Virginia Journal of Science
Volume 58, Number 1
Spring 2007

An Isotropic Metric

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ABSTRACT

An isotropic metric for a black hole and a better vacuum condition $\nabla^2 V_G = 0$ are presented which yield distinct terms for the energy densities of ordinary matter and gravitational fields in the Einstein tensor ($G^{44} = -g^2(2\nabla^2 V_G + (\nabla V_G)^2)$). This model resolves an inconsistency between electromagnetism and gravity in the calculation of field energy. Resolution of this inconsistency suggests a slight modification of the Einstein equation to $gG^{\mu\nu} = 8\pi G T^{\mu\nu}$.

INTRODUCTION

One can calculate the energy in the electric field of a shell of charge two different ways, which give the same result: One can integrate the square of the electric field over all space around the shell of charge, or one can integrate the work done in moving the charges to the shell. For electromagnetism, like charges repel. So one does positive work to assemble a sphere of charge, like compressing a spring. Thus an electric field has positive energy and positive mass. For gravity, like charges attract. So one does negative work to assemble a spherical shell of mass. Therefore, gravitational fields should have negative energy and negative mass. The vector or tensor nature of the field has no significance in this calculation since the energy stored is force through distance.

Analogies between gravitational and electromagnetic fields are usually explored in the linear approximation for gravitational fields. (See, for example, Chapter 3 of the text by Ohanian and Ruffini 1994) Such expositions may recognize that in the linear approximation, the laplacian of the field should be zero in the absence of matter, and that a gravitational field should contribute a term to the energy density which has a form of the square of the gradient of the gravitational potential. (e.g. ff. p.147-148.) However, when the full equations are developed, the curvature tensor $R^{\mu\nu}$ is assumed to be zero in the absence of matter. As a result the Einstein tensor $G^{\mu\nu}$ is also zero, implying that gravitational fields have no energy density. This assumption goes back a long way.

Almost all papers on gravitational fields over the last ninety years assume that mass density is always nonnegative. For example, the Einstein tensor for the Schwarzschild metric is zero, making its fields massless. The only papers that have admitted the concept of negative mass (e.g. Cattoen and Visser, 2005; Hochberg and Visser, 1998, and Morris and Thorne, 1988) have done so as a purely theoretical tool to explore the concept of wormholes, although weaknesses in this assumption have been identified (Barcelo and Visser, 2002). This assumption of nonnegative mass is the energy conditions used for all metrics in common use. Therefore, those metrics may be incorrect.

An isotropic metric for a black hole is presented here for which $G^{\mu\nu}$ has distinct terms for ordinary matter and gravitational fields. This $G^{\mu\nu}$ is derived in the usual way from the metric tensor $g_{\mu\nu}$. When $g_{\mu\nu}$ is isotropic, one can define a gravitational potential, and then $G^{\mu\nu}$ derived from it takes the form of a difference between the laplacian and the square of the gradient of the potential. The Einstein Tensor is a purely geometric quantity. Instead of attributing the entire $G^{\mu\nu}$ to ordinary matter, only the laplacian of the gravitational potential is attributed to ordinary matter. The remainder, which has the form of an energy density of a field is attributed to the contribution of gravitational fields. It should not be surprising that each term of $G^{\mu\nu}$ transforms properly under a lorentz transformation, because it is incidental whether each term is locally zero. An isotropic metric has the additional feature that world lines do not cross event horizons, thus avoiding interactions with regions where physical models break down.

METRIC AND EINSTEIN TENSOR

For an isotropic metric with gravitational distortion g, in the rest frame of the source of the gravitational field,

$$ds^2 = -(g_r)^2(dx^2 + dy^2 + dz^2) + dt^2/(g_t)^2 \quad . \tag{1}$$

Justification will be shown later for $g_r = g_t = g$. This metric differs from simply isotropic coordinates in that a sphere of radius r has a surface area of $4\pi g^2 r^2$ instead of $4\pi r^2$. Because the speed of light slows by a factor of g^2, the metric is *not* conformally flat, as will be shown later with the geodesics (Eq. 13).

The gravitational distortion affects momentum and energy as well as distance and time. For example, one could use a photon with a frequency matched to that of a clock in a gravitational well to carry information about the clock out of the well. Then, the gravitational distortion should affect energy the same as frequency. As a photon of energy M_γ climbs out of the gravitational potential, $g\cdot d(M_\gamma V_G) = -d(g M_\gamma)$, which yields $g = \exp(-V_G)$. Then the Einstein tensor, in spherical coordinates, derived from the isotropic metric of the length differential (Eq. 1) is:

$$G^{\mu\nu} = \begin{pmatrix} (\nabla V_G/g)^2 & 0 & 0 & 0 \\ 0 & -(\nabla V_G/rg)^2 & 0 & 0 \\ 0 & 0 & -(\nabla V_G/rg\sin\theta)^2 & 0 \\ 0 & 0 & 0 & G^{44} \end{pmatrix} \tag{2}$$

where $G^{44} = -g^2(2\nabla^2 V_G + (\nabla V_G)^2)$. From Einstein's equation, $G^{44} = 8\pi G T^{44}$, where T^{44} is the mass density. Because $V_G < 0$, $-g^2\nabla^2 V_G > 0$. This term represents mass density of ordinary matter. The term $-g^2(\nabla V_G)^2 < 0$ represents the mass density of gravitational fields. Both the gradient and laplacian are calculated with the metric scaling the coordinates in the usual manner.

