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Convergence of Pólya Algorithm and Continuous Metric Selections in Space of Continuous Functions

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We show that one can construct a continuous selection for the metric projection in the space of continuous functions by the Pólya algorithm. Moreover, the existence of a continuous selection for the metric projection is equivalent to the stable convergence of the Pólya algorithm. © 1995 Academic Press, Inc.

1. INTRODUCTION AND DEFINITIONS

Let T be a locally compact Hausdorff space and $C_0(T)$ be the Banach space of all real-valued continuous functions f on T which vanish at infinity (i.e., the set $\{t \in T: |f(t)| \geqslant \varepsilon\}$ is compact for every $\varepsilon > 0$). The norm on $C_0(T)$ is the supremum norm: $\|f\| := \sup_{t \in T} |f(t)|$. For any $f \in C_0(T)$ and a finite-dimensional subspace G of $C_0(T)$, the distance from f to G is $d(f, G) := \inf\{\|f - x\| : x \in G\}$. Consider the best approximation problem associated with G:

$$\text{find } g \text{ in } G \text{ such that } \|f - g\| = d(f, G). \tag{1.1}$$

The metric projection $\Pi_G(\cdot)$ from $C_0(T)$) onto G maps each f in $C_0(T)$ to the solution set of the best approximation problem (1.1); i.e.,

$$\Pi_G(f) := \{g \in G: \|f - g\| = d(f, G)\}.$$

The classical Haar theorem (cf. [28]) implies that $\Pi_G(f)$ is a singleton for all $f \in C_0(T)$ if and only if G is a Haar space (i.e., any nonzero element of G has at most $\dim G - 1$ zeros).

In general, $\Pi_G(f)$ has infinitively many elements and Π_G is a set-valued mapping from $C_0(T)$ to 2^G (subsets of G). In the theory of set-valued mappings, one important subject is the existence of continuous selections.

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The most well known result is the Michael selection theorem, which in essence states that a lower semicontinuous set-valued mapping has a continuous selection [29, 30]. The continuous selection constructed by Michael is the limit of a sequence of continuous ε-approximate selections which are defined by a partition of unity. Since $\Pi_G(f)$ is closed and convex for any $f \in C_0(T)$, Steiner point selector also provides a means for the construction of a selection for Π_G (cf. [38, 37, 35]). The definition of Steiner point selector involves the integration of the supporting function of $\Pi_G(f)$. Steiner point selector has many nice properties. In particular, it is continuous (or Lipschitz continuous) if Π_G is continuous (or Lipschitz continuous). Since G is finite-dimensional, Π_G is lower semicontinuous if and only if Π_G is continuous (cf. [39]). Therefore, Michael's selection or Steiner point selector provides a continuous selection for Π_G if Π_G is lower semicontinuous. However, the lower semicontinuity of Π_G is not necessary for the existence of continuous selections (cf. [10–12, 33, 26] for intrinsic characterizations of G whose metric projection has a continuous selection and related topics). In general, one can prove the existence of a continuous selection for Π_G by constructing a lower semicontinuous submapping of Π_G (cf. [4, 14, 22, 7, 27, 26]). This can also be accomplished by Brown's derived mappings of Π_G [8]. However, there is no simple constructive way to define a continuous selection for Π_G if such a selection is possible. The continuous selections constructed for Π_G, by Sommer and Nürnberger [32, 40–42], or by Li [17–21, 26], or by Blatter [1], or by Blatter and Fischer [2], or by Blatter and Schumaker [5], are based on the idea of the Chebyshev alternation property of uniform best approximations [28] and are quite complicated. So a natural question is whether there is a simple constructive way to define a continuous selection for Π_G when Π_G has continuous selections. The objective of this paper is to show that the Pólya algorithm is a solution; i.e., the limit of the best L_p-approximations as $p \to \infty$ produces a continuous selection for Π_G.

Suppose that there exists a finite Borel measure μ on T such that $\mu(U) > 0$ for every nonempty open subset U of T. Then we can "approximate" the supremum norm $\|\cdot\|$ by a sequence of strictly convex norms, known as L_p-norms, $\|f\|_p := (\int_T |f(t)|^p \, d\mu)^{1/p}$:

$$\lim_{p \to \infty} \|f\|_p = \|f\| \qquad \text{for} \quad f \in C_0(T).$$

Let $\Pi_G^p(f)$ be the best L_p-approximation to f from G; i.e., $\Pi_G^p(f)$ is the unique element in G such that

$$\|f - \Pi_G^p(f)\|_p = \inf_{g \in G} \|f - g\|_p.$$

It is well known that any accumulation point of $\{\Pi_G^p(f)\}$ as $p \to \infty$ is an element of $\Pi_G(f)$. In particular, if $\Pi_G(f)$ is a singleton, then $\lim_{p \to \infty} \Pi_G^p(f) = \Pi_G(f)$.

The idea of finding a best L_∞-approximation via best L_p-approximations dated back to G. Pólya in 1913 [34] and the process is now called the Pólya algorithm. We say that the Pólya algorithm (for G) converges if $\lim_{p \to \infty} \Pi_G^p(f)$ exists for every f in $C_0(T)$.

