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Characterization of Generalized Haar Spaces

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We say that a subset G of $C_0(T, \mathbb{R}^k)$ is rotation-invariant if $\{Qg : g \in G\} = G$ for any $k \times k$ orthogonal matrix Q. Let G be a rotation-invariant finite-dimensional subspace of $C_0(T, \mathbb{R}^k)$ on a connected, locally compact, metric space T. We prove that G is a generalized Haar subspace if and only if $P_G(f)$ is strongly unique of order 2 whenever $P_G(f)$ is a singleton. © 1998 Academic Press

1. INTRODUCTION

Let T be a locally compact Hausdorff space and G a finite-dimensional subspace of $C_0(T, \mathbb{R}^k)$, the space of vector-valued functions f on T which vanish at infinity, i.e., the set $\{t \in T : \|f(t)\|_2 \geqslant \varepsilon\}$ is compact for every $\varepsilon > 0$. Here $\|y\|_2 := (\sum_{i=1}^{k} |y_i|^2)^{1/2}$ denotes the 2-norm on the k-dimensional Euclidean space $\mathbb{R}^k$ (of column vectors). For f in $C_0(T, \mathbb{R}^k)$, the norm of f is defined as

$$\|f\| := \sup_{t \in T} \|f(t)\|_2.$$

The metric projection P_G from $C_0(T, \mathbb{R}^k)$ to G is given by

$$P_G(f) = \{g \in G : \|f - g\| = \mathrm{dist}(f, G)\}, \qquad \text{for} \quad f \in C_0(T, \mathbb{R}^k),$$

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where

$$\operatorname{dist}(f, G) = \inf\{\|f - g\| : g \in G\}.$$

A subspace G of $C_0(T, \mathbb{R}^k)$ is said to be a Chebyshev subspace if $P_G(f)$ is a singleton for every $f \in C_0(T, \mathbb{R}^k)$. In the Banach space of real-valued continuous functions $C_0(T) \equiv C_0(T, \mathbb{R}^1)$, it is well-known that G is an n-dimensional Chebyshev subspace of $C_0(T)$ if and only if G satisfies the Haar condition (i.e., every nonzero g in G has at most $(n-1)$ zeros). The Haar condition not only provides an intrinsic characterization of Chebyshev subspaces of $C_0(T)$, but also ensures strong unicity and Lipschitz continuity of the metric projection P_G, as shown in the following theorem.

THEOREM 1. *Suppose that G is an n-dimensional subspace of $C_0(T)$. Then the following are equivalent*:

(i) *G satisfies the Haar condition*;

(ii) *G is a Chebyshev subspace of $C_0(T)$*;

(iii) *for every f in $C_0(T)$, $P_G(f)$ is strongly unique, i.e., there exists a constant $\gamma(f) > 0$ such that*

$$\|f - g\| \geqslant \operatorname{dist}(f, G) + \gamma(f) \cdot \|g - P_G(f)\|, \qquad \text{for} \quad g \in G;$$

(iv) *for every f in $C_0(T)$, $P_G(f)$ is a singleton and P_G is Lipschitz continuous at f, i.e., there exists a constant $\lambda(f) > 0$ such that*

$$\|P_G(f) - P_G(h)\| \leqslant \lambda(f) \cdot \|f - h\|, \qquad \text{for} \quad h \in C_0(T).$$

Furthermore, if $T = [a, b]$ is a closed subinterval of $\mathbb{R}$, then all the above are equivalent to the following statement:

(v) *$P_G(f)$ is strongly unique whenever $P_G(f)$ is a singleton.*

The equivalence of (i) and (ii) is due to Haar [6]. Newman and Shapiro [11] proved that (i) implies (iii). Lipschitz continuity of P_G was proved by Freud in [5] and the equivalence condition (v) was given by McLaughlin and Sommers [10]. See [8] for more details. The above theorem summarizes the implications of the Haar condition in $C_0(T)$. One natural question is what are the implications of the Haar condition for a finite-dimensional subspace of the Banach space, $C_0(T, \mathbb{C})$, of all complex-valued continuous functions on T that vanish at infinity. Newman and Shapiro [11] proved that if $G := \{\sum_{i=1}^{n} c_i g_i(x) : c_i \in \mathbb{C}\}$ is an n-dimensional subspace of $C_0(T, \mathbb{C})$ and satisfies the Haar condition, then G is a Chebyshev subspace of $C_0(T, \mathbb{C})$ and, for every $f(x) \in C_0(T, \mathbb{C})$, there exists a constant $\gamma(f) > 0$ such that

$$\|f - g\|^2 \geqslant \operatorname{dist}(f, G)^2 + \gamma(f) \cdot \|g - P_G(f)\|^2, \qquad \text{for} \quad g \in G. \tag{1}$$

The inequality (1) is also referred to as strong unicity of order 2 and is equivalent to the following original form given by Newman and Shapiro:

$$\|f - g\| \geqslant \operatorname{dist}(f, G) + \beta(f) \cdot \|g - P_G(f)\|^2,$$
$$\text{for} \quad g \in G \quad \text{with} \quad \|g - P_G(f)\| \leqslant 1,$$

where $\beta(f)$ is some positive constant. Moreover, the Haar condition is also necessary for a finite-dimensional Chebyshev subspace of $C_0(T, \mathbb{C})$. In fact, an analog of (i)–(iv) of Theorem 1 holds for finite-dimensional Chebyshev subspaces of $C_0(T, \mathbb{R}^k)$, due to the following intrinsic characterization, which we call the generalized Haar condition, of finite-dimensional Chebyshev subspaces of $C_0(T, \mathbb{R}^k)$ given by Zukhovitskii and Stechkin [13].

DEFINITION 2. Let G be an n-dimensional subspace of $C_0(T, \mathbb{R}^k)$ and let m be the maximum integer less than n/k (i.e., $mk < n \leqslant (m+1)\,k$). Then G is called a generalized Haar space if

(i) every nonzero g in G has at most m zeros;

(ii) for any m distinct points t_i in T and any m vectors $\{x_1, ..., x_m\}$ in $\mathbb{R}^k$, there is a vector-valued function p in G such that $p(t_i) = x_i$ for $1 \leqslant i \leqslant m$.

The following analog in $C_0(T, \mathbb{R}^k)$ for parts (i)–(iv) of Theorem 1 was given in [1]. The equivalence (i) $\Leftrightarrow$ (ii) in the following theorem belongs to Zukhovitskii and Stechkin [13].

