory and the multi-period
dynamic (MPD) cash management model. In this Chapter, we will describe a multi-period dynamic (MPD) cash management model that is derived from the basic version of the Baumol model. Further, we will present an optimization technique that would help us obtain the optimal solution to the MPD cash management model.

7.2 Description of the Multi-Period Dynamic (MPD) Cash Management Model

In the basic version of Baumol’s cash management model, it is assumed that the available funds are already invested in securities that yield, perhaps, a higher return than the prevailing interest rate. This version of the model is discussed in many standard textbooks such as Fundamentals of Corporate Finance, 12th Edition, by Ross, Westerfield and Jordan (2019). (This textbook is adopted for the introductory finance course FIN 323 at ODU).

In the multi-period dynamic (MPD) model also, we assume that the available funds are invested in securities that yield a higher return than the prevailing interest rate. The similarity between the two models ends there. As a specific example of the MPD model, let us consider the hypothetical case of the Green Quantum Energy (GQE) company. Suppose that the GQE company has already developed and patented a new electric battery technology. It has also raised a total of $740 million from venture capital firms and private investors for the construction of the factory and the warehouses. The capital raised is already invested in a mutual fund managed by a well-known investment company with a proven record of higher returns. The company management has a separate fund established by the founders of the company for administrative expenses. The next step for the company is to build a factory to produce and sell its batteries in the open
market. It will take a year for the company to finish the construction of the factory and the warehouses.

The estimated costs for the four quarters of the coming year are as follows: Q1: $160 million, (b) Q2: $180 million, (c) Q3: $210 million, and (d) Q4: $190 million. The total costs for the year are equal to $740 million, an amount already invested in a brokerage account. The money has to be paid to the main contractor at the end of each quarter. It is the responsibility of the main contractor to pay all the sub-contractors involved in the construction work.

The company management can withdraw funds at the end of a quarter to pay the main contractor, only with approval of the board of directors of the company. The quarterly estimated costs for getting the approval of the board are as follows: Q1: $48,000 ($0.048 million), (b) Q2: $52,000 ($0.052 million), (c) Q3: $47,000 ($0.047 million), and (d) Q4: $32,000 ($0.032 million).

The investment company will charge the following rates for every dollar withdrawn in a quarter: (a) Q1: 1.0%  (b) Q2: 1.7%  (c) Q3: 0.8% and (d) Q4: 1.5%

The finance department has estimated that the interest rates at the end of each of the three quarters will be as follows: This rate indicates the opportunity cost on any cash carried forward to the next quarter. There will be no cash at the end of the fourth quarter. (a) Q1: 0.7%  (b) Q2: 1.6%  (c) Q3: 0.6% (d) Q4: rate not estimated

The objective is to determine the liquidation schedule that would cost the lowest.
Case A: In this case, the GQE company is required to make a payment to the main contractor at the end of every quarter by withdrawing the funds at the end of that quarter itself or using the cash carried forward from a previous quarter. Here the assumption is that the main contractor will not accept any delayed payments.

Mathematical notation:

\[ D_j, j = 1, 2, 3, 4 \]: Demand for cash at the end of quarter, \( j \)

\[ D_1 = 160, D_2 = 180, D_3 = 210, D_4 = 190. \] Total demand for the year = 740 million

\[ F_j, j = 1, 2, 3, 4 \]: Fixed cost for obtaining the Board’s approval at the end of quarter \( j \)

\[ F_1 = 0.048, F_2 = 0.052, F_3 = 0.047, F_4 = 0.032. \] (all numbers in millions of dollars)

\[ V_j, j = 1, 2, 3, 4 \]: Proportional charge for every dollar withdrawn at the end of quarter \( j \)

\[ V_1 = 0.010, V_2 = 0.017, V_3 = 0.008, V_4 = 0.015. \]

\[ r_j, j = 1, 2, 3 \]: Interest rate or opportunity cost for cash carried to the next quarter \( j+1 \)

\[ r_1 = 0.007, r_2 = 0.016, r_3 = 0.006. \]

Assume that cash on hand is zero at the beginning and end of the year, as well.

Though cash is withdrawn at the end of a quarter, the opportunity cost on the cash left behind at the end of any quarter is charged at the interest rate at the end of that quarter (and not at the interest rate at the end of the next quarter). The objective is to find the optimal liquidation schedule that will minimize the total costs involved in making the payments to the main contractor at the end of every quarter. The total costs include the fixed costs for obtaining the Board’s approval (akin to the brokerage fees), proportional charges paid to the investment
company on the funds withdrawn and the lost interest on the balances carried forward from one quarter to the next, after making the payment at the end of a quarter.

7.3 Representation of the MPD Model as a Shortest-Route-Problem

Before we can solve MPD cash management model, we need to understand its structural relationship to some well-known optimization problems problem that are solved with the help of the Wagner-Whitin algorithm. The fundamental principle underlying the Wagner-Whitin algorithm is that in the optimal solution the demand for cash in any quarter should be satisfied entirely from the liquidation of securities in that quarter itself or from the balance carried forward from the previous quarter. In other words, splitting the total demand between the two options and making two partial payments is not optimal. For instance, if the demand in a quarter is $210 million, it is not optimal to carry forward an amount of $70 million from the previous quarter and liquidate securities worth for the remainder of $140 million in the current quarter. The optimal procedure is either to carry forward the whole amount of $210 million from the previous quarter or liquidate securities worth the whole amount of $210 million in the current quarter. The optimal solution is to use just one of the two options to satisfy the demand. Dynamic programming (Denardo, 2003) technique combined with the Wagner-Whitin algorithm is used to obtain the optimal solution for the current cash management problem. This procedure is also closely related to the Shortest-Route-Algorithm which is most commonly used, as the name implies, to find the shortest route between two locations on a map. The route-map corresponding to the MPD cash management model is shown in Figure 7.1.
In the application of the Shortest-Route-Algorithm, every designated location (such as the center of a city) on the map is called a node. In Figure 7.1, we have five nodes. Each node represents the end of a quarter. Of course, the node marked 0 represents the end of the fourth quarter of the previous year or the beginning of the first quarter of the current year. The arc (or line) connecting any pair of nodes \((i, j)\) represents the distance (i.e., cost in dollars or time in hours) involved in reaching node \(j\) from node \(i\). If we are given a map with a large number of interconnected nodes, there is no simple way for us to identify the shortest route between the starting node, called the origin, and the ending node, called the destination, in advance. Mathematically, it can be proved that the only way to identify the shortest route between the “origin” and the “destination” is to find the shortest routes from the origin to all the intermediate nodes between the origin, the starting point, and the destination, the stopping point. In other words, if we were to travel by the
shortest route from the starting point A to the ending point E, the routes taken to pass through the intermediate points B, C, and D, should also be the shortest. In the MPD cash management model, our starting node is 0, the beginning of the first quarter of the current year. Our objective is to reach node 4 at the lowest total cost. In order to accomplish this goal, we have no choice but to find the lowest-cost option to reach each one of the intermediate nodes between 0 and 4.

7.4 Solution of the MPD Model as a Shortest-Route-Problem

\( Z(j) \): Minimum cost to meet the payment schedule from the beginning of the year to the end of the quarter \( j, j = 0, 1, 2, 3, 4 \).

\( \psi_k(j) \): Total cost of the alternative \( k, 1 \leq k \leq j \), to meet the payment obligations to the main contractor including the one at the end of the quarter \( j, j = 1, 2, 3, 4 \). From Figure 1, we can see that for any quarter \( j \), there are \( j \) alternatives routes or choices available to meet the payment schedule. (The number of alternative routes is equal to the number of arcs entering the node)

\( N(i, j) \): Total cost associated with one liquidation and the proportional charge on the amount withdrawn in quarter \( (i+1) \) and the lost interest for carrying the balance of cash forward till the end of quarter \( j \), where \( j \) could be \( (i+1) \) or higher. For example, the parameter \( N(1,3) \) represents the total cost of the liquidation \( F_2 \), and the related proportional charge on the amount withdrawn \( V_2(D_2 + D_3) \), and further the interest \( (r_2D_3) \) on the balance of cash carried to the end of quarter 3. From the amount of \( (D_2 + D_3) \) withdrawn at the end of quarter 2, the amount \( D_2 \) is immediately paid to the main contractor and the amount of \( D_3 \) is carried forward at interest rate \( r_2 \) to the end of the third quarter and then paid to the contractor. Or, \( N(1,3) = F_2 + V_2(D_2 + D_3) + r_2D_3 \). In other
words, the cost parameter $N(i, j)$ represents the total cost that accrues after quarter $i$ ends and quarter $(i + 1)$ begins and includes the costs till the end of quarter $j$, where $j$ could be just $(i + 1)$ or higher. In a generalized version of the problem with $T$ periods, the value of $i$ starts with 0 and the index $j$ can take any value between 1 and $T$. The evaluation of $N(i, j)$ plays a critical role in the solution of the problem, as demonstrated in the given problem.

We start with the first quarter and proceed to the last quarter. In the process we will obtain the optimal solutions for each of the quarters.

Let $Z(0)$ denote the total cost at the end of quarter 0 or beginning of quarter 1.

Obviously, $Z(0) = 0$, since no liquidations or payments are made at the beginning of the year.

Optimal solution for Quarter 1: There is only one choice.

In this case one withdrawal is made to make the payment of 160 million at the end of quarter 1.

$$
\psi_1(1) = Z(0) + N(0, 1) = Z(0) + F_1 + V_1 \times D_1 = 0 + 0.048 + 0.010 \times 160 = 1.648
$$

$$
Z(1) = \psi_1(1) = 1.648 \text{ million}
$$

Optimal solution for Quarter 2: There are two alternative routes to meet the payment schedules.

$$
\psi_1(2) = Z(0) + N(0, 2) = Z(0) + F_1 + V_1 \times (D_1 + D_2) + r_1 \times D_2
$$

Or, $\psi_1(2) = 0 + 0.048 + 0.010 \times 340 + 0.007 \times 180 = 4.708$

$$
\psi_2(2) = Z(1) + N(1, 2) = Z(1) + F_2 + V_2 \times D_2 = 1.648 + 0.052 + 0.017 \times 180 = 4.76
$$

$$
Z(2) = \min\{\psi_1(2), \psi_2(2)\} = \psi_1(2) = 4.708 \text{ million}
$$
The lowest cost alternative for meeting the payment schedule till the end quarter 2 is $\psi_1(2)$. This implies that the amount due at the end of quarter 2 is withdrawn at the end of quarter 1 itself.

According to the optimality principle of Wagner-Whitin algorithm, if any withdrawal is made at the end of any quarter, it should automatically include the amount due at the end of that quarter. Therefore, total amount withdrawn at the end of quarter 1 is equal to $(D_1 + D_2)$ and the amount $D_2$ is carried forward till the end of quarter 2.

Optimal solution for Quarter 3: There are three alternative routes to meet the payment schedules.

\[ \psi_1(3) = Z(0) + N(0, 3) = Z(0) + F_1 + \psi_1 \cdot (D_1 + D_2 + D_3) + r_1 \cdot (D_2 + D_3) + r_2 \cdot D_3 \]

Or, $\psi_1(3) = 0 + 0.048 + 0.01 \cdot 550 + 0.007 \cdot 390 + 0.016 \cdot 210 = 11.638$

\[ \psi_2(3) = Z(1) + N(1, 3) = Z(1) + F_2 + \psi_2 \cdot (D_2 + D_3) + r_2 \cdot D_3 \]

\[ \psi_2(3) = 1.648 + 0.052 + 0.017 \cdot 390 + 0.016 \cdot 210 = 11.69 \]

\[ \psi_3(3) = Z(2) + N(2, 3) = Z(2) + F_3 + \psi_3 \cdot D_3 = 4.708 + 0.047 + 0.008 \cdot 210 = 6.435 \]

\[ Z(3) = \min \{\psi_1(3), \psi_2(3), \psi_3(3)\} = \psi_3(3) = 6.435 \text{ million} \]

The lowest cost alternative for meeting the payment schedule till the end of quarter 3 is $\psi_3(3)$. This implies that the amount withdrawn at the end of quarter 3 is just equal to $D_3$. Now look at the cost components of $\psi_3(3)$. Since $\psi_3(3) = Z(2) + N(2, 3)$, we go back to quarter 2 and follow the optimal schedule indicated by $Z(2)$.