The limit of best L_p-approximations as $p \to \infty$ (if it exists) provides a natural selection for Π_G:

$$\operatorname{sba}(f) := \lim_{p \to \infty} \Pi_G^p(f). \tag{1.2}$$

Here we use the notation $\operatorname{sba}(f)$ to indicate that the limit of $\{\Pi_G^p(f)\}$ as $p \to \infty$ (if it exists) will be called the strict best approximation to f. When T consists of n isolated points, $C_0(T)$ can be identified with the n-dimensional Euclidean space $\mathbb{R}^n$ with the supremum norm, denoted by $l_\infty(n)$. In this case, Descloux [9] proved that the Pólya algorithm converges to the strict best approximation introduced by Rice. Therefore, one may consider (1.2) as an alternative definition of the strict best approximation introduced by Rice in $l_\infty(n)$ [36]. Rice [36] proved that $\operatorname{sba}(\cdot)$ is a continuous mapping from $l_\infty(n)$ to G; i.e., (1.2) does define a continuous selection for Π_G in $l_\infty(n)$. However, a counterexample given by Descloux shows that the Pólya algorithm does not converge for some finite-dimensional subspace G of $C[-1, 1]$ [9]; i.e., $\operatorname{sba}(f)$ is not always well-defined by (1.2). One main result of this paper is to prove that (1.2) does define a continuous selection for Π_G if Π_G has a continuous selection. The discovery of the connection between the convergence of the Pólya algorithm and the existence of continuous selections for Π_G is due to Sommer [43, 44], who proved that (1.2) defines a continuous selection for Π_G if Π_G has a continuous selection and G is a spline subspace of $C[a, b]$.

The paper is organized as follows. In Section 2, we show that there exist "best of the best" approximations in $\Pi_G(f)$ if Π_G has a continuous selection. This is the foundation for the stable convergence of the Pólya algorithm. Section 3 is devoted to the proof of the equivalence of the stable convergence of the Pólya algorithm and the existence of continuous selections for Π_G. As a consequence, we obtain that (1.2) does define a continuous selection for Π_G if Π_G has continuous selections. Finally, examples and comments are given in Section 4.

We conclude the section with definitions and notations used in this paper. The Pólya algorithm (for G) is said to converge stably if the limit $\lim_{h \to f, p \to \infty} \Pi_G^p(h)$ exists for any $f \in C_0(T)$. For any subset X of T, $\operatorname{int}(X)$ denotes the interior of X. For a subset F of $C_0(T)$, $E(F) := \{t \in T : |f(t)| = \|f\| \text{ for every } f \in F\}$ denotes the set of common extremum points of functions in F. The set of all zeros of a function f is expressed by $Z(f) := \{t \in T: f(t) = 0\}$. Let Q be a set-valued mapping from $C_0(T)$ to 2^G; i.e., $Q(f)$ is a subset of G for $f \in C_0(T)$. We say that Q has a continuous selection if there exists a continuous mapping S from $C_0(T)$ to G such that $S(f) \in Q(f)$ for all $f \in C_0(T)$. The set-valued mapping Q is lower semicontinuous at f if, for every open set U with $U \cap Q(f) \neq \varnothing$, $\{h \in C_0(T): U \cap Q(h) \neq \varnothing\}$ contains an open neighborhood of f. The mapping Q is said to be lower semicontinuous if Q is lower semicontinuous at f for every f in $C_0(T)$. When $Q(f) \subset \Pi_G(f)$, a function g^* in $Q(f)$ is called a local strict best approximation in $Q(f)$ if, for any $g \in Q(f)$,

$$\operatorname{int}\{t \in T: (f(t) - g^*(t))(g^*(t) - g(t)) \geq 0\} \supset E(f - Q(f)).$$

The meaning of local strict best elements will be clarified in Lemma 2.1. We also need the following definition of a relation $\prec$ on $C_0(T)$. For two functions f, h in $C_0(T)$ with $f \not\equiv h$, we say that f precedes h, written as $f \prec h$, if $E \subset \operatorname{int}\{t \in T: |f(t)| \leq |h(t)|\}$, where

$$E := \{t \in T: |h(t)| = \sup\{|h(x)|: f(x) \neq h(x)\}\}.$$

Intuitively, "$f \prec h$" means "$|f(t)| \leq |h(t)|$" in a neighborhood of the extremum points of h after we ignore the part of T where f and h are identical. Given a subset F of $C_0(T)$, a function f in F will be called a minimal element in F if there is no function h in F such that $h \prec f$. Note that we treat $\prec$ as if it is a partial order on $C_0(T)$ even though it is not.

2. Best of the Best Approximations

In this section we first show that local strict best approximations do deserve to be credited as "best of the best" approximations. Most results concerning local strict best approximations are known and proved in [22]. The main result in this section is Theorem 2.4, about the uniqueness of minimal elements in $f - \Pi_G(f)$. Its proof is based on manipulation of local strict best approximations.

To get a feel for local strict best approximations and minimal elements with respect to the relation $\prec$, we start with the following example.

