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G. Kerr

G. Melrose
Old Dominion University, gmelrose@odu.edu

J. Tweed
Old Dominion University

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Antiplane Shear of a Strip Containing a Staggered Array of Rigid Line Inclusions

G. KERR
New Mexico Tech
Socorro, NM 87801, U.S.A.

G. MELROSE AND J. TWEED
Old Dominion University
Norfolk, VA 23529, U.S.A.
Melrose@math.odu.edu

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Abstract—Motivated by the increased use of fibre-reinforced materials, we illustrate how the effective elastic modulus of an isotropic and homogeneous material can be increased by the insertion of rigid inclusions. Specifically, we consider the two-dimensional antiplane shear problem for a strip of material. The strip is reinforced by introducing two sets of ribbon-like, rigid inclusions perpendicular to the faces of the strip. The strip is then subjected to a prescribed uniform displacement difference between its faces, see Figure 1. It should be noted that the problem posed is equivalent to that of the uniform antiplane shear problem for an infinite two-dimensional material containing a staggered array of rigid inclusions (see [1] for a review of antiplane problems in the literature). The problem is reduced in standard fashion [2–6] to a mixed boundary value problem in a rectangular domain, whose closed form solution given in terms of integrals of Weierstrassian Elliptic functions, is obtained via triple sine series techniques. The effective shear modulus of the reinforced strip can now be calculated and compared with the shear modulus of a strip without inclusions. Also obtained are the stress singularity factors at the end tips of the inclusions. Numerical results are presented for several different reinforcement geometries.

Keywords—Antiplane, Rigid inclusions, Triple sine series.

1. INTRODUCTION

Consider an isotropic and homogeneous material with a constant shear modulus μ occupying the region R, given by

$$R = \{(x,y) \mid 0 \le x \le \pi,\ -\infty < y < \infty\},$$

reinforced with two sets of rigid inclusions located at

$$0 < a \le x \le b < \pi : y = 4n\lambda\pi, \qquad n = 0, \pm 1, \pm 2, \dots,$$

and

$$\pi - b \le x \le \pi - a : y = (4n+2)\lambda\pi, \qquad n = 0, \pm 1, \pm 2, \dots.$$

If the strip, which is assumed to be in equilibrium is loaded by prescribing a constant displacement difference of w_0 between the faces $x = 0$ and $x = \pi$, then [1] the sole nonzero displacement is

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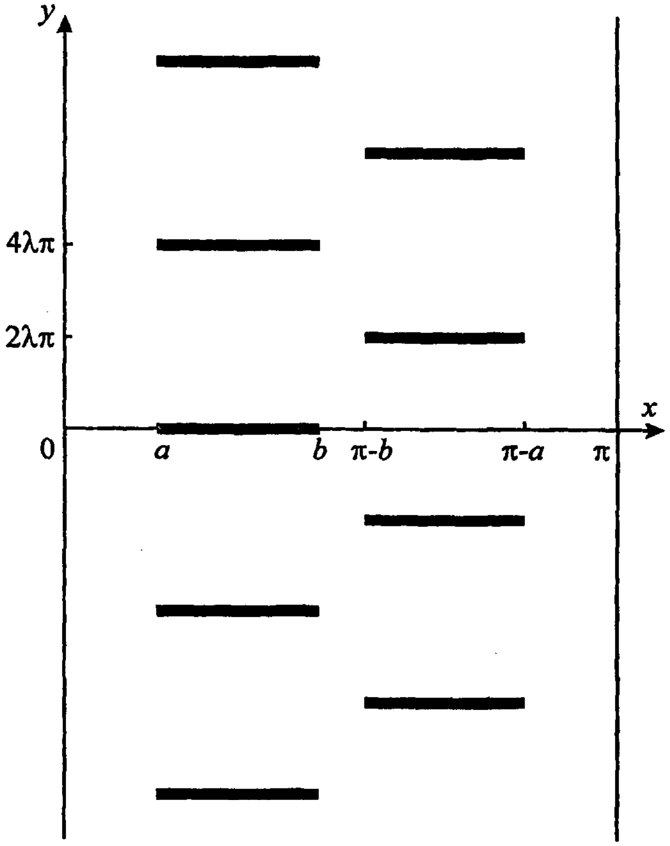