THEOREM 3. *Let G be a finite-dimensional subspace of $C_0(T, \mathbb{R}^k)$. Then the following are equivalent*:

(i) *G is a generalized Haar subspace.*

(ii) *G is a Chebyshev subspace of $C_0(T, \mathbb{R}^k)$.*

(iii) *P_G is strongly unique of order 2 at each f in $C_0(T, \mathbb{R}^k)$.*

(iv) *for every f in $C_0(T, \mathbb{R}^k)$, $P_G(f)$ is a singleton and P_G satisfies a Hölder continuity condition of order $\frac{1}{2}$.*

Here the Hölder condition is the analog in $C_0(T, \mathbb{R}^k)$ for Lipschitz continuity in Theorem 1. The metric projection P_G is said to satisfy a Hölder continuity condition of order $\frac{1}{2}$ at f if $P_G(\phi)$ is a singleton for every

ϕ in $C_0(T, \mathbb{R}^k)$ and there exists a positive number $\lambda = \lambda(f)$ such that $\|P_G(f) - P_G(h)\| \leqslant \lambda \, \|f-h\|^{1/2} \, (1 + \|f+h\|)^{1/2}$ for all h in $C_0(T, \mathbb{R}^k)$.

The main goal of this paper is to present an analog of part (v) of Theorem 1 for finite-dimensional subspaces in $C_0(T, \mathbb{R}^k)$. However, we can only do so under the assumption that G is rotation invariant.

Definition 4. A subspace G of $C_0(T, \mathbb{R}^k)$ is said to be rotation-invariant if $\{Qg : g \in G\} = G$ for any $k \times k$ orthogonal matrix Q.

Note that $C_0(T, \mathbb{C}) \equiv C_0(T, \mathbb{R}^2)$, since

$$f_1(x) + \mathbf{i} f_2(x) \equiv \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}.$$

Here $\mathbf{i} = \sqrt{-1}$. An n-dimensional subspace of $C_0(T, \mathbb{C})$ can be identified with a $(2n)$-dimensional subspace of $C_0(T, \mathbb{R}^2)$. In fact, one can prove that any rotation invariant finite-dimensional subspace in $C_0(T, \mathbb{R}^2)$ can be identified with a finite-dimensional subspace in $C_0(T, \mathbb{C})$ (cf. Lemma 9). In fact, we consider rotation-invariant subspaces of $C_0(T, \mathbb{R}^k)$ as the natural generalization of complex-valued function subspaces. Now we state the main theorem and present its proof in the next section.

Theorem 5. *Let G be a rotation-invariant finite-dimensional subspace of $C_0(T, \mathbb{R}^k)$, where T is a connected and locally compact metric space. If $P_G(f)$ is strongly unique with order 2 whenever $P_G(f)$ is a singleton, then G is a generalized Haar subspace.*

Remark. Theorem 5 holds for any space T which is connected, locally compact, first countable, and Hausdorff because these are the only properties of T used in the proof.

In Lemma 9, we will show that G is rotation-invariant if and only if G is the tensor product of k-copies of a subspace G_1 of $C_0(T)$, i.e., $G = G_1 \times \cdots \times G_1$. Thus, G is a rotation-invariant Chebyshev subspace of $C_0(T, \mathbb{R}^k)$ if and only if G is the tensor product of k-copies of a Haar subspace G_1 of $C_0(T)$.

Note that for $k = 1$ the result of McLaughlin and Sommers [10] follows from Theorem 5 and, in fact, Theorem 5 gives the following stronger result than that of McLaughlin and Sommers, since the strong unicity of $P_G(f)$ implies the strong unicity of order 2.

Corollary 6. *Suppose that G is an n-dimensional subspace of $C_0(T)$ and T is a connected and locally compact metric space. Then G is a Haar subspace if $P_G(f)$ is strongly unique of order 2 whenever $P_G(f)$ is a singleton.*

2. PROOF OF THE MAIN THEOREM

The proof of Theorem 5 will follow after five lemmas are given. We use $\langle x, y \rangle := \sum_{i=1}^{n} x_i y_i$ to denote the dot product of vectors x and y in $\mathbb{R}^k$, $\operatorname{supp}(\sigma) := \{t \in T : \sigma(t) \neq 0\}$ for any mapping $\sigma: T \to \mathbb{R}^k$, $Z(g) := \{t \in T : g(t) = 0\}$ for any function g in $C_0(T, \mathbb{R}^k)$, and $Z(K) := \bigcap_{g \in K} Z(g)$ for any subset K of $C_0(T, \mathbb{R}^k)$. For any subset K of T, $G|_K$ denotes the restriction of G on K as a subspace of $C(K, \mathbb{R}^k)$. The boundary and closure of K are denoted by $\operatorname{bd}(K)$ and $\operatorname{cl}(K)$, respectively. For a finite subset T_0 of T, let $\operatorname{card}(T_0)$ be the cardinality of T_0 (i.e., $\operatorname{card}(T_0)$ is the number of points in T_0). A mapping σ from T to $\mathbb{R}^k$ is called an annihilator of G if

$$\sum_{t \in \operatorname{supp}(\sigma)} \langle \sigma(t), g(t) \rangle = 0 \qquad \text{for} \quad g \in G.$$

LEMMA 7. *Suppose that $G \neq \{0\}$ is a finite-dimensional subspace of $C_0(T, \mathbb{R}^k)$. Then there exists a mapping σ from T to $\mathbb{R}^k$ that has the following properties*:

(a) $\operatorname{supp}(\sigma)$ *is a finite subset of* T;

(b) $\langle \sigma(t), g(t) \rangle \equiv 0$ *whenever* $\langle \sigma(t), g(t) \rangle \geqslant 0$ *for* $t \in \operatorname{supp}(\sigma)$;

(c) $G_\sigma := \{g \in G : \operatorname{supp}(\sigma) \subset Z(g)\}$ *satisfies the generalized Haar condition on* $T \backslash Z(G_\sigma)$.

(d) $\dim G_\sigma \geqslant 1$.