Let T be the closed interval $[-1, 1]$ and $G := \{|t|(\alpha + \beta t): \alpha, \beta$ are real numbers$\}$ be the subspace of $C[-1, 1]$ with basis functions $\{|t|, t|t|\}$. Then G is a weak Chebyshev subspace of $C[-1, 1]$ (cf. [15]) and any nonzero function in G has at most two zeros. Therefore, Π_G has a continuous selection (cf. [26, 31]).

Set $f(t) := 1 - |t|$ for $|t| \leqslant 1$. Then $d(f, G) = 1$. For $g(t) := |t|\,(\alpha + \beta t) \in G$, $f(t) - g(t) = 1 - |t|\,(1 + \alpha + \beta t)$ and

$$\begin{aligned}\|f - g\| &\geqslant \max\{|f(1) - g(1)|, |f(0) - g(0)|, |f(-1) - g(-1)|\}\\ &= \max\{|\alpha + \beta|, 1, |\alpha - \beta|\}.\end{aligned}$$

Obviously, a necessary condition for $g \in \Pi_G(f)$ is $|\alpha| + |\beta| \leqslant 1$. On the other hand, if $|\alpha| + |\beta| \leqslant 1$, then $1 + \alpha + \beta t \geqslant 0$. Moreover, $|f(t) - g(t)| \leqslant 1$ if $f(t) - g(t) \geqslant 0$ and $|f(t) - g(t)| \leqslant \max\{1, |t|\,(\alpha + \beta t)\} \leqslant 1$ if $f(t) - g(t) \leqslant 0$. Therefore, $\Pi_G(f) = \{|t|\,(\alpha + \beta t) : |\alpha| + |\beta| \leqslant 1\}$. Consider the two best approximations g_1, g_2 of f from G, where $g_1(t) := |t|$ and $g_2(t) := t \cdot |t|$. For any $g \equiv |t|\,(\alpha + \beta t) \in \Pi_G(f)$ with $\alpha \neq 1$, $(f(t) - g(t)) - (f(t) - g_1(t)) = g_1(t) - g(t) = |t|\,((1 - \alpha) - \beta t) \geqslant 0$ for t near 0; i.e.,

$$0 \in \operatorname{int}\{t \in T : |f(t) - g(t)| \geqslant |f(t) - g_1(t)|\} \qquad \text{for} \quad g \in \Pi_G(f). \tag{2.1}$$

Since $E(f - \Pi_G(f)) = \{0\}$, (2.1) implies that g_1 is a local strict best approximation in $\Pi_G(f)$ (cf. Lemma 2.1). Moreover, it follows from (2.1) that there is no $g \in \Pi_G(f)$ such that $f - g \prec f - g_1$; i.e., $f - g_1$ is a minimal element in $\Pi_G(f)$. However, $f - g_2 \not\prec f - g_1$ and $f - g_1 \not\prec f - g_2$. Intuitively, $f - g^*$ is a minimal element in $f - \Pi_G(f)$ if there is no function g in $\Pi_G(f)$ which approximates f "better than g^* in a neighborhood of $E(f - g^*)$."

The following lemma clarifies the meaning of local strict best approximations defined in the Introduction.

LEMMA 2.1. *Suppose that $Q(f)$ is a convex subset of $\Pi_G(f)$. A function g^* is a local strict best approximation in $Q(f)$, if and only if, for any $g \in Q(f)$,*

$$E(f - Q(f)) \subset \operatorname{int}\{t \in T : |f(t) - g^*(t)| \leqslant |f(t) - g(t)|\}.$$

Proof 2.1. First we claim that $E(f - Q(f)) \subset Z(g_1 - g_2)$ for any $g_1, g_2 \in Q(f)$. In fact, since $|f(t) - g(t)| = \|f - g\| = d(f, G)$ for $g := \frac{1}{2}(g_1 + g_2) \in Q(f)$ and $t \in E(f - Q(f))$, we have

$$\begin{aligned}d(f, G) &= |f(t) - \tfrac{1}{2}(g_1(t) + g_2(t))|\\ &\leqslant \tfrac{1}{2}(|f(t) - g_1(t)| + |f(t) - g_2(t)|) = d(f, G).\end{aligned}$$

Thus, the equality must hold, which implies $f(t) - g_1(t) = f(t) - g_2(t)$. That is, $t \in Z(g_1 - g_2)$ for any $t \in E(f - Q(f))$ and $g_1, g_2 \in Q(f)$.

Therefore, $f(t) - g^*(t) = f(t) - g(t)$ for $t \in E(f - Q(f))$ and $(f(t) - g^*(t))(f(t) - g(t)) > 0$ for t in a neighborhood of $E(f - Q(f))$. Also note that $|f(t) - g(t)| = |(f(t) - g^*(t)) + (g^*(t) - g(t))|$. Thus,

$$E(f - Q(f)) \subset \operatorname{int}\{t \in T : |f(t) - g^*(t)| \leqslant |f(t) - g(t)|\}$$

if and only if $(f(t)-g^*(t))(g^*(t)-g(t))\geqslant 0$ in a neighborhood of $E(f-Q(f))$. ∎

Remark. Note that g^* is "best of the best approximations" in $Q(f)$ with respect to the errors near the set of common extremum points. This is the reason why the terminology "local strict best approximation" is introduced. In [22], g^* was called a local maximal element.

