The Beauty of Numbers in Nature: Mathematical Patterns and Principles from the Natural World [Book Review]

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The Beauty of Numbers in Nature: Mathematical Patterns and Principles from the Natural World. *By Ian Stewart.* MIT Press, Cambridge, MA, 2017. $24.95. 224 pp., hardcover. ISBN 978-0-262-53428-4.

It came as quite a shock at the time. I cannot recall exactly when it happened, but it certainly caught me by surprise. For most of my life, and certainly from elementary school (or "primary" school in the UK) through high school I had become used to seeing posters and books illustrating the "standard" map of the world: the Mercator projection. Of course, Greenwich, London, sat at longitude 0° (and still does!). And certainly I knew that planet Earth is spheroidal and that this flat projection distorted the shapes and areas (especially near the polar regions), but somehow I was not prepared for the paradigmatic jolt I received when I encountered the Gall–Peters projection for the first time. "Wait, wait, the world isn't like that," I thought. "What's going on here?" It was just so, well, fascinatingly weird, but alas, I soon lost interest in pursuing that line of thought.... Fortunately, several decades later, I encountered a monograph that re-stimulated my interest in the mathematics of maps [1].

But I digress, notwithstanding Stewart's brief description of non-Euclidean geometry in this very context near the end of the book. In fact, all his descriptions are brief. This is not a criticism; the book is so wide-ranging in scope that he could do nothing else. It is richly populated with color photographs, diagrams, or artistic impressions of the phenomena under discussion. It is in fact an updated version of his 2001 book *What Shape Is a Snowflake?*, which I read at the time of its release, so unsurprisingly it has the same structure, i.e., parts and chapters, as before. However, a lot has happened in the mathematical and scientific worlds in the intervening sixteen years, and Stewart has rather successfully incorporated many subsequent developments into his thoughtful meanderings through nature. Stewart may be relatively rare among mathematicians in his ability to explain physical concepts with ease (if not always with complete accuracy; a physicist might have a few conniptions about his description of (i) raindrop "energy" and (ii) "centrifugal" force on p. 105). He discusses a breathtaking array of topics and phenomena, many linked via symmetry (and the breakage thereof).

Based on his earlier work, Stewart uses the generic snowflake as a template with which to examine many multifaceted mathematical principles that appear to undergird what we know of the world around us. I must confess to a certain amount of affection for this approach since I too have used another beautiful natural phenomenon—the rainbow—in this way recently [2]. Very near the beginning of the book, Stewart muses about the "no two snowflakes are alike" contention. By restricting his thought experiment to only to those differences that would be visible under a low-powered lens, say, being able to distinguish a hundred tiny features, then that results in about a nonillion $\left(2^{100} \approx 10^{30}\right)$ different shapes! But the temptation is great to share a story I've proffered before [3], so here goes. Thanks to the sharp eyes of a Minnesota man, it is possible that two identical snowflakes may finally have been observed. While out snowmobiling, he noticed a snowflake that looked familiar to him. Searching his memory, he realized it was identical to a snowflake he had seen as a child in Vermont. Weather experts, while excited, caution that this may be difficult to verify.

In the subsection entitled "Mathematics and Beauty" (pp. 100–101), Stewart ruminates on the idea that to many people mathematics and beauty are mutually exclusive, whereas, he claims, the relationship between the two is genuine (but elusive). As every reader of this will probably agree, it does get a little tiresome when well-meaning persons assume that mathematicians are (forgive me) glorified tax accountants, doing long "sums," and resort to comments about never being good at mathematics themselves, and how can you possibly understand all those squiggles? My stock answer is twofold: I point out that I struggled with mathematics as a child (and still have to work hard at it) and ask them if they read music. Whether or not they do, I point out (to my shame) that a musical score is just meaningless squiggles to me. But that doesn't prevent me from enjoying listening to classical music. In the same spirit, Stewart's book is exactly what is needed to clarify misunderstandings of this kind. Tangentially (yet still connecting mathematics and music) let me note how much I appreciated finding an interesting genre of beer mats (= coasters) some time ago while engaged in social lubrication in a pub near Reading, about thirty miles west of London. Some 3–4 inches square, each side consisted of white writing on a black background. On one side was a quote from the composer Claude Debussy: "Music is the arithmetic of sounds as optics is the geometry of light." On the reverse side was another quote, this time from Sid Vicious (the late great bass guitarist of the British band The Sex Pistols): "You just pick a chord, go twang and you've got music." And at the bottom of each side, in small letters, we find the statement: "Not everything in black and white makes sense. Guinness." Couple this with the title of a 2013 paper, "Why Do Bubbles Sink in Guinness?" [4], and I feel I can make the claim that some of the best applied mathematics can be initiated in an Irish or British pub! (The authors were at the University of Limerick.) For a study invoking that famous fluid in connection with magma and lava flow (with an interesting stability analysis), see [5].