Proof. We prove the lemma by induction. If $\dim(G) = 1$, then G does satisfy the generalized Haar condition on $T \backslash Z(G)$ and $\sigma(t) \equiv 0$ satisfies the conditions (a)–(d). Suppose that the lemma holds for subspaces of $C_0(T, \mathbb{R}^k)$ with dimension $< n$ and $\dim G = n$. Let $(m+1)$ be the smallest integer that is greater or equal to n/k. If G satisfies the generalized Haar condition on $T \backslash Z(G)$, let $\sigma(t) \equiv 0$. Then $G_\sigma = G$ and we are done. If G does not satisfy the generalized Haar condition on $T \backslash Z(G)$, then either there exist m points $t_1, ..., t_m$ in $T \backslash Z(G)$ such that $\dim G|_{\{t_1, ..., t_m\}} < km$ or there exists a nonzero function $g \in G$ such that $Z(g)$ contains $(m+1)$ points $t_1, ..., t_m, t_{m+1}$. In either case, there exists a finite subset T_0 of $T \backslash Z(G)$ and a nonzero function $g_0(t)$ such that $\dim G|_{T_0} < k \operatorname{card}(T_0)$ and $T_0 \subset Z(g_0)$. Since in the Banach space $C(T_0, \mathbb{R}^k)$ $\dim G|_{T_0} < k \operatorname{card}(T_0)$, there exists an annihilator τ of $G|_{T_0}$ with $\operatorname{supp}(\tau) \subset T_0$, so τ annihilates G also. Since $\operatorname{supp}(\tau) \subset T \backslash Z(G)$, $\dim G_\tau < \dim G$. Since $g_0 \in G_\tau$, $\dim G_\tau \geqslant 1$. Consider G_τ as a subspace defined on $T \backslash Z(G_\tau)$. By the induction assumption, there exists a mapping μ from $T \backslash Z(G_\tau)$ to $\mathbb{R}^k$ such that the conditions (a)–(d) hold for $\sigma \equiv \mu$ and $G \equiv G_\tau|_{T \backslash Z(G_\tau)}$.

Let $\sigma(t) := \tau(t)$ for $t \in \operatorname{supp}(\tau)$ and $\sigma(t) = \mu(t)$ for $t \notin \operatorname{supp}(\tau)$. Then it is easy to verify that the conditions (a)–(d) hold for σ. ▮

LEMMA 8. *Let K be a subset of G, $t_0 \in \operatorname{bd} Z(K)$, $t_0^i \in T \backslash Z(K)$ such that $t_0^i \to t_0$ as $i \to \infty$, and τ is a continuous mapping defined on $\{t_0, t_0^i : i = 1, 2, ...\}$. Then there exists a function $\bar{g} \in K$ and an index $\bar{\imath}$ such that $\langle \tau(t_0^i), g(t_0^i) \rangle = 0$ for $i \geqslant \bar{\imath}$ whenever*

$$\lim_{t \to \infty} \sup \frac{|\langle \tau(t_0^i), g(t_0^i) \rangle|}{|\langle \tau(t_0^i), \bar{g}(t_0^i) \rangle|^{3/2}} \leqslant 1, \tag{2}$$

where we define $0/0 := 0$.

Proof. Let $\langle \tau, K \rangle := \{\langle \tau(t), g(t) \rangle : g \in K\}$. Since $\dim \operatorname{span}\langle \tau, K \rangle|_{\{t_0^j t_0^{j+1}, ...\}}$ is a nonincreasing function of j and has finitely many values, there exists $\bar{\imath}$ such that $\dim \operatorname{span}\langle \tau, K \rangle|_{\{t_0^{\bar{\imath}}, t_0^{\bar{\imath}+1}, ...\}} = \dim \operatorname{span}\langle \tau, K \rangle|_{\{t_0^j, t_0^{j+1}, ...\}}$ for $j \geqslant \bar{\imath}$. That is, if $j \geqslant \bar{\imath}$ and $\langle \tau(t_0^i), g(t_0^i) \rangle = 0$ for $i \geqslant j$, then $\langle \tau(t), g(t) \rangle = 0$ on $T_0 := \{t_{\bar{\imath}}, t_{\bar{\imath}+1, ...}\}$.

Let g_1 be a nonzero function in K. (If $K = \{0\}$, then the lemma is trivially true.) If g_1 can not be used as $\bar{g}$, then there exists a function g_2 in K such that

$$\lim_{i \to \infty} \sup \frac{|\langle \tau(t_0^i), g_2(t_0^i) \rangle|}{|\langle \tau(t_0^i), g_1(t_0^i) \rangle|^{3/2}} \leqslant 1,$$

but $\langle \tau(t_0^i), g_2(t_0^i) \rangle \neq 0$ for infinitely many i's. By the choice of g_2, for all i sufficiently large, we have

$$|\langle \tau(t_0^i), g_2(t_0^i) \rangle| \leqslant 2\, |\langle \tau(t_0^i), g_1(t_0^i) \rangle|^{3/2}. \tag{3}$$

We claim that g_1 and g_2 are linearly independent. Let $c_1 g_1 + c_2 g_2 = 0$. Then for all i sufficiently large

$$\begin{aligned} 0 &= |c_1 \langle \tau(t_0^i), g_1(t_0^i) \rangle + c_2 \langle \tau(t_0^i), g_2(t_0^i) \rangle| \\ &\geqslant |c_1 \langle \tau(t_0^i), g_1(t_0^i) \rangle| - |c_2 \langle \tau(t_0^i), g_2(t_0^i) \rangle| \\ &\geqslant |c_1 \langle \tau(t_0^i), g_1(t_0^i) \rangle| - 2\, |c_2 \langle \tau(t_0^i), g_1(t_0^i) \rangle|^{3/2} \\ &= |\langle \tau(t_0^i), g_1(t_0^i) \rangle| (|c_1| - 2\, |c_2| \langle \tau(t_0^i), g_1(t_0^i) \rangle^{1/2}). \end{aligned} \tag{4}$$

Since $|c_2 \langle \tau(t_0^i), g_1(t_0^i) \rangle|^{1/2} \to 0$ as $i \to \infty$, it follows from (3) and (4) using those i for which $\langle \tau(t_0^i), g_2(t_0^i) \rangle \neq 0$ that $c_1 = 0$. Since g_2 is a nonzero function, $c_2 g_2 = 0$ implies $c_2 = 0$. Hence, g_1 and g_2 are linearly independent.