If one admits azimuthal as well as radial variation for g, then G^{44} still retains the same form. When expanded,

$$G^{44} = \left(2\frac{g_{,rr}}{g} + \frac{4g_{,r}}{rg} - \left(\frac{g_{,r}}{g}\right)^2\right) + \frac{1}{r^2}\left(2\frac{g_{,\theta\theta}}{g} + 2\frac{g_{,\theta}}{g}\cot\theta - \left(\frac{g_{,\theta}}{g}\right)^2\right). \tag{3}$$

A comma indicates partial derivatives with respect to the coordinates that follow it. The squared terms are the square of the gradient of the potential. All other terms are the laplacian of the potential.

MASS RECONCILIATION

The negative mass of the gravitational fields inferred by analogy with electromagnetic fields should be quantified. Suppose one attempts such a calculation, to determine the form for the gravitational distortion g in this model. In a vacuum, $\nabla^2 V_G = 0$, which is

$$2\frac{g_{,rr}}{g} + \frac{4g_{,r}}{rg} = 0 \quad . \tag{4}$$

With $g = 1$ and $V_G = -GM/r$ for large r, the solution to this equation is

$$g = 1 + \frac{GM}{r} \tag{5}$$

where M is the mass of the shell of matter. The remaining term in G^{44} is the mass of the gravitational fields. With spherical symmetry,

$$G^{44} = -g^2(\nabla V_G)^2 = -\left(\frac{g_{,r}}{g}\right)^2 \quad . \tag{6}$$

Total mass of the gravitational fields

$$M_G = 2\int_{r=r_0}^{\infty} \frac{G^{44}4\pi r^2 dr}{8\pi G} = \frac{1}{G}\int_{r=r_0}^{\infty} G^{44} r^2 dr \, . \tag{7}$$

If the energy of the gravitational fields is that which is released in assembling the shell of mass M, then $M_G = MV_G = -M \ln g$, in analogy with assembling a shell of electric charge. When one equates these two calculations of the mass,

$$-GM \ln g = -\int_{r=r_0}^{\infty} \left(\frac{g,_r}{g}\right)^2 r^2 dr \,. \tag{8}$$

$$\frac{-GMdr}{r^2} = \frac{dg}{g}. \quad \frac{GM}{r} = \ln g = -V_G \,. \tag{9}$$

This result does not satisfy the vacuum condition $\nabla^2 V_G = 0$. Apparently, the problem is over defined.

To solve this paradox, one might allow the distortion for the time coordinate, g_t, to differ from that for the space coordinate, g_r, and apply the vacuum condition only to g_r. With this substitution, G^{44} retains the desired form, $G^{44} = -g_t^2(2\nabla^2 V_G + (\nabla V_G)^2)$, and g_t appears only as a scaling factor, and not in the operators on V_G. If one admits further anisotropy, then $\nabla^2 V_G$ and $(\nabla V_G)^2$ no longer appear as distinct terms in any components of $G^{\mu\nu}$. As shown above, $g_r = 1 + GM/r$ to satisfy the vacuum condition. Then equating the two ways of calculating mass results in $g_t^2 = g_r^3$. It is reasonable to conclude that instead of $G^{\mu\nu} = 8\pi G T^{\mu\nu}$, the Einstein Equation should be

$$gG^{\mu\nu} = 8\pi G T^{\mu\nu} \,. \tag{10}$$

The left side of this equation retains its tensor properties because g is a scalar. This modification conforms with the scaling of energy by the distortion. As a bonus, because mass reconciliation then yields $g_t = g_r$, both g_t and g_r satisfy the vacuum condition. For the rest of this paper, $g_t = g_r = g$.

GEOMETRY NEAR A BLACK HOLE

As one descends into this black hole with isotropic gravitational distortion $g = 1 + GM/r$, it becomes increasingly self similar, since both M and r scale the same way with the gravitational distortion. Locally, the circumference asymptotically approaches $2\pi GM$, and the remaining distance to the event horizon asymptotically approaches GM.