By the Lazar–Morris–Wulbert lemma [16] (cf. also [6]), local strict best approximations of $\Pi_G(f)$ always exist if Π_G has a continuous selection. As a matter of fact, we can construct a lower semicontinuous submapping Π_G^* of Π_G via local strict best approximations. Let $\Pi_G^{(0)}(f):=\Pi_G(f)$ and $\Pi_G^{(k+1)}(f)$ be the set of all local strict best approximations in $\Pi_G^{(k)}(f)$ for $k\geqslant 0$. If $\dim G=n$, then $\Pi_G^{(k+1)}(f)=\Pi_G^{(k)}(f)$ for $k\geqslant n$. Moreover, $\Pi_G^{(n)}(f)\neq\varnothing$ for any $f\in C_0(T)$ and $\Pi_G^{(n)}$ is the derived mapping of Π_G introduced by Brown. The following lemma lists some useful properties of $\Pi_G^{(n)}$.

LEMMA 2.2 [22]. *Suppose that Π_G has a continuous selection and $\dim G=n$. Then*

(1) *The mapping $\Pi_G^{(n)}$ is lower semicontinuous.*

(2) *There exists $g^*\in\Pi_G^{(n)}(f)$ such that $E(f-g^*)=E(f-\Pi_G^{(n)}(f))$.*

(3) *For any two functions g_1, $g_2\in\Pi_G^{(n)}(f)$, $E(f-\Pi_G^{(n)}(f))\subset\operatorname{int} Z(g_1-g_2)$.*

(4) *If $g\in\Pi_G(f)\backslash\Pi_G^{(n)}(f)$, then there exists $g^*\in\Pi_G(f)$ such that*

$$E(f-g)\subset\operatorname{int}\{t\in T: |f(t)-g^*(t)|\leqslant|f(t)-g(t)|\},$$
$$E(f-g)\backslash\operatorname{int} Z(g^*-g)\neq\varnothing.$$

Remark. Intuitively, statement (4) means that, if g is not in $\Pi_G^{(n)}(f)$, then the error of approximation $|f(t)-g(t)|$ can be "strictly" reduced near the set of extremum points of $(f-g)$. That is, $f-g$ cannot be a minimal element in $f-\Pi_G(f)$.

The next lemma is a technical one which will be used in the proof of Theorem 2.4 for manipulation of local strict best approximations.

LEMMA 2.3. *Let $g\in\Pi_G(f)$. Then, for any positive number $\gamma>0$, $g\in\Pi_G(f_\gamma)\subset\Pi_G(f)$, where*

$$f_\gamma(t):=\begin{cases} g(t)+\gamma, & \text{if } f(t)-g(t)>\gamma\\ f(t), & \text{if } -\gamma\leqslant f(t)-g(t)\leqslant\gamma\\ g(t)-\gamma, & \text{if } f(t)-g(t)<-\gamma.\end{cases}$$

Proof 2.3. If $\gamma \geqslant \|f-g\|$, then $f_\gamma \equiv f$. Otherwise, let $g^* \in \Pi_G(f_\gamma)$. Then

$$\|f_\gamma - g^*\| \leqslant \|f_\gamma - g\| = \gamma.$$

If $\|f_\gamma - g^*\| < \gamma$, then it follows from the definition of f_γ that $(f(t)-g(t))(g^*(t)-g(t))>0$ whenever $|f(t)-g(t)| \geqslant \gamma$. In particular, $(f(t)-g(t))(g^*(t)-g(t))>0$ whenever $t \in E(f-g)$, which contradicts the Kolmogorov criterion for best approximations (cf. [28]). Hence, $d(f_\gamma, G)=\gamma$ and $g \in \Pi_G(f_\gamma)$. Moreover, $|f(t)-g^*(t)| = |f_\gamma(t)-g^*(t)| \leqslant \gamma \leqslant \|f-g\|$ if $|f(t)-g(t)| \leqslant \gamma$ and

$$\begin{aligned} |f(t)-g^*(t)| &\leqslant |f(t)-f_\gamma(t)| + |f_\gamma(t)-g^*(t)| \\ &\leqslant |f(t)-f_\gamma(t)| + \gamma = |f(t)-g(t)| \leqslant \|f-g\| \end{aligned}$$

if $|f(t)-g(t)| \geqslant \gamma$. Therefore, $\|f-g^*\| \leqslant \|f-g\|$ and $g^* \in \Pi_G(f)$. ∎

Finally, we are ready to prove the main theorem in this section: There exists at most one minimal element in $f-\Pi_G(f)$. The uniqueness of minimal elements in $f-\Pi_G(f)$ guarantees the stable convergence of $\{\Pi_G^p(h)\}$ as $h \to f$, $p \to \infty$ (cf. the proof of Theorem 3.1).

