2007

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Original Publication Citation

Grigoryan, H. R., & Radyushkin, A. V. (2007). Form factors and wave functions of vector mesons in holographic QCD. *Physics Letters B, 650*(5–6), 421-427. doi:https://doi.org/10.1016/j.physletb.2007.05.044

Available online at www.sciencedirect.com

ScienceDirect

Physics Letters B 650 (2007) 421–427

PHYSICS LETTERS B

www.elsevier.com/locate/physletb

Form factors and wave functions of vector mesons in holographic QCD

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Received 6 March 2007; received in revised form 9 May 2007; accepted 23 May 2007

Available online 26 May 2007

Editor: B. Grinstein

Abstract

Within the framework of a holographic dual model of QCD, we develop a formalism for calculating form factors of vector mesons. We show that the holographic bound states can be described not only in terms of eigenfunctions of the equation of motion, but also in terms of conjugate wave functions that are close analogues of quantum-mechanical bound state wave functions. We derive a generalized VMD representation for form factors, and find a very specific VMD pattern, in which form factors are essentially given by contributions due to the first two bound states in the Q^2-channel. We calculate electric radius of the ρ-meson, finding the value $\langle r_\rho^2\rangle_C = 0.53\ \text{fm}^2$.

1. Introduction

The AdS/CFT correspondence [1] conjectures equivalence of gravity theory on the anti-de Sitter space AdS_5 and a strongly coupled four-dimensional (4D) conformal field theory (CFT). The correspondence states that for every CFT operator $\mathcal{O}(x)$ there is a corresponding bulk field $\Phi(x,z)$ uniquely determined by the boundary condition (b.c.) $\Phi(x,z=0)$ at the ultraviolet (UV) 4D boundary of AdS space (x denotes the 4D coordinates and z stands for the fifth extra dimension). The addition of an infrared (IR) brane at $z=z_0$ breaks conformal invariance in the IR region, and allows one to have both particles and S-matrix elements. Due to the holographic equivalence between the broken CFT and the gravitational picture, the two theories have identical spectra and identical S-matrix elements [2]. In particular, the Kaluza–Klein modes on the gravity side can be interpreted as bound states in the 4D theory. The next conjecture is that the AdS/CFT correspondence can be extended to assert that any 5D gravity theory on AdS_5 is holographically dual to some strongly coupled, large-N_c 4D CFT (see, e.g., [2]). The goal of holographic models of quantum chromodynamics (QCD) is to find such a gravity theory for which the dual theory is as close to QCD as possible.

Holographic duals of QCD based on the AdS/CFT correspondence have been applied recently to hadronic physics (see, e.g., [3–14]). These models are able to incorporate essential properties of QCD such as confinement and chiral symmetry breaking, and have demonstrated in many cases success in determination of static hadronic properties, such as resonance masses, decay constants, chiral coefficients, etc. Dynamic properties (form factors) have been studied originally within the holographic approach of Ref. [3], and the connection between AdS/QCD approach of Refs. [3,4] and the usual light-cone formalism for hadronic form factors was proposed in [11] and discussed in [15]. The calculation of form factors of scalar and vector hadrons within the approach of Ref. [3] was performed in Refs. [16,17], and applied to study the universality of the ρ-meson couplings to other hadrons. The expressions for hadronic form factors given in Refs. [3,11,16] have an expected form of z-integral containing the product of two hadronic wave functions and a function describing the probing current. However, the hadronic functions used in Ref. [11] strongly differ from those in Refs. [3,16]. The latter give meson coupling constants through their derivatives at

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0370-2693
doi:10.1016/j.physletb.2007.05.044

$z = 0$ and satisfy Neumann b.c. at the IR boundary $z = z_0$, while the functions used in Ref. [11] satisfy Dirichlet b.c. at $z = z_0$, and are proportional (after extraction of the overall z^2 factor) to the meson coupling constants f_n at the origin. In these respects they are analogous to the bound state wave functions in quantum mechanics, which makes possible their interpretation in terms of light-cone variables proposed in Ref. [11].

The aim of this Letter is to study form factors and wave functions of vector mesons within the framework of the holographic QCD model described in Refs. [6–8] (which will be referred to as H-model). To this end, we consider a 5D dual of the simplest $N_f = 2$ version of QCD to be a Yang–Mills theory with the $SU(2)$ gauge group in the background of sliced AdS space, i.e., the 4D global $SU(2)$ isotopic symmetry of $N_f = 2$ QCD is promoted to a 5D gauge symmetry in the bulk. Note, that the AdS/QCD correspondence does not refer explicitly to quark and gluon degrees of freedom. Rather, one deals with the bound states of QCD which appear as infinitely narrow resonances. The counterparts in the correspondence relation are the vector current $J^a_\mu(x)$ with conformal dimension $\Delta = 3$ (in QCD, it may be visualized as $\bar q(x)t^a\gamma_\mu q(x)$), and the 5D gauge field $A^a_\mu(x,z)$.