Optimal solution for Quarter 4: There are four alternative routes to meet the payment schedules.

\[ \psi_1(4) = Z(0) + N(0, 4) = Z(0) + F_1 + \psi_1 \cdot (D_1 + D_2 + D_3 + D_4) \]

\[ + r_1 \cdot (D_2 + D_3 + D_4) + r_2 \cdot (D_3 + D_4) + r_3 \cdot D_4 \]
Or, \( \psi_1(4) = 0 + 0.048 + 0.01*740 + 0.007*580 + 0.016*400 + 0.006*190 = 19.048 \)

\( \psi_2(4) = Z(1) + N(1,4) = Z(1) + F_2 + V_2 \ast (D_2 + D_3 + D_4) + r_2 \ast (D_3 + D_4) + r_3 \ast D_4 \)

\( \psi_2(4) = 1.648 + 0.052 + 0.017 \ast 580 + 0.016 \ast 400 + 0.006 \ast 190 = 19.10 \)

\( \psi_3(4) = Z(2) + N(2,4) = Z(2) + F_3 + V_3 \ast (D_3 + D_4) + r_3 \ast D_4 \)

\( \psi_3(4) = 4.708 + 0.047 + 0.008 \ast 400 + 0.006 \ast 190 = 9.095 \)

\( \psi_4(4) = Z(3) + N(3,4) = Z(3) + F_4 + V_4 \ast D_4 = 6.435 + 0.032 + 0.015 \ast 190 = 9.317 \)

\[ Z(4) = \min \{ \psi_1(4), \psi_2(4), \psi_3(4), \psi_4(4) \} = \psi_3(4) = 9.095 \text{ million} \]

The lowest cost alternative for meeting the payment schedule till the end of quarter 4 is \( \psi_3(4) \). This implies that the amount due at the end of quarter 4 is also withdrawn at the end of quarter 3 itself.

So, the total amount withdrawn at the end of quarter 3 is equal to \( (D_3 + D_4) \) and the amount \( D_4 \) is carried forward till the end of quarter 4. Since \( \psi_3(4) = Z(2) + N(2,4) \), we go back to quarter 2 and follow the optimal schedule indicated by \( Z(2) \), which involves the withdrawal of \( (D_1 + D_2) \) and carrying forward an amount of \( D_2 \). The optimal schedule involves two liquidations, one at the end of quarter 1 and another at the quarter 3.

The advantage of the dynamic programming technique based on the Wagner-Whitin algorithm or the Shortest-Route-Algorithm is that it provides optimal solutions to all the sub-problems related to quarters 1, 2, and 3, before we reach the optimal solution for the final quarter 4. Of course, the disadvantage of the dynamic programming technique is that it is very cumbersome from a computational point of view. Further, we have so far considered only the case in which delayed payments are not allowed. Or, in the parlance of Inventory Control Theory, we considered the
case in which no backlogging of orders is allowed. Originally, Wagner and Whitin (1958) used
dynamic programming methodology to solve the MPD inventory problem. Mathematically, it can
be proved that in the Wagner-Whitin model, the demand for an item in any period can be satisfied
entirely from the production in that period or from the inventory of the previous period. No
backlogging is allowed in the original Wagner-Whitin model. Subsequently, Zangwill (1966)
extended the Wagner-Whitin model to include backlogging of the orders, by proving that it is
optimal to satisfy the demand in any period entirely from any one of the three available options: (a)
production in the current period, (b) inventory from the previous period, or (c) backlogging the
demand to a future period. In other words, there will be no splitting of the demand among these
three options. The optimal way to meet the demand is to use just one of the three options. Needless
to say, these inventory models can be adapted to cash management with suitable modifications. The
dynamic programming technique can be used to solve the case of delayed payments also. But the
computational burden will be very high compared to the previous case in which no delayed
payments are allowed. We can altogether avoid the use of dynamic programming since we now
have advanced software systems to solve these problems. In this context, it may be noted that it is
possible to have several different mathematical versions of the MPD model depending on the values
of the parameters, the types of securities involved and the conditions related to the delayed
payments. Solutions for two related versions of the MPD model that fall in the category of Linear
Programming and Network Flow problems (Bazaraa et al. 2009 and Winston, 2004) can be found in
Golden (1979) and Srinivasan and Kim (1986) In general, suppose that our MPD cash
management problem involves $T$ time periods, where $T$ is a relatively large number and we are
interested in obtaining the optimal solution only for the last period $T$. Further, assume that
delayed payments are allowed. Then, the use of dynamic programming is not warranted since we
do not need the optimal solutions for the intermediate periods 1 through \((T - 1)\). We can use zero-one integer programming methodology to solve large MPD cash management models even with the delayed payment option. A non-linear optimization model with multi-objective approach to cash management is described in Salas-Molina et al. (2018).

7.5 Solution of the MPD Model as a Zero-One Integer Programming Problem

The mathematical formulation of the optimization problem is given below:

It is called a 0-1 mixed integer programming problem since it involves variables that are allowed to take a value of 0 or 1, and nothing else.

Definition of variables in the model:

\(y_j, j = 1, 2, 3, 4\): \(y_j\) is an integer variable that takes a value of either 0 or 1.

If \(y_j = 1\), it implies that funds are withdrawn at the end of quarter \(j\).

If \(y_j = 0\), it implies that no funds are withdrawn at the end of quarter \(j\).

\(x_j, j = 1, 2, 3, 4\): \(x_j\) is the amount of money withdrawn at the end of quarter \(j\). It can assume any value between 0 and 740, the maximum amount available for withdrawal.

\(u_j, j = 1, 2, 3\): \(u_j\) is the amount of cash carried forward from end of quarter \(j\) to end of quarter \(j+1\) at the interest rate (or opportunity cost) \(r_j\), not at the rate of \(r_{j+1}\). The basic assumption is that the interest rate estimated at the end of a quarter applies from the end of that quarter to the end of next quarter. (This assumption can be changed easily, if necessary.)
$M_j, j = 1, 2, 3, 4$: Maximum amount of money that can be withdrawn at the end of any quarter.

In this problem, no specific limit is imposed on the amount of funds that can be withdrawn. This implies that $M_j = 740, j = 1, 2, 3, 4$. But in other problems, a specific limit can be imposed on each of $M_j, j = 1, 2, 3, 4$.

Minimize: $V_1x_1 + V_2x_2 + V_3x_3 + V_4x_4 + r_1u_1 + r_2u_2 + r_3u_3 + F_1y_1 + F_2y_2 + F_3y_3 + F_4y_4$

\[
x_1 - u_1 = D_1
\]
\[
u_1 + x_2 - u_2 = D_2
\]
\[
u_2 + x_3 - u_3 = D_3
\]
\[
u_3 + x_4 = D_4
\]
\[
x_1 - M_1y_1 \leq 0
\]
\[
x_2 - M_2y_2 \leq 0
\]
\[
x_3 - M_3y_3 \leq 0
\]
\[
x_4 - M_4y_4 \leq 0
\]

Data for scenario 1.

$D_j, j = 1, 2, 3, 4$: Demand for cash at the end of quarter, $j$

$D_1 = 160, D_2 = 180, D_3 = 210, D_4 = 190$. Total demand for the year = 740 million
$F_j, j = 1, 2, 3, 4$: Fixed cost for obtaining the Board’s approval, at the end of quarter $j$

$F_1 = 0.048, F_2 = 0.052, F_3 = 0.047, F_4 = 0.032$. (all numbers in millions of dollars)

$V_j, j = 1, 2, 3, 4$: Proportional charge for every dollar withdrawn, at the end of quarter $j$

$V_1 = 0.01, V_2 = 0.017, V_3 = 0.008, V_4 = 0.015$.

$r_j, j = 1, 2, 3$: Interest rate or opportunity cost for cash carried to the next quarter, $j+1$

$r_1 = 0.007, r_2 = 0.016, r_3 = 0.006$.

Assume that cash on hand is zero at the beginning and end of the year, as well.

### Table 7.1: Optimal Solution to Scenario 1 of the Problem

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<td>0</td>
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<td>9.095</td>
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</table>

The optimal solution to scenario 1 of the problem given above is as follows:

$x_1^* = 340; u_1^* = 180; x_3^* = 400; u_3^* = 190$; Total cost of the optimal solution: $9.095$ million

Recommendation: (a) Withdraw $340$ million and pay $160$ million to the main contractor at the end of quarter 1. Pay the remainder of $180$ million to the main contractor at the end of quarter 2.
(b) Withdraw $400 million and pay $210 million to the main contractor at the end of quarter 3. Pay the remainder of $190 million to the main contractor at the end of quarter 4.

**Case B:** In this case, the main contractor is willing to accept a delayed payment at the end of one of the quarters in the future, but with a penalty at a specified rate for each quarter.

Data for scenario 2.

\[ D_j, j = 1, 2, 3, 4 : \text{ Demand for cash at the end of quarter, } j \]

\[ D_1 = 160, D_2 = 180, D_3 = 210, D_4 = 190. \text{ Total demand for the year = 740 million} \]

\[ F_j, j = 1, 2, 3, 4 : \text{ Fixed cost for obtaining the Board’s approval, at the end of quarter, } j \]

\[ F_1 = 0.048, F_2 = 0.052, F_3 = 0.047, F_4 = 0.032. \text{ (all numbers in millions of dollars)} \]

\[ V_j, j = 1, 2, 3, 4 : \text{ Proportional charge for every dollar withdrawn, at the end of quarter, } j \]

\[ V_1 = 0.010, V_2 = 0.017, V_3 = 0.008, V_4 = 0.015. \]

\[ r_{j+1}, j = 1, 2, 3 : \text{ Interest rate or opportunity cost for cash carried to the next quarter, } j+1 \]

\[ r_1 = 0.007, r_2 = 0.016, r_3 = 0.006. \]

Assume that cash on hand is zero at the beginning and end of the year, as well.

Suppose that for every quarter of delay, the main contractor charges penalty of 0.5%.

\[ s_{j+1}, j = 1, 2, 3 : \text{ Rate of penalty for delaying payment till the end of next quarter, } j+1 \]

\[ s_1 = 0.005, s_2 = 0.005, s_3 = 0.005. \]

In addition to the variables defined in the previous model, we define the following variables:

\[ w_j, j = 1, 2, 3 : w_j \text{ is the payment delayed at the end of quarter } j, \text{ that will be paid in future from the} \]

funds withdrawn in quarter \( j+1 \) or later. The revised model is given below:
Minimize: \( V_x x_1 + V_x x_2 + V_x x_3 + V_x x_4 + r_1 u_1 + r_2 u_2 + r_3 u_3 + F_y y_1 + F_y y_2 + F_y y_3 + F_y y_4 \)
\[ + s_1 w_1 + s_2 w_2 + s_3 w_3 \]

\( x_1 - u_1 + w_1 = D_1 \)
\( u_1 + x_2 - w_1 - u_2 + w_2 = D_2 \)
\( u_2 + x_3 - w_2 - u_3 + w_3 = D_3 \)
\( u_3 + x_4 - w_3 = D_4 \)
\( x_1 - M_1 y_1 \leq 0 \)
\( x_2 - M_2 y_2 \leq 0 \)
\( x_3 - M_3 y_3 \leq 0 \)
\( x_4 - M_4 y_4 \leq 0 \)

**Table 7.2: Optimal Solution to Scenario 2 of the Problem**

<table>
<thead>
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<th>( u_1 )</th>
<th>( w_1 )</th>
<th>( x_2 )</th>
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<th>( x_3 )</th>
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<th>( x_4 )</th>
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<th>( y_2 )</th>
<th>( y_3 )</th>
<th>( y_4 )</th>
<th>( \text{Total Cost} )</th>
</tr>
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<td>0.017</td>
<td>0.016</td>
<td>0.005</td>
<td>0.008</td>
<td>0.006</td>
<td>0.005</td>
<td>0.015</td>
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<td>0.052</td>
<td>0.047</td>
<td>0.032</td>
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<td>$-740 &lt;= 0</td>
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<tr>
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<td>0.047</td>
<td>0</td>
<td>$8.375</td>
<td></td>
</tr>
</tbody>
</table>

The optimal solution to the problem given above is as follows:
\[ x_1^* = 160; \ w_2^* = 180; \ x_3^* = 580; \ u_3^* = 190; \]  
Total cost of the optimal solution: $8.375 million
Recommendation: (a) Withdraw $160 million and pay that amount to the main contractor at the end of quarter 1. (b) Withdraw $580 million at the end of quarter 3. Pay $180 million that was due at the end of quarter 2, but delayed until the end of quarter 3 including the penalty of 0.5%. Next, pay the amount of $210 million to the main contractor at the end of quarter 3. That leaves $190 million in cash. Carry it forward (incurring the opportunity cost at a rate of 0.6%) and pay it to the main contractor at the end of quarter 4. This option leads to savings of $0.72 million compared to the first option, whose cost was $9.095 million.