THEOREM 2.4. *Assume that Π_G has a continuous selection. Then there exists at most one element g^* in $\Pi_G(f)$ such that $f-g^*$ is a minimal element in $f-\Pi_G(f)$ (i.e., there is no $g \in \Pi_G(f)$ such that $f-g \prec f-g^*$).*

Proof 2.4. Suppose the contrary, that there are two distinct elements g_1 and g_2 in $\Pi_G(f)$ such that $f-g_i$ are both minimal in $f-\Pi_G(f)$. Let

$$\begin{aligned} \gamma_i &:= \sup\{|f(t)-g_i(t)| : g_1(t) \neq g_2(t)\}, \\ E_i &:= \{t \in T : |f(t)-g_i(t)| = \gamma_i\}. \end{aligned}$$

Then $\gamma_1 = \gamma_2$. In fact, if $\gamma_1 > \gamma_2$, then, for any $t^* \in E_1 \setminus \operatorname{int} Z(g_1-g_2)$, $|f(t^*)-g_2(t^*)| \leqslant \gamma_2 < \gamma_1 = |f(t^*)-g_1(t^*)|$. Therefore, there exists an open neighborhood $U(t^*)$ of t^* such that $|f(t)-g_2(t)| < |f(t)-g_1(t)|$ for $t \in U(t^*)$. Since $|f(t)-g_2(t)| = |f(t)-g_1(t)|$ for $t \in \operatorname{int} Z(g_1-g_2)$, for $U := \operatorname{int} Z(g_1-g_2) \cup (\bigcup_{t^* \in E_1 \setminus \operatorname{int} Z(g_1-g_2)} U(t^*))$, we have

$$|f(t)-g_2(t)| \leqslant |f(t)-g_1(t)| \qquad \text{for} \quad t \in U.$$

Since U is an open neighborhood of E_1, the above inequality means $f-g_2 \prec f-g_1$, which contradicts the fact that $f-g_1$ is a minimal element in $f-\Pi_G(f)$. Therefore, $\gamma_1 \leqslant \gamma_2$. Similarly, $\gamma_2 \leqslant \gamma_1$. Thus, γ_1 must be equal to γ_2.

Let

$$f^*(t) := \begin{cases} g_1(t) + \gamma_1, & \text{if } f(t) - g_1(t) > \gamma_1 \\ f(t), & \text{if } -\gamma_1 \leqslant f(t) - g_1(t) \leqslant \gamma_1 \\ g_1(t) - \gamma_1, & \text{if } f(t) - g_1(t) < -\gamma_1. \end{cases}$$

By $\gamma_1 = \gamma_2$ and the definition of γ_i, we know that $|f(t) - g_i(t)| > \gamma_i$ if and only if $f(t) - g_1(t) = f(t) - g_2(t)$. Hence, f^* can also be represented as

$$f^*(t) := \begin{cases} g_2(t) + \gamma_2, & \text{if } f(t) - g_2(t) > \gamma_2 \\ f(t), & \text{if } -\gamma_2 \leqslant f(t) - g_2(t) \leqslant \gamma_2 \\ g_2(t) - \gamma_2, & \text{if } f(t) - g_2(t) < -\gamma_2. \end{cases}$$

By Lemma 2.3, $g_i \in \Pi_G(f^*) \subset \Pi_G(f)$ for $i = 1, 2$.

Let $n := \dim G$. If $g_i \notin \Pi_G^{(n)}(f^*)$, then, by Lemma 2.2(4), there exists $g^* \in \Pi_G(f^*)$ such that $f^* - g^* \prec f^* - g_i$ and

$$\gamma_i = \|f^* - g_i\| = \sup\{|f^*(t) - g_i(t)| : g_i(t) \neq g^*(t)\}. \tag{2.2}$$

If $g_i \in \Pi_G^{(n)}(f^*)$ and $E(f^* - g_i) \setminus E(f^* - \Pi_G^{(n)}(f^*)) \neq \varnothing$, let $g^* \in \Pi_G^{(n)}(f^*)$ be such that (cf. Lemma 2.2(2))

$$E(f^* - g^*) = E(f^* - \Pi_G^{(n)}(f^*)).$$

Then (2.2) still holds. Moreover, since (cf. Lemma 2.2(3))

$$E(f^* - g^*) \subset E(f^* - \Pi_G^{(n)}(f^*)) \subset \operatorname{int} Z(g^* - g_i) \cap E(f^* - g_i),$$

we have $f^* - g^* \prec f^* - g_i$. By the definition of $\prec$, there exists an open neighborhood U_i of $E(f^* - g_i)$ such that $|f^*(t) - g^*(t)| \leqslant |f^*(t) - g_i(t)|$ for $t \in U$. Note that, for any $t \in U$,

$$\begin{aligned} |f(t) - g^*(t)| &\leqslant |f(t) - f^*(t)| + |f^*(t) - g^*(t)| \\ &\leqslant |f(t) - f^*(t)| + |f^*(t) - g_i(t)| \\ &= |f(t) - g_i(t)|. \end{aligned} \tag{2.3}$$

Set

$$\gamma_i^* := \sup\{|f(t) - g_i(t)| : g^*(t) \neq g_i(t)\},$$
$$E_i^* := \{t \in T : |f(t) - g_i(t)| = \gamma_i^*\}.$$

Let $t^* \in E(f^* - g_i) \setminus \operatorname{int} Z(g^* - g_i)$, which is not empty by (2.2). Then, $\gamma_i = |f^*(t^*) - g_i(t^*)| \leqslant |f(t^*) - g_i(t^*)| \leqslant \gamma_i^*$. Therefore, $E_i^* \subset E(f^* - g_i) \subset U$ and (2.3) implies that $f - g^* \prec f - g_i$, which is impossible.