We start with recalling the basic elements of the analysis of two-point functions $\langle JJ\rangle$ given in Refs. [6,7], and introduce a convenient representation for the A-field bulk-to-boundary propagator $\mathcal{V}(p,z)$ based on the Kneser–Sommerfeld formula [18] that gives $\mathcal{V}(p,z)$ as an expansion over bound state poles with the z-dependence of each pole contribution given by "ψ wave functions", that are eigenfunctions of the 5D equation of motion with Neumann b.c. at the IR boundary. Then we study the three-point function $\langle JJJ\rangle$ and obtain expression for transition form factors that involves ψ wave functions and the nonnormalizable mode factor $\mathcal{J}(Q,z)$. We write the latter as a sum over all bound states in the channel of electromagnetic current, which gives an analogue of generalized vector meson dominance (VMD) representation for hadronic form factors. As the next step, we introduce "ϕ wave functions" that strongly resemble wave functions of bound states in quantum mechanics (they satisfy Dirichlet b.c. at $z = z_0$, and their values at $z = 0$ give bound state couplings $g_5 f_n/M_n$, i.e., they have the properties necessary for the light-cone interpretation of AdS/QCD results proposed in Ref. [11]). We rewrite form factors in terms of ϕ functions, formulate predictions for ρ-meson form factors, and analyze these predictions in the regions of small and large Q^2.

The ρ-meson electric radius is calculated, and it is also shown that H-model predicts a peculiar VMD pattern when two (rather than just one) lowest bound states in the Q^2-channel play the dominant role while contributions from higher states can be neglected. This double-resonance dominance is established both for the ρ-meson form factor $F(Q^2)$ given by the overlap of the ψ wave functions (here we confirm the results obtained in Ref. [16] for the ρ-meson form factor considered there) and for the form factor $\mathcal{F}(Q^2)$ given by the overlap of the ϕ wave functions. Finally, we summarize our results.

2. Two-point function

Our goal is to analyze form factors of vector mesons within the framework of the holographic model of QCD based on AdS/QCD correspondence. As a 4D operator on the QCD side, we take the vector current $J^a_\mu(x) = \bar q(x)\gamma_\mu t^a q(x)$, to which corresponds a bulk gauge field $A^a_M(x,z)$ whose boundary value is the source for $J^a_\mu(x)$. We follow the conventions of the H-model [7], with the bulk fields in the background of the sliced AdS_5 metric

$$ds^2 = \frac{1}{z^2}\left(\eta_{\mu\nu}\,dx^\mu\,dx^\nu - dz^2\right), \quad 0 \leqslant z \leqslant z_0, \tag{1}$$

where $\eta_{\mu\nu} = \mathrm{Diag}(1,-1,-1,-1)$, and $z_0 \sim 1/\Lambda_{\mathrm{QCD}}$ is the imposed IR scale. The 5D gauge action in AdS_5 space, corresponding to $A^a_M(x,z)$, is

$$S_{\mathrm{AdS}} = -\frac{1}{4g_5^2}\int d^4x\,dz\,\sqrt{g}\,\mathrm{Tr}\left(F_{MN}F^{MN}\right), \tag{2}$$

where $F_{MN} = \partial_M A_N - \partial_N A_M - i[A_M, A_N]$, $A_M = t^a A^a_M$, ($t^a \in SU(2), a = 1,2,3$) and $M,N = 0,1,2,3,z$. Since the vector field $A^a_M(x,z)$ is taken to be non-Abelian, the 3-point function of these fields in the lowest approximation can be extracted directly from the Lagrangian.

Before calculating the 3-point function, we recall some properties of the 2-point function discussed in [7]. Consider the sliced AdS space with an IR boundary at $z = z_0$ and UV cutoff at $z = \epsilon$ (taken to be zero at the end of the calculations). In order to calculate the current–current correlator (or 2-point function) using the AdS/CFT correspondence, one should solve equations of motion, requiring the solution at the UV boundary ($z = 0$) to coincide with the 4D source of the vector current, calculate 5D action on this solution and then vary the action (twice) with respect to the boundary source. The task is simplified when the $A_z = 0$ gauge is imposed, and the gauge field is Fourier-transformed in 4D, $A_\mu(x,z) \Rightarrow \tilde A_\mu(p,z)$. Then

$$\tilde A_\mu(p,z) = \tilde A_\mu(p)\,\frac{V(p,z)}{V(p,\epsilon)}, \tag{3}$$

where $\tilde A_\mu(p)$ is the Fourier-transformed current source, and the 5D gauge field $V(p,z)$ is the so-called bulk-to-boundary propagator obeying

$$z\partial_z\left(\frac{1}{z}\,\partial_z V(p,z)\right) + p^2 V(p,z) = 0. \tag{4}$$

The UV b.c. $\tilde A_\mu(p,\epsilon) = \tilde A_\mu(p)$ is satisfied by construction. At the IR boundary (when $z = z_0$), we follow Ref. [7] (see also Ref. [16]) and choose the Neumann b.c. $\partial_z V(p,z_0) = 0$ which corresponds to the gauge invariant condition $F_{\mu z}(x,z_0) = 0$. Evaluating the bilinear term of the action on this solution leaves only the UV surface term

$$S^{(2)}_{\mathrm{AdS}} = -\frac{1}{2g_5^2}\int \frac{d^4p}{(2\pi)^4}\,\tilde A^\mu(p)\tilde A_\mu(p)\left[\frac{1}{z}\,\frac{\partial_z V(p,z)}{V(p,\epsilon)}\right]_{z=\epsilon}. \tag{5}$$