We will now consider another interesting scenario. Suppose that as soon as the first withdrawal of $160 million is made, the investment company agrees to reduce the proportional rate charged in the fourth quarter to 0.3%. Obviously, that will motivate the GQE company to keep the funds in the brokerage account until the end of the fourth quarter. How would that change the solution?

Data for scenario 3.

\( D_j, j = 1, 2, 3, 4: \) Demand for cash at the end of quarter, \( j \)

\[ D_1 = 160, D_2 = 180, D_3 = 210, D_4 = 190. \] Total demand for the year = 740 million

\( F_{j}, j = 1, 2, 3, 4: \) Fixed cost for obtaining the Board’s approval, at the end of quarter, \( j \)

\[ F_1 = 0.048, F_2 = 0.052, F_3 = 0.047, F_4 = 0.032. \] (all numbers in millions of dollars)

\( V_{j}, j = 1, 2, 3, 4: \) Proportional charge for every dollar withdrawn, at the end of quarter, \( j \)

\[ V_1 = 0.010, V_2 = 0.017, V_3 = 0.008, V_4 = 0.003. \] (Value of \( V_4 \) is reduced from 0.015 to 0.003)
$r_j, j = 1, 2, 3$ : Interest rate or opportunity cost for cash carried to the next quarter, $j+1$

$r_1 = 0.007, r_2 = 0.016, r_3 = 0.006$.

$s_j, j = 1, 2, 3$ : Rate of penalty for delaying the payment till the next quarter, $j+1$

$s_1 = 0.005, s_2 = 0.005, s_3 = 0.005$.

Assume that cash on hand is zero at the beginning and end of the year, as well.

In scenario 3, the optimal solution is as follows:

$x_1^* = 160; w_2 = 180; w_3^* = 390; x_4^* = 580$;  Total cost of the optimal solution: $6.27 million
Recommendation: (a) Withdraw $160 million and pay $160 million to the main contractor at the end of the first quarter. (b) Withdraw $580 million and pay $190 million to the main contractor at the end of quarter 4. That leaves $390 million. Pay at the end of quarter 4 the amount of $210 million and the penalty of 0.5% to the main contractor corresponding to the delayed payment from quarter 3. Further, pay at the end of quarter 4 the amount of $180 million and the penalty of 1.0% to the main contractor corresponding to the delayed payment from quarter 2. The reduction in the proportional charge by the investment company offers an opportunity for the GQE company to delay the second and third quarter payments till the end of quarter 4. In turn, this opportunity leads to savings of $2.105 million compared to the second option, whose cost was $8.375 million.

Microsoft Excel has a built-in Solver based on the methodology of zero-one integer programming. Here Microsoft Excel’s Solver module is used to generate the optimal solutions for the cash management problem of the GQE company.
CHAPTER 8

CONCLUSIONS AND SUGGESTIONS FOR FUTURE RESEARCH

8.1 Conclusions and Contributions of the Study

Every firm needs to have some cash on hand to meet its payment obligations to its suppliers and settle the accounts with its creditors. If the firm carries too much cash on hand, it is losing interest income it could have earned by investing some of it in the short-term securities traded in the financial markets. Of course, if it carries too little cash, it runs the risk of not being able to make a payment to a supplier on a due date. The latter scenario could indeed be catastrophic for some firms, if the financial markets perceive it as a default and start selling the firm’s shares at deep discounts. Clearly, for the firm, there is a trade-off between carrying too much or too little cash on hand in meeting its transactions demand for cash. In a similar fashion, instead of keeping too much cash on hand, an individual person can deposit some of it in a savings account and earn interest on it. On other hand, if a person keeps too little cash on hand, sometimes that person may not be able to pay even for day-to-day transactions. In other words, for an individual person too, there is a trade-off between carrying too much or too little cash on hand to meet the transactions demand for cash. For both the firms and the individual persons, an obvious goal is to determine the optimal amount of cash to carry on hand that would help them minimize the costs related to the brokerage transaction fees and lost interest, while meeting their regular payment obligations.

Trade-off problems, similar to the ones encountered in cash management by firms arose in the management of inventories of raw materials, intermediate and finished goods in the earlier part of the 20th century itself. It is interesting to note that several of the techniques that were developed for the optimal management of physical inventories were later applied to cash management. In his
classic paper entitled “The Transactions Demand for Cash: An Inventory Theoretic Approach,” Baumol (1952) presented a simple square-root-formula for estimating the optimal level of cash a firm needs to maintain in order to meet its transactions demand for cash while minimizing the related costs. In one of his classic papers, Tobin (1956) discussed another optimization model that incorporates the partial investment of cash before disbursing funds to meet the transactions demand for cash. However, as described in Baumol and Tobin (1989), Allais, a French economist who was also awarded a Nobel Prize in 1988, seven years after Tobin, developed the same formula in 1947. For this reason, the square-root-formula is called the BAT model in honor of these three renowned economists who independently developed it. It is worth noting that even today with all the advanced technologies, the results of the models developed by Baumol, Allais, and Tobin are applicable to the trade-offs in the cash management problems faced by corporations and individual persons as well.

The goal of the BAT model is to minimize the total costs involving the brokerage fees and the opportunity cost of interest lost on the cash held on hand. The brokerage fees are incurred in connection with the transactions for liquidating securities and converting them into cash. The opportunity cost of lost interest represents the income the firm could have earned by investing the cash in an interest-bearing asset instead of holding it on hand. In Chapter 1, the results published in the foundational papers of Baumol, Allais, and Tobin and the perspectives of their three seemingly different cash management models are briefly described. However, no unified treatment of their results has appeared in the literature. A contribution of this dissertation is to provide a proof of the equivalency of the Baumol, Allais and Tobin models and their unification into the BAT model, as it is now popularly called in standard textbooks (Ross et al. 2019).
The BAT model yields a square-root-formula that helps us determine the optimal level of cash to carry on hand. In practice, the results from the square-root-formula lead to fractional number of transactions and also fractional number of days (or weeks) in the cycle-time between two consecutive transactions. Therefore, the results are not useful from a managerial or implementation point of view. Mathematical techniques for obtaining integer solutions both for the number of transactions and the number of time periods between two consecutive transactions are discussed in Chapters 2 and 3. Perhaps, the methodology presented here for generating integer solutions may be considered one of the important contributions of this dissertation. The Baumol version of the BAT model is the basis for the problems studied in Chapter 2 and their main objective is to minimize the total costs related to the brokerage fees and lost interest. In contrast, the Tobin version of the BAT model is the starting point for the problems analyzed in Chapter 3 and their main objective is the maximization of interest income from the partial investment of funds, while meeting the transactions demand for cash. The results in these two chapters are derived from the inventory models presented in Chen, Dondeti and Mohanty (2019), which, in turn, are based on the theory of convex and concave functions.

In the basic version of the BAT model, there is no provision for short-term borrowing. However, in the case of individuals as well as corporations, it is sometimes beneficial to borrow money on a short-term basis and repay the loan as soon as the funds become available. An extended version of the BAT model with short-term borrowing is described first in Sastry (1970) and, later in Ogden and Sundaram (1998). An enhancement to the Sastry-Ogden-Sundaram (SOS) model that incorporates a bank’s requirement that the firm should buy insurance on the maximum amount borrowed during any time interval, in addition to paying interest on the borrowed money, is
discussed in Chapter 4. A penalty rate similar to the insurance rate is included in the theoretical models of Hadley and Whitin (1963) and Johnson and Montgomery (1974), but its impact on the final results of the model was not discussed in any detail, since the costs related to inventory maintenance account for only a small fraction of the total productions costs. In contrast, banks and other lending institutions often do require that the borrowing firms buy insurance on the maximum amount of money borrowed. The obvious reason is that the lenders want to get back all the money they loaned, should any of the borrowing firms default. For this reason, the impact of the insurance rate on the amount of the money a firm can borrow and the savings that may accrue to the firm is analyzed in detail. The results of this analysis presented in Chapter 4 may be considered another important contribution of this dissertation. Further, a generalized version of the BAT and SOS models with insurance requirement is described in Chapter 5. All the models discussed in the previous chapters may be considered special cases of this generalized version.

In Chapter 6, a single-period stochastic-demand cash management model is considered. In this model, the demand for cash is a random variable and the demand can vary over a wide range of values (i.e., from $20 million to $80 million). Suppose that a firm’s business is seasonal and it realizes all its profits just in one quarter. It needs to have all the needed cash on hand at the beginning of the quarter itself. If it borrows any funds during the quarter on an emergency basis from any bank, the interest rate will be very high. Analogous inventory models based on uniform and exponential distributions under a variety of scenarios have appeared in the literature. But in the cash management model studied here a beta distribution is considered more appropriate. Further, a hypothetical discrete distribution is also considered. Formulas are developed for the optimal amount of cash to be kept on hand for both cases, given the interest rate at the beginning
of the quarter and the interest rate charged by the bank when the funds are borrowed on an emergency basis, should such a need arise.

In the BAT model, it is assumed that the values of the parameters such as the demand rate for cash, the brokerage fee for the liquidation of any arbitrary amount of securities, the proportional charge for every dollar withdrawn, the interest rate that could be earned on a bank deposit, and the interest rate charged by a bank on the money loaned to the firm remain constant from one period to the next. Further, another implicit assumption underlying the BAT model is that the planning horizon is reasonably long. In other words, it is a static model. In contrast, we frequently encounter situations in which the parameter values change from one month to the next or from one quarter to the next, if not on a daily or weekly basis. This type of model that involves different parameter values in different periods is called a Multi-Period Dynamic (MPD) cash management model. In such a model, the demand rate for cash, the brokerage fees, interest rates and proportional charge rates vary from one period to the next. Further, there are two well-known variants of the MPD model. In the first case, a payment must be made when it is due; no delayed are allowed. In the second case, a payment can be delayed and settled in a subsequent period, but with an added penalty for the delay. Illustrative examples for both cases of the MPD model are solved in Chapter 7. The first MPD model in which no delayed payments are allowed is analogous to the well-known Wagner-Whitin (1958) Dynamic Lot Sizing model. An illustrative example of the first case is solved using the dynamic programming technique coupled with the Shortest-Route Algorithm. The second case of the MPD model, in which delayed payments are allowed, is formulated as a 0-1 mixed integer programming problem and solved using the Solver module of Microsoft Excel.
8.2 Suggestions for Future Research

Academicians and practitioners are always interested in the development of new and effective cash management models. Finance Directors of companies, big and small, are constantly looking for improved ways to manage their working capital needs. The BAT model is often the starting point in the discussion of cash management models in the classrooms and professional training seminars. Here are a few suggestions for future research in the area of cash management.

(1) In all the models studied in this dissertation, the brokerage fee $F$ and the interest rate $s$ charged by the bank are assumed to be independent of the amounts involved. An area to explore is the impact on the optimal cash balance, if $F$ and $s$ are increasing functions of the amounts liquidated or borrowed respectively.

(2) Similarly, another area to study is the increase or decrease in the proportional charge rates depending on the amounts invested or withdrawn.

(3) In case of stochastic demand models, the impact of probability distributions other than the beta distribution can be studied.

(4) In case of Multi-Period Dynamic (MPD) cash management models, there could be limits imposed on the amounts of the payments that could be delayed or there could be higher penalties for longer delays. The study of these scenarios coupled with the discounting of cash flows could yield interesting results.

(5) With modern technologies, instantaneous transfer of funds is possible. Matching cash inflows and outflows and forecasting the working capital needs is an important
responsibility of corporate finance directors. Integration of the cash management models with the cash-flow forecasting models of a firm could also be an interesting area to do research.

Of course, these are only a few of the many possible topics one can explore in the area of corporate cash management.
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Mineola, NY.


Parameters and Backlogging: Exact Results, a New Solution, and all Parameter Stability


130-140.