The contradiction proves that $g_i \in \Pi_G^{(n)}(f^*)$ and $E(f^* - g_i) = E(f^* - \Pi_G^{(n)}(f^*))$. Now, by Lemma 2.2(3), $E(f^* - g_i) \subset \operatorname{int} Z(g_1 - g_2)$. Note that $f^*(t) = f(t)$ if $t \notin E(f^* - g_i)$. Therefore,

$$\begin{aligned}\sup\{|f(t) - g_i(t)| : g_1(t) \neq g_2(t)\} &\leqslant \sup\{|f(t) - g_i(t)| : t \notin \operatorname{int} Z(g_1 - g_2)\}\\ &= \sup\{|f^*(t) - g_i(t)| : t \notin \operatorname{int} Z(g_1 - g_2)\}\\ &< \|f^* - g_i\| = \gamma_i,\end{aligned}$$

which is impossible. This completes the proof of Theorem 2.4. ∎

3. Pólya Algorithm and Continuous Selections

In this section, we first show that, for any accumulation point s^* of $\{\Pi_G(h)\}$ as $h \to f$ and $p \to \infty$, $f - s^*$ is a minimal element of $f - \Pi_G(f)$. Since $f - \Pi_G(f)$ has at most one minimal element (cf. Theorem 2.4), the Pólya algorithm must stably converge. This is the main effort in the proof of Theorem 3.1, which characterizes the existence of continuous metric selections by the stable convergence of the Pólya algorithm. As a consequence, the Pólya algorithm produces a continuous selection for Π_G if such a selection exists.

THEOREM 3.1. *Suppose that there exists a finite Borel measure μ on T such that $\mu(U) > 0$ for any nonempty open subset U of T. Then the metric projection Π_G has a continuous selection if and only if the Pólya algorithm is stably convergent; i.e., for any $f \in C_0(T)$, $\lim_{h \to f, p \to \infty} \Pi_G^p(h)$ exists.*

Proof 3.1. Let s^* be an accumulation point of $\{\Pi_G^p(h)\}$ as $h \to f$, $p \to \infty$; i.e., there exist $p_k \to \infty$ and $f_k \to f$, as $k \to \infty$, such that

$$s^* = \lim_{k \to \infty} g_k, \qquad \text{where} \quad g_k := \Pi_G^{p_k}(f_k) \qquad \text{for} \quad k = 1, 2, \ldots.$$

Now, it suffices to prove that $f - s^*$ is a minimal element in $f - \Pi_G(f)$. Since there exists at most one minimal element in $f - \Pi_G(f)$ (cf. Theorem 2.4), $f - s^*$ must be the minimal element. Therefore, the sequence $\{\Pi_G^{p_k}(f_k)\}$ has exactly one accumulation point; i.e., the limit $\lim_{h \to f, p \to \infty} \Pi_G^p(h)$ exists.

Assume the contrary, that there exists $g^* \in \Pi_G(f)$ such that $f - g^* \prec f - s^*$. Set

$$\begin{aligned}\gamma &:= \sup\{|f(t) - s^*(t)| : g^*(t) \neq s^*(t)\},\\ E &:= \{t \in T : |f(t) - s^*(t)| = \gamma\}.\end{aligned}$$

Then there exists an open neighborhood U_1 of E such that

$$|f(t)-g^*(t)| \leqslant |f(t)-s^*(t)| \qquad \text{for} \quad t \in U_1. \tag{3.1}$$

It follows from (3.1) that $f-s^*$ and $(f-s^*)-(f-g^*)$ $(\equiv g^*-s^*)$ have the same sign on U_1; i.e.,

$$(f(t)-s^*(t))(g^*(t)-s^*(t)) \geqslant 0 \qquad \text{for} \quad t \in U_1. \tag{3.2}$$

Also, there exists an open neighborhood U_2 of E such that

$$|f(t)-s^*(t)| \geqslant \frac{\gamma}{2} \qquad \text{for} \quad t \in U_2. \tag{3.3}$$

Let $U := U_1 \cap U_2$ and

$$6\delta := \gamma - \sup\{|f(t)-s^*(t)| : t \notin U,\ g^*(t) \neq s^*(t)\} > 0.$$

Set $\varepsilon := \min\{\delta, \gamma/4\}$. Then there is $k^* > 1$ such that

$$\|(f_k-g_k)-(f-s^*)\| < \varepsilon \qquad \text{for} \quad k \geqslant k^*. \tag{3.4}$$

By (3.3) and (3.4), we have

$$|f_k(t)-g_k(t)| > \frac{\gamma}{4} \qquad \text{for} \quad t \in U, \qquad k \geqslant k^*, \tag{3.5}$$

$$(f(t)-s^*(t))(f_k(t)-g_k(t)) \geqslant 0 \qquad \text{for} \quad t \in U, \qquad k \geqslant k^*. \tag{3.6}$$