The 2-point function of vector currents is defined by

$$\int d^4x\,e^{ip\cdot x}\left\langle J^a_\mu(x)J^b_\nu(0)\right\rangle = \delta^{ab}\,\Pi_{\mu\nu}(p)\,\Sigma\left(p^2\right), \tag{6}$$

where $\Pi_{\mu\nu}(p) \equiv (\eta_{\mu\nu} - p_\mu p_\nu/p^2)$ is the transverse projector. Varying the action (5) with respect to the boundary source produces

$$\Sigma(p^2) = -\frac{1}{g_5^2}\left(\frac{1}{z}\frac{\partial_z V(p,z)}{V(p,\epsilon)}\right)\bigg|_{z=\epsilon\to 0}. \tag{7}$$

(To get the tensor structure of (6) by a "naïve" variation, one should change $A^\mu A_\mu \to A^\mu \Pi_{\mu\nu}(p) A^\nu$ in Eq. (5).)

It is well known (see, e.g., [3,16]) that two linearly independent solutions of Eq. (4) are given by the Bessel functions $zJ_1(Pz)$ and $zY_1(Pz)$, where $P \equiv \sqrt{p^2}$. Taking Neumann b.c. for $V(p,z)$, one obtains

$$V(p,z) = Pz\left[Y_0(Pz_0)J_1(Pz) - J_0(Pz_0)Y_1(Pz)\right], \tag{8}$$

and, hence,

$$\Sigma(p^2) = \frac{\pi p^2}{2g_5^2}\left[Y_0(Pz) - J_0(Pz)\frac{Y_0(Pz_0)}{J_0(Pz_0)}\right]_{z=\epsilon\to 0}. \tag{9}$$

This expression is singular as $\epsilon \to 0$:

$$\Sigma(p^2) = \frac{1}{2g_5^2}p^2\ln(p^2\epsilon^2) + \cdots. \tag{10}$$

By matching to QCD result for $J_\mu^a = \bar{q}\gamma_\mu t^a q$ currents one finds $g_5^2 = 12\pi^2/N_c$ (cf. [6]).

The two-point function $\Sigma(p^2)$ has poles when the denominator function $J_0(Pz_0)$ has zeros, i.e., when Pz_0 coincides with one of the roots $\gamma_{0,n}$ of the Bessel function $J_0(x)$. These poles can be explicitly displayed by incorporating the Kneser–Sommerfeld expansion [18]

$$\frac{Y_0(Pz_0)J_0(Pz) - J_0(Pz_0)Y_0(Pz)}{J_0(Pz_0)} = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{J_0(\gamma_{0,n}z/z_0)}{[J_1(\gamma_{0,n})]^2(P^2z_0^2 - \gamma_{0,n}^2)}, \tag{11}$$

valid for $z \leqslant z_0$ (the case we are interested in). Taking formally $z=0$ gives a logarithmically divergent series reflecting the $\ln\epsilon$ singularity of the $z=\epsilon$ expression. Thus, some kind of regularization for this divergency of the sum is implied. Under this assumption,

$$\Sigma(p^2) = \frac{2p^2}{g_5^2 z_0^2}\sum_{n=1}^{\infty}\frac{[J_1(\gamma_{0,n})]^{-2}}{p^2 - M_n^2}, \tag{12}$$

where $M_n = \gamma_{0,n}/z_0$. Hence, the 2-point correlator of the H-model has poles when P coincides with one of M_n's. Given that the residues of all these poles are positive, the poles may be interpreted as bound states with M_n's being their masses. The coupling f_n^2 with which a particular resonance contributes to the total sum is determined by

$$f_n^2 = \lim_{p^2\to M_n^2}\left\{(p^2 - M_n^2)\Sigma(p^2)\right\}. \tag{13}$$

This prescription agrees with the usual definition $\langle 0|J_\mu^a|\rho_n^b\rangle = \delta^{ab} f_n \epsilon_\mu$ for the vector meson decay constants. In our case,

$$f_n^2 = \frac{2M_n^2}{g_5^2 z_0^2 J_1^2(\gamma_{0,n})}. \tag{14}$$

3. Three-point function

Consider now the trilinear term of the action calculated on the $V(q,z)$ solution:

$$S_{\mathrm{AdS}}^{(3)} = -\frac{\epsilon_{abc}}{2g_5^2}\int d^4x \int_\epsilon^{z_0}\frac{dz}{z}\left(\partial_\mu A_\nu^a\right)A^{\mu,b}A^{\nu,c}. \tag{15}$$

A naïve variation gives the result for the 3-point correlator $\langle J_a^\alpha(p_1)J_b^\beta(-p_2)J_c^\mu(q)\rangle$ that contains the isotopic Levi-Civita tensor ϵ_{abc}, the dynamical factor

$$W(p_1,p_2,q) \equiv \int_\epsilon^{z_0}\frac{dz}{z}\frac{V(p_1,z)}{V(p_1,\epsilon)}\frac{V(p_2,z)}{V(p_2,\epsilon)}\frac{V(q,z)}{V(q,\epsilon)}, \tag{16}$$

and the tensor structure

$$T^{\alpha\beta\mu} = \eta^{\alpha\mu}(q-p_1)^\beta - \eta^{\beta\mu}(p_2+q)^\alpha + \eta^{\alpha\beta}(p_1+p_2)^\mu$$

familiar from the QCD 3-gluon vertex amplitude. Restoring the transverse projectors $\Pi^{\alpha\alpha'}(p_1)$, etc. one can convert it into

$$\mathcal{T}^{\alpha\beta\mu} = \eta_{\alpha\beta}(p_1+p_2)_\mu + 2(\eta_{\alpha\mu}q_\beta - \eta_{\beta\mu}q_\alpha). \tag{17}$$