To calculate the geodesics, one integrates the local time for a photon to travel between two points, factors out the constant g^2c, and applies the Euler-Lagrange equations to the integrand.

$$\Delta T = \int \frac{dt}{g} = \frac{1}{g^2 c} \int g \sqrt{\left((r,_\theta)^2 + r^2\right)} d\theta \,. \tag{11}$$

The resulting geodesics are given by:

$$0 = r^2 r,_{\theta\theta} - 2r(r,_\theta)^2 - r^3 - \frac{g,_r}{g}\left((r,_\theta)^2 + r^2\right)^2 . \tag{12}$$

For $g = 1 + GM/r$,

$$r,_{\theta\theta} = \frac{2(r,_\theta)^2}{r} + r - \frac{GM}{r(r+GM)}\left((r,_\theta)^2 + r^2\right)^2 . \tag{13}$$

The right most term distinguishes these geodesics from those for flat space. It deflects the path of light toward the gravitational potential. The time for a photon to reach the event horizon at the center is infinite whether measured by an outside observer:

$$\Delta T = \frac{1}{g^2 c} \int_{r_1}^{r_2} g ds \approx \frac{GM}{g^2 c} \log \frac{r_2}{r_1} \tag{14}$$

or in a frame descending into the gravitational well:

$$\Delta T = \frac{1}{g^2 c} \int_{r_1}^{r_2} g^2 ds \approx \frac{GM}{g^2 c}\left(\frac{1}{r_2} - \frac{1}{r_1}\right). \tag{15}$$

A MASSLESS METRIC

For purposes of comparison, the following gravitational distortion describes a distribution of matter contrived to exactly cancel the negative mass of gravitational fields everywhere outside the event horizon:

$$g = \frac{r}{r - GM} = \frac{1}{1 - GM/r} . \tag{16}$$

Although $G^{44} = 0$ for this metric, G^{11}, G^{22}, and G^{33} are all nonzero. So, this $G^{\mu\nu}$ is not the same as for flat space.

An event horizon resides at $r = GM$. Substitution of g into the equations for time of travel (Eq. 14, 15) shows that objects still do not cross the event horizon. At a radius $2GM$, this metric has a waist, where the circumference is a minimum. Circumference $l_C = 2\pi rg = 2\pi r^2/(r\text{-}GM)$ which has a minimum value of $8\pi GM$. The total distance between two points at different depths

$$s = \int \frac{rdr}{r - GM} = \int_{r_1}^{r_2} \left(1 + \frac{GM}{r - GM}\right) dr = (r_2 - r_1) + \log\left(\frac{r_2 - GM}{r_1 - GM}\right) . \tag{17}$$

Putting $r_1 = 2GM$ at the waist and r_2 inside the waist shows that the circumference grows exponentially with depth:

$$s \approx \log\left(\frac{r_2}{GM} - 1\right) \tag{18}$$

$$l_c = \frac{2\pi r_2^{\,2}}{r_2 - GM} \approx \frac{2\pi GMe^{2s}}{e^s - 1} \approx 2\pi GMe^s . \tag{19}$$

Thus, the distribution of mass makes the space more expansive there than it would be if the matter were absent. One might interpret matter as an excess of volume within a surface area.

CONCLUSIONS AND IMPLICATIONS

Not only does an isotropic metric result in gravitational fields with negative mass, as one should expect, it offers a number of other advantages over the Schwarzschild metric. As shown above, an isotropic metric results in a very symmetric form for the Einstein tensor, with distinct terms for ordinary mass and gravitational fields. Objects do not cross event horizons. A large amount of free energy available to objects falling in the halo of a black hole might nucleate cosmoses. For example, the massless metric just shown illustrates how the presence of an energy density induces expansion. Since this metric is isotropic, it can accommodate the nucleation of isotropically expanding cosmoses in the halo of a black hole in a way that the Schwarzschild metric cannot. An isotropic metric also terminates electromagnetic field lines in a way that the

Schwarzschild metric and Kerr metrics cannot: Deep enough into the halo of the black hole, the circumference and surface area increase, thus causing electromagnetic field strengths decrease. Projected out, it appears that a charge density resides in the halo of a charged black hole. The termination of field lines provides cutoffs for fields, limits field energies, and might accommodate general relativistic models for the masses of the electron, muon, and tauon.

LITERATURE CITED

Barcelo, C. and M. Visser. 2002. Twilight for the energy conditions? International Journal of Modern Physics D. 11 (10): 1553-1560.

Cattoen, C. and M. Visser. 2005. Necessary and sufficient conditions for big bangs, bounces, crunches, rips, sudden singularities and extremality events. Classical and Quantum Gravity. 22 (23): 4913-4930.

Hochberg, D. and M. Visser. Null energy condition in dynamic wormholes. 1998. Physical Review Letters. 81 (4): 746-749.

Morris, M.S. and K.S. Thorne. 1988. Wormholes in Spacetime and Their Use for Interstellar Travel - A Tool for Teaching General-Relativity. American Journal of Physics 56 (5): 395-412.

Ohanian, H. and R. Ruffini. 1994. Gravitation and Spacetime. Second Edition. Norton, New York. 679 p.