APPENDIX A

MATHEMATICAL PROOFS OF THE PROPOSITIONS AND RESULTS
USED IN CHAPTERS 1 THROUGH 5

A.1 Review of the Basic Properties of Convex Functions

Let us examine the following equations retrieved from Chapters 1 and 2. In these equations, \( C \) and \( t \) are variables that are allowed to assume only positive values. Other algebraic symbols such as \( F \), \( M \), \( r \), \( D \), \( T \), and \( \lambda \) are constants with positive values (\( \lambda = Dr \)). The other symbols \( \psi \), \( \xi \), and \( \mu \) denote functions with \( C \) or \( t \) as independent variables.

\[
\psi(C) = \frac{FM}{C + rC}{2} \tag{1.5}
\]
\[
\xi(C) = \frac{(D / C)F + rC}{2} \tag{2.1}
\]
\[
\xi(C) = \frac{(DT / C)F + rTC}{2} \tag{2.2}
\]
\[
\psi(t) = F + \frac{(Dt)(rt)}{2} = F + \frac{Drt^2}{2} = F + \frac{\lambda t^2}{2} \tag{2.3}
\]
\[
\mu(t) = \frac{\psi(t)}{t} = \frac{F}{t + \lambda t} \tag{2.4}
\]

All the five equations listed above have one common property. Each one of them is a convex function. Let us re-state equation (2.4) as given below, where \( J \) and \( K \) are positive constants.

\[
\xi(x) = J / x + Kx / 2, \quad 0 < x < \infty \tag{A.1}
\]

It is immediately clear that for any value of \( x \), \( 0 < x < \infty \), \( \xi(x) \) will always be positive. Further, if the value of \( x \) is positive but close to 0, the value of \( J/x \) will be large, though the value of \( Kx/2 \) will be small. On the other hand, if the value of \( x \) tends to be larger, the value of \( Kx/2 \) will tend to be larger too, but \( J/x \) will be very small. In other words, the value of \( \xi(x) \) will always be positive and
can never reach a value of zero. Then, what is the lowest possible value for $\xi(x)$? Mathematically, simple differentiation will allow us to answer this question.

\[
\frac{d\xi}{dx} = \xi'(x) = -J / x^2 + K / 2 = 0 \tag{A.2}
\]

From (A.2), \[x^* = \sqrt{2J / K} \tag{A.3}\]

At \[x^* = \sqrt{2J / K} \], we get: \[\xi(x^*) = \sqrt{2JK} \tag{A.4}\]

Further, \[\frac{d^2\xi}{dx^2} = \xi''(x) = 2J / x^3 = \sqrt{K^3 / 2J} > 0 \text{ at } x = x^* \tag{A.5}\]

From (A.2), (A.3), (A.4) and (A.5), we can reach the following conclusions about the properties of a convex function. More details about the properties of convex functions can be found in Chiang and Wainwright (2005), and Stewart (2016). In general, the typical shape of the convex function $\xi(x)$ will be similar to the Figures 2.2 and 2.3 depicted in Chapter 2.

(1) The convex function $\xi(x)$ has a global minimum at a unique value of $x = x^*$.

(2) The first derivative $\xi'(x)$ is a monotonically increasing function of $x$, and the second derivative $\xi''(x)$ is always positive but monotonically decreasing with $x$.

(3) As the value of $x$ increases from a positive lower value towards $x^*$, the value of $\xi(x)$ will be positive, but will continue to decrease until it reaches the minimum value of $\xi(x^*)$. Further, beyond the value of $x^*$, as the value of $x$ goes up, so will the value of $\xi(x)$.

(4) For any $0 \leq \theta \leq 1$, and for any $x_1 > 0$, and $x_2 > 0$, every convex function $\xi(x)$ satisfies the following condition of inequality.
\[ \theta \xi(x_1) + (1 - \theta) \xi(x_2) \geq \xi(\theta x_1 + (1 - \theta)x_2) \]  \hspace{1cm} (A.6)

(5) If \( \xi_j(x_j), j = 1, 2, \ldots, n, \) are convex functions, then \( \xi(x) = \sum_{j=1}^{n} \xi_j(x_j) \) is also a convex function.

(6) Among all the integer values of \( x \) for which \( \xi(x) \) is defined, the minimum of \( \xi(x) \) will occur at one of the two integer values obtained by either rounding up or rounding down \( x^* \). In other words, the minimum of \( \xi(x) \) will occur at \( x = \lceil x^* \rceil \) or \( x = \lfloor x^* \rfloor \) if the search is limited to only integer values of \( x \).

A convex function that is continuous and satisfies the inequality condition (A.6) but is not differentiable because of several linear segments in the curve is called a piecewise linear convex function or a quasi-convex function. Nonetheless, it also has a global minimum at some value of \( x = \bar{x} \) (Bazaraa et al. 2013, and Winston, 2004). Geometrically, the inequality condition (A.6) implies that if we join any two points on the curve of the convex function with a straight line segment, then that segment of the straight line will lie above the curved path of the function that provides an alternate connection between \( x_1 \) and \( x_2 \). So far, our focus has been on the differentiability of \( \xi(x) \) at every value of \( x, \) \( 0 < x < \infty \). Suppose that our interest lies in identifying an integer value of \( x, \) \( 0 < x < \infty, \) at which \( \xi(x) \) attains its lowest value. The existence of a unique global minimum at a point \( x^* > 0 \) helps us identify another important property of the convex function. Based on the monotonicity properties of the first and second derivatives of \( \xi(x), \) we can conclude that if \( x^* \) is not an integer, then the minimum of \( \xi(x) \) will occur at one of the two integer values obtained by either rounding up or rounding down \( x^* \). In other words, we need to compare the values of \( \xi(x) \) at \( x = \lceil x^* \rceil \) and \( x = \lfloor x^* \rfloor \) and pick the lower one from the two options. There is no need to look at any other
integer values of \( x \) at all. Since we are often interested in finding
the integer number of transactions or integer values for the
cycle-times that yield the lowest-cost solutions, this property (No. 6 in the
list above) of the convex functions will prove very useful in our
analysis of cash management models.

**A.2 Review of the Basic Properties of Concave Functions**

In the previous Section we have looked at five seemingly different
convex functions, but all of them have the same basic properties. A
concave function may be considered as a mirror image or reflection of a
convex function against a horizontal line. In analyzing the properties of convex
functions, we look for the minimum value, whereas in the study of concave functions, our goal is
to identify the maximum value of the function. As an example, let us consider equation (1.9)
retrieved from Chapter 1. This equation, given below, represents the net revenue earned by
investing a portion of the funds available at the beginning of the year in the cash management
model of Tobin (1956). Obviously, the goal is to maximize the earnings or net revenue \( E(y) \).

\[
E(y) = Mr / 2 - Mr / 2 y - Fy
\]

In equation (1.9), \( y \) is the independent variable and \( E(y) \) is the dependent variable. Without loss
of generality, let us assume that \( N, J \) and \( K \) are positive constants and \( N \) is relatively larger than \( J \)
and \( K \). Then, a concave function as a counter-image of the convex function represented by (A.1)
can be expressed as follows:

\[
\zeta(x) = N - (J / x + Kx / 2) \quad 0 < x < \infty
\]

Or,

\[
\zeta(x) = N - \zeta(x) \quad 0 < x < \infty
\]
Our objective is to identify the value of \( x \) that would yield the highest value for \( \zeta(x) \). Clearly, the value of \( \zeta(x) \) will be maximum, if the value of \( \xi(x) \) is minimum. Again, simple differentiation will help us find the maximum possible value of \( \zeta(x) \).

\[
\frac{d\zeta}{dx} = \zeta'(x) = J / x^2 - K / 2 = 0
\]

(A.9)

From (A.9), \( x^* = \sqrt{2J / K} \)

(A.10)

At \( x^* = \sqrt{2J / K} \), we get: \( \zeta(x^*) = N - \sqrt{2JK} \)

(A.11)

Further, \( \frac{d^2\zeta}{dx^2} = \zeta''(x) = -2J / x^3 = -\sqrt{K^3 / 2J} < 0 \), at \( x = x^* \).

(A.12)

Interestingly, the value of \( x^* \) is the same as obtained in equations (A.1) and (A.7) regardless of whether it is a convex or concave function. However, the similarity ends there. We can now conclude from (A.9), (A.10), (A.11) and (A.12), that the concave function \( \zeta(x) \) has a global maximum at the unique value of \( x^* = \sqrt{2J / K} \) and \( \zeta(x^*) = N - \sqrt{2JK} \) (Chiang and Wainwright, 2005 and Stewart, 2016). In the case of concave functions also, it is possible to provide a list of the basic properties as done in the case of convex functions. But we will not do so. We will merely summarize them here.

We observe that the first and second derivatives of \( \zeta(x) \) move in opposite directions to those of \( \xi(x) \). In other words, the first derivative \( \zeta'(x) \) is a monotonically decreasing function of \( x \), and the second derivative \( \zeta''(x) \) is always negative but monotonically increasing with \( x \). Since \( \zeta(x) \) reaches its highest value at \( x^* \), we can also conclude that as the value of \( x \) increases from a positive lower value towards \( x^* \), the value of \( \zeta(x) \) will reach the maximum value at \( x = x^* \). Of course, then the value of \( \zeta(x) \) will continue to go down as \( x \) goes up beyond the value of \( x^* \). In general, the typical shape of
the concave function \( \zeta(x) \) will be similar to the Figures 3.4 and 3.5 depicted in Chapter 3. In this case too, the existence of a unique global maximum at a point \( x^* > 0 \) helps us identify another important property of the concave function. For any \( \theta, 0 \leq \theta \leq 1 \), and for any \( x_1 > 0 \), and \( x_2 > 0 \), every concave function \( \zeta(x) \) satisfies the following condition of inequality (Bazaraa et al. 2013, and Winston, 2004).

\[
\theta \zeta(x_1) + (1 - \theta) \zeta(x_2) \leq \zeta(\theta x_1 + (1 - \theta)x_2)
\]

(A.13)

It may be noted that in (A.13) the inequality sign is reversed compared to (A.6). Geometrically, the inequality condition (A.13) implies that if we join any two points on the curve of the concave function with a straight line segment, then that straight line segment will lie below the curved portion of the function which looks like an arch of an overhead bridge between \( x_1 \) and \( x_2 \). Further, based on the monotonicity properties of the first and second derivatives of \( \zeta(x) \), we can also conclude as we have done in the case of the convex functions, that if \( x^* \) is not an integer, then the maximum of \( \zeta(x) \) will occur at one of the two integer values obtained by either rounding up or rounding down the value of \( x^* \). In other words, we need to compare the values of \( \zeta(x) \) at \( x = \lceil x^* \rceil \) and \( x = \lfloor x^* \rfloor \) and pick the higher one from the two options. There is no need to look at any other integer values of \( x \) at all. Since we are often interested in finding the integer number of transactions or integer values for the cycle-times that yield the highest-profit solutions, this last property of the concave functions will prove very useful in our analysis of cash management models.

A.3 Proofs of Propositions 2.2 (3.2), 2.3 (3.3), 2.4 (3.4) and 2.5 (3.5)

In the derivation of the square-root-formula in Baumol (1952), the cycle-times or the time intervals between any two consecutive disbursements and consequently the amounts of the installment
payments are assumed to be equal to one another. However, no specific proof is provided by Baumol (1952) about the equality of the amounts of the disbursements or the corresponding time intervals. Tobin (1956) not only pointed out this fact but also presented a proof to that effect in the Appendix of his paper. The method used in Tobin (1956) to prove the equality of the cycle-times is somewhat unwieldy. A simpler proof, based on the theory of convex functions, is presented here to establish the same result. We will start with Proposition 2.2 retrieved from Chapter 2 which is essentially a re-statement of the problem discussed in Tobin (1956).