Gems. In this section I'll mention some of my favorite topics encountered along the way, and in some instances suggest related further reading. I was fascinated by the discussions of the following topics:

(i) The subject of scale (pp. 118–119) is an extremely important one; in its basic form it addresses the question "What happens as things change in size?" For an extensive and quite fascinating account of the many ramifications of this question, the book by West [6] is *the* place to start (see also [7]), but for students the best introduction is an essay by J. B. S. Haldane. In fact, a preliminary quiz I recently gave for a senior-level class on mathematical modeling that I am currently teaching posed the following question: "In Haldane's 1926 essay *On Being the Right Size* [8] he states that '*Five thousand mice weigh as much as a man. Their combined surface and food or oxygen consumption are about seventeen times a man's.*' Use surface area/volume arguments to justify this statement (or falsify it if you think it is wrong)." In this case a simple mathematical argument is worth a thousand words (and I did get some answers approaching the latter).

(ii) The discussion of the mathematics of music (pp. 120–121) also provided a tantalizing glimpse into the field of inverse problems, where Mark Kac's famous paper "Can One Hear the Shape of a Drum?" [9] is referenced. I cannot resist mentioning in this context J. B. Keller's article on inverse problems [10]. Early in the introduction, Keller poses three such inverse problems: "What is a question to which the answer is (i) Washington Irving, (ii) Nine W, and (iii) Chicken Sukiyaki. (I have taken a little liberty with his formulation by using the indefinite instead of the definite article to emphasize the general lack of uniqueness associated with many inverse problems.) In introducing these questions in class I try to encourage students to think outside the "mathematical box" for questions to these answers—the direct problem. Thus, "What is nine times W?," while technically correct, will not suffice. This can be used to great effect in the classroom despite the groans.

(iii) Symmetries of animal gaits (pp. 132–135) are a particular specialty of Stewart and his collaborators, and there is a very nice account of the development of one of his (collaborative) models [11], made especially interesting as it resulted in part from attending a rodeo! It is here that Stewart allows himself a little excursion into what the oceanographer Blair Kinsman has referred to as "private science" (as opposed to public science, that rather sanitized version which appears in print) [12]. The dichotomy is readily carried over to mathematics. Private science includes discussions of false starts, dead ends, frustrating delays, and, joy of joys, the "Eureka" moments that occasionally occur. The most important aspect of a mathematical model is its predictive capability—it must be testable. Stewart and his colleagues found their model predicted the existence of a new kind of quadruped gait—the jump—and after carefully reviewing a video of the rodeo event recognized this as most likely their "missing" gait—Eureka!

(iv) In a section on bizarre locomotion (pp. 142–143) Stewart addresses the tiny molecular motor that has evolved to make life easier for the bacterium E. coli. For a 1μm bacterium, swimming in water at 30μm/s the Reynolds number is about 3×10^{-5}, so viscosity is a major problem (though somewhat less so for Michael Phelps). E. M. Purcell's delightful article "Life at Low Reynolds' Number" [13] is well worth reading (as are [14] and [15]).

(v) Bifurcation and catastrophe are discussed on pp. 148–150. Stewart refers to the more modern terminology when describing multiple solutions and their stability, and thus he points out that the word "catastrophe" has rather gone out of vogue these days, and the less disaster-implying word is bifurcation. This is probably a good idea: I had an aunt who referred to it as calamity theory. I could never convince her that not everything Christopher Zeeman wrote about was calamitous. And while I am supposed to be reviewing Stewart's book, and in connection with the topics of multiple solutions, stability, emergent patterns, and critical transitions, I should mention that the book by Marten Scheffer [16] is a fine source of environmental applications.

(vi) Symmetry breaking and speciation (pp. 156–157).

(vii) Ice crystal/snowflake instability (p. 169) and the Mullins–Sekerka instability (p. 212)—see Bill Casselman's appendix to [3] and references therein.

(viii) Time travel, in particular, the "cumulative audience paradox," which was new to me. Stewart cites the example of the Battle of Hastings (p. 202) in writing "Major historical events would attract time-traveling tourists from the indefinitely far future. So, for example, the Battle of Hastings would have been surrounded by millions of spectators hoping to catch the death of King Harold. But we know, from historical records, that no such crowd was present." That's a very clever argument. Unless, of course, they all opted to watch the 18th recorded perihelion passage of Halley's comet from various unpopulated vantage points around the globe. . . .

A Surprising Omission.

Saturn's Polar Hexagon. The planet Saturn is mentioned relatively frequently throughout the book in connection with its rings, its satellites, and its gravitational influence on the orbit of Jupiter (and vice versa). Images from the Voyager 1 and 2 "fly-by" missions in 1980–1981 and the later Cassini mission reveal the presence of a persistent hexagonal pattern in Saturn's north polar regions (while as yet none has been observed at the south pole). Specifically, Saturn's circumpolar jet stream at latitude $\approx 77^\circ$ N is shaped by a prominent "wavenumber 6" perturbation [17]. In contrast to earlier models, the combination of the jet stream and the north polar vortex (which stabilizes a "jet-only" barotropic instability) appears to provide a reasonable explanation of this fascinating phenomenon. Given Stewart's penchant for the underlying themes of patterns, symmetry, and stability, this is a little disappointing.