If g_2 can not be used as $\bar{g}$, then there exists a function g_3 in K such that $\langle \tau(t_0^i), g_3(t_0^i)\rangle \neq 0$ for infinitely many i (possibly different from where $\langle \tau(t_0^i), g_2(t_0^i)\rangle \neq 0$) and

$$\limsup_{i\to\infty} \frac{|\langle \tau(t_0^i), g_3(t_0^i)\rangle|}{|\langle \tau(t_0^i), g_2(t_0^i)\rangle|^{3/2}} \leqslant 1. \tag{5}$$

By (2) and (5), for all i sufficiently large, we have

$$|\langle \tau(t_0^i), g_3(t_0^i)\rangle| \leqslant 2\, |\langle \tau(t_0^i), g_2(t_0^i)\rangle|^{3/2} \leqslant 4\, |\langle \tau(t_0^i), g_1(t_0^i)\rangle|^{9/4}. \tag{6}$$

Now suppose that $c_1 g_1 + c_2 g_2 + c_3 g_3 = 0$. By (4) and (6), we get that, for all i sufficiently large,

$$\begin{aligned} 0 &= |c_1\langle \tau(t_0^i), g_1(t_0^i)\rangle + c_2\langle \tau(t_0^i), g_2(t_0^i)\rangle + c_3\langle \tau(t_0^i), g_3(t_0^i)\rangle| \\ &\geqslant |\langle \tau(t_0^i), g_1(t_0^i)\rangle| (|c_1| - 2\,|c_2|\; |\langle \tau(t_0^i), g_1(t_0^i)\rangle|^{1/2} \\ &\quad - 4\,|c_3|\; |\langle \tau(t_0^i), g_1(t_0^i)\rangle|^{5/4}). \end{aligned}$$

Using those i for which $\langle \tau(t_0^i), g_1(t_0^i)\rangle \neq 0$ as above we obtain $c_1 = 0$. Then from $c_2 g_2 + c_3 g_3 = 0$ we obtain as above

$$\begin{aligned} 0 &= |c_2\langle \tau(t_0^i), g_2(t_0^i)\rangle + c_3\langle \tau(t_0^i), g_3(t_0^i)\rangle| \\ &\geqslant |\langle \tau(t_0^i), g_2(t_0^i)\rangle| (|c_2| - 2\,|c_3| \cdot |\langle \tau(t_0^i), g_2(t_0^i)\rangle|^{1/2}) \end{aligned}$$

and using those i for which $\langle \tau(t_0^i), g_2(t_0^i)\rangle \neq 0$ we obtain $c_2 = 0$. Since $g_3 \neq 0$ and $c_3 g_3 = 0$, we have $c_3 = 0$. Therefore, g_1, g_2, g_3 are linearly independent.

If no function in K can be used as $\bar{g}$ then continuing in this manner we can construct infinitely many linearly independent functions $g_1, g_2, \ldots$ in K. Since G is finite-dimensional, this is impossible. ∎

LEMMA 9. *Suppose that G is a rotation-invariant finite dimensional subspace of $C_0(T, \mathbb{R}^k)$. Then $m := n/k$ is an integer and G is the tensor product of k-copies of an m-dimensional subspace G_1 of $C_0(T)$, i.e.,*

$$G = \left\{ \sum_{i=1}^{k} g_i e_i \colon g_i \in G_1 \text{ for } 1 \leqslant i \leqslant k \right\},$$

where e_i is the ith canonical basis vector for $\mathbb{R}^k$ (i.e., all components of e_i are zero except that the ith component is 1).

Proof. If g is in G, then $g = \sum_{i=1}^{k} g_i e_i$ where $g_i \in C_0(T, \mathbb{R}^1)$. Let $G_i := \{g_i \colon g = \sum_{j=1}^{k} g_j e_j \in G\}$ for $i = 1, \ldots, k$. Then it is obvious that

$G \subset G_1 \times \cdots \times G_k$ ($\equiv \{\sum_{i=1}^k g_i e_i : g_i \in G_i\}$). For a fixed i, let Q_i be the $k \times k$ orthogonal matrix whose jth column is e_j for $j \neq i$ and $-e_i$ for $j = i$. For any $g_i \in G_i$, there exists $g_j \in G_j$ for $j \neq i$ such that $g := \sum_{j=1}^k g_j e_j \in G$. Then $Q_i g \in G$ and $g_i e_i = \frac{1}{2}(g - Q_i g) \in G$. Thus, $G_1 \times ... \times G_k \subset G$, which implies $G = G_1 \times \cdots \times G_k$.

Now we show that $G_i \equiv G_1$ for $1 \leqslant i \leqslant k$. Let B_i be the orthogonal matrix that as e_i has its first column, e_1 as its ith column, e_j as its jth column for $j \neq 1$ or i. For any $g_1 \in G_1$ and $g_i \in G_i$, we have $g_i e_1 = B_i(g_i e_i) \in G$ and $g_1 e_i = B_i(g_1 e_1) \in G$. Hence, $g_i \in G_1$ and $g_1 \in G_i$. So $G_1 = G_2 = \cdots = G_k$ and $G = \{\sum_{i=1}^k g_i e_i : g_i \in G_1 \text{ for } 1 \leqslant i \leqslant k\}$. ▮

LEMMA 10. *Suppose that G is a generalized Haar subspace of $C_0(T, \mathbb{R}^k)$ and $\dim G = mk$. Then, for any given m distinct points $t_1, ..., t_m$ in T and m vectors $x_1, ..., x_m$ in $\mathbb{R}^k$, there exists a function g in G such that $g(t_i) = x_i$ for $i = 1, ..., m$.*

Proof. We show that $\dim G|_{\{t_1, ..., t_m\}} = \dim G$. If not then there exists a $\bar{g} \neq 0$ in G such that $\bar{g}(t_i) = 0$, $i = 1, ..., m$. But this contradicts the fact that G is a generalized Haar set and therefore any function in G has at most $(m-1)$ zeroes. Since $\dim G = mk$ and $\dim C(\{t_1, ..., t_m\}, \mathbb{R}^k) = mk$ the result follows. ▮

Proof of Theorem 5. By Lemma 7, there exists a mapping σ from T into $\mathbb{R}^k$ such that the conditions (a)–(d) in Lemma 7 hold. It follows that if G is not a generalized Haar subspace, then $Z(G_\sigma) \neq \emptyset$. Since $Z(G_\sigma)$ is closed and T is connected, bd $Z(G_\sigma)$ contains at least one point, say t_0. Let $\{t_0^i\}_{i=1}^\infty$ be a sequence of distinct points in $T \backslash Z(G_\sigma)$ such that $\lim_{i \to \infty} t_0^i = t_0$.