Figure 1. A strip containing a staggered array of rigid line inclusions.

in the z-direction, and is denoted by $w(x, y)$. Furthermore, w is harmonic in R, except at the inclusion sites, and the nonzero stresses are given by

$$\sigma_{xz} = \mu \frac{\partial w}{\partial x}(x, y), \tag{1}$$

$$\sigma_{yz} = \mu \frac{\partial w}{\partial y}(x, y). \tag{2}$$

The equilibrium condition requires that

$$\int_{-\infty}^{\infty} \{\sigma_{xz}(\pi, y) - \sigma_{xz}(0, y)\}\, dy = 0. \tag{3}$$

To achieve the specified displacement difference, we require that

$$w(0, y) = 0, \qquad w(\pi, y) = w_0.$$

Both sets of inclusions will experience a uniform displacement: $w_0\phi_1/\pi$ (say, for the set at $y = 4n\lambda\pi$) and $w_0\phi_2/\pi$ (for the set at $y = (4n + 2)\lambda\pi$). Using a symmetry argument, however, it can be reasoned that the constants ϕ_1, ϕ_2 satisfy

$$\phi_1 + \phi_2 = \pi. \tag{4}$$

The actual value of ϕ_1 (and hence, ϕ_2) will be determined as part of the solution. It should be noted that if (4) was not assumed *a priori*, the problem could still be solved; (4) being a consequence of the more general problem. If we assume that w is of the form

$$w(x, y) = \frac{w_0}{\pi}\left[x + \phi(x, y)\right], \tag{5}$$

then, again using symmetry, we see that our problem is reduced to that of finding a function $\phi(x,y)$ harmonic in the rectangular domain $0 < x < \pi$, $0 < y < 2\lambda\pi$, and satisfying the following boundary conditions:

(i) $$\phi(0,y) = \phi(\pi,y) = 0, \qquad y \in (0, 2\lambda\pi);$$

(ii) $$\frac{\partial\phi}{\partial y}(x,0) = 0, \qquad x \in (0,a) \cup (b,\pi),$$
$$\phi(x,0) = \phi_1 - x, \qquad x \in (a,b);$$

(iii) $$\frac{\partial\phi}{\partial y}(x,2\lambda\pi) = 0, \qquad x \in (0,\pi - b) \cup (\pi - a, \pi),$$
$$\phi(x,2\lambda\pi) = \phi_2 - x, \qquad x \in (\pi - b, \pi - a),$$

where $\phi_1 + \phi_2 = \pi$.

2. THE SOLUTION OF THE ASSOCIATED MIXED BOUNDARY VALUE PROBLEM

Consider the function

$$\phi(x,y) = \sum_{n=1}^{\infty} A_n \frac{[\cosh(n(2\lambda\pi - y)) + (-1)^n \cosh(ny)]}{\sinh(2n\lambda\pi)} \sin(nx). \tag{6}$$

Clearly, ϕ is harmonic in the required domain and satisfies boundary Condition (i) automatically. Notice that $\phi(x,y)$ has the following symmetry:

$$\phi(x,y) = -\phi(\pi - x, 2\lambda\pi - y), \qquad x \in (0,\pi), \quad y \in (0, 2\lambda\pi). \tag{7}$$