**Proposition 2.2 (3.2):** For any fixed integer value of \( y \), the total cost function \( Z(y) \) is minimized at \( t^*_j = T / y, j = 1, 2, \ldots, y \), and the minimum value of the cost function

\[
Z(y) = Fy + \lambda T^2 / 2y.
\]

**Proof:** The relevant parameters are defined as follows:

- \( T \): Number of time periods (or days) in the planning horizon (always an integer).
- \( D \): Rate of disbursement of cash per time period (i.e., per day) in a continuous stream
- \( F \): Fixed brokerage fee for one transaction for liquidating securities, regardless of the amount
- \( r \): Interest rate per time period (i.e., per day); rate represents lost income on the cash withdrawn
- \( \lambda = Dr \)
- \( y \): Number of cycles or transactions; an integer variable, unless otherwise specified.
- \( t_j, j = 1, 2, 3, \ldots, y \), time interval between two consecutive transactions

Further, the total cost for any one of the time intervals \( t_j, j = 1, 2, 3, \ldots, y \), is given by:

\[
\psi(t_j) = F + (\lambda t_j^2 / 2)
\]  

(2.3)
Mathematically, we can re-state the optimization problem as given below:

\[
\text{Minimize } Z(t_1, t_2, \ldots, t_y) = \sum_{j=1}^{j=x} \psi(t_j)
\]

Subject to: \( t_1 + t_2 + \cdots + t_y = T \)

Define the parameters \( \theta_j, j=1,2,\ldots,y \), such that \( 0 \leq \theta_j \leq 1 \), and further, \( \theta_1 + \theta_2 + \cdots + \theta_y = 1 \). It is relatively easy to find the values of \( \theta_j, j=1,2,\ldots,y \), that meet these two conditions. It may be noted that the inequality condition (A.6) applies to a convex function with a single variable or several variables. Since the functions \( \psi(t_j) \) are convex and identical for all \( j, j=1,2,\ldots,y \), extending the inequality condition (A.6) to the function \( Z(t_1, t_2, \ldots, t_y) \), we get the following:

\[
\theta_1\psi(t_1) + \theta_2\psi(t_2) + \cdots + \theta_y\psi(t_y) \geq \psi(\theta_1t_1 + \theta_2t_2 + \cdots + \theta_yt_y)
\]

(A.14)

Let \( \theta_j = 1/y, j=1,2,\ldots,y \). Then, from (A.14) we get:

\[
(1/y)(\psi(t_1) + \psi(t_2) + \cdots + \psi(t_y)) \geq \psi((t_1 + t_2 + \cdots + t_y)/y)
\]

Or, \( (\psi(t_1) + \psi(t_2) + \cdots + \psi(t_y)) \geq y(\psi((t_1 + t_2 + \cdots + t_y)/y)) \)

(A.15)

However, \( t_1 + t_2 + \cdots + t_y = T \). Therefore, from (A.15), we have:

\[
(\psi(t_1) + \psi(t_2) + \cdots + \psi(t_y)) \geq y(\psi(T/y))
\]

(A.16)

From (A.16) we can conclude that the minimum possible value for \( Z(t_1, t_2, \ldots, t_y) \) is \( y(\psi(T/y)) \), for all \( t_j, 0 < t_j < T, j=1,2,\ldots,y \), regardless of what specific value each \( t_j \) may assume between the allowed bounds of 0 and \( T \).
If we let \( t_j = \frac{T}{y}, \ j = 1, 2, \ldots, y \), the left-hand-side of (A.16) is equal to \( y(\psi(T/y)) \), which is the same as the right-hand-side of (A.16). This implies that all the cycle times must be equal to one another and the optimal duration of each cycle, \( t_j^* = \frac{T}{y}, \ j = 1, 2, \ldots, y \).

We can also use Lagrange Multiplier Method (Chiang and Wainwright, 2005 and Stewart, 2016) to prove the same result. Let \( \tau \) denote the Lagrange Multiplier or dual variable.

Simply define the Lagrange function \( \eta(t_1, t_2, \cdots, t_y, \tau) = Z(t_1, t_2, \cdots, t_y) + (T - (t_1 + t_2 + \cdots + t_y))\tau \)

Or, \( \eta(t_1, t_2, \cdots, t_y, \tau) = \psi(t_1) + \psi(t_2) + \cdots + \psi(t_y) + (T - (t_1 + t_2 + \cdots + t_y))\tau \)

Here \( \tau \) is the Lagrange Multiplier. Now, we try to find the minimum value of \( \eta((t_1, t_2, \cdots, t_y, \tau)) \).

Differentiation of \( \eta((t_1, t_2, \cdots, t_y, \tau)) \) with respect to the \((y+1)\) variables yields the following results.

\[
\frac{\partial \eta}{\partial t_j} = \frac{d\psi(t_j)}{dt} - \tau = 0 \quad \text{for all} \ j = 1, 2, \ldots, y, \tag{A.17}
\]

From (2.3) given above in this Section, we get: \( \frac{d\psi(t_j)}{dt_j} = \lambda t_j \) for all \( j = 1, 2, \ldots, y, \)

Therefore, from (A.17), we have: \( \lambda t_j = \tau \) for all \( j = 1, 2, \ldots, y, \)

Or, \( t_j = \tau / \lambda \) for all \( j = 1, 2, \ldots, y, \) \tag{A.18}

But \( \frac{\partial \eta}{\partial \tau} = T - (t_1 + t_2 + \cdots + t_y) = 0. \) Or, \( T - \sum_{j=1}^{y} t_j = 0. \)

Since \( t_j = \tau / \lambda \) for all \( j = 1, 2, \ldots, y, \) we get: \( \tau = T \lambda / y \) \tag{A.19}

Therefore, \( t_j^* = \frac{\tau}{\lambda} = \frac{T}{y} \) for all \( j = 1, 2, \ldots, y. \) \tag{A.20}
Either way, the final result is that each cycle-time \( t'_j = T / y \), for all \( j, j = 1, 2, \ldots, y \).

**Proposition 2.3 (3.3):** Suppose that in a special case of the BAT model, only the number of transactions is required to be an integer, and the planning horizon \( H \) and the cycle time \( t \) are allowed to assume any positive and continuous values. Define the functions: \( \psi(t) = F + \lambda t^2 / 2 \); \( \mu(t) = F / t + \lambda t / 2 \); \( \mu(t) \) is minimized at \( t = t^* \); and, \( R(y) = Fy + (\lambda / 2)(H^2 / y) \); further, let \( y^* = H / t^* \). If \( y^* \) turns out to be an integer, then, let \( \pi = y^* \). Otherwise, if \( R([y^*]) \leq R([y^*]) \), let \( \pi = [y^*] \); or else, \( \pi = [y^*] \). Then, \( \pi \) is the optimal number of transactions, and for the entire planning horizon, the minimum cost \( R^*(\pi) = F\pi + (\lambda / 2)(H^2 / \pi) \).

**Proof:** First let us consider the function \( \mu(t) = F / t + \lambda t / 2 \).

Then, \( \frac{d\mu(t)}{dt} = F / t^2 - (\lambda / 2) = 0 \). Then, \( t^* = \sqrt{\frac{2F}{\lambda}} \)  \hspace{1cm} (A.21)

We see from (A.21) that the cost per unit time \( \mu(t) \) is minimized at \( t^* = \sqrt{\frac{2F}{\lambda}} \).

If \( y \) is the number of transactions during the planning horizon \( H \) and \( t \) is the cycle-time or time interval between two consecutive transactions, then \( yt = H, t = H / y, y = H / t \). Let \( R(y) \) denote the total cost for the time horizon \( H \) (which may or may not have an integer value). Since \( \mu(t) \) is the cost per unit time in any cycle of length \( t \), the total cost for the time horizon \( H \) involving \( y \) transactions is given by:
\[ R(y) = \mu(t)H = H(F / t + (\lambda / 2)t) = Fy + (\lambda / 2)H^2 / y \]  \hspace{1cm} (A.22)

For the moment we ignore the fact that \( y \) is an integer and differentiating \( R(y) \), we get:

\[ \frac{dR(y)}{dy} = R'(y) = F - (\lambda / 2)H^2 / y^2 = 0. \]  \hspace{1cm} (A.23)

Also, \[ \frac{d^2 R(y)}{dy^2} = R''(y) = \lambda H^2 / y^3 > 0 \quad \text{for all } y > 0 \]  \hspace{1cm} (A.24)

From (A.21) and (A.23) we have:

\[ \gamma^* = H(\sqrt{\lambda / 2F}) = H / t^* \]  \hspace{1cm} (A.25)

If the value of \( \gamma^* \) is an integer, in this case even if \( t^* \) is not an integer, it meets our requirement.

Then, we have the optimal solution: \( \pi = \gamma^* \), and \[ R'(\pi) = F\pi + (\lambda / 2)(H^2 / \pi). \]

From (A.23) and (A.24), it is clear that \( R(y) \) is a convex function and has a global minimum. This implies that if the value of \( \gamma^* \) is not an integer, we can identify the integer solution by merely rounding up or rounding down the value of \( \gamma^* \) and comparing the corresponding values of \( R(y) \).

If \( R([\gamma^*]) \leq R([\gamma^*]) \), let \( \pi = [\gamma^*] \); or else, \( \pi = [\gamma^*] \). Then, \( \pi \) is the optimal number of the transactions, and for the entire planning horizon, the minimum cost is given by:

\[ R'(\pi) = F\pi + (\lambda / 2)(H^2 / \pi). \]  \hspace{1cm} (A.26)

**Proposition 2.4 (3.4):** Define the set, \( Y = \{1, 2, \ldots, T\} \). In the BAT model, suppose that the number of cycles is fixed at some \( y \in Y \) and further, the cycle length (or duration in units of time) is restricted to only integer values. Then, in the corresponding optimal solution, no two cycles can differ by more than one period in length.
**Proof:** We will prove this Proposition through the use of numerical data. As in equation (2.3) of Chapter 2 or Section A.1, let \( \psi(t) \) represent the total cost for one cycle of duration \( t \) among the \( y \) cycles in the planning horizon \( T \) as given below.

\[
\psi(t) = F + \lambda t^2 / 2
\]  

(2.3)

As mentioned before, \( \psi(t) \) is a convex function. Then from (A.6), we get:

\[
\theta \psi(t_1) + (1 - \theta) \psi(t_2) \geq \psi(\theta t_1 + (1 - \theta)t_2)
\]  

(A.27)

Since the cost function \( \psi(t) \) is the same for all the \( y \) cycles, let \( t_1 = 13 \), and \( t_2 = 19 \).

From (A.27), if \( \theta = 1 / 2 \), we get: \( \psi(t_1) + \psi(t_2) \geq 2\psi((t_1 + t_2) / 2) \).

Or, \( \psi(13) + \psi(19) \geq 2\psi(16) \).  

(A.28)

Clearly, the total cost of the two cycles will be minimum, if \( t_1 = t_2 = 16 \).

Now, consider another scenario in which \( t_1 = 5 \), and \( t_2 = 18 \).

Again, from (A.27), if \( \theta = 1 / 2 \), we get: \( \psi(t_1) + \psi(t_2) \geq 2\psi((t_1 + t_2) / 2) \).

Or, \( \psi(5) + \psi(18) \geq 2\psi(11.5) \).

But, neither \( t_1 \) nor \( t_2 \) can take the value of 11.5, since both variables are required to be integers.

Some possible pairs of values are: (a) \( t_1 = 7, t_2 = 16 \), (b) \( t_1 = 9, t_2 = 14 \), (c) \( t_1 = 11, t_2 = 12 \).

Further, from (A.27) and the monotonicity property of \( \psi(t) \) we have the following inequality:
\psi(5) + \psi(18) \geq \psi(7) + \psi(16) \geq \psi(9) + \psi(14) \geq \psi(11) + \psi(12) \geq 2\psi(11.5). \quad (A.29)

The validity of (A.29) can be easily verified by assigning numerical values for \( F \) and \( t \) in equation (2.3) given above or for any convex function given in Section A.1. We can immediately conclude that in the optimal solution we must have \( t_1 = 11 \), and \( t_2 = 12 \). This implies that the cycle times are either equal or differ by one unit of time in the optimal solution. We will refer to the cycles that differ in length by no more than one unit of time as *near-equal-length* cycles. Chand (1982) has presented a proof of Proposition 2.4 by contradiction for the optimality of *near-equal-length* cycles. An constructive proof based on the theory of convex functions can be found in Chen, Dondeti, and Mohanty (2019).

**Proposition 2.5 (3.5):** Suppose that in the BAT model, \( Z(y) \) denotes the minimum cost for the time horizon of integer length \( T \), when the number of cycles is fixed at some \( y \in Y \) and the cycle length is restricted to only integer values. Define \( U = \lceil T / y \rceil \), \( m = T - Ly \), \( L = U - 1 \), and \( n = y - m = Uy - T \).

Then, \( Z(y) = m\psi(U) + n\psi(L) \). Or, \( Z(y) = Fy + (y/2)(mU^2 + nL^2) \).

**Proof:** Consider the first case in which \( (T \text{ mod } y) \neq 0 \). Let \( U = \lfloor T / y \rfloor \), and \( L = U - 1 \). Further, \( m = T - Ly \) and \( n = y - m \). Since \( U = L + 1 \), we get: \( mU + nL = (T - Ly)(L + 1) + (y - (T - Ly))L = T \).