For the factors corresponding to the hadronized channels, the Kneser–Sommerfeld expansion (11) gives

$$\frac{V(p,z)}{V(p,0)} \equiv \mathcal{V}(p,z) = -g_5\sum_{n=1}^{\infty}\frac{f_n\psi_n(z)}{p^2 - M_n^2}, \tag{18}$$

where p equals p_1 or p_2, and

$$\psi_n(z) = \frac{\sqrt{2}}{z_0 J_1(\gamma_{0,n})}zJ_1(M_n z) \tag{19}$$

is the "ψ wave function" obeying the same equation of motion (4) as $V(p,z)$ (with $p^2 = M_n^2$), satisfying the b.c.

$$\psi_n(0) = 0, \qquad \partial_z\psi_n(z_0) = 0, \tag{20}$$

and normalized according to

$$\int_0^{z_0}\frac{dz}{z}\left|\psi_n(z)\right|^2 = 1. \tag{21}$$

One remark is in order here. Since the "ψ wave functions" vanish at the origin and satisfy Neumann b.c. at the IR boundary, it is impossible to establish a direct analogy between $\psi_n(z)$'s and the bound state wave functions in quantum mechanics. For the latter, one would expect that they vanish at the confinement radius, while their values at the origin are proportional to the coupling constants f_n.

Taking a spacelike momentum transfer, $q^2 = -Q^2$ for the V/V factor of the EM current channel gives

$$\mathcal{J}(Q,z) = Qz\left[K_1(Qz) + I_1(Qz)\frac{K_0(Qz_0)}{I_0(Qz_0)}\right], \tag{22}$$

the nonnormalizable mode with Neumann b.c. (see also Ref. [16]). This factor can also be written as a sum of monopole contributions from the infinite tower of vector mesons:

$$\mathcal{J}(Q,z)=g_5\sum_{m=1}^{\infty}\frac{f_m\psi_m(z)}{Q^2+M_m^2}. \tag{23}$$

This decomposition, discussed in Ref. [16], directly follows from Eq. (18). Incorporating the representation for the bulk-to-boundary propagators given above we obtain

$$T\left(p_1^2,p_2^2,Q^2\right)=\sum_{n,k=1}^{\infty}\frac{f_nf_kF_{nk}(Q^2)}{(p_1^2-M_n^2)(p_2^2-M_k^2)}, \tag{24}$$

where $T(p_1^2,p_2^2,Q^2)=W(p_1,p_2,q)/g_5^2$, and

$$F_{nk}\left(Q^2\right)=\int_0^{z_0}\frac{dz}{z}\mathcal{J}(Q,z)\psi_n(z)\psi_k(z) \tag{25}$$

correspond to form factors for $n\to k$ transitions. This expression was also written in Ref. [16] for form factors considered there.

4. Wave functions

The formulas obtained above using explicit properties of the Bessel functions in the form of Kneser–Sommerfeld expansions, can also be derived from the general formalism of Green's functions. In particular, the Green's function for Eq. (4) can be written as

$$G(p;z,z')=\sum_{n=1}^{\infty}\frac{\psi_n(z)\psi_n(z')}{p^2-M_n^2}, \tag{26}$$

where $\psi_n(z)$'s are the normalized wave functions (19) that satisfy the Sturm–Liouville equation (4) with $p^2=M_n^2$ and Neumann b.c. (20). As discussed in Ref. [6], the bulk-to-boundary propagator is related to the Green's function by

$$\mathcal{V}(p,z')=-\left[\frac{1}{z}\partial_zG(p;z,z')\right]_{z=\epsilon\to 0}, \tag{27}$$

and the two-point function $\Sigma(P^2)$ is obtained from the Green's function using Eqs. (7), (27)

$$\Sigma\left(P^2\right)=\frac{1}{g_5^2}\left[\frac{1}{z'}\partial_{z'}\left[\frac{1}{z}\partial_zG(p;z,z')\right]\right]_{z,z'=\epsilon\to 0}. \tag{28}$$

Accordingly, the coupling constants are related to the ψ wave functions by

$$f_n=\frac{1}{g_5}\left[\frac{1}{z}\partial_z\psi_n(z)\right]_{z=0} \tag{29}$$

(cf. [6,16]). In view of this relation, it makes sense to introduce "ϕ wave functions"

$$\phi_n(z)\equiv\frac{1}{M_nz}\partial_z\psi_n(z)=\frac{\sqrt{2}}{z_0J_1(\gamma_{0,n})}J_0(M_nz), \tag{30}$$

which give the couplings g_5f_n/M_n as their values at the origin. In this respect, the "ϕ wave functions" are analogous to the bound state wave functions in quantum mechanics. Moreover, these functions satisfy Dirichlet b.c. $\phi_n(z_0)=0$ and are normalized by