Since G is rotation-invariant, it is easy to verify that G_σ is rotation-invariant and hence $\dim G_\sigma = km$ for some integer m. Choose m distinct points $\{t_1, ..., t_m\}$ in $T \backslash (\{t_0, t_0^i, i = 1, ...\} \cup Z(G_\sigma))$. Notice then that $\{t_1, ..., t_m\} \cap \operatorname{supp}(\sigma) = \emptyset$. Then by Lemma 10, for any vectors $x_1, ..., x_m$ in $\mathbb{R}^k$, there exists a function g in G_σ such that $g(t_j) = x_j$, for $j = 1, ..., m$. Since, for fixed i, $\dim G_\sigma|_{\{t_1^i, t_1, ..., t_m\}} < k(m+1)$, there exists an annihilator τ_i of G_σ such that $\operatorname{supp}(\tau_i) \subset \{t_0^i, t_1, ..., t_m\}$. By the interpolation property of G_σ on any m points of $T \backslash Z(G_\sigma)$, $\operatorname{supp}(\tau_i)$ must have $(m+1)$ points. Thus,

$$\operatorname{supp}(\tau_i) = \{t_0^i, t_1, ..., t_m\}.$$

Without loss of generality, we may assume that there exist unit vectors in $\mathbb{R}^k$, $\tau(t_0), \tau(t_1), ..., \tau(t_m)$, such that

$$\lim_{i \to \infty} \operatorname{sgn}(\tau_i(t_0^i)) = \tau(t_0)$$

and

$$\lim_{i \to \infty} \operatorname{sgn}(\tau_i(t_j)) = \tau(t_j), \qquad \text{for} \quad j = 1, ..., m.$$

If t_0 is in $\operatorname{supp}(\sigma)$, we may assume that $\operatorname{sgn}(\sigma(t_0)) = \tau(t_0)$. Otherwise, we can replace τ_i by $Q\tau_i$, where Q is an orthogonal matrix such that $Q\tau(t_0) = \operatorname{sgn}(\sigma(t_0))$. (Here the rotation-invariance of G is used.)

Let $\tau(t_0^i) = \operatorname{sgn}(\tau_i(t_0^i))$. Then τ is a continuous function on the closed set

$$A := \{t_0, t_0^i : i = 1, 2, ...\}. \tag{7}$$

Let $K = \{g \text{ in } G_\sigma : g \not\equiv 0$ and, for $1 \leqslant j \leqslant m$ either $g(t_j) = 0$ or $\langle g(t_j), \tau(t_j)\rangle > 0\}$. Since G_σ is a generalized Haar set on $T \backslash Z(G_\sigma)$, it follows that $K \neq \varnothing$ and if g is in K then for at least one j, $g(t_j) \neq 0$. Let $\bar{g}$ in K be the function given by Lemma 8.

Now follows a lengthy construction of a function f in $C_0(T, \mathbb{R}^k)$. First let $\bar{t} \in (\operatorname{supp}(\sigma) \cup \{t_1, ..., t_m\}) \backslash \{t_0\}$. Then, for t in a sufficiently small neighborhood of each such $\bar{t}$, define

$$\bar{f}(t) = \begin{cases} \operatorname{sgn}(\sigma(\bar{t}))(1 - \|g(\bar{t})\|_2) & \text{if} \quad \bar{t} \in \operatorname{supp}(\sigma) \backslash \{t_0\} \\ \tau(\bar{t})(1 - \|\bar{g}(t)\|_2) & \text{if} \quad \bar{t} = t_j \in \{t_1, ..., t_m\},\ \bar{g}(t_j) = 0 \\ \tau(\bar{t}) & \text{if} \quad \bar{t} = t_j \in \{t_1, ..., t_m\},\ \bar{g}(t_j) \neq 0. \end{cases}$$

Then, for $\varepsilon > 0$ small enough and t near $\bar{t}$ in $\operatorname{supp}(\sigma)$,

$$\begin{aligned} \|\bar{f}(t) - \varepsilon \bar{g}(t)\|_2 &= \|(1 - \|\bar{g}(t)\|_2) \operatorname{sgn} \sigma(\bar{t}) - \varepsilon \bar{g}(t)\|_2 \\ &\leqslant 1 - \|\bar{g}(t)\|_2 + \varepsilon \|\bar{g}(t)\|_2 \leqslant 1, \end{aligned}$$

and, similarly, $\|\bar{f}(t) - \varepsilon \bar{g}(t)\| \leqslant 1$ for t near t_j in $\{t_1, ..., t_m\}$ if $\bar{g}(t_j) = 0$. If $\bar{g}(t_j) \neq 0$, then $\langle \bar{g}(t_j), \tau(t_j)\rangle = \langle \bar{g}(t_j), \bar{f}(t_j)\rangle > 0$ and, by the continuity of $\bar{g}$, $\langle \bar{g}(t), \tau(t_j)\rangle > \delta > 0$ for t near t_j. Thus, for t near t_j,

$$\begin{aligned} \|\bar{f}(t) - \varepsilon \bar{g}(t)\|_2^2 &= \|\tau(t_j)\|^2 - 2\varepsilon \langle \bar{g}(t_j), \tau(t_j)\rangle + \varepsilon^2 \|\bar{g}(t)\|_2^2 \\ &\leqslant 1 - 2\,\delta\varepsilon + \varepsilon^2 \|\bar{g}\| < 1, \end{aligned}$$

if $\varepsilon > 0$ is small enough. Therefore, for a sufficiently small neighborhood W_1 of $[(\operatorname{supp}(\sigma) \cup \{t_1, ..., t_m\}) \backslash \{t_0\}$, $\bar{f}$ is continuous and

$$\|\bar{f}(t) - \varepsilon \bar{g}(t)\|_2 \leqslant 1, \tag{8}$$

if $t \in W_1$ and $\varepsilon > 0$ is small enough.