Obviously the length of each cycle will be either \( U \) or \( L \), \( L < T / y < U \). Now consider the second case in which \( (T \text{ mod } y) = 0 \). Then, we have: \( U = \lceil T / y \rceil = T / y \), \( L = U - 1 \), \( m = T - Ly = y \), and \( n = y - m = 0 \). Clearly, in this case, \( mU + nL = T \). In either case, we have \( m \) cycles of length \( U \) and \( n \) cycles of length \( L \). Note that among the \( y \) cycles, the \( m \) cycles, each of length \( U \) and the \( n \) cycles,
each of length \( L \) need not be consecutive. Of course, if \( (T \mod y) = 0 \), all the \( m \) cycles will be of length \( U \), and there will be no cycles of length \( L \). Either way, this partitioning procedure yields, for any given integer values of \( T \) and \( y \in Y \), a unique set of integer values for \( U, L, m \) and \( n \). Since the cycle lengths differ by no more than one period, they are all near-equal-length cycles and meet the condition for the optimality specified in Proposition 2.4. Hence the minimum cost \( Z(y) \) for the entire planning horizon of \( T \) periods partitioned into \( y \) cycles of near-equal-lengths is given by:

\[
Z(y) = my(U) + ny(L) = Fy + (\lambda / 2)(mU^2 + nL^2)
\]  

(A.30)

Papachristos and Ganas (1998) presented the closed-form expression for \( Z(y) \) given in (A.30) for an inventory model (similar to the BAT model) with no backlogging. For the extended case of the inventory model that permits backlogging (similar to the SOS model), a similar expression for \( Z(y) \) is presented in Ganas and Papachristos (2005). It may be noted that \( Z(y) \) is a piecewise linear convex function, since its values can be evaluated only for integer values of \( y = 1, 2, \ldots, T \). Then, how do we find the optimal number of cycles or transactions \( y^* \) or \( \pi \), that will yield the lowest cost? The algorithm described in Papachristos and Ganas (1998) or Ganas and Papachristos (2005) for finding the lowest cost solution \( Z^*(\pi) \) involves the partitioning of the set \( Y = \{1, 2, \ldots, T\} \) into several subsets and evaluating the first-order differences of \( Z(y) \). This is a cumbersome procedure and entails significant computational effort, especially if the value of \( T \) is large. In essence it does not exploit the convexity properties of the function \( Z(y) \). We can use the results of Proposition 2.3 to find the
optimal value of $Z(y)$. We will now describe the steps of the optimization algorithm that will help us find the optimal number of cycles $\pi$ and also the corresponding lowest cost $Z^*(\pi)$.

The Propositions 3.2, 3.3, 3.4 and 3.5 described in Chapter 3 are the same as the Propositions 2.2, 2.3, 2.4 and 2.5 that are proved here. They are repeated in Chapter 3 merely for ready reference, and therefore, their proofs are not given here.

A.4 Optimization Algorithm

According to equation (A.21) the cost per unit time in any cycle of length $t$, is minimized at $t = t^*$ as given below.

$$t^* = \frac{\sqrt{2F}}{\lambda}$$  \hspace{1cm} (A.21)

From equation (A.22) we have:

$$R(y) = \mu(t)H = H(F/t + (\lambda/2)t) = Fy + (\lambda/2)H^2/y$$ \hspace{1cm} (A.22)

In equation (A.22), $H$ is the time horizon which is allowed to assume positive but not necessarily integer values. **We now consider the cases where the planning horizon $T$ is strictly a positive integer. Further, we let $y^* = T/t^*$, unless otherwise indicated.**

**Case 1A: Values of $t^*$ and $y^*$ are Integers**

Suppose the value of $t^*$ from (A.21) is an integer and further, let $y^* = T/t^*$. If the value of $y^*$ also happens to be an integer, nothing more needs to be done. We have the optimal solution at hand. The values of $y^*$ and $t^*$ satisfy the requirements of optimality. Letting $\pi = y^*$, we get:
\[ Z^*(\pi) = F\pi + (\lambda / 2)(T^2 / \pi). \]  \hspace{1cm} (A.31)

However, in practice it may turn out that neither \( t^* \) nor \( y^* \) is an integer. Even then, equation (A.31) provides us with valuable information. Suppose we calculate the total cost for the non-integer value of \( y^* \), regardless of whether \( t^* \) is an integer or not as given below:

\[ Z(y^*) = Fy^* + (\lambda / 2)(T^2 / y^*). \]  \hspace{1cm} (A.32)

Since \( Z(y) \) is a convex function, and \( t \) and \( y \) are treated as continuous variables, the value of \( Z(y^*) \) obtained in equation (A.32) is a lower bound for \( Z(y) \). In other words, any further restrictions imposed on the values of \( y \) and \( t \) will only cause an increase in the value of \( Z(y) \) and never a decrease. In other words, the requirement that \( t \) and \( y \) must be integers in the optimal solution will certainly cost us more and not less than \( Z(y^*) \).

**Case 1B: Value of \( t^* \) is an Integer but not the Value of \( y^* \)**

Suppose the value of \( t^* \) is an integer, but we let \( y^* = T / t^* \), and \( y^* \) is not an integer.

Then, let \( y_1 = \lfloor y^* \rfloor \) and \( y_2 = \lceil y^* \rceil \).

Now we will partition the planning horizon \( T \) into \( y_i \) cycles with lengths \( U_i \) and \( L_i \) as described in Proposition 2.5 and find the associated cost as given below:

\[ Z(y_1) = m_1y_1(U_1) + n_1y_1(L_1) = Fy_1 + (\lambda / 2)(m_1U_1^2 + nL_1^2) \]

Similarily, we find

\[ Z(y_2) = m_2y_2(U_2) + n_2y_2(L_2) = Fy_2 + (\lambda / 2)(m_2U_2^2 + nL_2^2) \]

If \( Z(y_1) < Z(y_2) \), let \( \pi = y_1 \) and \( j = 1 \). Or else \( \pi = y_2 \) and \( j = 2 \).

Then, \( Z(\pi) \) is the optimal solution with the corresponding values of \( m_j, U_j, n_j, \) and \( L_j \).
Case 2A: Unique Solution when Value of $t^*$ is not an Integer

If $t^*$ is not an integer, then $y^*$ cannot be an integer, since $T$ is an integer and $y^* = T / t^*$. We now search for the optimal solution in the neighborhood of $t^*$ and $y^*$ by rounding up or rounding down their values. If $\mu([t^*]) < \mu([t^*])$, let $\tau = [t^*]$, or else, $\tau = [t^*]$. If $(T \mod \tau) = 0$, let $\pi = T / \tau$. If $(T \mod \tau) \neq 0$, define $y_a = [T/\tau]$, and $y_b = [T/\tau]$. If $Z(y_a) \leq Z(y_b)$, then $\pi = y_a$. Otherwise, $\pi = y_b$.

Either way, the optimal number of cycles is $\pi$, and the minimum cost for the entire planning horizon is $Z^*(\pi)$. In this scenario, there is a unique optimal solution. Once we know the optimal number of cycles $\pi$, we can immediately identify the corresponding values of $m, U, n$, and $L$.

Case 2B – Multiple Optimal Solutions when Value of $t^*$ is not an Integer

The given problem will have multiple optimal solutions only if $\mu([t^*]) = \mu([t^*])$. In that case, let $\tau_1 = [t^*]$, and $\tau_2 = [t^*]$. Further, suppose that $(T \mod \tau_1) = 0, \pi_1 = T / \tau_1$, and $(T \mod \tau_2) = 0, \pi_2 = T / \tau_2$. Let $Z^*(\pi_1) = F\pi_1 + (\lambda / 2)(T^2 / \pi_1)$, and $Z^*(\pi_2) = F\pi_2 + (\lambda / 2)(T^2 / \pi_2)$.

Since $\mu(\tau_1) = \mu(\tau_2)$, we have: $Z^*(\pi_1) = Z^*(\pi_2)$. In other words, both these options represent at least two of the several possible alternative optimal solutions. Further, $\mu(\tau_1) = F / \tau_1 + (\lambda / 2)\tau_1$ and $\mu(\tau_2) = F / \tau_2 + (\lambda / 2)\tau_2$. Since $\mu(\tau_1) = \mu(\tau_2)$, and $\tau_2 = \tau_1 - 1$, we get: $F - (\lambda / 2)\tau_1(\tau_1 - 1) = 0$. This implies that any $y$, $\pi_1 \leq y \leq \pi_2$, represents an optimal solution. Suppose there are $(q + 1)$ such different values of $y$. Then, in the corresponding partitions of the planning horizon $T$ into $y$ cycles, where $\pi_1 \leq y \leq (\pi_2 - 1)$, we will have cycle lengths $U = \tau_1$, and $L = \tau_2$, but obviously with different values of $m > 0$, and $n > 0$. There is also another partition in which all cycles have lengths
of only $\tau_2$. If we define the set $G = \{\pi_1, \pi_2, \ldots, \pi_q, \pi_L\}$, every element of the set $G$ will be an optimal solution, since all of them will have the same total cost.

**Case 3 – Neighborhood Search when Value of $t^*$ is not an Integer**

If $t^*$ is not an integer, simply let $\tau_1 = \lceil t^* \rceil$, $\tau_2 = \lfloor t^* \rfloor$, $y_S = \lceil T / \tau_1 \rceil$, and $y_E = \lfloor T / \tau_2 \rfloor$. By definition of the ceiling and floor functions, $\tau_1 > t^* > \tau_2$, since $t^*$ is not an integer. As before, suppose we let $y^* = T / t^*$, $y_1 = \lceil y^* \rceil$, and $y_2 = \lfloor y^* \rfloor$. Then we must have: $y_S \leq y_1 \leq y_2 \leq y_E$. Further, from the cases discussed above, we can conclude that any integer value of $y$ that would yield the lowest value of $Z(y)$ cannot be lower than $y_S$ and higher than $y_E$. In other words, it is sufficient if we start our search at $y = y_S$ and end our search $y = y_E$. The search would involve only a small set of values of $y$, since $\tau_1 - \tau_2 = 1$. The validity of this algorithm for the three cases follows from the convexity properties of the cost functions $\psi(t), \mu(t)$, and $Z(y)$ and Propositions 2.3, 2.4 and 2.5. A detailed proof of the validity of the different cases of the optimization algorithm can be found in Chen, Dondeti and Mohanty (2019).

The parameter $\lambda$ appears in several equations related to the cost functions $\psi(t), \mu(t)$, and $Z(y)$. If no short-term borrowing is allowed as is the case with the Baumol and Tobin models analyzed in Chapters 2 and 3, its value is defined as $\lambda = Dr$. In the Sastry-Ogden-Sundaram model discussed in Chapter 4, if we allow short-term borrowing, but do not require any insurance on the amount borrowed, the value of $\lambda = Drs / (r + s)$. In the Integrated Cash Management (ICM) model discussed in Chapter 5, if there is no requirement for the purchase of insurance on the amount borrowed, the
value of \( \lambda = (1 - \rho)Dr_s / (r + s) \). It is obvious that if \( \rho = 0 \), then \( \lambda = Drs / (r + s) \). Next, if \( s = \infty \), then the value of \( \lambda = Dr \). In other words, if the insurance rate \( e = 0 \) on the borrowed funds, it is relatively simple to find the optimal value of the cycle length \( t^* \) in terms of the parameters \( F \) and \( \lambda \) as given in equation (A.21) or in Section 5.4 wherein the special cases of the ICM model are discussed. But if the insurance rate \( e > 0 \), we cannot readily find a simple expression for \( Z(y) \) as in (A.30). It implies that finding the lowest value of \( Z(y) \) using the Optimization Algorithm given above may not be a simple task. The next question is whether the Optimization Algorithm is useful at all in finding the optimal solution for the ICM model when the insurance rate \( e > 0 \). The answer is in the affirmative and we need to prove that the average cost per unit time \( \mu(t) \) is convex even when the insurance rate \( e > 0 \).