$$\int_0^{z_0}dz\,z\left|\phi_n(z)\right|^2=1, \tag{31}$$

which strengthens this analogy. However, the elastic form factors $F_{nn}(Q^2)$ are given by the integrals

$$F_{nn}\left(Q^2\right)=\int_0^{z_0}\frac{dz}{z}\mathcal{J}(Q,z)\left|\psi_n(z)\right|^2 \tag{32}$$

involving ψ rather than ϕ wave functions. In fact, due to the basic Eq. (4), $\psi_n(z)$ wave functions can be expressed in terms of $\phi_n(z)$ as

$$\psi_n(z)=-\frac{z}{M_n}\,\partial_z\phi_n(z), \tag{33}$$

and we can rewrite the form factor integral as

$$F_{nn}\left(Q^2\right)=\int_0^{z_0}dz\,z\mathcal{J}(Q,z)\left|\phi_n(z)\right|^2+\frac{1}{M_n}\int_0^{z_0}dz\,\phi_n(z)\psi_n(z)\partial_z\mathcal{J}(Q,z). \tag{34}$$

Note, that the nonnormalizable mode

$$\frac{1}{z}\partial_z\mathcal{J}(Q,z)=-Q^2\left[K_0(Qz)-I_0(Qz)\frac{K_0(Qz_0)}{I_0(Qz_0)}\right] \tag{35}$$

corresponds to equation whose solutions are the functions $J_0(M_nz)$ satisfying Dirichlet b.c. at $z=z_0$. Expressing $\phi_n(z)$ in terms of $\partial_z\psi_n(z)$, integrating $|\psi_n(z)|^2$ by parts and using Eq. (4) for $\mathcal{J}(Q,z)$ gives

$$F_{nn}\left(Q^2\right)=\int_0^{z_0}dz\,z\mathcal{J}(Q,z)\left|\phi_n(z)\right|^2-\frac{Q^2}{2M_n^2}\int_0^{z_0}\frac{dz}{z}\mathcal{J}(Q,z)\left|\psi_n(z)\right|^2. \tag{36}$$

The second term contains the original integral for $F_{nn}(Q^2)$, and we obtain

$$F_{nn}\left(Q^2\right)=\frac{1}{1+Q^2/2M_n^2}\int_0^{z_0}dz\,z\mathcal{J}(Q,z)\left|\phi_n(z)\right|^2. \tag{37}$$

Notice, that the normalizable modes $\phi_n(z)$ in this expression correspond to Dirichlet b.c., while the nonnormalizable mode $\mathcal{J}(Q,z)$ was obtained using the Neumann ones.

Thus, we managed to get the expression for $F_{nn}(Q^2)$ form factors that contains ϕ instead of ψ wave functions. However, it contains an extra factor $1/(1+Q^2/2M_n^2)$, which brings us to the issue of different form factors of the ρ-meson and kinematic factors associated with them.

5. Form factors

Our result (25) contains only one function for each $n \to k$ transition, in particular $F_{nn}(Q^2)$ in the diagonal case. However, the general expression for the EM vertex of a spin-1 particle of mass M can be written (assuming P- and T-invariance) in terms of three form factors (see, e.g., [19], our G_2 is theirs $G_2 - G_1$):

$$\begin{aligned}&\langle\rho^+(p_2,\epsilon')\big|J^\mu_{\mathrm{EM}}(0)\big|\rho^+(p_1,\epsilon)\rangle\\&\quad=-\epsilon'_\beta\epsilon_\alpha\bigg[\eta^{\alpha\beta}\big(p_1^\mu+p_2^\mu\big)G_1\big(Q^2\big)\\&\qquad+\big(\eta^{\mu\alpha}q^\beta-\eta^{\mu\beta}q^\alpha\big)\big(G_1\big(Q^2\big)+G_2\big(Q^2\big)\big)\\&\qquad-\frac{1}{M^2}q^\alpha q^\beta\big(p_1^\mu+p_2^\mu\big)G_3\big(Q^2\big)\bigg].\end{aligned}\tag{38}$$

Comparing the tensor structure of this expression with (17), we conclude that H-model predicts $G_1^{(n)}(Q^2) = G_2^{(n)}(Q^2) = F_{nn}(Q^2)$, and $G_3^{(n)}(Q^2) = 0$ for form factors $G_i^{(n)}(Q^2)$ of nth bound state. It was argued (see [16]) that this is a general feature of AdS/QCD models for the ρ-meson form factors. Since $\mathcal{J}(Q = 0, z) = 1$, the diagonal form factors $F_{nn}(Q^2)$ in the H-model are normalized to unity, while the nondiagonal ones vanish for $Q^2 = 0$ (the functions $\psi_n(z)$ are orthonormal on $[0, z_0]$).

The form factors G_i are related to electric G_C, magnetic G_M and quadrupole G_Q form factors by

$$G_C = G_1 + \frac{Q^2}{6M^2}G_Q, \qquad G_M = G_1 + G_2,$$
$$G_Q = \left(1 + \frac{Q^2}{4M^2}\right)G_3 - G_2. \tag{39}$$

For these form factors, H-model gives

$$G_Q^{(n)}\big(Q^2\big) = -F_{nn}\big(Q^2\big), \qquad G_M^{(n)}\big(Q^2\big) = 2F_{nn}\big(Q^2\big),$$
$$G_C^{(n)}\big(Q^2\big) = \left(1 - \frac{Q^2}{6M^2}\right)F_{nn}\big(Q^2\big). \tag{40}$$

For $Q^2 = 0$, it correctly reproduces the unit electric charge of the meson, and "predicts" $\mu \equiv G_M(0) = 2$ for the magnetic moment and $D \equiv G_Q(0)/M^2 = -1/M^2$ for the quadrupole moment, which are just the canonical values for a pointlike vector particle [20].