Let $\bar{f}(t)$ and $\bar{h}(t)$ be defined on the closed set A (cf. (7)) by

$$\bar{h}(t)=\operatorname{sgn}(\tau_i(t))\equiv\tau(t),$$

and

$$\bar{f}(t)=\bar{h}(t)(1-|\langle\bar{h}(t),\,\bar{g}(t)\rangle|^{3/2}),$$

where $t=t_0, t_0^i$ for $i=1, 2, \ldots$. Now we show that there exists an index i_0 such that $\langle\bar{h}(t_0^i),\,\bar{g}(t_0^i)\rangle\neq 0$ for $i\geqslant i_0$. First observe that since $\bar{g}$ is in G_σ, τ_i annihilates G_σ, and $\operatorname{supp}(\tau_i)=\{t_0^i, t_1, \ldots, t_m\}$, we get

$$0=\langle\tau_i(t_0^i),\,\bar{g}(t_0^i)\rangle+\sum_{j=1}^{m}\langle\tau_i(t_j),\,\bar{g}(t_j)\rangle. \tag{9}$$

Since $\bar{g}$ is in K, by the definition, we have either $\bar{g}(t_j)=0$ or

$$0<\langle\bar{g}(t_j),\,\tau(t_j)\rangle=\lim_{t\to\infty}\frac{\langle\bar{g}(t_j),\,\tau_i(t_j)\rangle}{\|\tau_i(t_j)\|_2}. \tag{10}$$

However, (10) implies that $\langle\bar{g}(t_j),\,\tau_i(t_j)\rangle>0$ for i large enough whenever $\bar{g}(t_j)\neq 0$. Since there is at least one j with $\bar{g}(t_j)\neq 0$, it follows from (9) that $\langle\tau_i(t_0^i),\,\bar{g}(t_0^i)\rangle<0$ (i.e., $\langle\bar{h}(t_0^i),\,\bar{g}(t_0^i)\rangle<0$) for i large enough. Thus, for $i\geqslant i_0$, $\|\bar{f}(t_0^i)\|_2<1$ and $\|\bar{f}(t_0)\|_2=1$. Since $\lim_{i\to\infty}t_0^i=t_0$ and T is locally compact Hausdorff, there exist open sets W and V with compact closures such that $t_0\in V$, $[(\operatorname{supp}(\sigma)\cup\{t_1, \ldots, t_m\})\backslash\{t_0\}]\subset W$, and $\operatorname{cl}(W)\cap\operatorname{cl}(V)=\varnothing$. Choose i_0 large enough such that $t_0^i\in V$ for $i\geqslant i_0$. Choose $W\subset W_1$ so that (8) holds for $t\in W$ and $\varepsilon>0$ small enough. By relabeling of t_0^i, we may assume without loss of generality that $t_0^i\in V$ for all i and

$$\|\bar{f}(t_0^i)\|_2<1,\qquad\text{for}\quad i=1, 2, \ldots. \tag{11}$$

Now $\bar{h}$ can be extended from the closed set A (cf. (7)) to a continuous function $h(t)$ on the open set V with $A\subseteq V$ and $\|h(t)\|_2\equiv 1$, $t\in V$, by Tietze's Extension Theorem for locally compact Hausdorff spaces [12, p. 385] and the proof of Corollary 5.3 [4, p. 151]. Let $\bar{f}(t)=h(t)(1-|\langle h(t),\,\bar{g}(t)\rangle|^{3/2})$ for t in V. Since $B=\operatorname{cl}(V)\cup\operatorname{cl}(W)$ is compact, we can extend $\bar{f}$ from B to a function F on all of T with F in $C_c(T, \mathbb{R}^k)$ (the collection of functions in $C_0(T, \mathbb{R}^k)$ whose supports are compact) and $\|F(t)\|_2\leqslant 1$. Let

$$D:=\{t_0\}\cup\{t_1, \ldots, t_m\}\cup\operatorname{supp}(\sigma)\cup\{t_0^i\colon\langle h(_0^i),\,g(\bar{t}_0^i)\rangle\neq 0\}.$$

Then D is a G_δ set, there exists [4, p. 148] a function ϕ in $C_c(T, \mathbb{R})$ with $0 \leqslant \phi(t) \leqslant 1$ and $\phi^{-1}(1) = D$. Thus $f = \phi F$ is an extension of $\bar{f}$ from $W \cup V$ to T which satisfies the following conditions:

$$f(t) = \begin{cases} \operatorname{sgn}(\sigma(t)) & \text{for} \quad t \text{ in } \operatorname{supp}(\sigma), \\ \operatorname{sgn}(\tau(t)) & \text{for} \quad t \text{ in } \{t_1, ..., t_m\}, \end{cases} \tag{12}$$

$$\|f(t)\|_2 < 1 \quad \text{if } t \neq 0 \text{ and } t \in V, \qquad \text{and} \qquad \|f(t_0)\|_2 = 1, \tag{13}$$

$$\|f(t_0^i)\|_2 = 1 - |\langle h(t_0^i), \bar{g}(t_0^i)\rangle|^{3/2} \text{ if } \langle h(t_0^i), g(\bar{t}_0^i)\rangle \neq 0, \tag{14}$$

$$\|f(t)\|_2 \leqslant \|h(t)\|_2\,(1 - |\langle h(t), \bar{g}(t)\rangle|^{3/2}) \qquad \text{for} \quad t \in V, \tag{15}$$

$$\|h(t)\|_2 = 1 \qquad \text{if} \quad t \text{ is in } V, \tag{16}$$

$$h(t_0^i) = \tau_i(t_0^i) = \tau(t_0^i), \qquad i \geqslant i_0, \tag{17}$$

and

$$\|f(t) - \varepsilon \bar{g}(t)\|_2 \leqslant 1 \qquad \text{if} \quad \varepsilon \leqslant \varepsilon_0 \text{ and } t \notin V, \tag{18}$$

where $\varepsilon_0 > 0$ is a small positive number. Note that (18) was verified for $\bar{f}$ and t in W, now $\bar{f}(t)$ is replaced by $\phi(t)\,\bar{f}(t)$ for $0 < \phi(t) \leqslant 1$ and the same calculation shows (18) still holds for $t \in W$. However, $\sup\{\|f(t)\|_2 : t \notin (V \cup W)\} < 1$ since V and W are open sets containing the only points where f has norm 1. Thus, (18) holds for f and $t \notin V$.

We claim that $P_G(f) = 0$. First it is shown that if g is in $P_G(f)$ and $g \not\equiv 0$, then g is in K and thus in G_σ. If g is in $P_G(f)$, it is easy to verify that since $\|f - g\| \leqslant 1$ it follows that $\langle g(t), \sigma(t)\rangle \geqslant 0$ for t in $\operatorname{supp}(\sigma)$. Thus, by Lemma 7(b), $\langle g(t), \sigma(t)\rangle = 0$ for t in $\operatorname{supp}(\sigma)$. Thus, for t in $\operatorname{supp}(\sigma)$, we have $(g(t), f(t)) = 0$, and

$$1 = \|f\| \geqslant \|f(t) - g(t)\|_2^2 = \|f(t)\|_2^2 + \|g(t)\|_2^2 = 1 + \|g(t)\|_2^2.$$

As a result, $g(t) = 0$ for t in $\operatorname{supp}(\sigma)$ and g is in G_σ. Similarly, one can show that $\langle g(t_j), \tau(t_j)\rangle \geqslant 0$ for $j = 1, ..., m$, and $g(t_j) = 0$ whenever $\langle g(t_j), \tau(t_j)\rangle = 0$. Hence if $g \neq 0$, then g is in K.