Nitpicky Stuff. Sometimes the figures don't match the descriptions (e.g., on p. 104 the raindrops on leaves are not spheres); as someone who is something of a rainbow aficionado I feel honor-bound to point out that while Stewart's description of the double rainbow is correct, the diagram on p. 67 is wrong on several counts (one of them guaranteed to make some people apoplectic!). I leave it to the reader to determine why. On p. 128, referring to the eddies known as von Karman vortex streets, Stewart states, "The Earth's atmospheric vortices... are not shed by obstacles and so do not come in pairs." My immediate reaction upon reading this was, "Are too—mountains shed vortices!" In fact, he was referring to anticyclones, but as I indicated, vortex shedding does indeed occur in the atmosphere, as revealed by satellite photographs of Jan Mayan and its local (atmospheric) environment.[1] But these are all minor points and do not in any way detract from the beauty of the book.

Conclusion. Rabbi Abraham Joshua Heschel wrote, "Wonder is an act in which the mind confronts the mystery of the universe" [18]. At the end of his book (p. 215), Stewart invokes and expands on this idea: "I am a mathematician. I experience these wonders through a mind that has spent a lifetime learning how to detect patterns, how to understand patterns, how to analyze patterns, how to find new patterns... I do not believe that the universe is diminished through understanding... the universe is not a conjuror's magic, ruined if you know the trick. But more than all this, I'm aware of how little we truly know about our world... There is so much more to learn."

My review copy of this book is now well marked-up and annotated. But I have in mind a 9-year-old grandson who devours anything mathematical and for whom this book would be a perfect "slow time-release" gift; he will find the descriptions exciting and tantalizing even now at his young age, but especially in the months and years to come; the book is an impressive compendium which will surely induce wonder in anyone—young or old—who does more than skim its table of contents.

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Optimization and Differentiation. *By Simon Serovajsky*. CRC Press, Boca Raton, FL, 2017. $149.95. xxii+516 pp., hardcover. ISBN 978-14-987-5093-6.

Optimization and Differentiation by Simon Serovajsky is the author's distillation of his work on optimization with partial differential equations (PDEs) as constraints. These are infinite-dimensional optimization problems of the form

$$\min_{y,u} F(y) + G(u) \quad \text{s.t.} \quad A(y) = B(u),$$

where $F : Y \to \mathbb{R}$ and $G : U \to \mathbb{R}$ are functionals defined on some function spaces, $A : Y \to Z$ is a (possibly nonlinear) differential operator, and $B : U \to Z$ is a control operator. Here, F usually describes the discrepancy between the *state* y and some desired or measured state, and G is a penalty or regularization term required for ensuring the existence of a solution. More complicated formulations allow A to depend on the *control* u as well. The study of such problems has a long history; we mention only the monographs [3, 4, 5, 2, 1]. The first question is on the existence of a solution. Under suitable assumptions on F, G, A, and B, this can be shown by Tonelli's direct method of the calculus of variations. (Briefly, if F and G are bounded from below, the problem admits a finite infimum and hence there exist minimizing sequences for y and u, which are bounded by virtue of F and G and therefore contain weakly converging subsequences. If F and G are weakly lower semicontinuous and A and B are weak-to-weak continuous, the limits are the desired solutions.) One is then interested in characterizing these minimizers by necessary optimality systems, i.e., that an appropriate derivative of the functional vanishes at a minimizer $(\bar{y}, \bar{u})$. The difficulty lies in the equality constraint. The three most common approaches of treating this are, in ascending order of abstractness:

(i) Define a solution mapping $S : u \mapsto y$ solving $A(y) = B(u)$; show its differentiability by considering solutions $y, \tilde{y}$ for two different $u, \tilde{u}$, forming the difference quotient, identifying a linear PDE satisfied by the difference up to a higher-order term, and passing to the limit $\tilde{u} \to u$; apply the chain rule to obtain

$$S'(\bar{u})^* F'(S(\bar{u})) + G'(\bar{u}) = 0.$$

(ii) Consider the equation as an abstract equality constraint $e(y, u) = 0$ in Z; form the Lagrangian

$$L(y, u, p) = F(y) + G(u) + \langle p, e(y, u)\rangle_Z;$$

set the partial derivatives L_y, L_u, L_p with respect to y, u, p to zero.

(iii) Consider the equation as an abstract equality constraint $e(y, u) = 0$ and apply the implicit function theorem

$$e_y(y(u), u)y'(u) + e_u(y(u), u) = 0$$

to compute $y'(u) = S'(u)$ and proceed as in (i).

If additional inequality constraints are present or if one of the functionals is not differentiable, these optimality conditions become variational inequalities or involve subdifferentials; see, e.g., [1]. In each case, one obtains that the derivative of the solution mapping involves the solution of a linearized