Another interesting combination of form factors

$$\mathcal{F}\big(Q^2\big) = G_1\big(Q^2\big) + \frac{Q^2}{2M^2}G_2\big(Q^2\big) - \left(\frac{Q^2}{2M^2}\right)^2 G_3\big(Q^2\big) \tag{41}$$

appears if one takes the "$+++$" component of the 3-point correlator (obtained, e.g., by convoluting it with $n_\alpha n_\beta n_\mu$, where $n^2 = 0$, $(np_1) = 1$, $(nq) = 0$ [15]). The H-model result (37) for $\mathcal{F}(Q^2)$ is particularly simple:

$$\mathcal{F}_{nn}\big(Q^2\big) = \int_0^{z_0} dz\, z\,\mathcal{J}(Q,z)\big|\phi_n(z)\big|^2. \tag{42}$$

Thus, it is the form factors $\mathcal{F}_{nn}(Q^2)$ that are the most direct analogues of diagonal bound state form factors in quantum mechanics.

6. Low-Q^2 behavior

Our expression for $\mathcal{F}_{nn}(Q^2)$ is close to that proposed for a generic meson form factor in the holographic model of Ref. [11]. There, the authors used $\mathcal{K}(Qz) \equiv QzK_1(Qz)$ as the q-channel factor. Indeed, the difference between $\mathcal{J}(Q, z)$ and $\mathcal{K}(Qz)$ is exponentially small when $Qz_0 \gg 1$, but the two functions radically differ in the region of small Q^2, where the function $\mathcal{K}(Qz)$ displays the logarithmic branch singularity

$$\mathcal{K}(Qz) = 1 - \frac{z^2Q^2}{4}\big[1 - 2\gamma_E - \ln\big(Q^2z^2/4\big)\big] + \mathcal{O}\big(Q^4\big), \tag{43}$$

that leads to incorrect infinite slope at $Q^2 = 0$. To implant the AdS/QCD information about the hadron spectrum in the q-channel one should use $\mathcal{J}(Q, z)$ that corresponds to a tower of bound states in the q-channel. The lowest singularity in this case is located at $Q^2 = -M_1^2$. Since it is separated by a finite gap from zero, the form factor slopes at $Q^2 = 0$ are finite.

To analyze the form factor behavior in the $Qz_0 \ll 1$ limit, we expand

$$\mathcal{J}(Q,z)|_{Qz_0\ll 1} = 1 - \frac{z^2Q^2}{4}\left[1 - \ln\frac{z^2}{z_0^2}\right] + \mathcal{O}\big(Q^4\big). \tag{44}$$

As expected, the result is analytic in Q^2. For the lowest transition (i.e., for the ρ-meson form factor), explicit numbers are as follows:

$$\mathcal{F}_{11}\big(Q^2\big) \approx 1 - 0.692\frac{Q^2}{M^2} + 0.633\frac{Q^4}{M^4} + \mathcal{O}\big(Q^6\big), \tag{45}$$

where $M = M_1 = m_\rho$. Another small-Q^2 expansion

$$F_{11}\big(Q^2\big) \approx 1 - 1.192\frac{Q^2}{M^2} + 1.229\frac{Q^4}{M^4} + \mathcal{O}\big(Q^6\big), \tag{46}$$

can be either calculated from the original expression (32) involving ψ-functions or by dividing $\mathcal{F}_{11}(Q^2)$ by $(1 + Q^2/2M^2)$. The latter approach easily explains the difference in slopes of these two form factors at $Q^2 = 0$. Finally, for the electric form factor, we obtain

$$G_C^{(1)}\big(Q^2\big) \approx 1 - 1.359\frac{Q^2}{M^2} + 1.428\frac{Q^4}{M^4} + \mathcal{O}\big(Q^6\big). \tag{47}$$

For the electric radius of the ρ-meson this gives

$$\langle r_\rho^2\rangle_C = 0.53\ \mathrm{fm}^2, \tag{48}$$

the value that is very close to the recent result (0.54 fm^2) obtained within the Dyson–Schwinger equations (DSE) approach [21]. Lattice gauge calculations [22] indicate a similar value in the $m_\pi^2 \to 0$ limit.