Now we show that for any nonzero g in $P_G(f)$,

$$\limsup_{i \to \infty} \frac{|\langle \tau(t_0^i), g(t_0^i)\rangle|}{|\langle \tau(t_0^i), \bar{g}(t_0^i)\rangle|^{3/2}} > 2. \tag{19}$$

If not, then

$$\limsup_{i \to \infty} \frac{|\langle \tau(t_0^i), \frac{1}{2} g(t_0^i)\rangle|}{|\langle \tau(t_0^i), \bar{g}(t_0^i)\rangle|^{3/2}} \leqslant 1.$$

Since $g \in K$, it is easy to verify that $\frac{1}{2} g \in K$. By Lemma 8, $\langle \tau(t_0^i), g(t_0^i) \rangle = 0$, for $i \geqslant \bar{\imath}$. (We may assume that $\bar{\imath} \geqslant i_0$.) Now τ_i annihilates G_σ, $g \in G_\sigma$, and $\operatorname{supp}(\tau_i) = \{t_0^i, t_1, ..., t_m\}$. Thus,

$$0 = \sum_{t \in \operatorname{supp}(\tau_i)} \langle g(t), \tau_i(t) \rangle = \langle \tau_i(t_0^i), g(t_0^i) \rangle + \sum_{j=1}^{m} \langle \tau_i(t_j), g(t_j) \rangle,$$

and by the definition of $\tau(t)$,

$$0 = \langle \tau(t_0^i), g(t_0^i) \rangle = \langle \operatorname{sgn}(\tau_i(t_0^i), g(t_0^i)) \rangle = \frac{\langle \tau_i(t_0^i), g(t_0^i) \rangle}{\|\tau_i(t_0^i)\|}.$$

Hence, $\langle \tau_i(t_0^i), g(t_0^i) \rangle = 0$. Since τ_i is an annihilator of G_σ, $g \in K \subset G_\sigma$, and $\operatorname{supp}(\tau_i) = \{t_0^i, t_1, ..., t_m\}$, we obtain

$$\sum_{j=1}^{m} \langle \tau_i(t_j), g(t_j) \rangle = \sum_{t \in \operatorname{supp}(\tau_i)} \langle \tau_i(t), g(t) \rangle = 0. \tag{20}$$

Since g is in K, $g(t_j) = 0$ or $\langle g(t_j), \tau(t_j) \rangle > 0$ for $j = 1, ..., m$. If $\langle g(t_j), \tau(t_j) \rangle > 0$ then for i sufficiently large $\langle \tau_i(t_j), g(t_j) \rangle > 0$. Thus from (20) it follows that $g(t_j) = 0$, $j = 1, ..., m$. But then $g \equiv 0$ since G_σ is a generalized Haar set on $G \backslash Z(G_\sigma)$ and this contradicts the assumption that $g \not\equiv 0$, and thus (19) holds.

Now with nonzero g in $P_G(f)$ from (19) it follows that for infinitely many indices i,

$$|\langle \tau(t_0^i), g(t_0^i) \rangle| > 2\, |\langle \tau(t_0^i), \bar{g}(t_0^i) \rangle|^{3/2}. \tag{21}$$

Since τ_i is an annihilator of G_σ, the above inequality implies that, for infinitely many i's

$$\|\tau_i(t_0^i)\|_2 \langle \tau(t_0^i), g(t_0^i) \rangle = -\sum_{j=1}^{m} \langle \tau_i(t_j), g(t_j) \rangle < 0. \tag{22}$$

Thus,

$$\begin{aligned} \|f(t_0^i) - g(t_0^i)\|_2^2 &= \|\langle \tau(t_0^i), f(t_0^i) - g(t_0^i) \rangle\, \tau(t_0^i)\|_2^2 \\ &\quad + \|f(t_0^i) - g(t_0^i) - \langle \tau(t_0^i), f(t_0^i) - g(t_0^i) \rangle\, \tau(t_0^i)\|_2^2 \\ &= |\, \|f(t_0^i)\|_2 - \langle \tau(t_0^i), g(t_0^i) \rangle|^2 \\ &\quad + \|g(t_0^i) - \langle \tau(t_0^i), g(t_0^i) \rangle\, \tau(t_0^i)\|_2^2, \end{aligned} \tag{23}$$

where the first equality is an orthogonal decomposition of the error vector and then we use the definition of $f(t)$ to simplify the expression.

We continue the estimate of $\|f(t_0^i)-g(t_0^i)\|_2^2$ by using indices i for which (21) and (22) hold. Then

$$\begin{aligned}\|f(t_0^i)-g(t_0^i)\|_2^2 &\geqslant (\|f(t_0^i)\|_2+|\langle \tau(t_0^i), g(t_0^i)\rangle|)^2\\ &\geqslant \|f(t_0^i)\|_2^2+2\,\|f(t_0^i)\|_2\,|\langle \tau(t_0^i), g(t_0^i)\rangle|. \qquad (24)\end{aligned}$$

Note that $\phi(t_0^i)=1$ and $f(t_0^i)=h(t_0^i)(1-|\langle h(t_0^i), \bar{g}(t_0^i)\rangle|^{3/2})$. Thus,

$$\begin{aligned}\|f(t_0^i)\|_2^2 &= (1-|\langle h(t_0^i), \bar{g}(t_0^i)\rangle|^{3/2})^2\\ &= (1-|\langle \tau(t_0^i), \bar{g}(t_0^i)\rangle|^{3/2})^2\\ &\geqslant 1-2\,|\langle \tau(t_0^i), \bar{g}(t_0^i)\rangle|^{3/2}. \qquad (25)\end{aligned}$$

Since $\|f(t_0^i)\|_2\to 1$ and $|\langle \tau(t_0^i), \bar{g}(t_0^i)\rangle|^{1/2}\to 0$ as $i\to\infty$, we have $2\,\|f(t_0^i)\|_2\geqslant 1$ for i sufficiently large. Then, by (24), (25), and (21), we get that for infinitely many i's,

$$\|f(t_0^i)-g(t_0^i)\|_2^2\geqslant 1-2\,|\langle \tau(t_0^i), \bar{g}(t_0^i)\rangle|^{3/2}+|\langle \tau(t_0^i), g(t_0^i)\rangle|>1.$$

This is impossible, since $g\in P_G(f)$. The contradiction proves our claim that $P_G(f)=\{0\}$.