We can derive from (3.2), (3.5), and (3.6) that for $t \in U$ and $k \geqslant k^*$,

$$\begin{aligned} &|(f_k(t)-g_k(t))-\lambda(g^*(t)-s^*(t))| \\ &\qquad = |f_k(t)-g_k(t)| - \lambda\, |g^*(t)-s^*(t)|, \end{aligned} \tag{3.7}$$

where $\lambda := \varepsilon/\|g^*-s^*\|$. Let $V := \{t \in U : |f(t)-s^*(t)| > \gamma-\varepsilon\}$. Then V is also an open neighborhood of E. Since $E \setminus \operatorname{int} Z(g^*-s^*) \neq \varnothing$, there is $t^* \in V \setminus Z(g^*-s^*)$. Let

$$2\eta := |g^*(t^*)-s^*(t^*)| > 0.$$

Then there exists an open neighborhood W of t^* in V such that

$$|g^*(t)-s^*(t)| \geqslant \eta \qquad \text{for} \quad t \in W.$$

From now on, assume $k \geqslant k^*$. Then, for $g_{\lambda,k} := g_k + \lambda(g^* - s^*)$, we have

$$\int_{(Z(g^*-s^*)\cup U)\setminus W} |f_k - g_k|^{p_k}\, d\mu \geqslant \int_{(Z(g^*-s^*)\cup U)\setminus W} |f_k - g_{\lambda,k}|^{p_k}\, d\mu, \tag{3.8}$$

since $|f_k - g_k| \geqslant |f_k - g_{\lambda,k}|$ for $t \in Z(g^* - s^*) \cup U$ (cf. (3.7)). If $t \in T\setminus (Z(g^* - s^*) \cup U)$, then

$$\begin{aligned}|f_k(t) - g_{\lambda,k}(t)| &\leqslant |f(t) - s^*(t)| + \varepsilon + \lambda\, \|g^* - s^*\| \\ &\leqslant \gamma - 6\delta + 2\varepsilon \leqslant \gamma - 4\delta,\end{aligned}$$

which implies that

$$\begin{aligned}&\int_{T\setminus(Z(g^*-s^*)\cup U)} |f_k - g_{\lambda,k}|^{p_k}\, d\mu \\ &\qquad \leqslant (\gamma - 4\delta)^{p_k} \cdot \mu\{T\setminus(Z(g^* - s^*) \cup U)\}.\end{aligned} \tag{3.9}$$

Note that, if $h_1(t) \geqslant \gamma - 3\varepsilon$ and $h_2(t) \geqslant \lambda \cdot \eta$ for $t \in W$, then

$$\begin{aligned}\int_W (h_1 + h_2)^p\, d\mu - \int_W h_1^p\, d\mu &= \int_W ((h_1 + h_2)^p - h_1^p)\, d\mu \\ &= \int_W p(h_1 + \theta h_2)^{p-1}\, h_2\, d\mu \\ &\geqslant p \cdot \lambda \cdot \eta \cdot \mu(W) \cdot (\gamma - 3\varepsilon)^{p-1},\end{aligned} \tag{3.10}$$

where $\theta := \theta(t)$ is a number between 0 and 1 whose existence is guaranteed by the mean value theorem. But, for $t \in W$, we have

$$|f_k(t) - g_{\lambda,k}(t)| \geqslant |f(t) - s^*(t)| - \varepsilon - \lambda\, |g^*(t) - s^*(t)| \geqslant \gamma - 3\varepsilon$$

and $\lambda\, |g^*(t) - s^*(t)| \geqslant \lambda \cdot \eta$. Therefore, by (3.7) and (3.10),

$$\begin{aligned}\int_W |f_k - g_k|^{p_k}\, d\mu &= \int_W (|f_k - g_{\lambda,k}| + \lambda\, |g^* - s^*|)^{p_k}\, d\mu \\ &\geqslant \int_W |f_k - g_{\lambda,k}|^{p_k}\, d\mu + p_k \cdot \lambda \cdot \eta \cdot \mu(W) \cdot (\gamma - 3\varepsilon)^{p_k - 1}.\end{aligned}$$

Since

$$\lim_{k\to\infty} \frac{(\gamma - 4\delta)^{p_k} \cdot \mu\{T\setminus(Z(g^* - s^*) \cup U)\}}{p_k \cdot \lambda \cdot \eta \cdot \mu(W) \cdot (\gamma - 3\varepsilon)^{p_k - 1}} = 0,$$

for k large enough,

$$\int_W |f_k - g_k|^{p_k}\, d\mu > \int_W |f_k - g_{\lambda,k}|^{p_k}\, d\mu + (\gamma - 4\delta)^{p_k} \cdot \mu\{T \backslash (Z(g^* - s^*) \cup U)\}. \tag{3.11}$$

It follows from (3.8), (3.9), and (3.11) that, for k large enough,

$$\int_T |f_k - g_k|^{p_k}\, d\mu > \int_T |f_k - g_{\lambda,k}|^{p_k}\, d\mu;$$

i.e., $\|f_k - g_k\|_{p_k} > \|f_k - g_{\lambda,k}\|_{p_k}$ for k large enough. This contradicts the definition of $g_k := \Pi_G^{p_k}(f_k)$.