7. Vector meson dominance patterns

Numerically, the result $1.359/M^2$ for the slope of $G_C^{(1)}(Q^2)$ is larger than the simple VMD expectation $1/M^2$. In fact, a part of this larger value is due to the factor $(1 - Q^2/6M^2)$ relating $G_C^{(1)}(Q^2)$ and $F_{11}(Q^2)$, which is kinematic to some extent. The $F_{11}(Q^2)$ form factor, however, can be written in the generalized VMD representation (cf. [16])

$$F_{11}(Q^2) = \sum_{m=1}^{\infty} \frac{F_{m,11}}{1 + Q^2/M_m^2}, \tag{49}$$

with the coefficients $F_{m,11}$ given by the overlap integrals

$$F_{m,11} = 4 \int_0^1 d\xi\, \xi^2 \frac{J_1(\gamma_{0,m}\xi) J_1^2(\gamma_{0,1}\xi)}{\gamma_{0,m} J_1^2(\gamma_{0,m}) J_1^2(\gamma_{0,1})}, \tag{50}$$

apparently having a purely dynamical origin. The coefficients $F_{m,11}$ satisfy the sum rule

$$\sum_{m=1}^{\infty} F_{m,11} = 1, \tag{51}$$

that provides correct normalization $F_{11}(Q^2 = 0) = 1$. Numerically, the unity value of the form factor $F_{11}(Q^2)$ for $Q^2 = 0$ is dominated by the first bound state that gives 1.237. The second bound state makes a sizable correction by -0.239, while adding a small 0.002 contribution from the third bound state fine-tunes 1 beyond the 10^{-3} accuracy. Contributions from higher bound states to the form factor normalization are negligible at this precision.

The slope of $F_{11}(Q^2)$ at $Q^2 = 0$ is given by the sum of $F_{m,11}/M_m^2$ coefficients. Now, the dominance of the first bound state is even more pronounced: the Q^2 coefficient $1.192/M^2$ in Eq. (46) is basically contributed by the first bound state that gives $1.237/M^2$, with small $-0.045/M^2$ correction from the second bound state. Other resonances are not visible at the three-digit precision.

Thus, for small Q^2, H-model predicts a rather peculiar pattern of VMD for $F_{11}(Q^2)$ (observed originally in Ref. [16] for a form factor considered there): strong dominance of the first q-channel bound state, whose coupling $F_{1,11}$ exceeds 1, with the second resonance (having the negative coupling $F_{2,11}$) compensating this excess.

Similarly, the $\mathcal{F}_{11}(Q^2)$ form factor has the generalized VMD representation with coefficients $\mathcal{F}_{m,11}$ given by the overlap integrals

$$\mathcal{F}_{m,11} = 4 \int_0^1 d\xi\, \xi^2 \frac{J_1(\gamma_{0,m}\xi) J_0^2(\gamma_{0,1}\xi)}{\gamma_{0,m} J_1^2(\gamma_{0,m}) J_1^2(\gamma_{0,1})}. \tag{52}$$

Now, $\mathcal{F}_{1,11} \approx 0.619$, $\mathcal{F}_{2,11} \approx 0.391$, $\mathcal{F}_{3,11} \approx -0.012$, $\mathcal{F}_{4,11} \approx 0.002$, etc. In this case also, the value of the $\mathcal{F}_{11}(Q^2)$ form factor for $Q^2 = 0$ is dominated by the first two bound states. For the slope of the form factor at $Q^2 = 0$, the dominance of the first bound state is again more pronounced: the Q^2 coefficient $0.692/M^2$ in Eq. (45) is basically contributed by the first bound state that gives $0.619/M^2$, with a small $0.074/M^2$ correction from the second bound state and a tiny $-0.001/M^2$ contribution from the third one.

Thus, for $\mathcal{F}_{11}(Q^2)$, H-model gives again a two-resonance dominance pattern, with the coupling $\mathcal{F}_{2,11}$ of the second resonance being now just somewhat smaller than the coupling $\mathcal{F}_{1,11}$ of the first resonance, both being positive. The relation between the two VMD patterns follows from Eq. (37):

$$F_{m,11} = \frac{\mathcal{F}_{m,11}}{1 - M_m^2/2M_1^2}. \tag{53}$$

In particular, it gives $F_{1,11} = 2\mathcal{F}_{1,11}$, and negative sign for $F_{2,11}$. It also determines that if higher coefficients $\mathcal{F}_{m,11}$ are small then $F_{m,11}$'s are even smaller.

8. Large-Q^2 behavior

Eq. (37) tells us that asymptotically $F_{11}(Q^2)$ is suppressed by a power of $1/Q^2$ compared to $\mathcal{F}_{11}(Q^2)$, which is known to behave like $1/Q^2$ for large Q^2 [11,15]. The absence of $1/Q^2$ term in the asymptotic expansion for $F_{11}(Q^2)$ means that the coefficients $F_{m,11}$ defined in Eq. (49) satisfy the "superconvergence" relation

$$\sum_{m=1}^{\infty} M_m^2 F_{m,11} = 0, \tag{54}$$

reflecting a "conspiracy" [16] between the poles. Writing $M_m^2 F_{m,11} \equiv A_m M^2$, we obtain that $A_1 \approx 1.237$, $A_2 \approx -1.261$, $A_3 \approx 0.027$ (our results for the ratios A_2/A_1, A_3/A_1 agree with the calculation of Ref. [16]). Again, the sum rule is practically saturated by the first two bound states, which give contributions that are close in magnitude but opposite in sign.