### A.5 Convexity of the Cost Function with Insurance Requirement

From Chapter 5, we have the following equations:

The average cost per unit time \( \mu \) is given by:

\[
\mu = \psi(t) / t = F / t + B^2 s / (2Q(1 - \rho)) + C^2 r / (2Q(1 - \rho)) + eB / t
\]

Or, \( \mu = FD / Q + B^2 s / (2Q(1 - \rho)) + r(Q(1 - \rho) - B)^2 / (2Q(1 - \rho)) + eBD / Q \) \hspace{1cm} (5.3)

Partial differentiation of \( \mu \) with respect to \( B \) and \( Q \) yields the following results.

\[
\frac{\partial \mu}{\partial B} = Bs / (Q(1 - \rho)) - r(Q(1 - \rho) - B) / (Q(1 - \rho)) + eD / Q = 0
\]

Or, \( B = (1 - \rho)(rQ - eD) / (r + s) \) \hspace{1cm} (5.4)

Also, \( \frac{d \mu}{dQ} = -FD / Q^2 - B^2 s / (2Q^2 (1 - \rho)) + r(1 - \rho) / 2 - rB^2 / (2Q^2 (1 - \rho)) - eBD / Q^2 = 0 \)

Or, \( rQ^2 (1 - \rho)^2 = 2FD(1 - \rho) + (r + s)B^2 + 2eBD(1 - \rho) \) \hspace{1cm} (5.5)
Or, \[ Q^* = \sqrt{2(r + s)FD / (1 - \rho) - e^2D^2} / (rs) \] (5.6)

Or, \[ Q^* = \left( \frac{2FD}{r(1 - \rho)} - \frac{e^2D^2}{r(r + s)} \right) / \left( \sqrt{\frac{r + s}{s}} \right) \]

Then from (5.4), \[ B^* = (1 - \rho)(rQ^* - eD) / (r + s) \] (5.7)

\[ \frac{\partial^2 \mu}{\partial B^2} = s / (Q(1 - \rho)) + r / (Q(1 - \rho)) = (r + s) / (Q(1 - \rho)) > 0 \] (A.33)

\[ \frac{\partial^2 \mu}{\partial Q\partial B} = -Bs / (Q^2(1 - \rho)) - Br / (Q^2(1 - \rho)) - eD / Q^2 \] (A.34)

\[ \frac{\partial^2 \mu}{\partial Q^2} = 2FD / Q^3 + B^2s / (Q^3(1 - \rho)) + B^2r / (Q^3(1 - \rho)) + 2eBD / Q^3 > 0 \] (A.35)

\[ \frac{\partial^2 \mu}{\partial B\partial Q} = -Bs / (Q^2(1 - \rho)) - Br / (Q^2(1 - \rho)) - eD / Q^2 \] (A.36)

From (A.34) and (A.36) it is easy to see that \[ \frac{\partial^2 \mu}{\partial Q\partial B} = \frac{\partial^2 \mu}{\partial B\partial Q} \] as they should be.

Let \( \Omega \) denote the determinant of the Hessian Matrix (Chiang and Wainwright, 2005).

Then, \[ \Omega = \left( \frac{\partial^2 \mu}{\partial B^2} \right) \left( \frac{\partial^2 \mu}{\partial Q^2} \right) - \left( \frac{\partial^2 \mu}{\partial B\partial Q} \right)^2 \]

Or, \[ \Omega = \left( \frac{(r + s)}{Q(1 - \rho)} \right) \left( \frac{B^2(r + s)}{Q^3(1 - \rho)} + \frac{2FD + 2eDB}{Q^3} \right) - \left( \frac{B(r + s)}{Q^2(1 - \rho)} + \frac{eD}{Q^2} \right)^2 \]

Or, \[ \Omega = \left( \frac{B^2(r + s)}{Q^2(1 - \rho)} \right) + \left( \frac{2FD + 2eDB(r + s)}{Q^3(1 - \rho)} \right) - \left( \frac{B(r + s)}{Q^2(1 - \rho)} \right)^2 - \left( \frac{2B(r + s)eD}{Q^2(1 - \rho)^2} \right) - \left( \frac{e^2D^2}{Q^2} \right) \]

Or, \[ \Omega = \left( \frac{2FD(r + s)}{Q^4(1 - \rho)} \right) - \left( \frac{e^2D^2}{Q^4} \right) \]

Or, \[ \Omega = (1 / Q^4)(2FD(r + s) / (1 - \rho) - e^2D^2) \]
Since \( rsQ^2 = 2(r + s)FD / (1 - \rho) - e^2D^2 \) from equation (5.5b), we get:

\[
\Omega = (1 / Q^4)(2FD(r + s) / (1 - \rho) - e^2D^2) = rsQ^2 / Q^4
\]

Or, \( \Omega = rs / Q^2 > 0 \) \( \text{(A.37)} \)

From (A.33), (A.35) and (A.37) we can conclude that the Hessian Matrix is positive definite and the second-order derivatives and the determinant of the Hessian Matrix satisfy the requirements for the convexity of the cost function (Chiang and Wainwright, 2005). In turn, we can conclude that the function \( Z(y) \), representing the total cost for the entire planning horizon of \( T \) periods will be a piecewise linear convex function. But as before, we need the value of \( t^* \) as a starting point in finding the lowest cost solution. First, we calculate the optimal amount of the security \( Q^* \) to be converted into cash through liquidation, using equation (5.6) given below.

\[
Q^* = \sqrt{(2(r + s)FD / (1 - \rho) - e^2D^2) / (rs)} \tag{5.6}
\]

Then, the optimal length of the cycle, \( t^* = Q^* / D \) \( \text{(A.38)} \)

Further, from (A.38) we get \( y^* = T / t^* \). \( \text{(A.39)} \)

At this stage though we have the values of \( t^* \) and \( y^* \), the evaluation of the cost for a cycle of length \( U \) or \( L \) and consequently the value of \( Z(y) \) is not a simple as in equation (A.30). Therefore, we will resort to the \textit{Neighborhood Search} procedure which falls under Case 3 of the Optimization Algorithm. The steps for finding the optimal solution are given below:
Regardless of whether \( t^* \) is an integer or not, simply let \( \tau_1 = \lceil t^* \rceil \), \( \tau_2 = \lfloor t^* \rfloor \), \( y_S = \lceil T / \tau_1 \rceil \), and \( y_E = \lfloor T / \tau_2 \rfloor \). Let \( y \) be any integer between \( y_S \) and \( y_E \). Partition time horizon \( T \) into \( y \) cycles as described in Proposition 2.5. Then we must have: \( mU + nL = T \). In other words, we have \( m \) cycles of length \( U \) and \( n \) cycles of length \( L \).

If \( Q_U \) denotes the total demand for cash for the cycle of length \( U \), we get: \( Q_U = DU \).

If \( B_U \) denotes the maximum amount borrowed during the cycle of length \( U \), from (5.7) we get:
\[
B_U = (1 - \rho)(rQ_U - eD) / (r + s)
\]

If \( \psi(U) \) denotes the total cost for the cycle of length \( U \), from (5.3) we get:
\[
\psi(U) = U\mu = U(FD / Q_U + B_U^2 / (2Q_U(1 - \rho)) + r(Q_U(1 - \rho) - B_U)^2 / (2Q_U(1 - \rho)) + eB_U D / Q_U) \quad (A.40)
\]

Similarly, for any cycle of length \( L \), we calculate the following:
\[
Q_L = DL. \quad B_L = (1 - \rho)(rQ_L - eD) / (r + s)
\]
\[
\psi(L) = L\mu = L(FD / Q_L + B_L^2 / (2Q_L(1 - \rho)) + r(Q_L(1 - \rho) - B_L)^2 / (2Q_L(1 - \rho)) + eB_L D / Q_L) \quad (A.41)
\]

Then
\[
Z(y) = m\psi(U) + n\psi(L) \quad (A.42)
\]

In this fashion, find \( Z(y) \) for every \( y, y_S \leq y \leq y_E \). Then the minimum value of \( Z(y) \) among this set of values will be the optimal solution. It may be noted that if \( e = 0 \), we can calculate the parameter \( \lambda \) using the equation \( \lambda = (1 - \rho) Drs / (r + s) \). Then, obviously, it is a little simpler to find the optimal solution. The numerical problems solved in Chapters 1 through 5 illustrate the application of the Optimization Algorithm and the different scenarios that could arise in the solution of different cash management models.
APPENDIX B

MATHEMATICAL PROOFS OF THE RESULTS USED IN CHAPTER 6

B.1 Demand Following a Beta Distribution

Assume that the demand for cash follows a Beta Distribution

The probability density function (pdf) corresponding to the Beta distribution with parameters $\alpha$ and $\beta$ is defined as follows (Balakrishnan and Nevzorov, 2003):

$$
\xi(u) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1}(1-u)^{\beta-1}, \quad 0 \leq u \leq 1
$$

Also, the expected value of $u$, $E(u) = \frac{\alpha}{\alpha + \beta}$

Or, an alternative definition, if the random variable can assume values in the interval $[a, b]$ is given below:

$$
\xi(w) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{w-a}{b-a}\right)^{\alpha-1} \left(\frac{b-w}{b-a}\right)^{\beta-1} \left(\frac{1}{b-a}\right), \quad 0 < a \leq w \leq b
$$

Let $\alpha = 4, \beta = 2$, and $\alpha + \beta = 6$.

$$
\xi(w) = 20 \left(\frac{w-a}{b-a}\right)^3 \left(\frac{b-w}{b-a}\right) \left(\frac{1}{b-a}\right), \quad 0 < a \leq w \leq b
$$

Then, $E(w) = \int_{a}^{b} w \xi(w) dw$

Let $w = a + (b-a)u$. Then, we have the limits for $u$: $0 \leq u \leq 1$. Also, $dw = (b-a)du$. 
Substitution yields: \((w - a) / (b - a) = u\), and \((b - w) / (b - a) = 1 - u\). Further, we get:

\[
\tilde{\xi}(u) = 20u^3(1 - u) = 20(u^3 - u^4). \quad \text{Then,} \quad E(u) = \alpha / (\alpha + \beta) = 2 / 3.
\]

\[
E(w) = \int_0^1 \tilde{\xi}(u) du + (b - a) \int_0^1 u \tilde{\xi}(u) du = a + (b - a)E(u) = a + (b - a)(2 / 3) = a / 3 + 2b / 3
\]

Alternately, \(E(w) = E[a + (b - a)u] = a + (b - a)E(u) = a + (b - a)(2 / 3) = a / 3 + 2b / 3\).

Before determining the optimal amount of cash \(Q\) that would minimizing the total cost, let us consider the two extreme options:

Option (1): Let \(Q = b\). Clearly, there would always be not only enough funds to make the single payment of \(w, a \leq w \leq b\), but also there would be a surplus of \((b - w)\) that would have earned interest at the rate of \(r\) percent per the period.

Then, \(E[\psi(b)] = \int_a^b (b - w) \tilde{\xi}(w) dw = r(b - E(w)) = r(b - a / 3 - 2b / 3) = r(b - a) / 3\)

Option (2): Let \(Q = a\). Clearly, there would always be a shortage of funds, and to make the single payment of \(w, a \leq w \leq b\), we need to borrow an amount of \((w - a)\) that would require us to pay interest at the rate of \(s\) percent per the period.

Then, \(E[\psi(a)] = \int_a^b (w - a) \tilde{\xi}(w) dw = s(E(w) - a) = s(a / 3 + 2b / 3 - a) = 2s(b - a) / 3\)

Assuming that \(s > r / 2\), for any \(Q, a < Q < b\), we must have: \(E[\psi(b)] < E[\psi(Q)] < E[\psi(a)]\).