Of course, as in the case of (4),(7) could have been assumed as part of the symmetry of the problem. Equation (7) shows us that

$$\left.\begin{aligned} \phi(x,0) &= -\phi(\pi - x, 2\lambda\pi) \\ \phi_y(x,0) &= \phi_y(\pi - x, 2\lambda\pi) \end{aligned}\right\}, \qquad 0 < x < \pi, \tag{8}$$

which leads us to the conclusion that boundary Conditions (ii) and (iii) will both be satisfied if $\{A_n\}_1^\infty$ are given by the triple sine series

$$\begin{aligned} &\sum_{n=1}^{\infty} nA_n \sin nx = 0, && x \in (0,a) \cup (b,\pi), \\ &\sum_{n=1}^{\infty} \omega_n A_n \sin nx = \phi_1 - x, && x \in (a,b), \end{aligned} \tag{9}$$

in which

$$\omega_n = \coth(2n\lambda\pi) + (-1)^n \operatorname{csch}(2n\lambda\pi). \tag{10}$$

Boundary Condition (ii) suggests that $\phi_y(x,0)$ take the form

$$\phi_y(x,0) = -\sum_{n=1}^{\infty} nA_n \sin nx = -H[(b-x)(x-a)]\, g(x), \qquad x \in [0,\pi], \tag{11}$$

where $H(x)$ is the heaviside step function, and $g(x)$ is an, as yet, undetermined function. From (11), it is clear that the coefficients $\{A_n\}_1^\infty$ are given by

$$A_n = \frac{2}{n\pi} \int_a^b g(t) \sin(nt)\, dt. \tag{12}$$

Additionally, since each inclusion is in equilibrium, we require that

$$\int_a^b g(x)\,dx = 0. \tag{13}$$

Substituting (12) into the second of the equations in (9), it is now evident that $g(t)$ satisfies the singular integral equation

$$\frac{1}{\pi}\int_a^b M(x,t)\,g(t)\,dt = \phi_1 - x, \qquad x \in (a,b), \tag{14}$$

with subsidiary condition (13) and kernel

$$M(x,t) = 2\sum_{n=1}^{\infty} \frac{[\coth(2n\lambda\pi) + (-1)^n \operatorname{csch}(2n\lambda\pi)]}{n} \sin(nx)\sin(nt). \tag{15}$$

Since (14) is a Carlemann-type equation [7], it is solved indirectly by differentiating both sides with respect to x. So, we first note [8] that

$$\frac{\partial}{\partial x} M(x,t) = -2\eta t + 2\zeta(t) + \frac{\wp'(t)}{\wp(t) - \wp(x)}, \tag{16}$$

where

$$\zeta(x) = \zeta\left(x, \pi, \pi\left[\frac{1}{2} + i\lambda\right]\right) \tag{17}$$

is the Weierstrass Zeta function,

$$\wp(x) = \wp\left(x, \pi, \pi\left[\frac{1}{2} + i\lambda\right]\right) \tag{18}$$

is the Weierstrass p-function, and in terms of the theta function

$$\vartheta_1(x,q) = 2\sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin((2n+1)x), \tag{19}$$

η is a constant given by

$$\eta = -\frac{1}{12}\frac{\vartheta_1'''(0,q)}{\vartheta_1'(0,q)} \tag{20}$$

and

$$q = e^{i\pi[(1/2)+i\lambda]}. \tag{21}$$

The Weierstrass p-function $\wp(x) = \wp(x, \omega_1, \omega_3)$ appearing above has one real parameter $\omega_1 = \pi$ and one complex parameter $\omega_3 = \omega_1[1/2 + i\lambda]$, $\lambda > 0$. In this case [8], the function is real valued and monotone decreasing, from infinity to $e_1 = \wp(\pi, \pi, \pi[1/2 + i\lambda])$, as x increases from 0 to π. Therefore, differentiating both sides of (14) with respect to x, we obtain the related Cauchy-type integral equation

$$\frac{1}{\pi}\int_a^b g(t)\frac{\wp'(t)}{\wp(t) - \wp(x)}\,dt = -1 + B, \qquad x \in (a,b), \tag{22}$$

where B is given by

$$B = \frac{1}{\pi}\int_a^b [2\eta t - 2\zeta(t)]\,g(t)\,dt. \tag{23}$$

Introducing the change of variable $\tau = \wp(t)$ reduces (22) to the Airfoil equation, whose solution may be found in [7]. Reverting back to the original variable, we discover that $g(t)$ is given by