In case of $\mathcal{F}(Q^2)$, the two lowest bound states both give positive $\mathcal{O}(1/Q^2)$ contributions at large Q^2. In Ref. [15], it was shown that the asymptotic normalization of $\mathcal{F}_{11}(Q^2)$ exceeds the VMD expectation M_1^2/Q^2 by a factor of 2.566. We can infer this normalization from the values of the coefficients $\mathcal{F}_{m,11}$ defined in Eq. (52). Writing $M_m^2 \mathcal{F}_{m,11} \equiv \mathcal{A}_m M_1^2$, we obtain that $\mathcal{A}_1 \approx 0.619$, $\mathcal{A}_2 \approx 2.061$, $\mathcal{A}_3 \approx -0.150$, $\mathcal{A}_4 \approx 0.054$. Note, that the total result is dominated by the second bound state, which is responsible for about 80% of the value. The lowest bound state contributes only about 25%, while the higher states give just small corrections.

It is worth noting that the large-Q^2 behavior of $\mathcal{F}_{11}(Q^2)$ is determined by the large-Qz_0 form of $\mathcal{J}(Q, z)$: it can be (and was) calculated using $\mathcal{K}(Qz)$, the free-field version of $\mathcal{J}(Q, z)$. As a result, the value of the asymptotic coefficient (2.566 in case of $\mathcal{F}_{11}(Q^2)$) is settled by the sum rule

$$\sum_{m=1}^{\infty} M_m^2 \mathcal{F}_{m,11} = |\phi_1(0)|^2 \int_0^{\infty} d\chi\, \chi^2 K_1(\chi) = 2|\phi_1(0)|^2, \tag{55}$$

that should be satisfied by any set of coefficients $\mathcal{F}_{m,11}$. A particular distribution of "2.566" among the bound states is governed by the specific q-channel dynamics (in our case, by the choice of the Neumann b.c. for $\mathcal{J}(Q, z)$ at $z = z_0$). Thus, in the dynamics described by $\mathcal{J}(Q,z)$, the large value of the asymptotic coefficient is explained by large contribution due to the second bound state.

It was shown in Ref. [15] that the asymptotic $1/Q^2$ behavior for $\mathcal{F}_{11}(Q^2)$ is established only for $Q^2 \sim 10\ \text{GeV}^2$, and one may question the applicability of the H-model for such large Q^2. The discussion of this problem, however, is beyond the scope of the present Letter.

9. Summary

In this Letter, we described the formalism that allows to study form factors of vector mesons in the holographic QCD model of Refs. [6–8] (H-model). An essential ingredient of our approach is a systematic use of the Kneser–Sommerfeld representation that explicitly displays the poles of two- and three-point functions and describes the structure of the corresponding bound states by eigenfunctions of the 5D equation of motion, the "ψ wave functions". These functions vanish at $z=0$ and satisfy Neumann b.c. at $z=z_0$, which prevents a direct analogy with bound state wave functions in quantum mechanics. To this end, we introduced an alternative description in terms of "ϕ wave functions" that satisfy Dirichlet b.c. at $z=z_0$ and have finite values at $z=0$ which determine bound state couplings $g_5 f_n/M_n$. Thus, the ϕ wave functions have the properties necessary for the light-cone interpretation proposed in Ref. [11] and discussed also in Ref. [15].

Analyzing the three-point function, we derived expressions for bound state form factors both in terms of ψ and ϕ wave functions, and obtained specific predictions for form factor behavior at small and large values of the invariant momentum transfer Q^2. In particular, we calculated the electric radius of the ρ meson, and obtained the value $\langle r_\rho^2 \rangle_C = 0.53\ \text{fm}^2$ that practically coincides with the recent result [21] obtained within the DSE approach. Our result is also consistent with the $m_\pi^2 \to 0$ extrapolation of the recent lattice gauge calculation [22].

We derived a generalized VMD representation both for the $F_{11}(Q^2)$ form factor (the expression for which coincides with a model ρ-meson form factor considered in Ref. [16]) and for the $\mathcal{F}_{11}(Q^2)$ form factor introduced in the present Letter, and demonstrated that H-model predicts a very specific VMD pattern, in which these form factors are essentially given by contributions due to the first two bound states in the Q^2-channel, with the higher bound states playing a negligible role. We showed that, while the form factor slopes at $Q^2=0$ in this picture are dominated by the first bound state, the second bound state plays a crucial role in the large-Q^2 asymptotic limit. In particular, it provides the bulk part of the negative contribution necessary to cancel the naïve VMD $1/Q^2$ asymptotics for the $F_{11}(Q^2)$ form factor (corresponding to the overlap integral involving the ψ functions), and it dominates the asymptotic $1/Q^2$ behavior of the $\mathcal{F}(Q^2)$ form factor (given by the overlap of the ϕ functions).

A possible future application of our approach is the analysis of bound state form factors in the model of Ref. [12] in which the hard-wall boundary conditions at the $z=z_0$ IR boundary are substituted by an oscillator-type potential. This model provides the $M_n^2 \sim n\Lambda^2$ asymptotic behavior of the spectrum of highly excited mesons, which is more consistent with the semiclassical limit of QCD [23] than the $M_n^2 \sim n^2\Lambda^2$ result of the H-model.

Acknowledgements

H.G. would like to thank J. Erlich for illuminating discussions, A.W. Thomas for valuable comments and support, J.L. Goity and R.J. Crewther for stimulating conversations, J.P. Draayer for support at Louisiana State University, and G.S. Pogosyan and S.I. Vinitsky for support at JINR, Dubna. A.R. thanks J.J. Dudek for attracting attention to Ref. [22].

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