Next we show that $P_G(f)$ is not strongly unique of order 2 by estimating $\|f-\varepsilon\bar{g}\|$. By the definition of $f(t)$, for $\varepsilon>0$ small enough, $\|f(t)-\varepsilon\bar{g}(t)\|_2\leqslant 1$ if $t\notin V$ (a neighborhood of t_0) (cf. (18)). If $P_G(f)$ is strongly unique of order 2, then there exists a positive constant γ such that

$$\|f-\varepsilon\bar{g}\|^2\geqslant \operatorname{dist}(f, G)^2+\gamma\operatorname{dist}(\varepsilon\bar{g}, P_G(f))^2,$$

i.e.,

$$\|f-\varepsilon\bar{g}\|^2\geqslant 1+\gamma\varepsilon^2\,\|\bar{g}\|^2. \qquad (26)$$

Let $t_\varepsilon\in V$ be such that

$$\|f(t_\varepsilon)-\varepsilon\bar{g}(t_\varepsilon)\|_2=\|f-\varepsilon\bar{g}\|>1.$$

Since $\|f(t)\|_2<1$ for $t\in V$ and $t\neq t_0$, it follows that $t_\varepsilon\to t_0$ as $\varepsilon\to 0^+$. Note that

$$\begin{aligned}&\|f(t_\varepsilon)-\varepsilon\bar{g}(t_\varepsilon)\|_2^2\\ &\quad=\|f(t_\varepsilon)\|_2^2-2\varepsilon\langle f(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle+\varepsilon^2\|\bar{g}(t_\varepsilon)\|_2^2\\ &\quad\leqslant 1-|\langle h(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle|^{3/2}-2\varepsilon\langle f(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle+\varepsilon^2\,\|\bar{g}(t_\varepsilon)\|_2^2.\end{aligned}$$

By the above equality, (26), and $\bar{g}(t_\varepsilon) \to 0$ as $\varepsilon \to 0$, we obtain that, for $\varepsilon > 0$ small enough,

$$1 + \gamma\varepsilon^2 \, \|\bar{g}\|^2 \leqslant \|f - \varepsilon\bar{g}\|^2$$
$$\leqslant 1 - 2\varepsilon\langle f(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle - |\langle h(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle|^{3/2} + \tfrac{1}{2}\gamma\varepsilon^2 \, \|\bar{g}\|^2,$$

which implies that

$$-2\varepsilon\langle f(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle - |\langle h(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle|^{3/2} \geqslant \tfrac{1}{2}\gamma\varepsilon^2 \, \|\bar{g}\|^2.$$

As a consequence, $\langle f(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle < 0$ and

$$2\varepsilon \, |\langle f(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle| \geqslant |\langle h(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle|^{3/2} + \tfrac{1}{2}\gamma\varepsilon^2 \, \|\bar{g}\|^2.$$

Since $f(t_\varepsilon) = \alpha h(t_\varepsilon)$ for some $0 \leqslant \alpha \leqslant 1$, the above inequality implies

$$2\varepsilon \, |\langle h(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle| \geqslant |\langle h(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle|^{3/2} + \tfrac{1}{2}\gamma\varepsilon^2 \, \|\bar{g}\|^2. \tag{27}$$

Since $|\langle h(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle|^{1/2} \to 0$, for $\varepsilon > 0$ small enough,

$$\frac{\gamma \, \|\bar{g}\|^2}{2} \, |\langle h(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle|^{-1/2} > 1. \tag{28}$$

By (27) and (28),

$$2\varepsilon \, |\langle h(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle| > \frac{2}{\gamma \, \|\bar{g}\|^2} \, |\langle h(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle|^2 + \tfrac{1}{2}\gamma\varepsilon^2 \, \|\bar{g}\|^2. \tag{29}$$

Equivalently, we have

$$\left(\sqrt{\frac{2}{\gamma \, \|\bar{g}\|^2}} \, |\langle h(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle| - \sqrt{\frac{\gamma \, \|\bar{g}\|^2}{2}} \, \varepsilon \right)^2 < 0,$$

which is impossible. Therefore, $P_G(f)$ is not strongly unique of order 2.

REFERENCES

1. M. Bartelt and W. Li, Haar theory in vector-valued continuous function spaces, *in* "Approximation Theory VIII," Vol. 1, "Approximation and Interpolation" (C. K. Chui, and L. L. Schumaker, Eds.), pp. 39–46, World Scientific, New York, 1995.
2. V. I. Berdyshev, On the uniform continuity of metric projection, *in* "Approximation Theory," Banach Center Publications, Vol. 4, pp. 35–41, PWN–Polish Scientific Publishers, Warsaw, 1979.
3. F. Deutsch, Best approximation in the space of continuous vector-valued functions, *J. Approx. Theory* **53** (1988), 112–116.

4. J. Dugundgi, "Topology," Allyn and Bacon, Boston, 1967.
5. G. Freud, Eine Ungleichung für Tschebysheffische approximation Polynome, *Acta. Sci. Math. (Szeged)* **19** (1958), 162–164.
6. A. Haar, Die Minkowski Geometrie und die Annaherung an stetige Funktionen, *Math. Ann.* **78** (1918), 294–311.
7. R. Holmes and B. Kripke, Smoothness of approximation, *Michigan Math. J.* **15** (1968), 225–248.
8. W. Li, Strong uniqueness and Lipschitz continuity of metric projection: A generalization of the classical Haar theory, *J. Approx. Theory* **65** (1989), 164–184.
9. P. K. Lin, Strongly unique best approximation in uniformly convex Banach spaces, *J. Approx. Theory* **56** (1989), 101–107.
10. H. McLaughlin and K. Sommers, Another characterization of Haar subspaces, *J. Approx. Theory* **14** (1975), 93–102.
11. D. J. Newman and H. S. Shapiro, Some theorems on Chebyshev approximation, *Duke Math. J.* **30** (1963), 673–684.
12. W. Rudin, "Real and Complex Analysis," McGraw–Hill, New York, 1980.
13. S. I. Zukhovitskii and S. B. Stechkin, On approximation of abstract functions with values in Banach space, *Dokl. Akad. Nauk. SSSR* **106** (1956), 773–776.