$$g(t) = \frac{C + (B-1)\,[\wp(a) + \wp(b) - 2\wp(t)]}{2\Delta(t)}, \tag{24}$$

where C is an arbitrary constant and where

$$\Delta(t) = \sqrt{[\wp(a) - \wp(t)]\,[\wp(t) - \wp(b)]}. \tag{25}$$

Now, define

$$\begin{aligned} I_{mn} &= \frac{1}{\pi}\int_a^b \frac{t^m[\wp(a) + \wp(b) - 2\wp(t)]^n}{\Delta(t)}\,dt, \\ J_{mn} &= \frac{1}{\pi}\int_a^b \frac{\{\zeta(t)\}^m\,[\wp(a) + \wp(b) - 2\wp(t)]^n}{\Delta(t)}\,dt, \\ K_0 &= \int_0^a \frac{1}{\Delta_1(t)}\,dt, \end{aligned} \tag{26}$$

where

$$\Delta_1(t) = \sqrt{[\wp(t) - \wp(a)]\,[\wp(t) - \wp(b)]}. \tag{27}$$

Then substituting (24) into (13) and (23) reveals that

$$C = \frac{-I_{01}}{I_{00}\,(\eta I_{11} - J_{11} - 1) - I_{01}\,(\eta I_{10} - J_{10})} \tag{28}$$

and

$$B = \frac{I_{00}\,(\eta I_{11} - J_{11}) - I_{01}\,(\eta I_{10} - J_{10})}{I_{00}\,(\eta I_{11} - J_{11} - 1) - I_{01}(\eta I_{10} - J_{10})}. \tag{29}$$

Now the solution of (22) must also be a solution of (14). Hence, substituting (24) into (14), we discover [6] that this will indeed be the case provided:

$$\phi_1 = \frac{2\,(CI_{10} + (B-1)I_{11})\,(J_{10} - \eta I_{10}) + K_0(CI_{00} + (B-1)\,I_{01}) + 2I_{10}}{2I_{00}}. \tag{30}$$

3. THE EFFECTIVE SHEAR MODULUS

In the absence of the inclusions, it is clear that the specified displacement $w(0, y) = 0$, $w(\pi, y) = w_0$ would require an applied surface traction of

$$\sigma_{xz}^0 = \frac{w_0\mu}{\pi}. \tag{31}$$

To induce the same displacement in the reinforced material would require an applied traction of

$$\sigma_{xz}^1 = \frac{1}{2\lambda\pi}\int_0^{2\lambda\pi} \sigma_{xz}(\pi, y)\,dy, \tag{32}$$

or in terms of the integrals (26)

$$\sigma_{xz}^1 = \frac{w_0\mu}{\pi}\left[1 - \frac{1}{2\lambda\pi}\,\{CI_{10} + (B-1)I_{11}\}\right]. \tag{33}$$

Hence, the effective shear modulus μ_e is given by

$$\frac{\mu_e}{\mu} = \frac{\sigma_{xz}^1}{\sigma_{xz}^0} = 1 + \frac{1}{2\lambda\pi}\,[I_{11} - BI_{11} - CI_{10}]. \tag{34}$$

Figures 2 and 3 show the variation of the scaled effective shear modulus μ_e/μ with $(b-a)/\pi$ for several values of λ. In Figure 2, the center of the line inclusion in the plane $y = 0$ is fixed at 0.5π. In Figure 3, the center line is fixed at 0.2π.