Obviously, when \(Q > w\), we have a surplus of funds \((Q - w)\), and a shortage of \((w - Q)\), when \(w > Q\).
Then, the expected interest cost, at the respective interest rates $r$ and $s$, is given by:

$$E[\psi(Q)] = \int_a^b (Q-w)\xi(w)dw + \int_0^b (w-Q)\xi(w)dw$$

for any $Q, a < Q < b$. \hfill (B.1)

**Method A – Finding $Q^*$ from Total Cost**

Let $Q = a + (b-a)x$. Then, we get;

$$E[\psi(Q)] = 20(b-a)r\int_0^x (x-u)(u^3 - u^4)du + 20(b-a)s\int_x^1 (u-x)(u^3 - u^4)du$$

$$E[\psi(Q)] = 20(b-a)\left[r\int_0^x (u^3 - u^4)du - sx\int_x^1 (u^3 - u^4)du - r\int_0^x (u^4 - u^5)du + s\int_x^1 (u^4 - u^5)du\right]$$

$$E[\psi(Q)] = 20(b-a)\left[(r+s)(x^5 / 20 - x^6 / 30) + s(1 / 30 - x / 20)\right]$$

Then, differentiating $E[\psi(Q)]$ with respect to $x$ and setting the derivative to 0, we get:

$$\frac{dE}{dx} = 20(b-a)\left[(r+s)(x^4 / 4 - x^5 / 5) - s / 20\right] = 0$$

Or, $20(x^4 / 4 - x^5 / 5) = s / (r+s)$

But, $\int_0^x \xi(u)du = \int_0^x 20u^3(1-u)du = 20(x^3 / 4 - u^5 / 5) = P(u \leq x) = F(x)$

Further, $\frac{d^2E}{dx^2} = 20(b-a)\left[(r+s)(x^3 - x^4)\right] = 20(b-a)\left[(r+s)x^3(1-x)\right] > 0$, since $0 < x < 1$. 
Therefore, $E[\psi(Q)]$ has a global minimum, and the minimum occurs at $x^*$ such that $F(x^*) = s / (r + s)$

Or, the corresponding $Q^* = a + (b - a)x^*$. The cost function $E[\psi(Q)]$ is a convex function since it has a global minimum at some value of $Q$, $a \leq Q \leq b$, (or, in general, at a value $Q$, $-\infty < Q < \infty$)

**Method B -- Finding $Q^*$ without the Total Cost (Using the Rule of Differentiating Integrals)**

$$E[\psi(Q)] = \int_a^b (Q - w)\xi(w)dw + \int_s^Q (w - Q)\xi(w)dw \quad (B.1)$$

From Leibniz rule for differentiating an integral (Stefanica, 2008, and Winston, 2004) we get:

$$\frac{d}{dy} \int_{p(y)}^{q(y)} \varphi(y, t)dt = \varphi(y, q(y)) \frac{dq(y)}{dy} - \varphi(y, p(y)) \frac{dp(y)}{dy} + \int_{p(y)}^{q(y)} \frac{\partial \varphi(y, t)}{\partial y} dt$$

Therefore, $\frac{dE[\psi(Q)]}{dQ} = \frac{d}{dQ} \left[ \int_a^b (Q - w)\xi(w)dw \right] + \frac{d}{dQ} \left[ \int_s^Q (w - Q)\xi(w)dw \right] \quad (B.2)$

Since $a$ and $b$ are constants, $\frac{da}{dQ} = \frac{db}{dQ} = 0$. Of course, $\frac{dQ}{dQ} = 1$.

Also, $(Q - w) = (w - Q) = 0$, when $w = Q$.

Applying Leibniz rule for (B.2), we get,

$$\frac{dE[\psi(Q)]}{dQ} = \int_a^Q \xi(w)dw - \int_s^Q \xi(w)dw = rF(Q) - s[1 - F(Q)]$$

where $F(Q)$ is cumulative distribution function (CDF) at $w = Q$. 
Setting the first derivative to 0, we get: \( F(Q^*) = s / (r + s) \)

Further, \( \frac{d^2 E}{dQ^2} = r \xi(Q) + s \xi(Q) > 0, \quad a < Q < b. \)

This implies, we get the global minimum, at \( w = Q^* \).

**Method C -- Finding \( Q^* \) without the Total Cost (Using the Rule of Integration by Parts)**

\[
E[y(Q)] = r \int_a^Q (Q - w) \xi(w) \, dw + s \int_Q^b (w - Q) \xi(w) \, dw \quad \text{(B.1)}
\]

From the Rule of Integration by Parts (Stefanica, 2008, and Winston, 2004), we have:

\[
\int \phi(v) \phi'(v) \, dv = \phi(v) \phi(v) - \int \phi(v) \phi'(v) \, dv, \quad \text{where} \quad \frac{d\phi(v)}{dv} = \phi'(v), \quad \text{and} \quad \frac{d\phi(v)}{dv} = \phi'(v).
\]

Let \( F(w) = \int_a^w \xi(w) \, dw \), or \( F'(w) = \xi(w) \, dw \), and \( \zeta(w) = \int_a^w F(w) \, dw \).

Since \( \xi(w) \) is the probability density function of beta distribution, obviously, \( F(w) \) is the Cumulative Distribution Function, CDF of \( \xi(w) \) and \( 0 \leq F(w) \leq 1 \).

It may be noted that depending on the values of the parameters \( \alpha \) and \( \beta \), it is sometimes easy to obtain a closed-form expression for \( F(w) \), and sometimes it is not.

Further, \( F(a) = 0 \), and \( F(b) = 1 \). Also, \( \zeta(w) \) has a finite value for any \( w, \ a \leq w \leq b \).

The simple results to be noted are: \( \frac{d\xi(w)}{dw} = F(w) \), \( \frac{d(Q - w)}{dw} = -1 \), and \( \frac{d(w - Q)}{dw} = 1 \).
Also, \[ E[\psi(Q)] = r \int_a^Q (Q - w)\xi(w)dw + s \int_a^b (w - Q)\xi(w)dw \]

By applying the rule of Integration by Parts, we get the following expression for \( E[\psi(Q)] \).

\[ E[\psi(Q)] = r \left[ (Q - w)F(w) \right]_a^Q + r \int_a^Q F(w)dw + s \left[ (w - Q)F(w) \right]_Q^b - s \int_Q^b F(w)dw \]

But, \( \left[ (Q - w)F(w) \right]_a^Q = (Q - Q)F(Q) - (Q - a)F(a) = 0. \)

Similarly, \( \left[ (w - Q)F(w) \right]_Q^b = (b - Q)F(b) - (Q - Q)F(Q) = (b - Q). \)

Therefore, \( E[\psi(Q)] = r \int_a^Q F(w)dw + s(b - Q) - s \int_Q^b F(w)dw \)

\[ E[\psi(Q)] = r \left[ \zeta(Q) - \zeta(a) \right] + s(b - Q) - s \left[ \zeta(b) - \zeta(Q) \right] \]

Differentiating \( E[\psi(Q)] \), we get:

\[ \frac{dE}{dQ} = r \frac{d\zeta}{dQ} - s + s \frac{d\zeta}{dQ} \quad \text{Or,} \quad \frac{dE}{dQ} = rF(Q) - s + sF(Q) \]

Setting the first derivative to 0, we get: \( F(Q^*) = s / (r + s) \)

Further, \( \frac{d^2E}{dQ^2} = r\xi(Q) + s\xi(Q) > 0, \quad a < Q < b. \)

This implies, we get the global minimum, at \( w = Q^*. \)
B.2 Demand Following a Discrete Distribution

Assume that the demand for cash follows a Discrete Distribution

Let \( 0 < w_1 < w_2 < \cdots < w_n \) be the possible discrete values for the random variable \( w \). Further, let 
\[
P(w = w_j) = \xi_j,
\]
represent the probability that the variable \( w \) will assume the value \( w_j, \ j = 1, 2, \ldots, n \).

Obviously, \( \xi_1 + \xi_2 + \cdots + \xi_n = 1 \) and \( F(w_j) < F(w_k) \), if \( w_j < w_k \), where \( F(w) \) represents the cumulative probability of the random variable \( w \). For convenience, we assume that \( (w_k - w_j) \geq 2 \), for any \( k > j \).

As before, let \( r \) represent the interest rate that could be earned if the surplus funds were invested. Further, let \( s \) represent the interest rate that has to be paid on the funds borrowed for the period.

Again, we consider two extreme options, before determining the optimal amount of cash \( Q \) that would minimize the total cost (Winston, 2004).

Option (1): Let \( Q = b \). Clearly, there would always be not only enough funds to make the single payment of \( w, a \leq w \leq b \), but also there would be a surplus of \( (b - w) \) that would have earned interest at the rate of \( r \) percent per the period.

Then,
\[
E[\psi(b)] = r \sum_{j=1}^{n} (b - w_j) \xi_j = r(b - E(w))
\]

Option (2): Let \( Q = a \). Clearly, there would always be a shortage of funds, and to make the single payment of \( w, a \leq w \leq b \), we need to borrow an amount of \( (w - a) \) that would require us to pay interest at the rate of \( s \) percent per the period.

Then,
\[
E[\psi(a)] = s \sum_{j=1}^{n} (w_j - a) \xi_j = s(E(w) - a)
\]
Since these two represent the extreme options, for any \( a < Q < b \), the expected cost \( E[\psi(Q)] \) will be between \( E[\psi(a)] \) and \( E[\psi(b)] \), regardless of which is higher or lower between the two.

Obviously, when \( Q > w \), we have a surplus of funds \((Q - w)\), and a shortage of \((w - Q)\), when \( w > Q \).

Since \( w \) is a discrete variable, select a \( Q \) such that \( w_k < Q \) and \( Q + 1 = w_{k+1} \). (Note that \((w_k - w_j) \geq 2\))

Then, the expected interest cost, at the respective interest rates \( r \) and \( s \), is given by:

\[
E[\psi(Q)] = r \sum_{j=1}^{j=k} (Q - w_j)\xi_j + s \sum_{j=k+1}^{j=n} (w_j - Q)\xi_j
\]

for any \( Q \), \( w_k < Q \) and \( Q + 1 = w_{k+1} \).

We have already seen that when \( w \) follows a continuous distribution (such as beta), \( E[\psi(Q)] \) is a convex function of \( Q \) and it has global minimum. Of course, it is always possible to create a discrete version of a continuous probability distribution. Similarly, it is also possible to create an approximate version of a continuous distribution from a discrete distribution, since both types of distributions satisfy the basic property \( F(w_1) < F(w_2) \) for any \( w_1 < w_2 \). This implies that even in the case of the discrete distribution, there exists an optimal value \( Q^* \) at which the function \( E[\psi(Q)] \) not only reaches its minimum, but also exhibits the property of convexity. Of course, our next step is to find \( Q^* \).

An important property of a convex function \( \phi(z) \) is that if its minimum occurs at \( z \), then the two following conditions must hold: \( \phi(z-1) \geq \phi(z) \) and \( \phi(z) \leq \phi(z+1) \).

Therefore, if the minimum of \( E[\psi(Q)] \) occurs at some \( Q \), then we must have:

\[
E[\psi(Q-1)] \geq E[\psi(Q)] \quad \text{and} \quad E[\psi(Q)] \leq E[\psi(Q+1)].
\]

Or equivalently, \( E[\psi(Q-1)] - E[\psi(Q)] \geq 0 \), and \( E[\psi(Q+1)] - E[\psi(Q)] \geq 0 \).

Clearly, \( E[\psi(Q)] = r \sum_{j=1}^{j=k} (Q - w_j)\xi_j + s \sum_{j=k+1}^{j=n} (w_j - Q)\xi_j \) for any \( Q \), \( w_k < Q \), and \( Q + 1 = w_{k+1} \).
Similarly, $E[\psi(Q + 1)] = r \sum_{j=1}^{j=k} (Q + 1 - w_j) \xi_j + s \sum_{j=k+1}^{j=n} (w_j - Q - 1) \xi_j$, $w_k < Q$, and $Q + 1 = w_{k+1}$,

$$E[\psi(Q + 1)] - E[\psi(Q)] = r \sum_{j=1}^{j=k} \xi_j - s \sum_{j=k+1}^{j=n} \xi_j = r[P(w < Q)] - s[P(w \geq Q)].$$

Therefore, the optimal value of $Q$ must be such that $r[P(w < Q)] - s[1 - F(Q)] \geq 0$.

Since $F(Q) = P(w \leq Q)$, $F(Q) \geq P(w < Q)$. Hence, $r[F(Q)] - s[1 - F(Q)] \geq 0$. Or, $F(Q) \geq s / (r + s)$.

But, if $F(Q) \geq s / (r + s)$, for $Q < w_{k+1}$, then it will be automatically true for any $Q \geq w_j, k+1 \leq j \leq n$.

Our goal is to identify $Q$ such that $E[\psi(Q + 1)] - E[\psi(Q)] \geq 0$, and also, $E[\psi(Q - 1)] - E[\psi(Q)] \geq 0$.

In other words, $Q^*$ is the optimal value, if it is the smallest value of $Q$ such that $F(Q^*) \geq s / (r + s)$.

Clearly, the value of the function $E[\psi(Q)]$ goes down from $a$ to $Q^*$, and then it goes up from $Q^*$ to $b$. 
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Recent Publications
   EOQ model: An extension for a shorter planning horizon with integer requirements for order
   cycle times and the number of order-cycles,” *Journal of Academy of Business and

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