See the proof of Corollary 3.2 for proof of the fact that the stable convergence of the Pólya algorithm implies the existence of continuous selections. ∎

COROLLARY 3.2. *Suppose that there exists a finite Borel measure μ on T such that $\mu(U) > 0$ for any nonempty open subset U of T. If the metric projection Π_G has a continuous selection, then the strict best approximation* sba$(\cdot)$ *in* (1.2) *is well-defined and is a continuous selection for Π_G. Moreover,* sba(f) *is independent of the choice of μ.*

Proof 3.2. It follows from Theorem 3.1 that, for any $\varepsilon > 0$, there exist $\delta > 0$ and $p^* > 1$ such that

$$\|\Pi_G^p(h) - \text{sba}(f)\| \leqslant \varepsilon \qquad \text{for} \quad p \geqslant p^*, \qquad \|h - f\| \leqslant \delta. \tag{3.12}$$

Let $p \to \infty$ in (3.12). By Theorem 3.1, $\lim_{p \to \infty} \Pi_G^p(h) = \text{sba}(h)$. Therefore, $\|\text{sba}(h) - \text{sba}(f)\| \leqslant \varepsilon$ for $\|h - f\| \leqslant \delta$. Thus, sba$(\cdot)$ is a continuous selection for Π_G. From the proof of Theorem 3.1 we know that sba(f) is the unique element in $\Pi_G(f)$ such that $f - \text{sba}(f)$ is a minimal element in $f - \Pi_G(f)$. Since the minimal element of $f - \Pi_G(f)$ is independent of μ, so is sba(f). ∎

4. EXAMPLES AND COMMENTS

Let I be an index set which is at most countable with the discrete topology. Then $C_0(I)$ is either $l_\infty(n)$ if I is finite or c_0 (the Banach space of all sequences converging to 0). Consider a finite-dimensional subspace G of $C_0(I)$. Since the boundary for any subset of I is empty, Π_G has a continuous selection (cf. [26]). For any $w(i) > 0$ with $\sum_{i \in I} w(i) < \infty$, define

$$\mu(J) := \sum_{i \in J} w(i) \qquad \text{for} \quad J \subset I,$$

which is a finite Borel measure on I satisfying the condition: $\mu(J) > 0$ for any nonempty subset J of I. The corresponding L_p-norm is

$$\|f\|_p = \left(\sum_{i \in I} w(i) \, |f(i)|^p \right)^{1/p}.$$

By Theorem 3.1, the corresponding best L_p-approximations converge to the strict best approximation as $p \to \infty$. Note that the individual values of $w(i)$ are irrelevant to the limit.

Let $[a, b]$ be a closed interval of the real line and $w(t)$ be a positive Lebesgue measurable function such that $\int_a^b w(t) \, dt < \infty$. Then the weighted Lebesgue measure μ can be defined as

$$\mu(U) := \int_U w(t) \, dt$$

and the corresponding L_p-norm is

$$\|f\|_p = \left(\int_a^b w(t) \, |f(t)|^p \, dt \right)^{1/p}.$$

If G is a finite-dimensional subspace of $C[a, b]$ and Π_G has a continuous selection, then the best L_p-approximations converge to the strict best approximation as $p \to \infty$ and the limit is irrelevant to the choice of $w(t)$.

It was a pleasant surprise for the author to find out that an 80-year-old idea of Pólya provides a simple solution to the problem of constructing a continuous selection for the metric projection in $C_0(T)$.

There are quite a few attempts to generalize Rice's definition of strict best approximations to $C_0(T)$, where T is not a finite set. In her thesis [45], Stover defines strict best approximations in $C[a, b]$ via discretization of $[a, b]$: Let $\{X_n\}$ be a sequence of finite subsets of $[a, b]$ such that $X_n \subset X_{n+1}$ for $n \geqslant 1$ and $\lim_{n \to \infty} \sup_{a \leqslant c \leqslant b} \inf_{t \in X_n} |t - c| = 0$. If there is a unique limit of the sequence of strict best approximations to $f|_{X_n}$ in $C(X_n)$, independent of the choice of the sequence $\{X_n\}$, then this limit is called the strict best approximation to f. She proved that Π_G has a continuous selection if and only if every $f \in C[a, b]$ has a strict best approximation. Also, one could extend Rice's definition of strict best approximations via extremal signature and local strict best approximations as done by Strauss [46, 47], or via alternation signatures and local strict best approximations as in [26]. The similarity between Rice's and Stover's definitions makes Stover's definition an attractive one. But (1.2) seems to be the simplest even though it does not explicitly provide "best of the best" feature in its form. We strongly believe that all the definitions concerning strict best approximations are equivalent when Π_G has a continuous selection, due to the fact that they all involve the feature of local strict best approximations one way or another.

Also, it would be interesting to know whether the strict best approximation is Lipschitz continuous if Π_G has a Lipschitz continuous selection. Note that Π_G has a Lipschitz continuous selection if and only if Π_G is Lipschitz continuous (cf. [25]). When $C_0(T) \equiv l_\infty(n)$, Finzel has proved that sba($\cdot$) is a Lipschitz continuous selection for Π_G [13].

Finally, we would like to point out that some of the results in this paper were announced in [23], which unfortunately is unreadable due to a serious typographic error: int($\cdot$) was printed as "$\int(\cdot)$" ($\equiv$\int($\cdot$) in TEX).

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