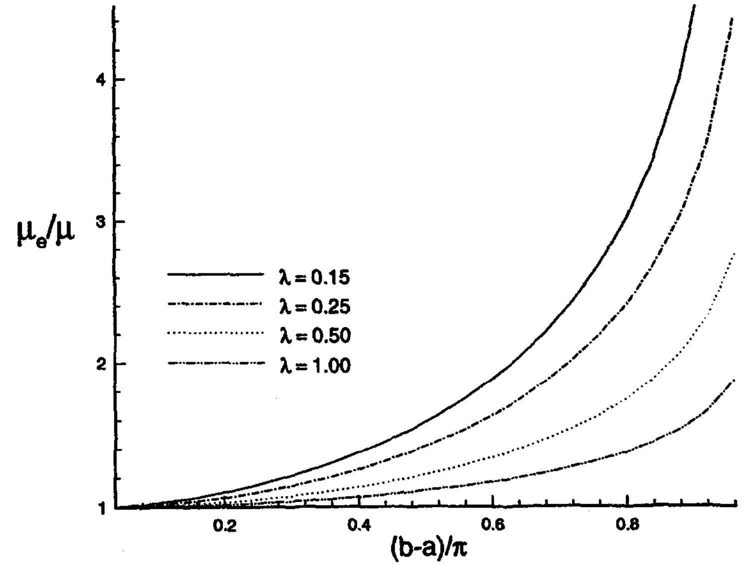

Figure 2. The variation of μ_e/μ with $(b-a)/\pi$ when $(b+a)/2\pi = 0.5$ for several values of λ.

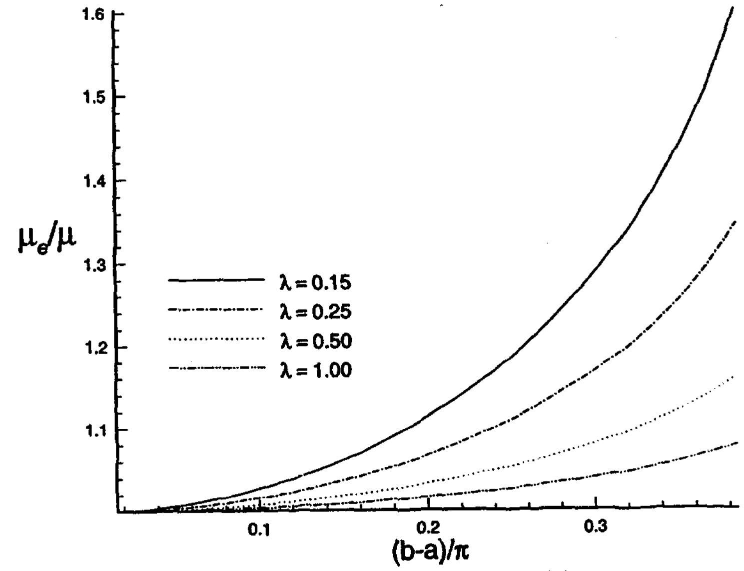

Figure 3. The variation of μ_e/μ with $(b-a)/\pi$ when $(b+a)/2\pi = 0.2$ for several values of λ.

4. STRESS SINGULARITY FACTORS

It is, of course, well known that the reinforced composite will be most susceptible to tear at the end tips of the inclusions, where the stress field is singular. Therefore, we define the stress singularity factors as

$$\mathrm{SSF}_a = \lim_{x \to a^-} \sqrt{2(a-x)}\,\sigma_{xz}(x,0) \tag{35}$$

and

$$\mathrm{SSF}_b = \lim_{x \to b^+} \sqrt{2(x-b)}\,\sigma_{xz}(x,0). \tag{36}$$

Combining (1) and (5), we note that

$$\sigma_{xz}(x,0) = \frac{\mu\omega_0}{\pi}\left(1 + \phi_x(x,0)\right),$$

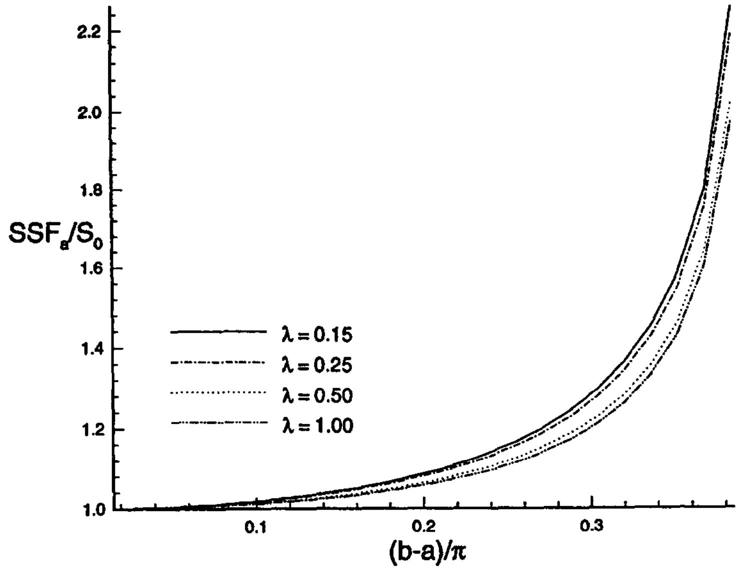

Figure 4. The variation of SSF_a/S_0 with $(b-a)/\pi$ when $(b+a)/2\pi = 0.2$ for several values of λ.

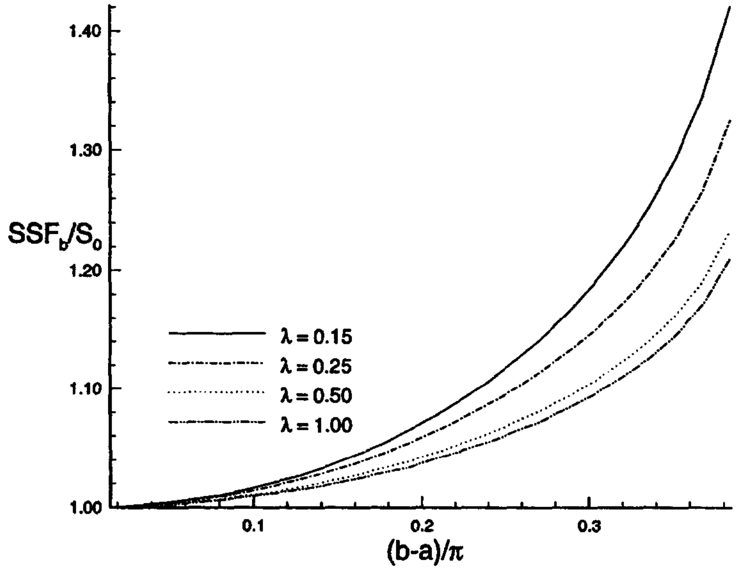

Figure 5. The variation of SSF_b/S_0 with $(b-a)/\pi$ when $(b+a)/2\pi = 0.2$ for several values of λ.

and for $x \in (0,a) \cup (b,\pi)$, from (6) and (24)

$$\phi_x(x,0) = -1 + \operatorname{sgn}(a-x)\frac{C + (B-1)\left[\wp(a) + \wp(b) - 2\wp(x)\right]}{2\Delta_1(x)}. \tag{37}$$

Therefore, we find that

$$\mathrm{SSF}_a = \frac{\mu\omega_0}{\pi}\sqrt{\frac{-1}{2\wp'(a)}}\left[\frac{C}{\sqrt{\wp(a)-\wp(b)}} - (B-1)\sqrt{\wp(a)-\wp(b)}\right] \tag{38}$$

and

$$\mathrm{SSF}_b = \frac{\mu\omega_0}{\pi}\sqrt{\frac{-1}{2\wp'(b)}}\left[\frac{-C}{\sqrt{\wp(a)-\wp(b)}} - (B-1)\sqrt{\wp(a)-\wp(b)}\right]. \tag{39}$$

To plot the stress singularity factors, we scale them by $S_0 = \sigma^0_{xz}\sqrt{(b-a)/2}$ which is the stress singularity factor for a single inclusion in an infinite sheet of material under the same load. Figures 4 and 5 show such plots with the center line of the inclusion fixed at 0.2π.

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